

Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen
Band: 17 (1957)

Artikel: A general theory of deformations of membrane shells
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DOI: <https://doi.org/10.5169/seals-15752>

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A General Theory of Deformations of Membrane Shells¹⁾

Une théorie générale des déformations des voiles minces sans moments fléchissants

Eine Allgemeintheorie für die Verformung von Membranschalen

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Introduction

Current work in the theory of shells concerns itself mostly with an elaboration of the bending theory. Efforts are being made to apply the simplified versions of this theory to new geometric shapes and to examine critically the simplifications that have been introduced. At the same time, however, the authors of this paper feel that membrane theory has not been exploited to its limits if it has not been used to calculate deformations as well as stresses for all configurations where it is applicable. The present paper is intended to close this gap.

The knowledge of the deformation of shells has proven useful in a variety of engineering applications and is indispensable in the analysis of shells where the boundary conditions are given in terms of deformations and in externally statically indeterminate problems.

Convenient systems of coordinates for the formulation of the deformation problem are the cartesian one for shells of translation and the cylindrical one for shells of revolution. This same choice of coordinates has been made in Pucher's formulation of the membrane stress problem and accounts for the general applicability of this method. In many formal aspects the present calculation of membrane deformations is an extension of the Pucher method and we therefore begin with a brief review of the latter.

¹⁾ Paper based on Ph. D. Dissertation at Stanford University by F. T. GEYLING, 1953 (Ref. 1).

I. The Pucher Solution of the Membrane Stress Problem²⁾

Fig. 1 shows a shell of arbitrary configuration whose middle surface is given as $z(x, y)$ in the cartesian system of coordinates. A few coordinate lines $x = \text{constant}$ and $y = \text{constant}$ are indicated on it. In fig. 2, a magnified view is shown of a differential element of the shell, this element being bounded by arcs of coordinate lines. It is loaded by the external forces p_x, p_y, p_z , per unit area of the shell and held in equilibrium by the skew system of membrane forces N_x, N_y , and N_{xy} per unit length of coordinate line. The equality of

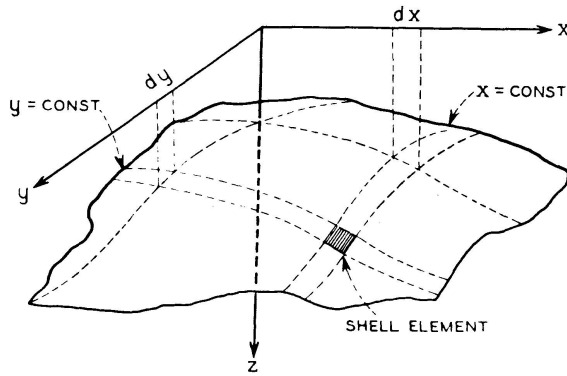


Fig. 1.

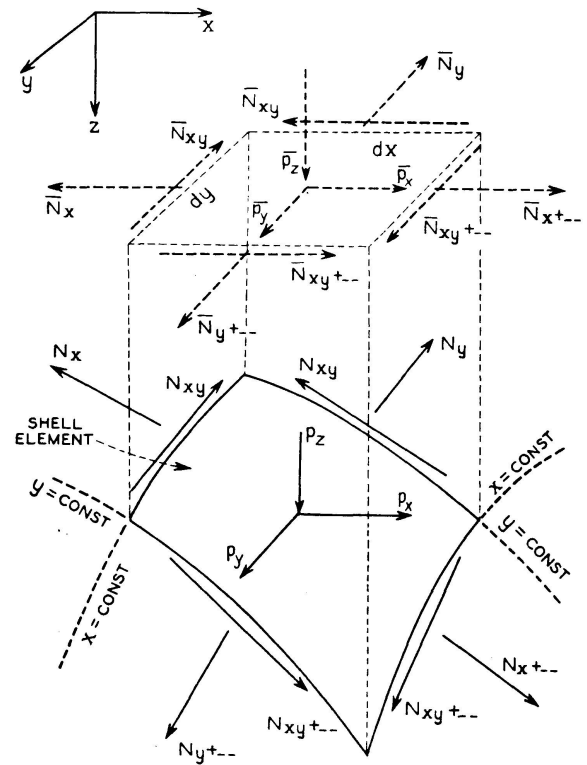


Fig. 2.

the shear resultants, $N_{xy} = N_{yx}$ follows readily. In the Pucher method we write the equilibrium equations for the shell element in terms of the horizontal components of the membrane forces per unit length of horizontal line element dx or dy ; these are denoted as $\bar{N}_x, \bar{N}_y, \bar{N}_{xy}$, and shown in fig. 2. Likewise, the external loads are expressed per unit horizontal area of $dx dy$ and denoted as $\bar{p}_x, \bar{p}_y, \bar{p}_z$. This set of forces will be referred to as the projected forces of the shell element³⁾.

²⁾ Ref. 6 and ref. 3, chapter 4.

³⁾ Throughout this paper partial derivatives with respect to x and y will be indicated by subscripts, e. g.,

$$\frac{\partial^2 z}{\partial x^2} = z_{xx} \quad \text{and} \quad \frac{\partial}{\partial y} () = ()_y, \quad (0)$$

where the quantity in parentheses may also be the marginal number for a formula. The only exceptions will be \bar{N}_x, \bar{N}_y and \bar{N}_{xy} ; the strains ϵ_x, ϵ_y , and γ_{xy} ; and \bar{p}_x, \bar{p}_y , and \bar{p}_z .

It is readily seen that the equilibrium equations for the shell element in the x and y directions in terms of the projected forces turn out to be identical to the equilibrium conditions of plane stress or plane strain, namely,

$$\frac{\partial \bar{N}_x}{\partial x} + \frac{\partial \bar{N}_{xy}}{\partial y} = -\bar{p}_x, \quad (1)$$

$$\frac{\partial \bar{N}_y}{\partial y} + \frac{\partial \bar{N}_{xy}}{\partial x} = -\bar{p}_y. \quad (2)$$

The equilibrium equation in the z -direction is

$$\frac{\partial \bar{N}_x}{\partial x} z_x + \bar{N}_x z_{xx} + \frac{\partial \bar{N}_y}{\partial y} z_y + \bar{N}_y z_{yy} + \frac{\partial \bar{N}_{xy}}{\partial x} z_y + \frac{\partial \bar{N}_{xy}}{\partial y} z_x + 2 \bar{N}_{xy} z_{xy} = -\bar{p}_z, \quad (3)$$

which, by way of further comparison with the plane stress problem, takes the place of the compatibility equation there. As in the plane theory of elasticity, we introduce the Airy stress function φ according to definitions

$$\bar{N}_x = \varphi_{yy} - \int \bar{p}_x dx, \quad \bar{N}_y = \varphi_{xx} - \int \bar{p}_y dy, \quad \bar{N}_{xy} = -\varphi_{xy}, \quad (4)$$

which satisfies the equilibrium equations (1) and (2) automatically. Equation (3) is left to be satisfied by the stress function and takes the form

$$L(\varphi) = \varphi_{yy} z_{xx} - 2 \varphi_{xy} z_{xy} + \varphi_{xx} z_{yy} = -\bar{p}_z + \bar{p}_x z_x + \bar{p}_y z_y + z_{xx} \int \bar{p}_x dx + z_{yy} \int \bar{p}_y dy, \quad (5)$$

where L will be called the Pucher operator.

The boundary conditions for φ indicate the type of support which the shell is given. In roof structures one may expect three alternative edge constraints as shown in fig. 3.

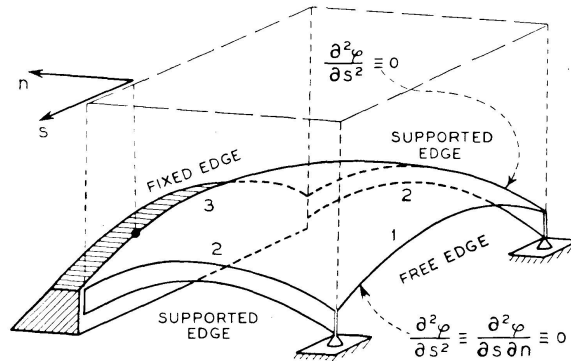


Fig. 3.

1. An edge is completely free. Both normal and shear forces are required to vanish along it, i. e.,

$$\frac{\partial^2 \varphi}{\partial s^2} \equiv \frac{\partial^2 \varphi}{\partial s \partial n} \equiv 0,$$

where n and s are the normal and tangential variables along the horizontal projection of the edge.

2. An edge is supported by a shear diaphragm. We will denote it as a "supported edge". The diaphragm is usually denied all rigidity transverse to its plane so that the membrane force normal to that edge is required to vanish, i. e.,

$$\frac{\partial^2 \varphi}{\partial s^2} \equiv 0.$$

The diaphragm, however, is expected to carry whatever shear forces the shell discharges upon it. These forces are dependent on the statically indeterminate interaction between the shell and the edge member. We simplify this boundary condition by accepting

$$\frac{\partial^2 \varphi}{\partial s \partial n}$$

as obtained from φ along the supported edge.

3. An edge is completely fixed. Both the normal and shear forces are results of the interaction between the shell and the edge abutment and again we simplify these boundary conditions by accepting whatever edge stresses the membrane solution produces. That is, we accept

$$\frac{\partial^2 \varphi}{\partial s^2} \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial s \partial n}$$

as found from φ .

II. The Elasto-Kinematic Relations of the Deformation Theory

Fig. 4 shows the differential shell element once more, this time with the positive displacement components u , v , and w and the geometric parameters α , β , and ω indicated on it. We shall next express the displacement components in terms of the membrane strains by using the kinematic laws and shall finally employ the elastic relations to express them in terms of the membrane forces.

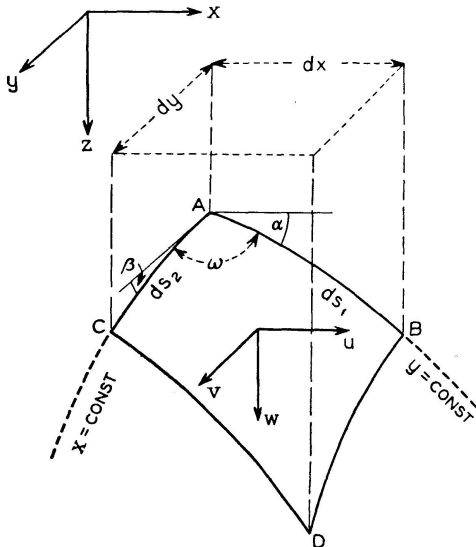


Fig. 4.

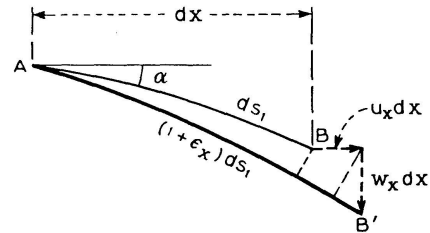


Fig. 5.

A. The Kinematic Relations

Fig. 5 shows the edge $AB = ds_1$ of the shell element which is subjected to the strain ϵ_x . This strain is caused by the x and z components of the relative displacement between the points A and B while the y component of displacement does not make a first order contribution. From fig. 5 and after some manipulation we obtain

$$\epsilon_x = u_x \cos^2 \alpha + w_x \sin \alpha \cos \alpha. \quad (6)$$

Correspondingly, we write for the strain of coordinate lines $x = \text{const.}$

$$\epsilon_y = v_y \cos^2 \beta + w_y \sin \beta \cos \beta. \quad (7)$$

The shear strain, γ_{xy} of the shell element is the change in the angle ω . A decrease of ω in the first quadrant of the xy system shall represent a positive γ_{xy} . Fig. 6 shows that the two arc elements ds_1 and ds_2 are transformed by the incremental displacement components at B and C into the heavily

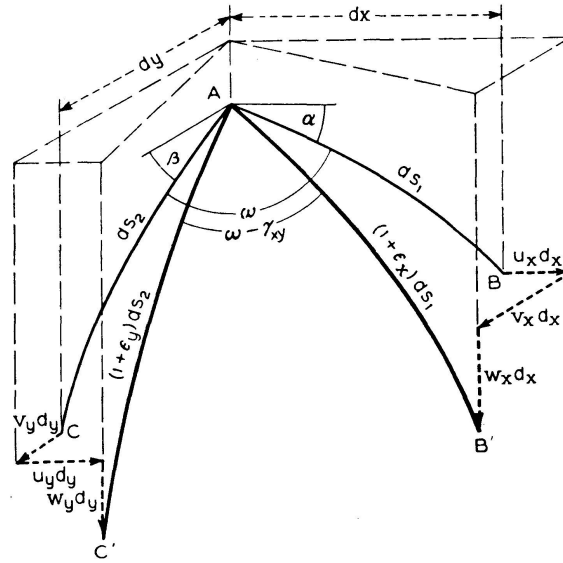


Fig. 6.

drawn arc elements which now enclose the angle $\omega - \gamma_{xy}$. To obtain an equation involving γ_{xy} we assume AC' and AB' to be straight lines and vectors and write their dot-product

$$\begin{aligned} (\vec{AC'}) \cdot (\vec{AB'}) &= ds_1 ds_2 (1 + \epsilon_x) (1 + \epsilon_y) \cos(\omega - \gamma_{xy}) \\ &= (1 + u_x) dx u_y dy + (1 + v_y) dy v_x dx \\ &\quad + (\tan \alpha + w_x) dx (\tan \beta + w_y) dy. \end{aligned} \quad (8)$$

Using the kinematic expressions for ϵ_x and ϵ_y , setting $\cos \gamma_{xy} = 1$ and $\sin \gamma_{xy} = \gamma_{xy}$, retaining only terms in the first power of one of the derivatives or of γ_{xy} as first order small quantities, and simplifying we obtain

$$\begin{aligned} \gamma_{xy} = \frac{1}{\sin \omega} & (u_y \cos \alpha \cos \beta - u_x \sin \alpha \cos^2 \alpha \sin \beta \\ & + v_x \cos \beta \cos \alpha - v_y \sin \beta \cos^2 \beta \sin \alpha \\ & + w_x \cos^3 \alpha \sin \beta + w_y \cos^3 \beta \sin \alpha). \end{aligned} \quad (9)$$

B. The Elastic Relations

Fig. 7 shows a shell element in normal view where it appears as a parallelogram subjected to the skew system of membrane forces N_x , N_y , and N_{xy} . The extensional strains ϵ_x and ϵ_y are also in skewed directions to each other. To derive the elastic relation for ϵ_x we define the normal set of forces S_x and

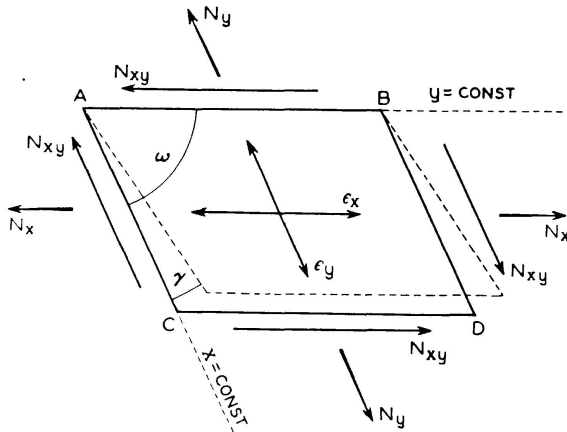


Fig. 7.

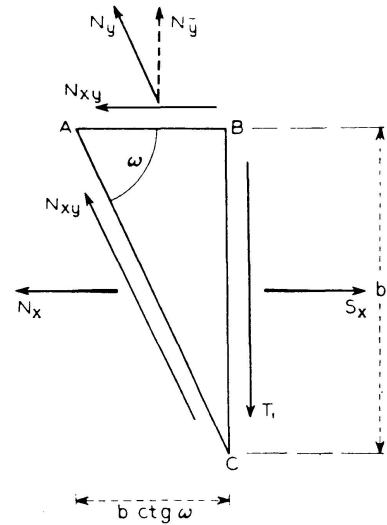


Fig. 8.

$N_{\bar{y}}$ in terms of the original membrane forces as shown in fig. 8. We have

$$\epsilon_x = \frac{1}{Et} (S_x - \nu N_{\bar{y}}) = \frac{1}{Et} \left(N_x \frac{1}{\sin \omega} + 2 N_{xy} \operatorname{ctg} \omega + N_y \sin \omega (\operatorname{ctg}^2 \omega - \nu) \right), \quad (10)$$

and by analogy,

$$\epsilon_y = \frac{1}{Et} \left(N_y \frac{1}{\sin \omega} + 2 N_{xy} \operatorname{ctg} \omega + N_x \sin \omega (\operatorname{ctg}^2 \omega - \nu) \right), \quad (11)$$

where ν is Poisson's ratio. To obtain the elastic relation for γ_{xy} we define an orthogonal set of strains ϵ_x , $\epsilon_{\bar{y}}$, $\bar{\gamma}_{xy}$ in terms of the orthogonal forces S_x , $N_{\bar{y}}$ and T_1 (fig. 8) as follows

$$\epsilon_x = \frac{1}{Et} (S_x - \nu N_{\bar{y}}), \quad \epsilon_{\bar{y}} = \frac{1}{Et} (N_{\bar{y}} - \nu S_x), \quad \bar{\gamma}_{xy} = \frac{2(1+\nu) T_1}{Et}. \quad (12)$$

Fig. 9 shows the relation between ϵ_x , $\epsilon_{\bar{y}}$, $\bar{\gamma}_{xy}$ and γ_{xy} from which there follows

$$\gamma_{xy} \frac{b}{\sin \omega} = \bar{\gamma}_{xy} b \sin \omega + \epsilon_x b \operatorname{ctg} \omega \sin \omega - \epsilon_{\bar{y}} b \cos \omega, \quad (13)$$

or in terms of the skew membrane forces

$$\gamma_{xy} = \frac{(1+\nu)}{Et} (N_x \cos \omega + 2 N_{xy} + N_y \cos \omega). \quad (14)$$

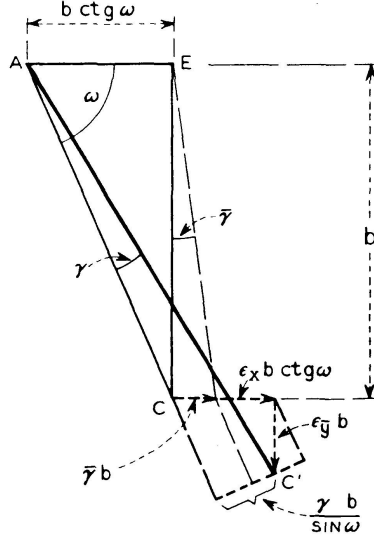


Fig. 9.

C. The Elasto-Kinematic Relations

We equate the expressions we got by the kinematic and the elastic laws for ϵ_x and do the same for ϵ_y . Noting that

$$N_x = \bar{N}_x \frac{\cos \beta}{\cos \alpha}, \quad N_y = \bar{N}_y \frac{\cos \alpha}{\cos \beta}, \quad \text{and} \quad N_{xy} = \bar{N}_{xy},$$

and expressing all trigonometric functions in terms of z and its derivatives we find

$$u_x = -w_x z_x + f, \quad (15)$$

$$v_y = -w_y z_y + g, \quad (16)$$

where f and g are given below. Proceeding correspondingly with the expressions for γ_{xy} and using the results for u_x and v_y from equations (15) and (16) we have

$$u_y + v_x = -w_x z_y - w_y z_x + k. \quad (17)$$

Equations (15), (16) and (17) shall henceforth be referred to as the elasto-kinematic relations. The terms f , g , and k stand as short notations for the following quantities

$$f = \frac{1}{Et(1+z_x^2+z_y^2)^{1/2}} \{ \bar{N}_x (1+z_x^2)^2 + 2 \bar{N}_{xy} z_x z_y (1+z_x^2) + \bar{N}_y [z_x^2 z_y^2 - \nu(1+z_x^2+z_y^2)] \},$$

$$g = \frac{1}{Et(1+z_x^2+z_y^2)^{1/2}} \{ \bar{N}_y (1+z_y^2)^2 + 2 \bar{N}_{xy} z_x z_y (1+z_y^2) + \bar{N}_x [z_x^2 z_y^2 - \nu(1+z_x^2+z_y^2)] \},$$

$$k = \frac{2}{Et(1+z_x^2+z_y^2)^{1/2}} \{ \bar{N}_x z_x z_y (1+z_x^2) + \bar{N}_{xy} [(1+\nu)(1+z_x^2+z_y^2) + 2z_x^2 z_y^2] + \bar{N}_y z_x z_y (1+z_y^2) \}.$$

III. Differential Equations and Expressions for w , u , and v

A. The Equations for w , u , and v

In order to obtain a differential equation for w alone we write $(15)_y$ and $(17)_x$ and eliminate u_{xy} between the two expressions. Taking $\frac{\partial}{\partial y}$ of the resulting equation and writing $(16)_{xx}$ we can eliminate v_{xxy} from these two. In the resulting equation all third order derivatives of w cancel out leaving w subject only to the Pucher operator as follows

$$L(w) \equiv w_{xx} z_{yy} - 2w_{xy} z_{xy} + w_{yy} z_{xx} = k_{xy} - f_{yy} - g_{xx} \equiv R. \quad (19)$$

To derive a differential equation for u alone we write $(15)_x$ and $(15)_y$ and solve for w_{xx} and w_{xy} respectively. Eliminating v_{xy} from differentiated forms of (16) and (17) we obtain an expression for w_{yy} . Substituting the above expressions for derivatives of w into (19) and simplifying we get

$$L(u) + 2u_x(z_{xy}^2 - z_{xx}z_{yy})/z_x = 2f(z_{xy}^2 - z_{xx}z_{yy})/z_x + f_y(z_{yy} - 2z_{xy}) + (k_y - g_x)z_{xx} + Rz_x. \quad (20)$$

An analogous equation can be written for v .

B. Expressions for u and v derived from w

In many problems the boundary conditions for the displacements will make it most convenient to solve for w first and then use this result to calculate u and v . To obtain an expression for u in terms of w we integrate (15) and get

$$u = \int (f - w_x z_x) dx + j(y), \quad (21)$$

where $j(y)$ is a function of y alone and will be controlled by the condition (17). We write $(17)_y$ and $(16)_x$ and eliminate v_{xy} from these two equations to obtain an expression for u_{yy} . Another expression for u_{yy} is obtained from $(21)_{yy}$. Equating these two expressions and solving for $j(y)$ we get

$$j(y) = \iint \{ k_y - g_x - w_x z_{yy} - w_{yy} z_x - [\int (f - w_x z_x) dx]_{yy} \} dy dy + u_1 y + u_2, \quad (22)$$

where u_1 and u_2 are constants. The integrand of the double integral in $j(y)$ will be automatically independent of x if the single integral term is left inside it, as can be shown by forming $\frac{\partial}{\partial x}$ of the double integrand and obtaining $R - L(w)$, which is zero.

Corresponding results for v are

$$v = \int (g - w_y z_y) dy + i(x), \quad (23)$$

$$i(x) = \{ \iint k_x - f_y - w_y z_{xx} - w_{xx} z_y - [\int (g - w_y z_y) dy]_{xx} \} dx dx + v_1 x + v_2. \quad (24)$$

From condition (17) it can be shown that $u_1 = -v_1$. The constants u_1 , u_2 , v_1 , and v_2 represent rigid body displacements of the shell.

IV. Edge Conditions Imposed Upon Membrane Displacements by a Shear Diaphragm

In this paper we shall limit ourselves to considering membrane deformation problems for shells with either free or supported edges. Since a free edge imposes no boundary conditions on the membrane deformations we proceed to derive boundary conditions in terms of u , v , and w from the mechanical actions of a shear diaphragm upon a shell. For this purpose we introduce the following

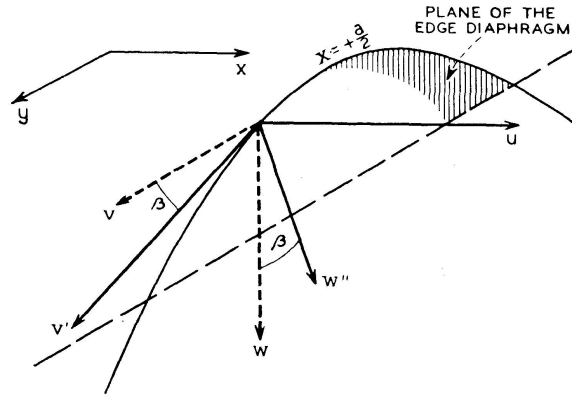


Fig. 10.

more relevant edge displacement components in terms of u , v , and w . Consider an edge with $x = \text{const.}$ as shown in fig. 10. We have u as before and

$$v' = v \cos \beta + w \sin \beta, \quad (25)$$

$$w'' = w \cos \beta - v \sin \beta. \quad (26)$$

Along edges with $y = \text{const.}$ we use v as before and

$$u' = u \cos \alpha + w \sin \alpha, \quad (27)$$

$$w' = w \cos \alpha - u \sin \alpha. \quad (28)$$

If we assume that a shear diaphragm has no rigidity normal to its own plane there will be no control imposed on displacement components u along edges $x = \text{const.}$ and no control on displacement components v along edges $y = \text{const.}$ Since the edge members carry the shell by tangential shear forces, their principal constraining action on the membrane deformations appears to be their control of u' and v' . These displacement components must then be

equal for the shell and the edge member in order to ensure tangential compatibility. From this fact we derive a boundary condition for w along $x = \text{const.}$ as follows. Solving equation (25) for v , differentiating it with respect to y and simplifying, then expressing v_y in terms of equation (16) we find

$$w = \frac{1}{z_{yy}} [v' (1 + z_y^2)^{1/2}]_y - \frac{g}{z_{yy}}. \quad (29)$$

Correspondingly, we have along edges $y = \text{const.}$

$$w = \frac{1}{z_{xx}} [u' (1 + z_x^2)^{1/2}]_x - \frac{f}{z_{xx}}. \quad (30)$$

Should it be desired to obtain boundary conditions for v along edges $x = \text{const.}$, we may eliminate w from equations (25) and (16) and thus obtain

$$v = \frac{z_y}{z_{yy}} \left\{ g - z_y \left[\frac{v'}{z_y} (1 + z_y^2)^{1/2} \right]_y \right\}. \quad (31)$$

Its counterpart along edges $y = \text{const.}$ is

$$u = \frac{z_x}{z_{xx}} \left\{ f - z_x \left[\frac{u'}{z_x} (1 + z_x^2)^{1/2} \right]_x \right\}. \quad (32)$$

Boundary conditions based on compatibility between the shell and edge members in terms of w' and w'' can also be formulated. However, their applicability to membrane deformation problems seems rather restricted and therefore they will not be shown in detail⁴).

V. Membrane Deformations of the Elliptic Paraboloid

As an example for shells of positive Gaussian curvature we consider those whose middle surfaces are elliptic paraboloids described by:

$$z = \frac{x^2}{h_1} + \frac{y^2}{h_2} \quad (\text{see fig. 11}). \quad (33)$$

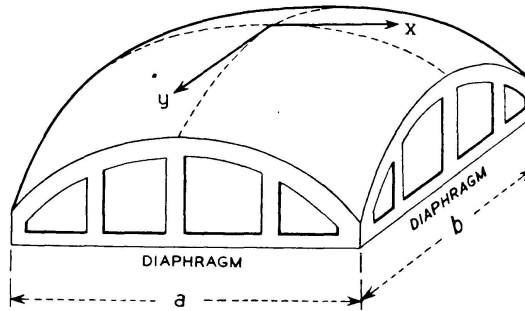


Fig. 11.

⁴) See ref. 1, pp. 67 to 69.

A. Inextensional Deformations

We begin with a consideration of deformations which are possible without membrane stresses being generated in the shell. Thus, if $\bar{N}_x \equiv \bar{N}_y \equiv \bar{N}_{xy} \equiv 0$, we have $f \equiv g \equiv k \equiv R \equiv 0$. For this case equation (19) for w reduces to

$$\frac{1}{h_2} w_{xx} + \frac{1}{h_1} w_{yy} = 0, \quad (34)$$

which can be transformed to the Laplace equation by a change of variables such as

$$\xi = \left(\frac{h_2}{h_1} \right)^{1/2} x,$$

so that convenient use can be made of results from potential theory in the deformation problems for such shells.

We can now take a piece of an elliptic paraboloid without any edge members on it and of random shape and prescribe any distribution of w displacements along its edges. An inextensional deformation w_0 will then exist for the shell satisfying these boundary conditions since it is merely a solution of the Dirichlet problem which we have stated. We also note from the maximum principle of potential theory that the upward or downward deflections w within the shell will nowhere be larger than the maximum up or down deflections prescribed along the edges. A local dent in the interior of the shell will not be possible as an inextensional deformation. We see that such shells of positive Gaussian curvature offer no membrane resistance to a wide variety of edge distortions, but must remember that this result holds only for small deflections since the kinematic relations (15), (16), and (17) were linearized accordingly. Our group of inextensional deformations therefore represents a much larger class than would be obtained from a study of the applicability of elliptic paraboloids as a problem of differential geometry.

Now consider a rectangular piece of the elliptic paraboloid with four fixed unstrained edge members. From the boundary conditions (29) and (30) we get with $f \equiv g \equiv 0$ and $u' \equiv v' \equiv 0$, $w \equiv 0$ on all edges. From the Dirichlet problem it is obvious that w vanishes identically throughout the shell. No inextensional deformation whatever will be possible for the elliptic paraboloid (and any other shell with $K > 0$) with fixed and unstrained edges.

Next, let us subject the elliptic paraboloid with four edge members to the twisting deformation

$$w = \frac{-4e}{ab} xy.$$

This deformation obviously satisfies equation (34) and results in the following displacements u and v from the formulas (21) to (24) where we have set $u_1 = u_2 = v_1 = v_2 = 0$:

$$u = \frac{4ey}{ab} \left(\frac{x^2}{h_1} + \frac{y^2}{3h_2} \right) \quad \text{and} \quad v = \frac{4ex}{ab} \left(\frac{y^2}{h_2} + \frac{x^2}{3h_1} \right).$$

These deformations will result in a warping of the shear diaphragms, in their rotation within their own plane about centers located at z_1 and z_2 , and finally in a shear distortion, ψ , of the rectangular edge layout of the shell. All of these results are illustrated in fig. 12.

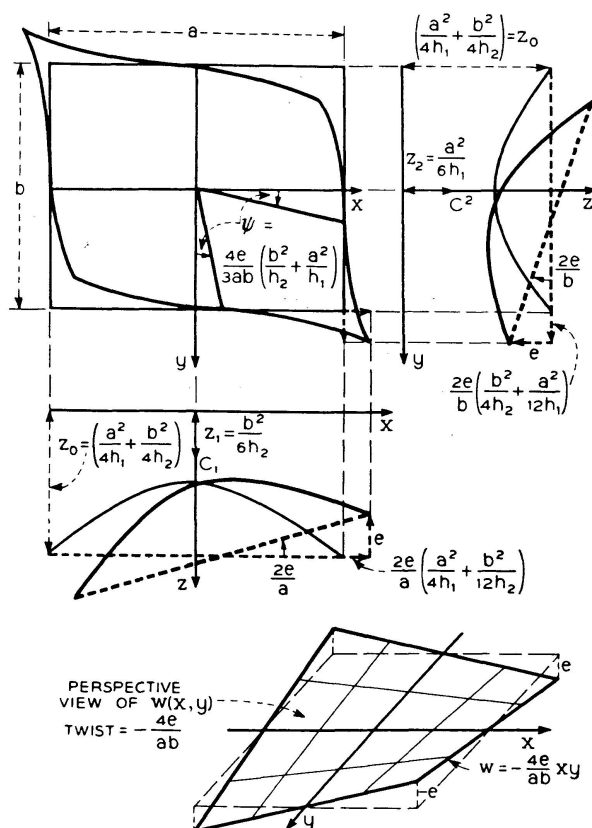


Fig. 12.

The inextensional deformation that we just found reveals the interesting fact that an elliptic paraboloid with four edge members is insensitive to uneven settlement of the corner supports if at the same time, it is allowed to go through an accompanying shear distortion of its edge layout. Conversely, such a shell will offer resistance against a shear deformation of its edge assembly only if its corners are constrained to remain in one plane.

B. Extensional Deformations

We consider the elliptic paraboloid supported at all four edges and subjected to the load distribution

$$\bar{p}_z = p \cos \frac{\pi x}{a} \cos \frac{\pi y}{a}.$$

We take the edge layout of the shell as a square and for ease of numerical calculations we simplify the equation for the paraboloid to

$$z = \frac{x^2 + y^2}{a}.$$

Also, wherever Poisson's ratio, ν , occurs it shall be taken as 0.2. Equation (19) for this case reduces to

$$\nabla^2 w = \frac{a}{2} R.$$

To obtain the membrane forces for the shell we refer to the Pucher solution of this case⁵⁾. Computations for R based on the membrane forces from this solution are best carried out numerically⁶⁾. The boundary values for w follow from equations (29) and (30), where we set $u' \equiv v' \equiv 0$, since we neglect the deformations of the edge members due to the shear forces from the shell. The boundary value problem for w is conveniently solved by relaxation methods and the results of these computations are given in table 1 for one octant of the shell in terms of the dimensionless variable

$$W = \frac{Et}{a^2 p} w,$$

where E = Young's modulus and t = thickness of the shell.

Table 1.. Values of $W = \frac{Et}{a^2 p} w$ in One Octant of the x, y System

y/a	x/a					
	0.0	0.1	0.2	0.3	0.4	0.5
0.0	1.145	1.087	0.921	0.663	0.342	0
0.1		1.034	0.880	0.640	0.339	0.014
0.2			0.761	0.573	0.329	0.054
0.3				0.461	0.308	0.114
0.4					0.265	0.189
0.5						0.255

Fig. 13 gives a perspective view of W for the third quadrant of the xy system.

We proceed to calculate u and v according to the method outlined by the formulas 21 to 24⁷⁾.

Table 2 shows the results for u in terms of the dimensionless variable

$$U = \frac{Et}{a^2 p} u.$$

⁵⁾ Ref. 1, p. 14.

⁶⁾ For details of numerical techniques the reader is referred to ref. 1, p. 97.

⁷⁾ For an adaptation of these formulas to numerical procedures the reader is referred to ref. 1, p. 59.

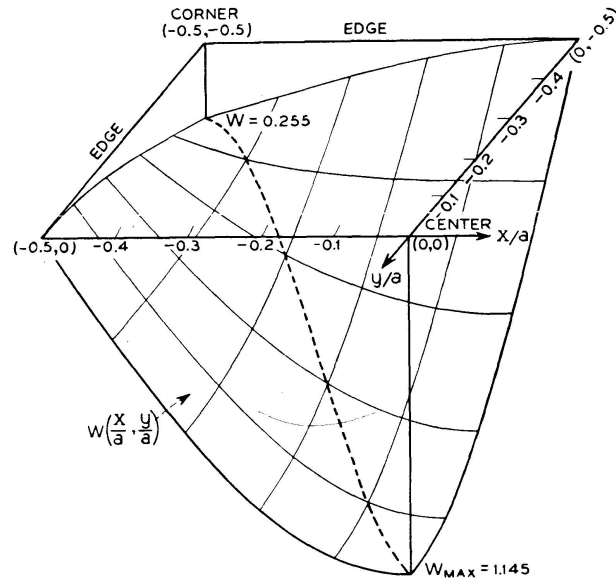


Fig. 13.

Table 2. Values of $U = \frac{Et}{a^2 p} u$ for the First Quadrant of the x, y System

y/a	x/a					
	0.0	0.1	0.2	0.3	0.4	0.5
0.0	0	-0.036	-0.047	-0.013	+0.078	+0.223
0.1	0	-0.034	-0.045	-0.015	+0.069	+0.206
0.2	0	-0.028	-0.040	-0.019	+0.045	+0.157
0.3	0	-0.021	-0.032	-0.026	+0.009	+0.080
0.4	0	-0.011	-0.022	-0.032	-0.035	-0.019
0.5	0	-0.001	-0.011	-0.034	-0.074	-0.127

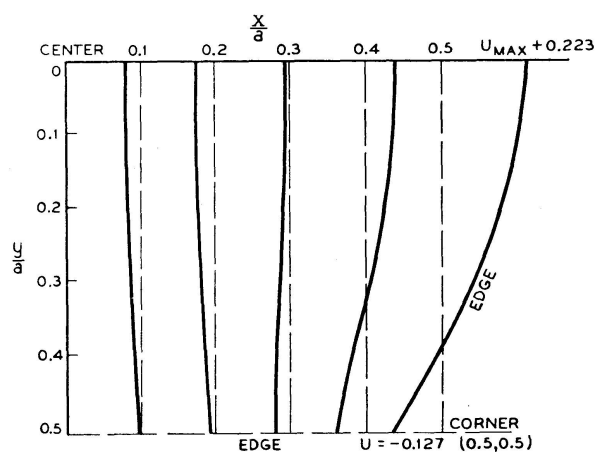


Fig. 14.

Fig. 14 illustrates the distribution of U for one quadrant of the shell and it indicates clearly the way in which the edge members warp.

VI. Deformations of the Hyperbolic Paraboloid

A. Inextensional Deformations

As an example for shells of negative Gaussian curvature, we consider the hyperbolic paraboloid, whose middle surface is given by

$$z = \frac{y^2}{h_2} - \frac{x^2}{h_1} \quad (\text{see fig. 15}). \quad (35)$$

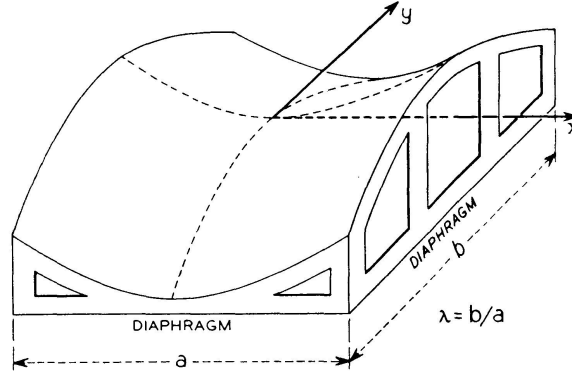


Fig. 15.

Equation (19) for w becomes

$$\frac{1}{h_1} w_{yy} - \frac{1}{h_2} w_{xx} = 0,$$

which is of the hyperbolic type and can be written in the canonical form

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = 0, \quad (36)$$

where $\xi = \left(\frac{h_2}{h_1}\right)^{1/2} x + y$ and $\eta = \left(\frac{h_2}{h_1}\right)^{1/2} x - y$. The general solution of (36) is $w = w_1(\xi) + w_2(\eta)$ which represents two forms of cylindrical bending about straight line generators $\xi = \text{const.}$, and $\eta = \text{const.}$ in the shell. Clearly, the middle surface of the shell is unstrained, and we are indeed dealing with inextensional deformations.

To build up the solutions for rectangular domains, consider fig. 16 where a hyperbolic paraboloid is bounded by the edges 1, 2, 3, 4. For simplicity, we take $h_1 = h_2$ and thus $\xi = x + y$, $\eta = x - y$. These characteristic coordinates are also shown in the figure. Now suppose that the boundary conditions $w_0 = w_1(\xi) + w_2(\eta)$ and $w_{0,y} = \frac{dw_1(\xi)}{d\xi} - \frac{dw_2(\eta)}{d\eta}$ are specified on edge 1 so that values for w_1 and w_2 are given on that line. These will be propagated unchanged through the shell, viz. $w(\xi)$ unchanged along generators $\xi = \text{const.}$ and $w(\eta)$ along $\eta = \text{const.}$ At edge 2 we solve for $w_2(\eta)$ from the boundary condition for w_0 with the $w_1(\xi)$ which was carried over from edge 1. This $w_2(\eta)$ now

travels to edge 4. The converse of the situation on edge 2 holds true for edge 3. On edge 4 some boundary value distribution $w_0 = w_1(\xi) + w_2(\eta)$ follows from the total procedure. So far, the solution has been built up as for an initial value — boundary value problem. It would be convenient to supplant the condition for w_0 on edge 1 by any continuous boundary condition for w_0 on edge 4 and thus stating a boundary value problem. In many cases this is not possible. Consider a square domain as in fig. 17. The entire region consists of rectangular loops of generators as shown in the figure. Along these generators we may have $w_1(\xi) = -w_2(\eta)$ so that $w_2 \equiv 0$ on all edges but not necessarily

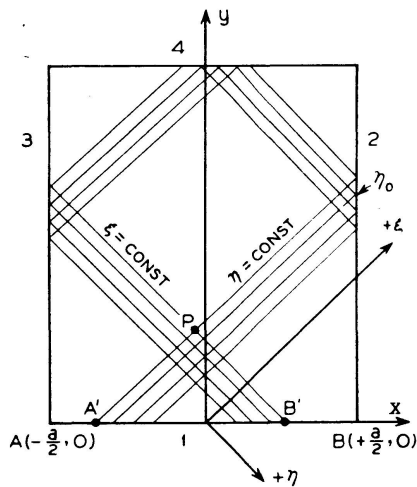


Fig. 16.

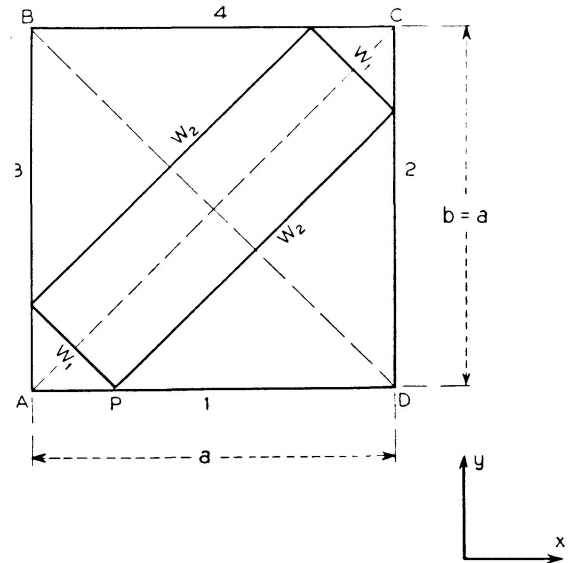


Fig. 17.

vanishing in the interior of the region. We can now imagine larger domains made up of such squares whose w_0 will be antisymmetric continuations across all dividing lines between the squares. Clearly, the ratio $\lambda = b/a$ will be of the form $\frac{q}{r}$, where q and r are integers, i. e., λ must be a rational number. The density of the set of rational numbers seems to make every value of λ a “near-critical” one, however, one can see that such ratios as $\frac{1001}{1000}$ require small distances between nodal lines in the pattern of w_0 . For finite amplitudes of w_0 this means that the changes of curvature and therefore bending moments in the deformed shell are no longer negligible. Such deformations, then, must be dismissed from the membrane theory and we shall consider only such critical values of λ which are ratios of small integers. For these λ , inhomogeneous boundary conditions can be specified on all edges if they satisfy certain symmetry conditions⁸⁾. However, these problems will have no unique solutions, since nontrivial w_0 have been shown above to exist for $w_0 \equiv 0$ on all edges.

⁸⁾ See ref. 1, pp. 114 and 115.

For noncritical values of λ it can be shown⁹⁾ that unique solutions exist for the inhomogeneous boundary value problem for w_0 , and of course no nontrivial solutions exist for the homogeneous problem.

So far we have not assumed that the rectangular domain ab of the hyperbolic paraboloid is confined along the edges. Now, consider that it is supported by fixed, unstrained shear diaphragms on all sides, calling for the boundary condition $w_0 \equiv 0$ everywhere. Then we know from the foregoing that this condition will not preclude inextensional deflections w_0 in the interior of the shell if λ has a critical value. This fact represents a serious practical drawback of hyperbolic shells as compared with the structural performance of elliptic ones.

Finally, one can subject the hyperbolic paraboloid to an inextensional twist $w_0 = \frac{4e}{ab}xy$ and make similar observations about the results in fig. 18 as we did with fig. 12. In particular, the shear distortion of the edge layout is

$$\psi = \frac{4e}{3ab} \left(\frac{a^2}{h_1} - \frac{b^2}{h_2} \right),$$

which will vanish, in distinction from elliptic cases, for $\frac{a^2}{b^2} = \frac{h_1}{h_2}$. Under these circumstances, other pure twists, w_0 , like the one in fig. 19 are also possible.

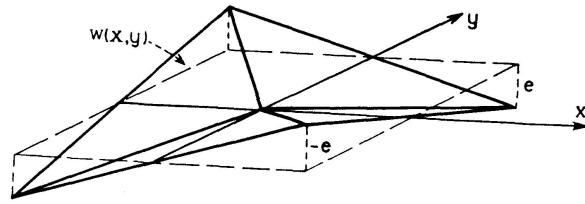
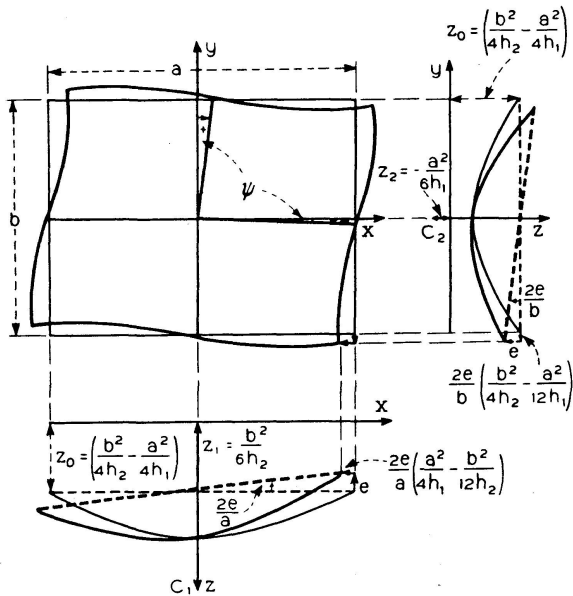


Fig. 19.

Fig. 18.

B. Extensional Deformations

We consider the hyperbolic paraboloid with $h_1 = h_2 = b$ and $\lambda = 2$ under the uniform loading $\bar{p}_z \equiv p$. It is supported by shear diaphragms on all edges (fig. 20).

⁹⁾ See ref. 1, p. 115.

To solve for the membrane forces in this shell, we note that equation (5) in its canonical form becomes

$$\frac{\partial^2 \varphi}{\partial \xi \partial \eta} = \mathfrak{F}(\xi, \eta) = -\frac{bp}{8}.$$

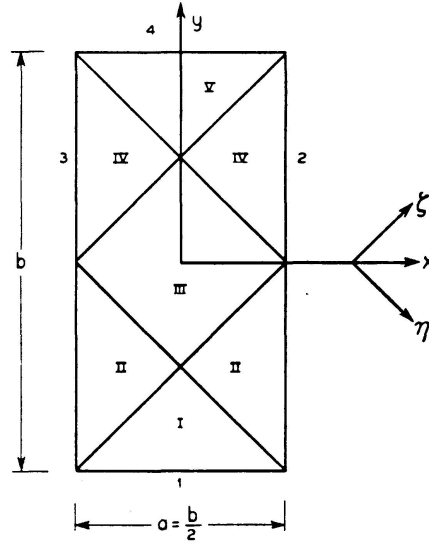


Fig. 20.

The boundary conditions are $\varphi_{xx} \equiv 0$ at 1 and $\varphi_{yy} \equiv 0$ at 2, 3, which may be ensured by $\varphi \equiv 0$ on 1, 2, 3. At 4 we should also like to have $\varphi_{xx} \equiv 0$, but due to the hyperbolic nature of the problem we must try to meet this condition by adjusting φ_y on 1. We assume some values for the latter to start with. The total rectangular domain is separated into five characteristic zones as delineated by a set of dividing characteristics (fig. 20). The solution φ is now constructed in each of these zones by the Riemann integration method where we use the formulas

$$\varphi(P) = \frac{1}{2}[\varphi(A) + \varphi(B)] + \frac{1}{2} \int_A^B \varphi_y dx - \int_{ABP} \int \mathfrak{F} d\xi d\eta$$

for regions like fig. 21 and

$$\varphi(P) = \varphi(A) + \varphi(C) - \varphi(B) - \int_{ABCP} \int \mathfrak{F} d\xi d\eta.$$

for regions like fig. 22. By using the new edge values arising in each zone as initial values for the next, we ensure continuity of φ throughout the rectan-

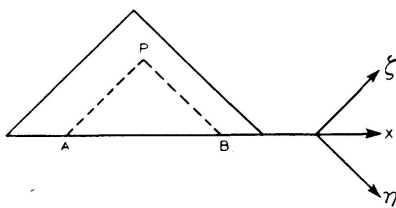


Fig. 21.

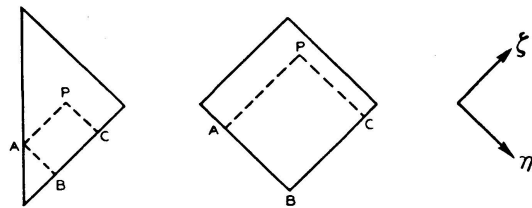


Fig. 22.

gular domain. However, the continuity of any of its derivatives at the dividing characteristics does not necessarily follow unless these derivatives are taken in the direction of the latter¹⁰). Consequently, the resulting membrane forces may be discontinuous at the dividing characteristics, a fact which proves of vital interest in the following discussion of deformations¹¹).

The canonical form of equation (19) for w reads

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = \Re(\xi, \eta), \quad (36)$$

where \Re , containing the membrane forces, is discontinuous at the dividing characteristics. Since the boundary conditions (29) and (30) also contain the membrane forces, they call for discontinuities in w at the corners of ab . These, however, were arbitrarily smoothed out in the present example, to retain some semblance of w with physical reality. Using these boundary conditions on the edges 1, 2, 3 and some initial condition of w_y on edge 1, a solution of equation (36) was constructed by the method just outlined in finding φ . The boundary values of w on edge 4 turned out to be identical to the ones on 1. Since $\lambda=2$ is a critical value for the given shell, non-trivial inextensional deformations with homogeneous boundary conditions are possible. They are odd in y and can be used to eliminate the antisymmetric part of the solution already found, so that the resulting w is even in y as can be reasonably demanded with a pattern of membrane forces that has the same property¹²). Fig. 23 shows the results in terms of a contour map of $W = \frac{Et}{pb^2} w$ (E = Young's modulus, t = thickness of shell) for one quadrant in the x, y system and table 3 gives corresponding numerical values. The figure shows that discontinuities in W_x and W_y occur along dividing characteristics.

With w known, the horizontal displacements $U = \frac{Et}{pb^2} u$ and $V = \frac{Et}{pb^2} v$ can be found in each characteristic zone by formulas (21) to (24) and their adaptations for numerical work. Since the deformed shapes of zones II and IV (fig. 20) no longer fit into the space between the distorted edges of the neighboring ones they had to be given some arbitrary orientation. The resulting vector field for U and V in one quadrant of the x, y system is given in fig. 24. It chiefly points out the considerable incompatibilities in tangential displacements along the dividing characteristics which are essentially due to discontinuities of the membrane strains in these directions.

¹⁰) For the details and results of this calculation the reader is referred to ref. 1, pp. 30—35.

¹¹) Indeed, this feature greatly decreases the value of membrane analysis in hyperbolic cases and in the authors' opinion has not been given proper recognition in such work as ref. 7. An interesting analysis by S. BAN (ref. 8) avoids this difficulty by treating only hyperbolic paraboloid panels which do not exceed a characteristic zone.

¹²) For details see again ref. 1, pp. 121—129.

Table 3. Values of $W = \frac{Et}{pb^2}w$ in the Fourth Quadrant of the x, y System

y/b	x/b					
	0.00	0.05	0.10	0.15	0.20	0.25
0.00	0.1188	0.1224	0.1332	0.1523	0.1817	0.2236
-0.05	0.1193	0.1228	0.1338	0.1534	0.1833	0.2038
-0.10	0.1205	0.1242	0.1358	0.1565	0.1694	0.1896
-0.15	0.1221	0.1262	0.1390	0.1463	0.1600	0.1805
-0.20	0.1240	0.1287	0.1312	0.1398	0.1546	0.1766
-0.25	0.1261	0.1230	0.1267	0.1366	0.1535	0.1779
-0.30	0.1187	0.1206	0.1252	0.1373	0.1569	0.1849
-0.35	0.1191	0.1211	0.1276	0.1421	0.1654	0.1982
-0.40	0.1279	0.1305	0.1384	0.1520	0.1799	0.2186
-0.45	0.1466	0.1499	0.1597	0.1766	0.2013	0.2471
-0.50	0.1768	0.1809	0.1932	0.2143	0.2446	0.2850

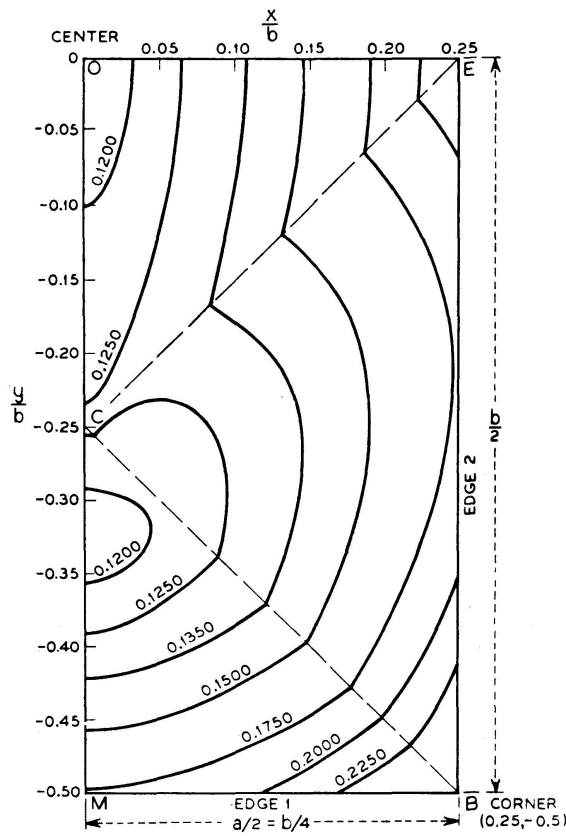


Fig. 23.

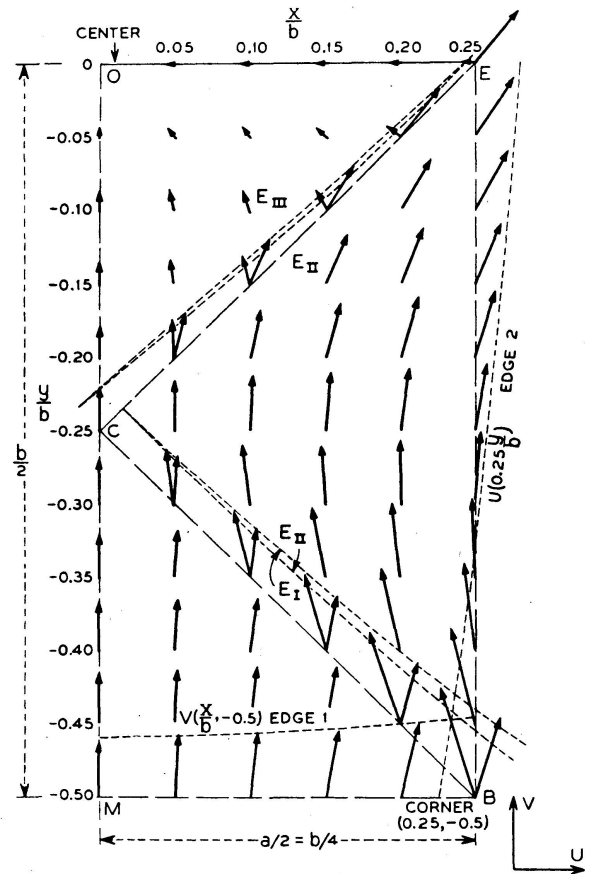


Fig. 24.

In concluding, we may say that the membrane solutions for inhomogeneous, hyperbolic problems where discontinuities occur, hardly yield any realistic results. Direct recourse must be taken to developing an appropriate bending theory. In elliptic cases, however, the analysis of membrane deformations

produces quite acceptable results especially if the bending corrections necessitated by the uncontrolled w' and w'' displacements at the boundaries can be proven restricted to the edge regions of the shell.

Bibliography

1. F. GEYLING, "A General Theory of Deformations of Membrane Shells", Ph. D. Dissertation, Stanford University, 1953.
2. W. FLÜGGE, «Statik und Dynamik der Schalen», Berlin, Springer, 1934.
3. — "Stresses in Shells" (in preparation), New York, McGraw-Hill.
4. — «Das Relaxationsverfahren in der Schalen-Statik», Beiträge zur angewandten Mechanik (Girkmann-Federhofer Anniversary Volume), Vienna, Deuticke, 1950, p. 17.
5. — «Zur Membrantheorie der Drehschalen negativer Krümmung», Z. ang. Math. Mech., Vol. 27 (1947), p. 65.
6. A. PUCHER, «Über den Spannungszustand in gekrümmten Flächen», Beton und Eisen, Vol. 33 (1934), p. 298.
7. F. AIMOND, "Treatise on Statics of Parabolic-Hyperboloidal Shells Not Stiff in Bending", (French), Publ. Intern. Assoc. Bridge Struct. Eng., Vol. 4 (1936), pp. 1—112.
8. S. BAN, "The Statics of Hyperbolic Paraboloid Shells", (German), Publ. Intern. Assoc. Bridge Struct. Eng., Vol. 13, (1953), pp. 1—16.

Summary

The paper develops a membrane theory of deformations for shells of arbitrary shapes besides surfaces of revolution and cylinders which have been treated in the past. Cartesian coordinates and projected membrane forces are used whereby many of the formal details parallel those of the Pucher method of solution for membrane stress problems. This method is reviewed for reference.

After a derivation of the kinematic and elastic relations the elasto-kinematic equations between the displacement components u , v , w , and the membrane forces \bar{N}_x , \bar{N}_y , \bar{N}_{xy} follow as

$$\begin{aligned} u_x &= -w_x z_x + f; & v_y &= -w_y z_y + g; & \text{and} \\ u_y + v_x &= -w_x z_y - w_y z_x + k; \end{aligned}$$

where f , g , k contain the membrane forces. Elimination of u and v from these equations yields a differential equation for the vertical deflection w :

$$L(W) \equiv w_{xx}z_{yy} - 2w_{xy}z_{xy} + w_{yy}z_{xx} = R,$$

where the operator L is typical of the Pucher method. R contains the membrane forces. Assuming w as (found) u and v may be obtained, by quadratures. As an alternative, differential equations similar to the one for w are developed for u and v whose second order operators are again L .

In most structural applications the edges of a shell are supported by verti-

cal shear diaphragms. The various constraining actions of such an edge member are formulated as boundary conditions for the deformations. Amongst them the most important and useful one, in terms of w , is that there be no tangential sliding between the edge of the shell and the diaphragm. Shells with positive Gaussian curvature are studied first and particular attention is given to the deformations of an elliptic paraboloid with four edge diaphragms under uneven settlement of the corner supports, surface loading, and other circumstances. The results of these investigations are applicable to structural practice.

The corresponding problems are then developed for the hyperbolic paraboloid as an example for shells of negative Gaussian curvature. A study of its inextensional deformations reveals that inextensional deflections with homogeneous boundary conditions may occur in such a shell if a critical or eigenvalue condition is satisfied. The same shell with uniform surface loading is discussed as an example for extensional deformations. This investigation serves mainly to point out the discontinuities along dividing characteristics in such hyperbolic membrane problems which make their solutions rather unrealistic.

The shell shows that the membrane theory produces quite acceptable results for the deformations in elliptic cases but it shows also that in hyperbolic cases the results of similar computations are of a rather limited value and call for the development of the bending theory.

Résumé

Dans cette étude, les auteurs développent une théorie membranaire des déformations des voûtes minces de forme arbitraire, à l'exception des surfaces de révolution des cylindres, qui ont fait l'objet d'études antérieures. Grâce à l'emploi des coordonnées cartésiennes et des projections des efforts de membrane, de nombreux éléments formels correspondent à la méthode de Pucher pour la solution du problème des tensions membranaires. Cette méthode est à la base de la présente étude.

Après avoir établi les relations cinématiques et élastiques, les auteurs déterminent les équations élasto-cinématiques entre les composantes u , v , w des déplacements et les efforts membranaires \bar{N}_x , \bar{N}_y , \bar{N}_{xy} , sous les formes suivantes :

$$\begin{aligned} u_x &= -w_x z_x + f; & v_y &= -w_y z_y + g & \text{et} \\ u_y + v_x &= -w_x z_y - w_y z_x + k \end{aligned}$$

en désignant par f , g et k les efforts membranaires. En éliminant u et v de ces équations, on obtient une équation différentielle relative à la flexion verticale w :

$$L(W) \equiv w_{xx} z_{yy} - 2w_{xy} z_{xy} + w_{yy} z_{xx} = R$$

l'opérateur L étant caractéristique de la méthode de Pucher. R contient les efforts membranaires. Si l'on considère w comme connu (trouvé), on obtient u et v par quadrature. On peut également, en variante, établir pour u et v des équations différentielles semblables à celle qui concerne w et dont l'opérateur de deuxième ordre est également L .

Dans la plupart des applications aux ouvrages, les bords des voûtes minces sont supportés par des éléments plans verticaux. Les différentes actions perturbatrices qui s'exercent sur un tel élément de bordure sont formulées à titre de conditions marginales pour l'étude des déformations. Parmi ces conditions, la plus importante et la plus utile, suivant laquelle aucun glissement ne doit se produire entre le bord de la voûte mince et l'élément plan, est exprimée en w .

Les auteurs étudient tout d'abord les voûtes minces comportant une courbure positive de Gauss. L'attention est particulièrement attirée sur les déformations d'un parabolôïde elliptique comportant quatre éléments plans de bordure, avec tassement non uniforme des appuis aux angles, charge superficielle et autres conditions. Les résultats de ces investigations peuvent être appliqués à la construction pratique.

Des problèmes analogues sont ensuite étudiés pour le parabolôïde hyperbolique, à titre d'exemple d'une voûte mince présentant une courbure négative de Gauss. L'étude de ses déformations exemptes d'allongement montre que dans une telle voûte, il peut se produire des flexions sans allongement, avec conditions marginales homogènes, lorsqu'une condition critique ou particulière est remplie. A titre d'exemple de déformations avec allongements, les auteurs étudient la même voûte mince dans le cas d'une charge superficielle uniformément répartie. Cette investigation est essentiellement destinée à mettre en évidence les défauts de coordination des autres caractéristiques discriminatoires dans ces problèmes relatifs aux membranes hyperboliques, défaut qui rend problématique la possibilité d'application pratique de la solution de ces problèmes. La voûte mince montre que la théorie des membranes donne des résultats entièrement acceptables pour les déformations, dans le cas des ellipses; elle montre toutefois aussi que dans le cas des hyperboles, les résultats des calculs analogues ne présentent qu'une valeur très limitée et exigent une transformation de la théorie de la flexion.

Zusammenfassung

In der vorliegenden Arbeit wird eine Membrantheorie für Formänderungen von Schalen beliebiger Form, außer Rotationsflächen und Zylindern, die früher behandelt wurden, entwickelt. Dank der Verwendung von Kartesischen Koordinaten und projizierten Membrankräften entsprechen viele formale Einzelheiten der Methode von Pucher zur Lösung der Membranspannungsprobleme. Diese Methode dient als Grundlage.

Nach einer Ableitung der kinematischen und elastischen Beziehungen

erhält man die elasto-kinematischen Gleichungen zwischen den Komponenten u , v , w der Verschiebungen und den Membrankräften \bar{N}_x , \bar{N}_y , \bar{N}_{xy} zu

$$u_x = -w_x z_x + f; \quad v_y = -w_y z_y + g \quad \text{und} \\ u_y + v_x = -w_x z_y - w_y z_x + k,$$

worin f , g und k die Membrankräfte enthalten. Eliminiert man aus diesen Gleichungen u und v , so erhält man eine Differentialgleichung für die vertikale Durchbiegung w :

$$L(W) \equiv w_{xx} z_{yy} - 2w_{xy} z_{xy} + w_{yy} z_{xx} = R,$$

wobei der Operator L typisch für die Methode von Pucher ist. R enthält die Membrankräfte. Nimmt man w als bekannt an (gefunden), so erhält man u und v durch Quadratur. Als Alternative werden für u und v zu derjenigen für w ähnliche Differentialgleichungen aufgestellt, deren Operator zweiter Ordnung ebenfalls L ist.

In den meisten baulichen Anwendungen werden die Schalenränder von vertikalen Binderscheiben getragen. Die verschiedenen Randstörungen eines solchen Randgliedes werden als Randbedingungen für die Formänderungen formuliert. Unter diesen ist die wichtigste und nützlichste Bedingung ausgedrückt in w , daß zwischen Schalenrand und Scheibe kein Gleiten stattfinden darf. Zuerst werden Schalen mit positiver Gaußscher Krümmung untersucht. Spezielle Aufmerksamkeit wird den Formänderungen eines elliptischen Paraboloids mit vier Randscheiben bei ungleichmäßiger Setzung der Eckstützen, Flächenbelastung und andern Umständen gewidmet. Die Ergebnisse dieser Untersuchungen sind in der Konstruktionspraxis anwendbar.

Die entsprechenden Probleme werden dann für das hyperbolische Paraboloid als Beispiel einer Schale mit negativer Gaußscher Krümmung entwickelt. Eine Untersuchung über ihre dehnungslosen Formänderungen zeigt, daß die dehnungslosen Durchbiegungen mit homogenen Randbedingungen in einer solchen Schale vorkommen können, falls eine kritische oder Eigenwertsbedingung erfüllt ist. Dieselbe Schale wird unter gleichmäßig verteilter Flächenlast als ein Beispiel für Formänderungen mit Dehnungen diskutiert. Diese Untersuchung dient hauptsächlich dazu, den Mangel an Zusammenhang weiterer Unterscheidungsmerkmale in solchen hyperbolischen Membranproblemen hervorzuheben, was ihre Lösung für die praktische Anwendung fragwürdig macht. Die Schale zeigt, daß die Membrantheorie ganz annehmbare Resultate für die Formänderungen im Fall von Ellipsen gibt; sie zeigt aber auch, daß im Fall von Hyperbeln die Ergebnisse ähnlicher Berechnungen nur einen sehr beschränkten Wert haben und den Ausbau der Biegelehre fordern.