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## **Analysis of Thin Elastic Shallow Segmental Shells**

*Etude des voiles minces élastiques à faible courbure*

*Berechnung von dünnen und flachen elastischen Segmentschalen*

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### **Introduction**

An approximate procedure of stress analysis is presented in this treatise for elastic thin shallow segmental shells of constant thickness. The usual limitations imposed upon the thin shell behavior and material by the classical shell theory are also assumed to be applicable in the following development.

Segmental shells find frequent practical application in structural and mechanical engineering. Many of these shells possess geometrical proportions which are consistent with the fundamental assumptions of the theory of shallow shells and therefore admit a simplified procedure of stress analysis. In general the proposed solution is quite appropriate for shallow segmental shells which have their middle surfaces described by generators that are shallow segments of second degree curves. The solution is developed from the linear generalized shallow shell equations, which were formulated by MUSHTARI<sup>1)</sup> and VLASOV<sup>2)</sup> and recently used by the author in the treatment of spherical calotte shells<sup>3)</sup>.

VLASOV has pointed out that for most shallow thin shells the curvatures of the parametric coordinate lines can be treated as approximately constant. This simplification perpetrates some errors into the final solution, which however are of minor importance for most practical cases. He also came to the conclusion that this approach was quite tenable for the stress analysis of shallow thin shells with a rise to span ratio  $\leq \frac{1}{5}$ .

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<sup>1)</sup> Izvestiya Fiziko-Matematicheskovo Obshchestva pri Kazanskom Universitete, 11, 1938, Series 8.

<sup>2)</sup> Obshchaya Teoriya Obolochek i ee Prilozheniya v Tekhnike, Moscow, 1949.

<sup>3)</sup> Stress and Strain in Thin Shallow Spherical Calotte Shells, Publ. Int. Assoc. Bridge and Struct. Engng., Zurich, v. 17, 1957, pp. 139—160.

## Fundamental Solution for Shallow Shells

### *Geometry of Shallow Shells of Rotation*

The fundamental quadratic form of the middle surface of rotational shallow shells in cylindrical manifold is given by

$$ds^2 = dr^2 + r^2 d\theta + dz^2. \quad (1)$$

An element of the shell's middle surface is measured by coordinates  $r$  and  $\theta$  which designate the radius and its angle from a reference position embedded in the base plane and by coordinate  $z$  normal to the base plane enclosed by the shell.

Coordinate  $z$  of a rotationally symmetrical shell can be expressed as a function of radius  $r$

$$z = f(r) = f.$$

Then

$$dz = \frac{df}{dr} dr = f' dr.$$

Equation (1) becomes

$$ds^2 = [1 + (f')^2] dr^2 + r^2 d\theta^2. \quad (2)$$

The normal curvatures of the coordinate lines for the shallow rotational shell are given approximately by

$$\frac{1}{R_r} \doteq \pm \frac{\frac{d^2 z}{dr^2}}{\left[1 + \left(\frac{dz}{dr}\right)^2\right]^{3/2}} \quad (3)$$

and

$$\frac{1}{R_\theta} \doteq \pm \frac{\frac{dz}{dr}}{r \left[1 + \left(\frac{dz}{dr}\right)^2\right]^{1/2}},$$

where  $R_r$  and  $R_\theta$  designate the radii of the parametric coordinate lines  $\theta = \text{constant}$  and  $r = \text{constant}$  respectively.

At this stage the restriction

$$|f'|^2 \ll 1 \quad (4)$$

is proposed as a measure of the shallowness for the shell of rotation.

If (4) is assumed to be valid then (2) becomes

$$ds^2 \doteq dr^2 + r^2 d\theta^2 \quad (5)$$

and the approximate normal curvatures (3) are expressible by

$$\begin{aligned} \frac{1}{R_r} &\doteq \pm f'', \\ \frac{1}{R_\theta} &\doteq \pm \frac{f'}{r}, \end{aligned} \quad (6)$$

where

$$f'' = \frac{d^2 f}{dr^2}.$$

### Geometry of Shallow Rotational Shells Generated by Second Degree Curves:

When the shell's generator is a segment of a second degree curve

$$z = f(r) = I(1 - Sr^2), \quad (7)$$

then

$$f'' = -2IS$$

and according to (6)

$$\begin{aligned} \frac{1}{R_r} &\doteq \pm 2IS = \frac{1}{R} = c, \\ \frac{1}{R_\theta} &\doteq \pm 2IS = \frac{1}{R} = c, \end{aligned} \quad (8)$$

where  $R$  denotes a constant radius and  $c$  a constant.

Hence in case of shallow shells whose middle surfaces have been generated by shallow segments of curve (7) and which obey restriction (4), the principal curvatures are approximately equal, i. e.

$$\frac{1}{R_r} \doteq \frac{1}{R_\theta} \doteq \frac{1}{R}. \quad (9)$$

### Geometry of Shallow Spherical Shell:

The middle surface of the spherical shell is given in MONGE's form by

$$z = f(r) = R \left[ \left( 1 - \frac{r^2}{R^2} \right)^{1/2} - 1 \right] + H,$$

where  $H$  designates the crown height of the shell measured normal to the base enclosed by the shell and  $R$  denotes the constant radius of its middle surface.

Then

$$f' = \frac{dz}{dr} = - \frac{\frac{r}{R}}{\left( 1 + \frac{r^2}{R^2} \right)^{1/2}}.$$

For spherical segments the following order of magnitude relationship can be specified as a measure of shallowness

$$\left| \frac{r}{R} \right|^2 \ll 1.$$

Consequently

$$|f'|^2 \doteq \left| \frac{r}{R} \right|^2 \ll 1,$$

hence the line segment (2) can again be represented approximately by

$$ds^2 \doteq dr^2 + r^2 d\theta^2$$

and the principal normal curvatures are obviously

$$\frac{1}{R_r} = \frac{1}{R_\theta} = \frac{1}{R} = c. \quad (10)$$

*Solution for Segmental Shells*

As is evident upon inspection of the previous section shells of rotation generated by shallow segments of circular or second degree curves admit approximately constant radii of principal curvatures, i. e.

$$\frac{1}{R_r} = \frac{1}{R_\theta} = c,$$

and their middle surface can be described sufficiently accurately by

$$ds^2 = dr^2 + r^2 d\theta^2.$$

In this case the fundamental differential equation for such shallow shells can be expressed by<sup>4)</sup>

$$\nabla^2 \left[ \nabla^2 V - i \frac{c \sqrt{12(1-\nu^2)}}{h} V \right] = \frac{p_n}{D} - \frac{2c}{D} \Gamma - i \omega \nabla^2 [(1-\nu) \Gamma + E h (\gamma T)], \quad (11)$$

where

$\Gamma$  External load intensity potential function.

$p_\theta = -\frac{1}{r} \frac{\partial \Gamma}{\partial \theta}$  Circumferential component of load intensity.

$p_r = -\frac{\partial \Gamma}{\partial r}$  Radial component of load intensity.

$p_n$  Normal component of load intensity.

$\gamma$  Coefficient of linear thermal expansion.

$T = T(r, \theta)$  Temperature distribution function that describes the differential thermal increase or decrease from a stress and strain free temperature level of the shell, bringing about an extension or contraction of shell's middle surface.

$\nu$  POISSON'S ratio.

$E$  YOUNG'S modulus.

$D = \frac{E h^3}{12(1-\nu^2)}$  Cylindrical flexural rigidity.

$h$  Constant shell thickness.

$V = w + i \left( \frac{\sqrt{12(1-\nu^2)}}{E h^2} \right) F.$

$w$  Normal displacement function.

$F$  Stress function.

$i = \sqrt{-1},$

$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$

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<sup>4)</sup> See ref. [3], p. 143.

For shallow shells the loading intensities  $p_r$  and  $p_\theta$  are mostly of minor importance, hence they can be suppressed by setting

$$\Gamma = 0.$$

As the segmental shells considered here are assumed closed at the apex, the following solution satisfies equation (11) when  $\Gamma=0$  and  $T(r, \theta) = T_0 = \text{constant}$ <sup>5)</sup>

$$w = \frac{p_0}{E h c^2} + \frac{\gamma_0 T_0}{c} + A_0^1 \text{ber } \mu + A_0^2 \text{bei } \mu + E_0^1 + \sum_{n=1}^{\infty} [A_n^1 \text{ber}_{mn} \mu + A_n^2 \text{bei}_{mn} \mu + C_n^1 r^n] \sin m n \theta \quad (12)$$

$$\text{and } F = \frac{p_0 r^2}{4 c} + \frac{E h^2}{\sqrt{12(1-\nu^2)}} \left\{ A_0^1 \text{bei } \mu - A_0^2 \text{ber } \mu + E_0^2 + \sum_{n=1}^{\infty} [A_n^1 \text{bei}_{mn} \mu - A_n^2 \text{ber}_{mn} \mu + C_n^2 r^{mn}] \right\} \sin m n \theta, \quad (13)$$

where  $m$  Free constant to be evaluated from meridional boundary conditions.

$p_n = p_0$  Constant normal component of load intensity.

$A_n^1, A_n^2, C_n^1, C_n^2, E_n^1, E_n^2$  Integration constants.

$$\mu = \lambda r,$$

$$\lambda = \frac{c \sqrt{12(1-\nu^2)}}{h}.$$

### Sectional Quantities

In the light of the coordinate system used for the approximate analysis of shallow segmental shells, the stress resultants and stress couples appear in the following form<sup>6)</sup>:

$$\begin{aligned} N_{rr} &= \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r} \frac{\partial F}{\partial r}, \\ N_{\theta\theta} &= \frac{\partial^2 F}{\partial r^2}, \\ N_{r\theta} &= N_{\theta r} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial F}{\partial \theta} \right), \\ Q_r &= -D \frac{\partial}{\partial r} (\nabla^2 w), \\ Q_\theta &= -D \frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 w), \end{aligned} \quad (14)$$

<sup>5)</sup> See ref. [3].

<sup>6)</sup> See ref. [3].

$$\begin{aligned}
 M_{rr} &= -D \left[ \frac{\partial^2 w}{\partial r^2} + \nu \left( \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \right], \\
 M_{\theta\theta} &= -D \left[ \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \nu \left( \frac{\partial^2 w}{\partial r^2} \right) \right], \\
 M_{\theta r} &= M_{r\theta} = -D(1-\nu) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} \right).
 \end{aligned} \tag{14}$$

See fig. 1.

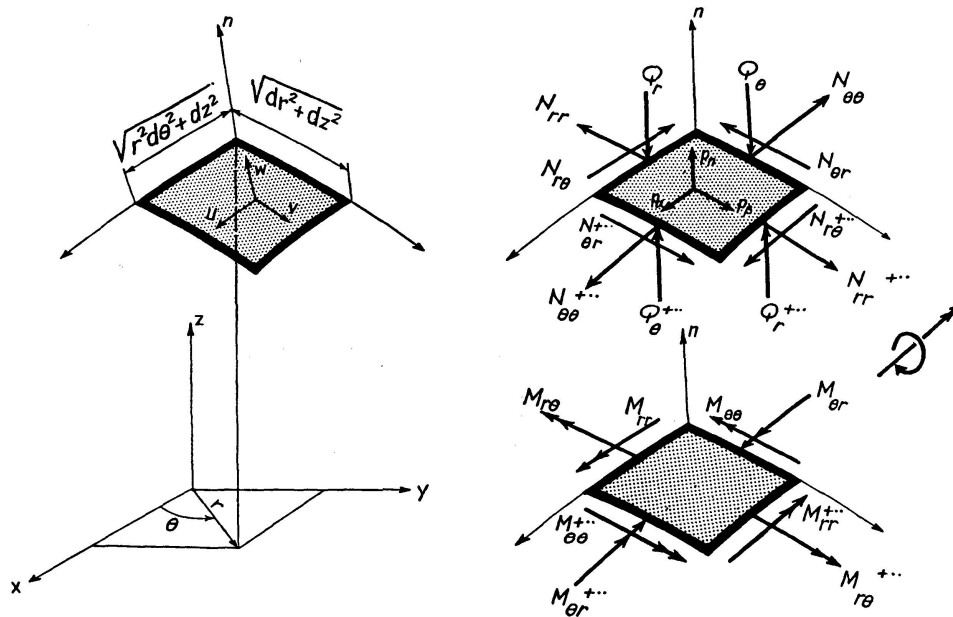


Fig. 1. Differential Element of Thin Shell Showing Components of Surface Load Intensities, Stress Resultants, Stress Couples, Coordinates and Displacements.

### Boundary Conditions Pertaining to Segmental Shells

#### *Segmental Shells Supported at its Periphery*

A segmental shell shown in fig. 2 that is simply supported at its meridional and circumferential edges admits the following practical boundary conditions:

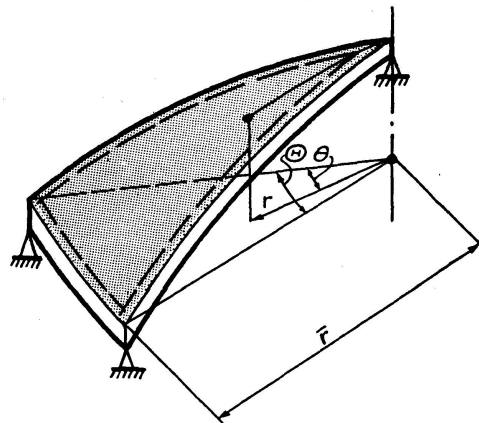


Fig. 2. Segmental Shell Supported at Three Edges.

At  $\theta = \Theta, 0$ :

$$N_{\theta\theta} = 0, \quad (15)$$

$$M_{\theta\theta} = 0, \quad (16)$$

$$w = 0, \quad (17)$$

$$\epsilon_r = 0, \quad (18)$$

where

$$\epsilon_r = \frac{N_{rr} - \nu N_{\theta\theta}}{E h}$$

designates the radial strain in the shell. Boundary conditions (15) to (18) can all be satisfied by

$$n m \Theta = n \pi, \quad (n = 1, 3, 5 \dots).$$

Hence

$$m = \frac{\pi}{\Theta}$$

and

$$n = (2k + 1), \quad (k = 0, 1, 2 \dots).$$

At  $r = \bar{r}$ :

$$N_{rr} + r c Q_r = 0$$

or more approximately

$$N_{rr} = 0, \quad (19)$$

$$M_{rr} = 0, \quad (20)$$

$$\epsilon_\theta = 0, \quad (21)$$

$$w = 0, \quad (22)$$

where

$$\epsilon_\theta = \frac{N_{\theta\theta} - \nu N_{rr}}{E h}$$

designates the circumferential strain in the shell.

In order to deal more effectively with the boundary conditions at  $r = \bar{r}$  particular solutions contained in (12) and (13) are expanded into FOURIER series in terms of  $\theta$ , whence

$$w_0 = \frac{4}{\pi} \left[ \frac{p_0}{E h c^2} + \frac{\gamma T_0}{c} \right] \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin \frac{(2k+1)\pi}{\Theta} \theta, \quad (23)$$

$$F_0 = \frac{p_0 r^2}{\pi c} \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin \frac{(2k+1)\pi}{\Theta} \theta. \quad (24)$$

Introduction of series (23) and (24) in (12) and (13) respectively yields

$$w = \sum_{k=0}^{\infty} \left\{ \left( \frac{4 p_0}{\pi E h c^2} + \frac{4 \gamma T_0}{\pi c} \right) \frac{1}{(2k+1)} + A_k^1 \text{ber}_{\bar{m}} \mu + A_k^2 \text{bei}_{\bar{m}} \mu + C_k^1 r^{\bar{m}} \right\} \sin \bar{m} \theta \quad (25)$$

and

$$F = \sum_{k=0}^{\infty} \left\{ \left( \frac{p_0 r^2}{\pi c} \right) \frac{1}{(2k+1)} + \frac{E h^2}{\sqrt{12(1-\nu^2)}} [A_k^1 \text{bei}_{\bar{m}} \mu - A_k^2 \text{ber}_{\bar{m}} \mu + C_k^2 r^{\bar{m}}] \right\} \sin \bar{m} \theta, \quad (26)$$

where 
$$\bar{m} = \frac{(2k+1)\pi}{\Theta}.$$

Insertion of the series solutions (25) and (26) into boundary conditions (19)—(22) and making use of (14) gives four linear simultaneous equations

$$A_k^1 \psi_1 + A_k^2 \psi_2 + C_k^2 \psi_3 = \Psi_1, \quad (27)$$

$$A_k^1 \psi_4 + A_k^2 \psi_5 + C_k^1 \psi_6 = \Psi_2, \quad (28)$$

$$A_k^1 \psi_7 + A_k^2 \psi_8 + C_k^2 \psi_9 = \Psi_3, \quad (29)$$

$$A_k^1 \psi_{10} + A_k^2 \psi_{11} + C_k^1 \psi_{12} = \Psi_4, \quad (30)$$

where

$$\psi_1 = -\frac{1}{\omega} \left( \frac{\bar{m}}{r} \right)^2 \text{bei}_{\bar{m}} \bar{\mu} + \frac{1}{\omega} \left( \frac{\lambda}{r} \right) \text{bei}'_{\bar{m}} \bar{\mu},$$

$$\psi_2 = \frac{1}{\omega} \left( \frac{\bar{m}}{r} \right)^2 \text{ber}_{\bar{m}} \bar{\mu} - \frac{1}{\omega} \left( \frac{\lambda}{r} \right) \text{ber}'_{\bar{m}} \bar{\mu},$$

$$\psi_3 = \frac{\bar{m}}{\omega} (1 - \bar{m}) \bar{r}^{\bar{m}-2},$$

$$\psi_4 = \nu \left[ \left( \frac{\lambda}{r} \right) \text{ber}'_{\bar{m}} \bar{\mu} - \left( \frac{\bar{m}}{r} \right)^2 \text{ber}_{\bar{m}} \bar{\mu} \right] + \lambda^2 \text{ber}''_{\bar{m}} \bar{\mu},$$

$$\psi_5 = \nu \left[ \left( \frac{\lambda}{r} \right) \text{bei}'_{\bar{m}} \bar{\mu} - \left( \frac{\bar{m}}{r} \right)^2 \text{bei}_{\bar{m}} \bar{\mu} \right] + \lambda^2 \text{bei}''_{\bar{m}} \bar{\mu},$$

$$\psi_6 = (1 - \nu) \bar{m} (\bar{m} - 1) \bar{r}^{\bar{m}-2},$$

$$\psi_7 = \left( \frac{\lambda^2}{\omega} \right) \text{bei}''_{\bar{m}} \bar{\mu},$$

$$\psi_8 = -\left( \frac{\lambda^2}{\omega} \right) \text{ber}''_{\bar{m}} \bar{\mu},$$

$$\psi_9 = \frac{\bar{m} (\bar{m} - 1)}{\omega} \bar{r}^{\bar{m}-2},$$

$$\psi_{10} = \text{ber}_{\bar{m}} \bar{\mu},$$

$$\psi_{11} = \text{bei}_{\bar{m}} \bar{\mu},$$

$$\psi_{12} = \bar{r}^{\bar{m}},$$

$$\Psi_1 = \frac{p_0}{\pi c} \left[ (2k+1) \left( \frac{\pi}{\Theta} \right)^2 - \frac{2}{(2k+1)} \right],$$

$$\Psi_2 = \frac{4\nu}{\pi \bar{r}^2} \left[ \frac{p_0}{E h c^2} + \frac{\gamma T_0}{c} \right],$$

$$\Psi_3 = -\frac{2}{(2k+1)} \frac{p_0}{\pi c},$$

$$\begin{aligned}\Psi_4 &= -\frac{4}{\pi} \left[ \frac{p_0}{E h c^2} + \frac{\gamma T_0}{c} \right] \frac{1}{(2k+1)}, \\ \bar{\mu} &= \lambda \bar{r}, \\ \text{ber}_{\bar{m}}'' \mu &= \frac{d^2}{d\mu^2} (\text{ber}_{\bar{m}} \mu) = -\frac{1}{\mu} \text{ber}_{\bar{m}}' \mu + \left( \frac{\bar{m}}{\mu} \right)^2 \text{ber}_{\bar{m}} \mu - \text{bei}_{\bar{m}} \mu, \\ \text{bei}_{\bar{m}}'' \mu &= \frac{d^2}{d\mu^2} (\text{bei}_{\bar{m}} \mu) = -\frac{1}{\mu} \text{bei}_{\bar{m}}' \mu + \left( \frac{\bar{m}}{\mu} \right)^2 \text{bei}_{\bar{m}} \mu + \text{ber}_{\bar{m}} \mu.\end{aligned}$$

Solving equations (27), (28), (29) and (30) for the integration constants yields:

$$A_k^1 = \frac{D_{Ak}^1}{D}, \quad A_k^2 = \frac{D_{Ak}^2}{D}, \quad C_k^1 = \frac{D_{Ck}^1}{D}, \quad C_k^2 = \frac{D_{Ck}^2}{D}, \quad (31)$$

where

$$\begin{aligned}D_{Ak}^1 &= -\psi_3 \{ \Psi_2 \psi_8 \psi_{12} + \Psi_3 \psi_6 \psi_{11} - \Psi_4 \psi_8 \psi_6 - \Psi_3 \psi_5 \psi_{12} \} - \\ &\quad - \psi_9 \{ \Psi_1 \psi_5 \psi_{12} + \Psi_4 \psi_2 \psi_6 - \Psi_1 \psi_6 \psi_{11} - \Psi_2 \psi_2 \psi_{12} \}, \\ D_{Ak}^2 &= -\psi_3 \{ \Psi_3 \psi_4 \psi_{12} + \Psi_4 \psi_6 \psi_7 - \Psi_3 \psi_6 \psi_{10} - \Psi_2 \psi_7 \psi_{12} \} - \\ &\quad - \psi_9 \{ \Psi_2 \psi_1 \psi_{12} + \Psi_1 \psi_6 \psi_{10} - \Psi_4 \psi_1 \psi_6 - \Psi_1 \psi_4 \psi_{12} \}, \\ D_{Ck}^1 &= -\psi_3 \{ \Psi_4 \psi_4 \psi_8 + \Psi_3 \psi_5 \psi_{10} + \Psi_2 \psi_7 \psi_{11} - \Psi_2 \psi_8 \psi_{10} - \Psi_3 \psi_4 \psi_{11} - \Psi_4 \psi_5 \psi_7 \} - \\ &\quad - \psi_9 \{ \Psi_4 \psi_5 \psi_1 + \Psi_2 \psi_2 \psi_{10} + \Psi_1 \psi_4 \psi_{11} - \Psi_1 \psi_5 \psi_{10} - \Psi_2 \psi_1 \psi_{11} - \Psi_4 \psi_2 \psi_4 \}, \\ D_{Ck}^2 &= -\psi_6 \{ \Psi_4 \psi_1 \psi_8 + \Psi_3 \psi_2 \psi_{10} + \Psi_1 \psi_7 \psi_{11} - \Psi_1 \psi_8 \psi_{10} - \Psi_3 \psi_1 \psi_{11} - \Psi_4 \psi_2 \psi_7 \} - \\ &\quad - \psi_{12} \{ \Psi_3 \psi_1 \psi_5 + \Psi_2 \psi_2 \psi_7 + \Psi_1 \psi_4 \psi_8 - \Psi_1 \psi_5 \psi_7 - \Psi_2 \psi_1 \psi_8 - \Psi_3 \psi_2 \psi_4 \}, \\ \text{and} \quad D &= -\psi_3 \{ \psi_{12} [\psi_4 \psi_8 - \psi_5 \psi_7] + \psi_6 [\psi_7 \psi_{11} - \psi_8 \psi_{10}] \} - \\ &\quad - \psi_9 \{ \psi_1 [\psi_5 \psi_{12} - \psi_6 \psi_{11}] + \psi_2 [\psi_6 \psi_{10} - \psi_4 \psi_{12}] \}.\end{aligned}$$

Once the constants  $A_k^1$ ,  $A_k^2$ ,  $C_k^1$  and  $C_k^2$  have been evaluated as indicated above the problem of the segmental shell has been formally solved.

### *Segmental Shells Supported only at the Meridional Edges*

The shallow segmental shell shown in fig. 3 is supported only along the meridional edges  $\theta=0$  and  $\theta=\Theta$ . At the parallel boundary  $r=\bar{r}$  a reaction capable edge member has been omitted.

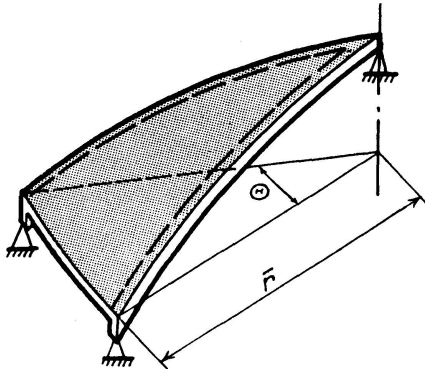


Fig. 3. Segmental Shell Supported at Meridional Edges.

Boundary conditions at the parallel edge are modified to imply a free edge while the boundary conditions (15)–(18) at the meridional edges are satisfied by solution series (25) and (26).

Hence at  $r = \bar{r}$ :

$$N_{rr} = 0, \quad (32)$$

$$M_{rr} = 0, \quad (33)$$

$$Q_r + \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta} = 0, \quad (34)$$

$$N_{r\theta} = 0. \quad (35)$$

Substitution of series solutions (25) and (26) in boundary conditions (32) to (35) and applying relationships (14) yields

$$A_k^1 \psi_1 + A_k^2 \psi_2 + C_k^2 \psi_3 = \Psi_1, \quad (36)$$

$$A_k^1 \psi_4 + A_k^2 \psi_5 + C_k^1 \psi_6 = \Psi_2, \quad (37)$$

$$A_k^1 \psi_{13} + A_k^2 \psi_{14} + C_k^1 \psi_{15} = 0, \quad (38)$$

$$A_k^1 \psi_{16} + A_k^2 \psi_{17} + C_k^2 \psi_{18} = 0, \quad (39)$$

where  $\psi_{13} = -\lambda^3 \text{ber}_{\bar{m}}''' \bar{\mu} + \left(\frac{\bar{m}}{\bar{r}}\right)^2 \lambda \text{ber}_{\bar{m}}' \bar{\mu} - 2 \left(\frac{\bar{m}^2}{\bar{r}^3}\right) \text{ber}_{\bar{m}} \bar{\mu} - \left(\frac{\lambda^2}{\bar{r}}\right) \text{ber}_{\bar{m}}'' \bar{\mu} +$

$$+ \frac{\lambda}{\bar{r}^2} \text{ber}_{\bar{m}}' \bar{\mu} + (1 - \nu) \bar{m} \left[ \left(\frac{\lambda}{\bar{r}^2}\right) \text{bei}_{\bar{m}}' \bar{\mu} - \frac{1}{\bar{r}^3} \text{bei}_{\bar{m}} \bar{\mu} \right],$$

$$\psi_{14} = -\lambda^3 \text{bei}_{\bar{m}}''' \bar{\mu} - \left(\frac{\lambda^2}{\bar{r}}\right) \text{bei}_{\bar{m}}'' \bar{\mu} + \left(\frac{\lambda}{\bar{r}^2}\right) \text{bei}_{\bar{m}}' \bar{\mu} + \left(\frac{\bar{m}}{\bar{r}}\right)^2 \lambda \text{bei}_{\bar{m}} \bar{\mu} +$$

$$+ 2 \left(\frac{\bar{m}^2}{\bar{r}^3}\right) \text{bei}_{\bar{m}} \bar{\mu} + (1 - \nu) \bar{m} \left[ \frac{\lambda}{\bar{r}^2} \text{bei}_{\bar{m}}' \bar{\mu} - \frac{1}{\bar{r}^3} \text{bei}_{\bar{m}} \bar{\mu} \right],$$

$$\psi_{15} = (1 - \nu) \bar{m} (\bar{m} - 1) \bar{r}^{\bar{m}-3},$$

$$\psi_{16} = \left(\lambda \frac{\bar{m}}{\bar{r}}\right) \text{bei}_{\bar{m}}' \bar{\mu} - \left(\frac{\bar{m}}{\bar{r}^2}\right) \text{bei}_{\bar{m}} \bar{\mu},$$

$$\psi_{17} = -\lambda \frac{\bar{m}}{\bar{r}} \text{ber}_{\bar{m}}' \bar{\mu} + \frac{\bar{m}}{\bar{r}^2} \text{ber}_{\bar{m}} \bar{\mu},$$

$$\psi_{18} = \bar{m} (\bar{m} - 1) \bar{r}^{\bar{m}-2},$$

$$\text{ber}_{\bar{m}}''' \mu = \frac{d^3}{d\mu^3} (\text{ber}_{\bar{m}} \mu) = \frac{\bar{m}^2 + 2}{\mu^2} \text{ber}_{\bar{m}}' \mu - \text{bei}_{\bar{m}}' \mu - 2 \left(\frac{\bar{m}^2}{\mu^3}\right) \text{ber}_{\bar{m}} \mu + \frac{1}{\mu} \text{bei}_{\bar{m}} \mu,$$

$$\text{bei}_{\bar{m}}''' \mu = \frac{d^3}{d\mu^3} (\text{bei}_{\bar{m}} \mu) = \frac{\bar{m}^2 + 2}{\mu^2} \text{bei}_{\bar{m}}' \mu + \text{ber}_{\bar{m}}' \mu - 2 \left(\frac{\bar{m}^2}{\mu^3}\right) \text{bei}_{\bar{m}} \mu - \frac{1}{\mu} \text{ber}_{\bar{m}} \mu.$$

The solution of linear simultaneous equations (36), (37), (38) and (39) for integration constants yields:

$$A_k^1 = \frac{\bar{D}_{Ak}^1}{\bar{D}}, \quad A_k^2 = \frac{\bar{D}_{Ak}^2}{\bar{D}}, \quad C_k^1 = \frac{\bar{D}_{Ck}^1}{\bar{D}}, \quad C_k^2 = \frac{\bar{D}_{Ck}^2}{\bar{D}}, \quad (40)$$

where

$$\begin{aligned}
 \bar{D}_{Ak}^1 &= \psi_3 [\Psi_2 \psi_{15} \psi_{17}] + \psi_{18} [\Psi_1 (\psi_5 \psi_{15} - \psi_6 \psi_{14}) - \Psi_2 \psi_4 \psi_{15}], \\
 \bar{D}_{Ak}^2 &= -\psi_3 [\Psi_2 \psi_{15} \psi_{16}] + \psi_{18} [\Psi_2 \psi_1 \psi_{15} + \Psi_1 (\psi_6 \psi_{13} - \psi_4 \psi_{15})], \\
 \bar{D}_{Ck}^1 &= -\Psi_2 \psi_3 \psi_{16} [\psi_{13} - \psi_{14}] + \psi_{18} [\Psi_2 (\psi_2 \psi_{13} - \psi_1 \psi_{14}) + \Psi_1 (\psi_4 \psi_{14} - \psi_5 \psi_{13})], \\
 \bar{D}_{Ck}^2 &= -\Psi_1 [\psi_{15} (\psi_5 \psi_{16} - \psi_4 \psi_{17}) + \psi_6 (\psi_3 \psi_{17} - \psi_{14} \psi_{16})] + \Psi_2 \psi_{15} [\psi_2 \psi_{16} - \psi_1 \psi_{17}], \\
 \text{and } \bar{D} &= -\psi_3 [\psi_{15} (\psi_5 \psi_{16} - \psi_4 \psi_{17}) + \psi_6 (\psi_{13} \psi_{17} - \psi_4 \psi_{16})] + \\
 &\quad + \psi_{16} [\psi_1 (\psi_5 \psi_{15} - \psi_6 \psi_{14}) + \psi_2 (\psi_6 \psi_{13} - \psi_4 \psi_{15})].
 \end{aligned}$$

Substitution of the calculated integration constants (40) in series (25) and (26) yields the normal displacement function and stress function that are sufficient to describe the state of stress and strain prevalent in the skin of the shell subject to surface loading.

### Sectional Quantities

When solution functions (25) and (26) are introduced in relations (14) the following sectional stress resultants and stress couples result:

$$\begin{aligned}
 N_{rr} &= \sum_{k=0}^{\infty} \left\{ -\frac{p_0}{\pi c} \left[ (2k+1) \left( \frac{\pi}{\Theta} \right)^2 - \frac{2}{(2k+1)} \right] + A_k^1 \left[ -\frac{1}{\omega} \left( \frac{\bar{m}}{r} \right)^2 \text{bei}_{\bar{m}} \mu + \right. \right. \\
 &\quad \left. \left. + \frac{1}{\omega} \left( \frac{\lambda}{r} \right) \text{bei}'_{\bar{m}} \mu \right] + A_k^2 \left[ \frac{1}{\omega} \left( \frac{\bar{m}}{r} \right)^2 \text{ber}_{\bar{m}} \mu - \frac{1}{\omega} \left( \frac{\lambda}{r} \right) \text{ber}'_{\bar{m}} \mu \right] + \right. \\
 &\quad \left. + C_k^2 \left[ \frac{\bar{m}}{\omega} (1 - \bar{m}) r^{\bar{m}-2} \right] \right\} \sin \bar{m} \theta, \\
 N_{\theta\theta} &= \sum_{k=0}^{\infty} \left\{ \frac{2p_0}{\pi c} \left[ \frac{1}{2k+1} \right] + A_k^1 \left[ \left( \frac{\lambda^2}{\omega} \right) \text{bei}''_{\bar{m}} \mu \right] - A_k^2 \left[ \left( \frac{\lambda^2}{\omega} \right) \text{ber}''_{\bar{m}} \mu \right] + \right. \\
 &\quad \left. + C_k^2 \left[ \frac{\bar{m}}{\omega} (\bar{m} - 1) r^{\bar{m}-2} \right] \right\} \sin \bar{m} \theta, \\
 N_{r\theta} &= \sum_{k=0}^{\infty} \left\{ -A_k^1 \left[ \left( \frac{\bar{m}}{\omega} \frac{\lambda}{r} \right) \text{bei}_{\bar{m}} \mu - \left( \frac{\bar{m}}{\omega} \frac{1}{r^2} \right) \text{bei}_{\bar{m}} \mu \right] + \right. \\
 &\quad \left. + A_k^2 \left[ \left( \frac{\bar{m}}{\omega} \frac{\lambda}{r} \right) \text{ber}'_{\bar{m}} \mu - \left( \frac{\bar{m}}{\omega} \frac{1}{r^2} \right) \text{ber}_{\bar{m}} \mu \right] - C_k^2 \left[ \frac{\bar{m}}{\omega} (\bar{m} - 1) r^{\bar{m}-2} \right] \right\} \cos \bar{m} \theta, \\
 Q_r &= -D \sum_{k=0}^{\infty} \left\{ A_k^1 \left[ \lambda^3 \text{ber}'''_{\bar{m}} \mu - \lambda \left( \frac{\bar{m}}{r} \right)^2 \text{ber}'_{\bar{m}} \mu + 2 \left( \frac{\bar{m}^2}{r^3} \right) \text{ber}_{\bar{m}} \mu + \left( \frac{\lambda^2}{r} \right) \text{ber}''_{\bar{m}} \mu - \right. \right. \\
 &\quad \left. \left. - \left( \frac{\lambda}{r^2} \right) \text{ber}'_{\bar{m}} \mu \right] + A_k^2 \left[ \lambda^3 \text{bei}'''_{\bar{m}} \mu - \lambda \left( \frac{\bar{m}}{r} \right)^2 \text{bei}'_{\bar{m}} \mu - 2 \left( \frac{\bar{m}^2}{r^3} \right) \text{bei}_{\bar{m}} \mu + \right. \right. \\
 &\quad \left. \left. + \left( \frac{\lambda^2}{r} \right) \text{bei}''_{\bar{m}} \mu - \left( \frac{\lambda}{r^2} \right) \text{bei}'_{\bar{m}} \mu \right] \right\} \sin \bar{m} \theta,
 \end{aligned}$$

$$Q_{\theta} = -D \sum_{k=0}^{\infty} \left\{ A_k^1 \left[ \left( \lambda^2 \frac{\bar{m}}{r} \right) \text{ber}_{\bar{m}}'' \mu - \left( \frac{\bar{m}}{r} \right)^3 \text{ber}_{\bar{m}} \mu + \left( \lambda \frac{\bar{m}}{r} \right) \text{ber}_{\bar{m}}' \mu \right] + \right. \\ \left. + A_k^2 \left[ \left( \lambda^2 \frac{\bar{m}}{r} \right) \text{bei}_{\bar{m}}'' \mu - \left( \frac{\bar{m}}{r} \right)^3 \text{bei}_{\bar{m}} \mu + \left( \lambda \frac{\bar{m}}{r} \right) \text{bei}_{\bar{m}}' \mu \right] \right\} \cos \bar{m} \theta,$$

$$M_{rr} = -D \sum_{k=0}^{\infty} \left\{ A_k^1 \left[ \lambda^2 \text{ber}_{\bar{m}}'' \mu + \nu \left( \frac{\lambda}{r} \text{ber}_{\bar{m}}' \mu - \frac{\bar{m}^2}{r^2} \text{ber}_{\bar{m}} \mu \right) \right] + \right. \\ \left. + A_k^2 \left[ \lambda^2 \text{bei}_{\bar{m}}'' \mu + \nu \left( \frac{\lambda}{r} \text{bei}_{\bar{m}}' \mu - \frac{\bar{m}^2}{r^2} \text{bei}_{\bar{m}} \mu \right) \right] + \right. \\ \left. + C_k^1 [\bar{m}(\bar{m}-1) + \nu \bar{m}(1-\bar{m})] r^{\bar{m}-2} \right\} \sin \bar{m} \theta,$$

$$M_{\theta\theta} = -D \sum_{k=0}^{\infty} \left\{ A_k^1 \left[ \left( \frac{\lambda}{r} \right) \text{ber}_{\bar{m}}' \mu - \left( \frac{\bar{m}}{r} \right)^2 \text{ber}_{\bar{m}} \mu + \nu \lambda^2 \text{ber}_{\bar{m}}'' \mu \right] + \right. \\ \left. + A_k^2 \left[ \left( \frac{\lambda}{r} \right) \text{bei}_{\bar{m}}' \mu - \left( \frac{\bar{m}}{r} \right)^2 \text{bei}_{\bar{m}} \mu + \nu \lambda^2 \text{bei}_{\bar{m}}'' \mu \right] + \right. \\ \left. + C_k^1 [\bar{m}(1-\bar{m}) + \nu \bar{m}(\bar{m}-1)] r^{\bar{m}-2} \right\} \sin \bar{m} \theta,$$

$$M_{r\theta} = -D(1-\nu) \sum_{k=0}^{\infty} \left\{ A_k^1 \left[ \left( \lambda \frac{\bar{m}}{r} \right) \text{ber}_{\bar{m}}' \mu - \left( \frac{\bar{m}}{r^2} \right) \text{ber}_{\bar{m}} \mu \right] + \right. \\ \left. + A_k^2 \left[ \left( \lambda \frac{\bar{m}}{r} \right) \text{bei}_{\bar{m}}' \mu - \left( \frac{\bar{m}}{r^2} \right) \text{bei}_{\bar{m}} \mu \right] + \right. \\ \left. + C_k^1 [\bar{m}(\bar{m}-1) r^{\bar{m}-2}] \right\} \cos \bar{m} \theta.$$

### Numerical Results

A shallow thin segmental shell of the type shown in fig. 2 that admits the following information

$$c = \frac{1}{R},$$

$$R = 100 \text{ meters},$$

$$h = 0,10 \text{ meters},$$

$$\bar{r} = 1,75 \text{ meters},$$

$$\lambda = 0,59,$$

$$\Theta = \frac{\pi}{6},$$

$$\nu = 0,30,$$

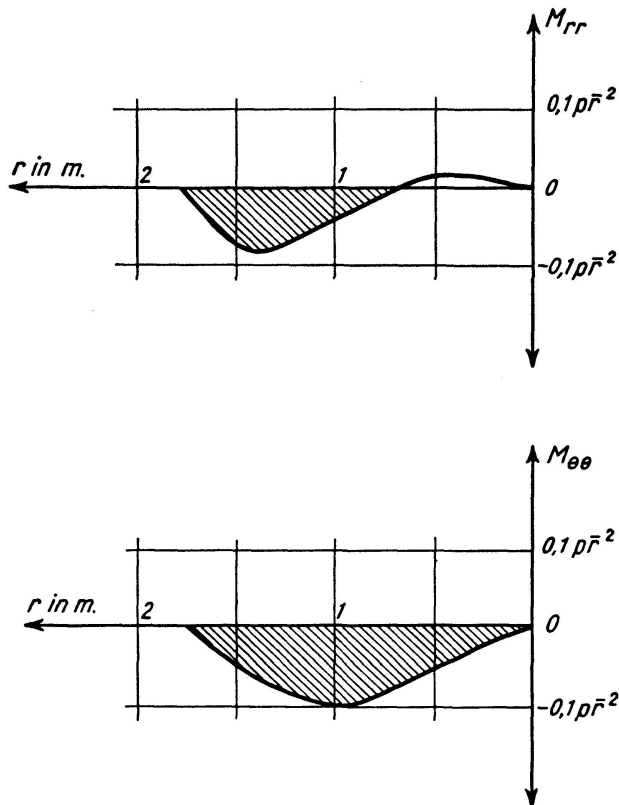


Fig. 4. Distribution of Stress Couples  $M_{rr}$  and  $M_{\theta\theta}$  along the Radius  $r$ .

is subjected to a uniform normal surface pressure of intensity  $p$ . The resulting stress couples prevailing within the skin of the segmental shell along the radius  $r$  at  $\theta = \frac{\Theta}{2}$  are shown in fig. 4.

### Summary

A method of stress analysis is presented in this paper for thin elastic shells that are meridional segments of rotational surfaces generated by shallow segments of second degree curves.

The MUSHTARI-VLASOV shallow shell theory served as the foundation for the development. The analysis does not neglect the interaction between transverse bending and in the surface acting stress resultants of the shell. The ensuing solution is expressed by a trigonometric series coupled with Kelvin functions of  $\bar{m}$ 'th order. The usefulness of this analysis is limited to segmental shells spanning circumferential angles  $\Theta$  which conform to the restriction  $\frac{\pi}{\Theta} = \text{integer}$ . Numerical results are also given on an extremely shallow segmental shell stressed by a uniformly distributed normal load intensity.

### Résumé

L'auteur expose une méthode de détermination des contraintes dans les voiles minces élastiques, en forme de segments, délimités par des sections

méridiennes dans une surface de révolution engendrée par des éléments à faible courbure de courbes du second degré.

La théorie de MUSHTARI-VLASOV pour les voiles à faible courbure sert de base au développement. L'étude ne néglige pas l'action réciproque de la flexion transversale et des efforts agissant dans la surface du voile. La solution qui s'ensuit est exprimée par une série de fonctions trigonométriques et par des fonctions de KELVIN d'ordre  $\bar{m}$ . L'étude s'applique uniquement à des voiles en forme de segments dont l'angle au centre  $\Theta$  satisfait la condition  $\frac{\pi}{\Theta} =$  nombre entier. On donne des résultats numériques pour un voile à courbure très faible, sollicité par une charge normale, uniformément répartie.

### Zusammenfassung

Diese Abhandlung beschreibt eine Methode der Spannungsberechnung an dünnen elastischen Rotationsschalen, erzeugt durch ein flaches Segment einer Kurve zweiten Grades und begrenzt durch Meridianschnitte.

Als Grundlage diente die Theorie von MUSHTARI-VLASOV für flache Schalen. Die Berechnung vernachlässigt die gegenseitige Beeinflussung zwischen der Querbiegung und den Resultierenden der in der Schalenoberfläche wirkenden Spannungen nicht. Die Lösung wird in Form einer mit KELVINSchen Funktionen  $\bar{m}$ -ter Ordnung gekoppelten trigonometrischen Reihe gegeben. Die Brauchbarkeit der Berechnung beschränkt sich auf Segmentschalen, deren Zentriwinkel  $\Theta$  der Bedingung  $\frac{\pi}{\Theta} =$  ganze Zahl genügen. Es werden numerische Resultate angegeben für eine sehr flache Schale, beansprucht durch eine gleichmäßig verteilte Normalbelastung.