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Distribution of Loads in Two Types of Railway Bridges

Répartition des charges sur deux types de ponts de chemin de fer

Lastverteilung in zwei Eisenbahnbrückentypen

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Notation

a = spacing between centers of longitudinal beams, in feet.
a' = distance between edge beam and nearest longitudinal diaphragm, in feet.
b = the total width of the bridge, in feet, measured from center to center of edge girders.
g = distance between rail centers, taken as 5 feet in numerical computations.
k = spring constant.
l = length of bridge, center to center of bearings.
s = spacing of transverse members, in feet.
x, y = rectangular coordinate axes.
E = modulus of elasticity for a longitudinal member.
E_c = modulus of elasticity for a transverse member.
I = moment of inertia of a longitudinal member.
I_c = moment of inertia of a transverse member.
M = bending moment.
$M_{r,r+1}$ = couple at the end r in beam extending from r to $r+1$.
N = number of transverse members.
γ = $\left(\frac{K}{4EI}\right)^{1/4}$.
λ = $\frac{N E_c I_c l^4}{l E I a^3 \pi^4}$, a dimensionless parameter.
ξ = x coordinate of position of a concentrated load.

Other symbols are defined wherever necessary.

Introduction

The present study is an investigation of the longitudinal and lateral distribution of loads in certain types of railway bridges in common use; the aims of the investigation are to make a theoretical analysis and to furnish formulas and tables summarizing the results.

The terms "longitudinal" and "lateral" distribution arise basically out of the kinds of problem encountered in the design of two types of railroad bridges. These are the two general types investigated. In the first type the problem of interest is the "longitudinal" (parallel to traffic) distribution of a train of wheel loads and in the second type the problem is the "transverse" (perpendicular to direction of traffic) distribution of a train of wheel loads. The definition of "longitudinal" and "lateral" will be made clear in the course of the next few paragraphs.

Type I. This type consists of a large number of transverse beams supported on heavy longitudinal edge girders. It is necessary, for economical design of this type, for a wheel load or wheel loads to be distributed among as many of the transverse beams as possible. The problem is thus one of investigating the propagation of the load or loads in the direction parallel to the traffic or the

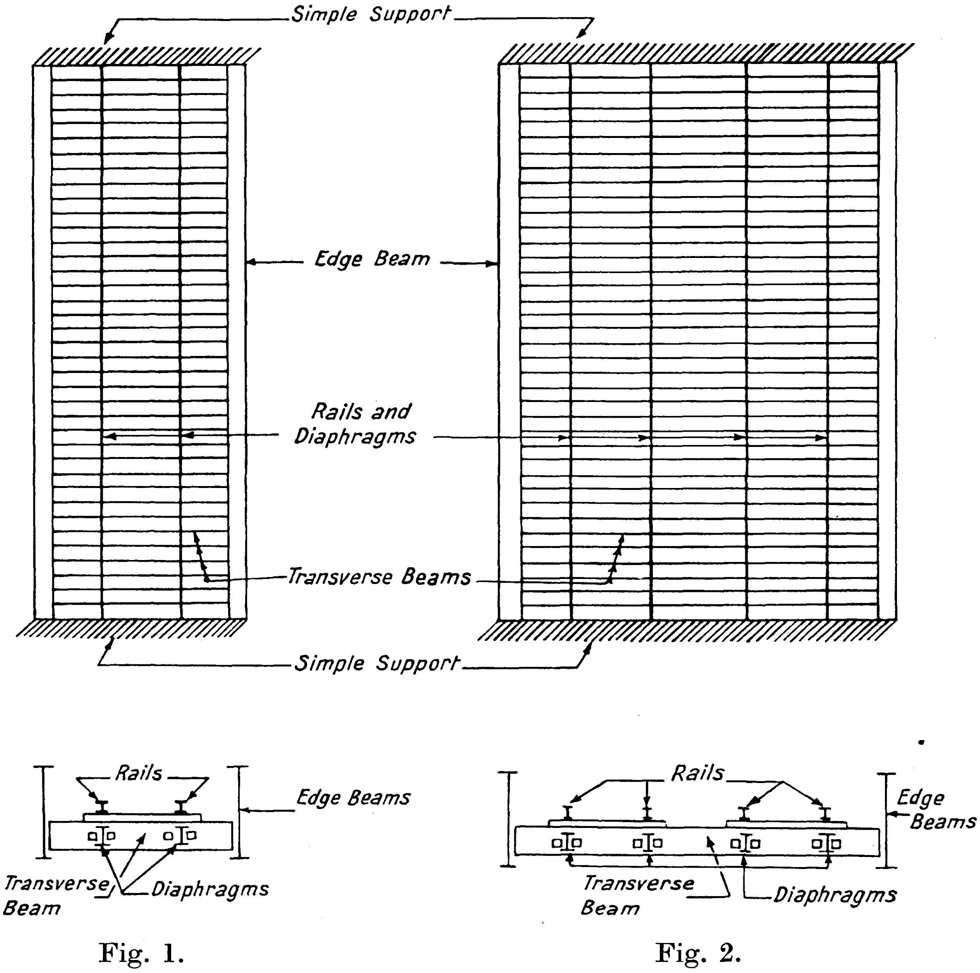

Fig. 1. Fig. 2.

"longitudinal distribution". This distribution is achieved to a considerable extent by the rails themselves. However, several other devices are employed in order to achieve a more uniform distribution. An important and widely used method is by means of longitudinal "diaphragms". As generally used, these diaphragms consist of plates or short I beams, placed across two adjacent transverse beams and attached to the webs of the latter by clip angles, the diaphragms running the full length of the bridge. Such diaphragms may be placed on the same vertical line as the running rails, or anywhere else over the transverse section as the designer may think proper.

Other methods which to a greater or lesser extent assist in the longitudinal distribution of the loads consist of ballasting the floor, placing of a steel floor plate, or placing a concrete liner several inches in thickness on the bridge floor. One or more of these devices may be used in any particular bridge.

This paper confines itself to investigating the distribution of loads in bridges of the open-deck type, that is to say, those which do not carry either ballast or concrete liner. However, in the light of experimental results, the theory may be modified to take approximate account of these additional factors.

There are several variations of this basic type. In a single track bridge, the diaphragms may be immediately underneath the rails as mentioned above, or a single diaphragm may run along the longitudinal center line of the bridge. Similar variations are also possible in a double track bridge. And further, as has been noted above, the diaphragms may be placed anywhere over the transverse section, depending on the judgment of the designer.

In order to keep these variations within limits, two types are investigated as shown in figs. 1 and 2 and designated, respectively, types I a and I b.

Evidently the results of the analysis would depend entirely on the assumptions made regarding the behaviour of the rails and the diaphragms as component parts of the whole structure. In order to keep these assumptions and therefore the limitations of the analysis in mind it is convenient to list them:

1. The bridge is assumed to be simply supported at the ends; all bridges investigated are right bridges.

2. The transverse beams are assumed to be closely spaced and therefore taken as approximating a continuous distribution. Without this simplification the problem becomes intractable mathematically in a general form.

3. The rails and the longitudinal diaphragms are supposed to behave as continuous beams. This is perhaps the most questionable of the assumptions. It may be said that this assumption has the effect of overestimating the stiffness of the longitudinal diaphragms, that is, they in effect achieve a better longitudinal distribution than they may be expected to do in practice. This follows from the nature of the diaphragms. Their behavior must depend upon the action of the clip angles and the elasticity of the rivets that connect them to the transverse beams. However, diaphragms are customarily treated as continuous beams and this assumption gives the simplest mathematical picture

of the action of the diaphragms. One may take a smaller stiffness to account for the partial effectiveness of the connections.

4. The torsional stiffness of all members is considered negligible. This assumption is amply justified for all practical purposes.

5. Concrete slabs and ballast are not taken into account. The introduction of slabs changes the picture completely while the effect of ballast must necessarily be determined by experimental methods. On the other hand, steel floor-plates may be taken into account by combining their effect with that of the transverse beams. This amounts to saying that while the torsional stiffness of a concrete slab cannot be neglected, that of a steel plate may be, without appreciable error.

With the above limitations, the exact solution of the problem is not difficult. However, the mathematical solution yields results which are not convenient for design purposes, and approximate solutions have to be devised. Furthermore, the exact solution leads to very slowly converging infinite series and numerical computation becomes quite laborious.

Type II. The second basic type of bridge investigated consists of a number of equally spaced, identical longitudinal beams. Timber ties are placed across the top of these beams. It is necessary, for the most efficient use of the longitudinal beams, that the wheel loads be distributed as uniformly as possible over the cross section of the bridge. In an ideal bridge, each of the longitudinal beams will carry exactly $1/n$ of the total statical moment over any cross section, where n is the number of beams. Such a bridge is, of course, impossible in practice and the problem here is one of investigating the propagation of the wheel loads in the direction transverse to the direction of traffic, that is, of "lateral distribution". This distribution is achieved, to a considerable extent, by the ties themselves. However, several other devices are employed in order to achieve a more uniform distribution. A widely used method is by means of transverse "diaphragms". Generally, these diaphragms consist of a plate or short I beam placed across two adjacent longitudinal beams and attached to the webs of the latter by clip angles. A series of these short diaphragms constitute a transverse diaphragm extending the full width of the bridge. Such diaphragms may be placed anywhere along the length of the bridge, but are usually widely spaced compared to the ties.

Other methods which to a greater or lesser degree assist in the lateral distribution of the loads consist of ballasting the floor, placing of a steel floor plate or placing of a concrete deck. One or more of these methods may be used in any given bridge.

This paper, as mentioned above, confines itself primarily to investigating bridges of the open-deck type. However, in the light of experimental results, if these are available, the theory may be modified to take approximate account of such facts as ballast and concrete deck.

The variations within this basic type consist in a change in the number of longitudinal beams or in the number of diaphragms, or both. However, in order to reduce the number of variables as well as to simplify the general problem both from the mathematical and design standpoints it is convenient to add the total stiffness of the diaphragms to that of the ties in all cases. This is because even in bridges with transverse diaphragms the ties may safely be assumed to constitute the members primarily effective in the lateral distribution of loads, as may be seen by a comparison of the relative flexural rigidity of diaphragms to ties in any existing bridge. The variations in this type of bridge will consist, after the above simplification, merely in a change in the number of longitudinal beams.

Three variations of type II are investigated, consisting of 4, 5 and 6 longitudinal beams. These are referred to, respectively, as type II a, II b and II c.

The assumptions made in the analysis are as follows:

1. The bridge is assumed to be simply supported at the ends; all bridges investigated are right bridges.

2. The ties are always sufficiently closely spaced to approximate a continuous distribution. The flexural rigidity of the diaphragms is added to that of the ties. The stiffness is thus the average for the actual spacing.

3. The diaphragms and ties are assumed to behave as continuous beams and the ties are supposed to extend the full width of the bridge. The first assumption is justified in the case of the ties but probably not for the diaphragms. The second assumption is justified in single track bridges of the open deck type. In double-track bridges the assumption may not always be strictly correct.

4. The torsional stiffness of all members is treated as negligible.

5. Concrete slabs and effect of ballast are not taken into account. If a floor plate exists its flexural rigidity may be added to that of the ties.

6. All longitudinal girders are identical and equally spaced.

With the assumptions above the formal solution is a simple one. Again the exact solution is not convenient for design purposes. By the use of tables and charts, however, the solution of specific problems may be simplified.

From the foregoing it is evident that this paper makes no attempt at a comprehensive study of the subject. Because of the complexity of the general problem it has been found necessary to ignore various phases.

Historical Review

The problem of lateral and longitudinal distribution of loads in a railroad bridge is a special and somewhat simple case of a more general problem. In the general problem the structure consists of an arbitrary number of longitudinal girders interconnected cross-wise by an arbitrary number of transverse girders. On this network is placed a deck slab, usually of reinforced concrete.

Rollings loads may move both laterally and longitudinally on the bridge. The bridge may be continuous over several supports.

The difficulties in the "exact" analysis of such a structure are numerous. Some of the difficulties are as follows:

The structure is actually a ribbed plate and no mathematical treatment of the problem exists. The closest approximations to the actual bridge are obtained by treating it as an anisotropic plate or as a system of interconnected girders in which the slab acts as the top flange of the girders. This second method neglects the twisting resistance of the place, while the first method assumes a continuous distribution of the flexural rigidities of the girders.

If the bridge does not possess a deck plate, the greatest difficulty arises in the treatment of a finite number of cross girders. The employment of trigonometric series is not possible, and one has to have recourse to the various methods of solving statically indeterminate structures. Whatever the method employed, an exact solution must find all the indeterminate quantities which are at least equal to the total number of nodes. Since these may be, in any given case, very large in number, the exact solutions are necessarily laborious, even if certain quantities involved in the computation are furnished in the form of tables.

Charles Massonnet [2, 3] gives an excellent historical summary of the various investigations of the general problem. The first investigators in this field include F. Engesser, A. Ostenfeld and others. The method of using the "characteristic loads" of a network of girders has been used by Bleich and Melan. For a numerical approach to the problem relaxation methods have been used, but are very laborious except in the simplest cases.

All the exact methods lead to the construction of influence *surfaces* for moments, shears, etc. To compute the moments one has to compute the volume under the influence surface.

The exact methods are of little practical utility. They serve, however, as valuable guides to the error involved in approximate methods. Among the most important of the approximate methods is the one attributed to Engesser, in which the cross girders are treated as infinitely rigid. According to Massonnet this method gives reasonably accurate results in certain instances. Leonhardt has improved upon Engesser's method by taking into account the actual flexural rigidity of the cross girder.

Charles Massonnet has recently treated the problem as an anisotropic plate [2].

The above historical review is given very briefly as the solutions there do not have a direct bearing upon the problems under investigation in the present paper. They are however invaluable as guides in tackling the present research. In the first place it is clear that diaphragms are treated in all cases as "cross girders" with a definite flexural rigidity. This, while questionable as representing the true action of these members, seems necessary in order to formulate a tractable mathematical problem. Secondly the treatment of a *small* number of transverse girders (or "diaphragms", the terms are used interchangeably) leads to difficulties while a large number of such girders may be approximated by a continuous distribution and therefore analyzed by the use of trigonometric series.

Fortunately, in the types of railroad bridges under study, the transverse members are generally large in number, as indicated in the Introduction. The exception to this rule arises only where a number of longitudinal girders are connected by a very small (that is widely separated) number of transverse diaphragms. But in such cases, the distribution is achieved primarily by the ties, which should accordingly be classed as "diaphragms"; the effect of the actual diaphragms may be approximately taken into account by adding their flexural rigidity to that of the ties.

It is also clear from the above review that the closest theoretical approach to a railroad bridge with a deck slab is that of the anisotropic plate. In the longitudinal direction the component parts are the longitudinal girders (assumed distributed uniformly over the width of the bridge) and the deck slab while in the transverse direction they consist of the ties and the deck slab. The picture in an actual bridge is, however, complicated by the fact that the slab and the ties do not usually have the same boundaries. For this reason, the effect of the slab is not taken into account in this paper, and open deck bridges are exclusively considered in the analysis.

Longitudinal Distribution of Loads

Case Ia. Two Longitudinal Diaphragms

In the following analysis the rails are treated as the longitudinal diaphragms. If actual diaphragms exist immediately underneath the rails, the moments of inertia of such diaphragms may be added to that of the rails, it being assumed that a diaphragm and the rail above it act as two separate longitudinal beams.

Consider any transverse section of the bridge (fig. 3).

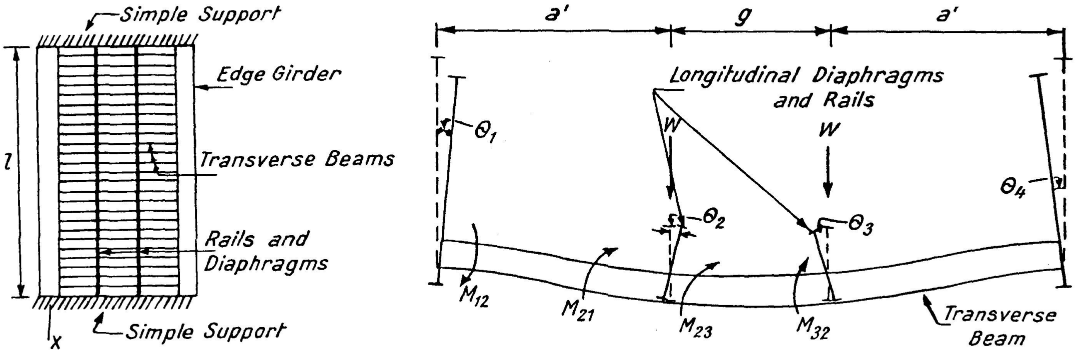

Fig. 3.

Assume an arbitrary set of deflections $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$ positive downward, and positive (clockwise) rotation of all longitudinal members. The transverse beams are supposed to be sufficiently closely spaced to approximate a continuous distribution. If N is the number of transverse beams, $E_c I_c$ the flexural rigidity of each beam, then the following slope-deflection equations may be written where the M's are the bending moments per unit length of bridge:

$$M_{12} = \frac{N E_c I_c}{l a'}\left[4\Theta_1 + 2\Theta_2 - 6\frac{(y_2 - y_1)}{a'}\right],$$

$$M_{21} = \frac{N E_c I_c}{l a'}\left[2\Theta_1 + 4\Theta_2 - 6\frac{(y_2 - y_1)}{a'}\right],$$

$$M_{23} = \frac{N E_c I_c}{l g}\left[4\Theta_1 + 2\Theta_3 - 6\frac{(y_3 - y_2)}{g}\right].$$

Due to symmetry, $y_3 = y_2$, $\Theta_3 = -\Theta_2$.

Hence the equations become:

$$M_{12} = \frac{N E_c I_c}{l a'}\left[4\Theta_1 + 2\Theta_2 - 6\frac{(y_2 - y_1)}{a'}\right],$$

$$M_{21} = \frac{N E_c I_c}{l a'}\left[2\Theta_1 + 4\Theta_2 - 6\frac{(y_2 - y_1)}{a'}\right],$$

$$M_{23} = \frac{N E_c I_c}{l g} 2\Theta_2.$$

If $V_1(x)$, $V_2(x)$ denote the downward shears produced on the girders (or diaphragms) by the transverse beams, then:

$$V_1(x) = -\frac{(M_{12} + M_{21})}{a'},$$

$$V_2(x) = \frac{(M_{12} + M_{21})}{a'} - \frac{(M_{23} + M_{32})}{a'} = \frac{(M_{12} + M_{21})}{a'}.$$

Hence:

$$\begin{aligned} E I_1 \frac{d^4 y_1}{d x^4} &= -\frac{N E_c I_c}{l a'^2}\left[6\Theta_1 + 6\Theta_2 - 12\frac{(y_2 - y_1)}{a'}\right], \\ E I_2 \frac{d^4 y_2}{d x^4} &= -\frac{N E_c I_c}{l a'^2}\left[6\Theta_1 + 6\Theta_2 - 12\frac{(y_2 - y_1)}{a'}\right] + \text{External Load} \end{aligned} \tag{a}$$

Neglecting the torsional stiffnesses of the girders and diaphragms, it follows that:

$$M_{12} = M_{21} + M_{23} = 0.$$

These result in the following:

$$\begin{aligned} 4\Theta_1 + 2\Theta_2 - 6\frac{(y_2 - y_1)}{a'} &= 0, \\ \frac{3 y_1}{a'} - \frac{3 y_2}{a'} + \Theta_1 + \frac{(2g + a')}{g}\Theta_2 &= 0. \end{aligned} \tag{b}$$

The unknown $y_r(x)$ and $\Theta_r(x)$ may be taken in the form:

$$\left.\begin{aligned} y_r &= \sum_{n=1}^{\infty} a_{nr} \sin \frac{n\pi x}{l} \\ \Theta_r &= \sum_{n=1}^{\infty} b_{nr} \sin \frac{n\pi x}{l} \end{aligned}\right\} r = 1, 2. \tag{c}$$

(c) satisfy the boundary conditions of simple support.

The wheel loads W may be represented by the series:

$$W = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l}. \tag{d}$$

Substituting the series (c) into the equations (b) there results:

$$4 b_{n1} + 2 b_{n2} - 6 \frac{(a_{n2} - a_{n1})}{a'} = 0,$$

$$\frac{3}{a'}(a_{n1} - a_{n2}) + b_{n1} + \frac{(2g + a')}{g} b_{n2} = 0.$$

The above two equations when solved for the b_{nr} yield:

$$\begin{aligned} b_{n1} &= \frac{3(a' + g)}{a'(2a' + 3g)}(a_{n2} - a_{n1}), \\ b_{n2} &= \frac{3g}{a'(2a' + 3g)}(a_{n2} - a_{n1}). \end{aligned} \tag{e}$$

Substituting the series (c) into equations (a) and eliminating the b_{nr} by means of (e), there result the following equations, the external load in the second of equations (a) being replaced by the series (d):

$$E I_1 \frac{n^4 \pi^4}{l^4} a_{n1} = -\frac{N E_c I_c}{l a'^2} \frac{6(a_{n1} - a_{n2})}{(2a' + 3g)},$$

$$E I_2 \frac{n^4 \pi^4}{l^4} a_{n2} = \frac{N E_c I_c}{l a'^2} \frac{6(a_{n1} - a_{n2})}{(2a' + 3g)} + C_n.$$

Denoting $\frac{6 N E_c I_c}{(2a' + 3g) E I_1 l a'^2}$ by λ_1, and $\frac{6 N E_c I_c}{(2a' + 3g) E I_2 l a'^2}$ by λ_2.

These equations become:

$$\begin{aligned} a_{n1}\left(\frac{n^4 \pi^4}{l^4} + \lambda_1\right) - \lambda_1 a_{n2} &= 0, \\ a_{n2}\left(\frac{n^4 \pi^4}{l^4} + \lambda_2\right) - \lambda_2 a_{n1} &= \frac{C_n}{E I_2}. \end{aligned} \tag{f}$$

Solving these equations simultaneously:

$$a_{n1} = \frac{C_n \lambda_1}{E I_2 \left\{\frac{n^8 \pi^8}{l^8} + \frac{n^4 \pi^4}{l^4}(\lambda_1 + \lambda_2)\right\}},$$

$$a_{n2} = \frac{C_n \left(\frac{n^4 \pi^4}{l^4} + \lambda_1\right)}{E I_2 \left\{\frac{n^8 \pi^8}{l^8} + \frac{n^4 \pi^4}{l^4}(\lambda_1 + \lambda_2)\right\}}.$$

Bending Moments

In general, only the bending moments in the transverse beams are required. In such cases it is convenient to have expressions giving these moments directly. Thus

$$
\begin{aligned}
M_{21} &= \frac{N E_c I_c}{l a'} \frac{6 (y_1 - y_2)}{(2 a' + 3 g)}, \\
&= \frac{-6 N E_c I_c}{l a' (2 a' + 3 g)} \sum_{n=1}^{\infty} \frac{C_n}{E I_2} \left\{ \frac{1}{\frac{n^4 \pi^4}{l^4} + \lambda_1 + \lambda_2} \right\} \sin \frac{n \pi x}{l}, \\
&= (\lambda_1 + \lambda_2) \lambda_2 a' \frac{l^8}{\pi^8} \sum_{n=1}^{\infty} \frac{C_n}{n^4 \left\{ n^4 + (\lambda_1 + \lambda_2) \frac{l^4}{\pi^4} \right\}} \sin \frac{n \pi x}{l} - \lambda_2 a' \frac{l^4}{\pi^4} \sum_{n=1}^{\infty} \frac{C_n}{n^4} \sin \frac{n \pi x}{l}.
\end{aligned}
$$

In the case of concentrated loads W placed at $x = \xi$,

$$
C_n = \frac{2 W}{l} \sin \frac{n \pi \xi}{l}.
$$

Substituting this in the above, the expression becomes

$$
\begin{aligned}
M_{21} = {} & 2 (\lambda_1 + \lambda_2) \lambda_2 a' \frac{l^7}{\pi^8} \sum_{n=1}^{\infty} \frac{\sin \frac{n \pi \xi}{l} \sin \frac{n \pi x}{l}}{n^4 \left\{ n^4 + (\lambda_1 + \lambda_2) \frac{l^4}{\pi^4} \right\}} \\
& - 2 \lambda_2 a' \frac{l^3}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin \frac{n \pi \xi}{l} \sin \frac{n \pi x}{l}.
\end{aligned} \tag{1}
$$

Case Ib. Four Longitudinal Diaphragms (Double Track Bridge)

In double track bridges, it is customary to use four diaphragms these being placed directly underneath the rails. Such an arrangement is assumed here and a symmetrical loading condition, which in general will yield the maximum bending moment in the transverse beams is contemplated.

Referring to fig. 4 and proceeding as in the case of two longitudinal diaphragms, the deflections y_1, y_2 and y_3 may be expressed as follows:

$$
y_r = \sum_{n=1}^{\infty} a_{nr} \sin \frac{n \pi x}{l} \qquad r = 1, 2, 3,
$$

where

$$
a_{n1} = \frac{\frac{C_n}{E I_2} \left\{ \lambda_1 \frac{(3c+2g)}{a' g} + \frac{2 \lambda_1 \lambda_2}{g^4} \frac{l^4}{\pi^4 n^4} \left[(3c+6g+2a')(3c+2g) - 9(c+g)^2 \right] \right\}}{\frac{n^8 \pi^8}{l^8} + \frac{n^4 \pi^4}{l^4} \left\{ \lambda_1 \frac{(3c+2g)}{a' g} + \lambda_2 \left[\frac{(3c+2g)}{a' g} + \frac{6(c+g)}{g^2} + \frac{6 a' c + 12 a' g + 4 a'^2}{g^3} \right] \right\} + \frac{\lambda_2 (2 \lambda_1 + \lambda_2)}{g^4} \left[(3c+2g)(3c+6g+2a') - 9(c+g)^2 \right]},
$$

$$a_{n3} = \frac{\frac{C_n}{E I_2}\left\{\frac{n^4\pi^4}{l^4} + \lambda_1 \frac{(3c+2g)}{a' g} + \lambda_2\left[\frac{3c+2g}{a' g} + \frac{9(c+g)}{g^2} + \frac{6a'c+12a'g+4a'^2}{g^3}\right] + \frac{\lambda_2(2\lambda_1+\lambda_2)}{g^4}\left[(3c+2g)(3c+6g+2a') - 9(c+g)^2\right]\right.}{\frac{n^8\pi^8}{l^8} + \frac{n^4\pi^4}{l^4}\left\{\lambda_1\frac{(3c+2g)}{a' g} + \lambda_2\left[\frac{(3c+2g)}{a' g} + \frac{6(c+g)}{g^2} + \frac{6a'c+12a'g+4a'^2}{g^3}\right]\right\} + \frac{2\lambda_1\lambda_2 l^4}{g^4\pi^4 n^4}\left[(3c+2g)(3c+bg+2a') - 9(c+g)^2\right]\right\}}.$$

and y_2 may be obtained from the relation

$$y_2 = -\left(\frac{\lambda_2}{\lambda_1} y_1 + y_3\right) + \frac{2}{E I_2}\frac{l^4}{\pi^4}\sum_{n=1}^{\infty}\frac{C_n}{n^4}\sin\frac{n\pi x}{l},$$

in which

$$\lambda_1 = \frac{3 N E_c I_c}{l a'^2 E I_1\left\{\frac{3(g+2c)}{2a'} + \frac{3c+2g}{g}\right\}}, \qquad \lambda_2 = \frac{3 N E_c I_c}{l a'^2 E I_2\left\{\frac{3(g+2c)}{2a'} + \frac{3c+2g}{g}\right\}}.$$

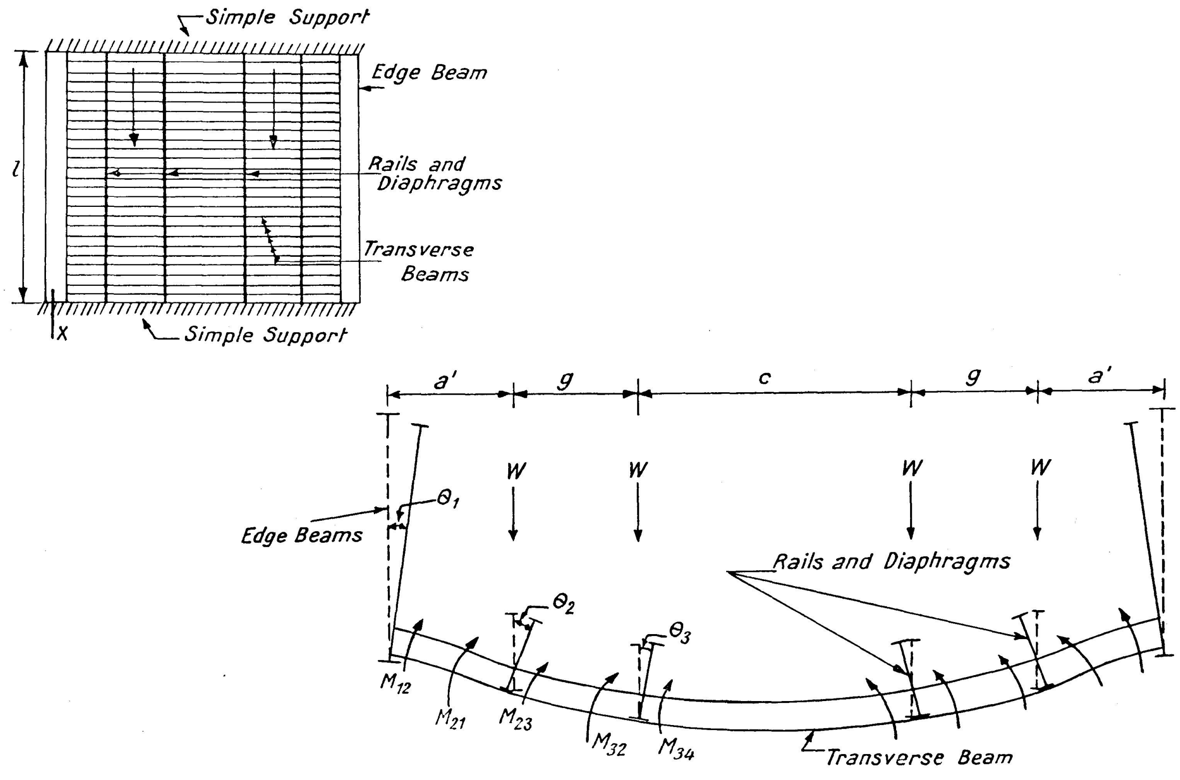

Fig. 4.

Approximate Solutions for Longitudinal Distribution

The analyses presented above result in infinite series which are, in most practical instances, very slowly converging. This slow convergence is due, in the main, to the large relative stiffness of transverse beams to rail (or longitudinal diaphragm). The method given above possesses the advantage of being general and the same procedure may be employed for any number of longitudinal diaphragms, whether these consist of rails, or actual diaphragms or both, the only limit being the increasing complexity of the algebra involved.

The particular cases discussed above are, however, susceptible of a much simpler solution by assuming that the edge girders are infinitely rigid and the bridge is infinitely long. The longitudinal diaphragms may then be looked upon as infinite beams on an elastic foundation made up of the transverse beams. It may be shown by actual solution of a bridge of practical proportions that the above assumptions have a negligible effect upon the final result.

Assume, as noted above, that the supporting girders are infinitely rigid; within certain limits, to be specified below, depending on the proportions of the members, it is sufficient to assume that the bridge is infinitely long.

HETENYI [1] gives the following equation for the deflection of an infinitely long beam on an elastic foundation:

$$y = \frac{P\gamma}{2k} e^{-\gamma x} (\cos \gamma x + \sin \gamma x).$$

Where P = the concentrated load on the beam at $x = 0$.
k = the "spring constant", defined as the load required for unit downward deflection for unit length of spring.
$\gamma = \left(\frac{k}{4EI}\right)^{1/4}$.
EI = Flexural rigidity of beam.

HETENYI further notes that a beam of finite length may be treated as one of infinite length without appreciably impairing the accuracy of the solution, provided that:

$$l \geqq \frac{2\pi}{\gamma}.$$

It is evident that the problems treated above can be solved if the values of k for the various cases can be determined.

Ia. Single Track Bridge. Two Longitudinal Diaphragms

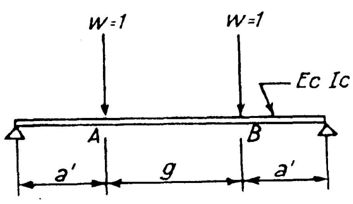

Fig. 5.

Referring to fig. 5 and assuming the ends are simply supported, the deflection at A due to unit loads at A and B is

$$\delta = a'^2 \frac{(2a' + 3g)}{6 E_c I_c}.$$

The flexural rigidity of the transverse beams per unit length of bridge is $\frac{N E_c I_c}{l}$. Hence the deflection of the foundations due to unit loads at A and B is given

by: $$\delta = l a'^2 \frac{(2a' + 3g)}{6 N E_c I_c},$$

whence
$$k = \frac{6\,N\,E_c\,I_c}{l\,a'^2\,(2\,a' + 3\,g)}.$$

Loads and Bending Moments in Transverse Beams

The load on the spring is the deflection times the spring constant. Hence the load on the transverse beams per unit length of bridge, due to wheel loads P is: $p' = P\,k\,y$

or
$$p' = \frac{P\gamma}{2} e^{-\gamma x} (\cos\gamma x + \sin\gamma x),$$

where
$$\gamma = \left(\frac{k}{4\,E\,I}\right)^{1/4} = \left\{\frac{3}{2}\,\frac{N}{l}\,\frac{E_c\,I_c}{E\,I}\,\frac{1}{a'^2\,(2\,a' + 3\,g)}\right\}^{1/4}. \qquad \text{(b)}$$

Equation (b) gives the concentrated loads p' on the transverse beams at a distance x from the applied wheel loads P. If the beam spacing is s then the total load on each beam may be obtained, approximately, as $p'\,s$. The exact value would involve the integral of expression (b) and is not warranted. Thus

$$p = p'\,s = \frac{P\gamma}{2} s\,e^{-\gamma x} (\cos\gamma x + \sin\gamma x). \qquad \text{(c)}$$

The maximum bending moment in the transverse beam due to the loads P is given by:

$$M = p\,a' = \frac{P\,\gamma\,s}{2} a'\,e^{-\gamma x} (\cos\gamma x + \sin\gamma x). \qquad \text{(d)}$$

In order to facilitate preparation of tables the length l of the bridge may be eliminated from the expression for γ by the following approximation:

$$(N-1)\,s = l, \quad \text{or} \quad \frac{(N-1)}{l} = \frac{1}{s}.$$

Approximately $N/l = 1/s$ since $N \gg 1$.
The expression for γ thus becomes:

$$\gamma = \left\{\frac{3}{2}\,\frac{E_c\,I_c}{E\,I}\,\frac{1}{s\,a'^2\,(2\,a' + 3\,g)}\right\}^{1/4}.$$

Bending Moment due to a Train of Loads

In practice, the bending moment in a transverse beam is caused by a train of loads and not due to a single load. Thus, it would be necessary to compute the effect of each one of the train of loads by means of the expression (c). However, the effect of a load on a beam decreases rapidly with the distance due to the negative exponential in expression (c). This suggests that if some "standard" loading, which is flexible enough to take account of slight variations in the actual loading, is assumed, then the designer's task will be greatly lightened without unduly affecting the precision of the solution.

The standard loading here suggested consists of three equal axle loads each of $2\,P$, which will usually be the heaviest drivers, spaced a distance z

apart. The maximum bending moment due to this standard loading may be obtained by a simple extension of the results of the previous section. The maximum effect of this train on any beam will be when the load on the central driver $2P$ is directly over the beam. The expression for the maximum load on the beam becomes:

$$p = P\gamma s\{\tfrac{1}{2} + e^{-\gamma z}(\cos\gamma z + \sin\gamma z)\} \tag{e}$$

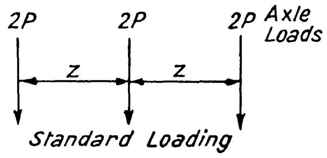

Fig. 6.

and the expression for the maximum bending moment in any transverse beam is:

$$M = p a' = P a' \gamma s\{\tfrac{1}{2} + e^{-\gamma z}(\cos\gamma z + \sin\gamma z)\}, \tag{f}$$

or

$$M = P a' s \gamma \beta_{\gamma z}, \tag{g}$$

where

$$\beta_{\gamma z} = \{\tfrac{1}{2} + e^{-\gamma z}(\cos\gamma z + \sin\gamma z)\}.$$

Values of $\beta_{\gamma z}$ may be obtained from table 4 given in the Appendix and tables of γ for any given value of $I_c/I s$ and a' are furnished in table 1 in the Appendix. g, the gauge, is taken in all cases as 5 feet, this being closely the distance center of rails generally employed in American and many other railroads.

Case Ic. Double Track Bridge. Four Longitudinal Diaphragms

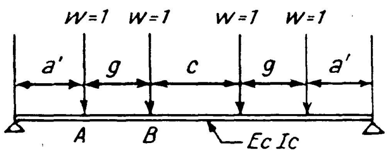

Fig. 7.

Referring to fig. 7 and assuming that the condition at the supporting girders approximates one of simple support, the deflection at A due to all the unit loads shown is:

$$\delta_A = \frac{l a'}{6 N E_c I_c}(4 a'^2 + 12 a' g + 6 a' c + 3 g^2 + 3 g c).$$

The deflection at B due to the unit loads shown is

$$\delta_B = \frac{l}{6 N E_c I_c}(4 a'^2 + 12 a'^2 g + 9 a' g c + 6 a'^2 c + 9 a' g^2 + 3 c g^2 + 2 g^3).$$

Thus

$$\left.\begin{aligned} k_A &= \frac{6 N E_c I_c}{l a'(4 a'^2 + 12 a' g + 6 a' c + 3 g^2 + 3 g c)}, \\ k_B &= \frac{6 N E_c I_c}{l(4 a'^3 + 12 a'^2 g + 9 a' g c + 6 a'^2 c + 9 a' g^2 + 3 c g^2 + 2 g^3)}. \end{aligned}\right\} \tag{g}$$

In this case the loads delivered by the rails on the transverse beam will not all be equal. The two exterior loads will be given by $k_A y_A$ and the two interior loads by $k_B y_B$

where

$$y_A = \frac{P\gamma_A}{2 k_A} e^{-\gamma_A x}(\cos\gamma_A x + \sin\gamma_A x),$$

$$y_B = \frac{P\gamma_B}{2 k_B} e^{-\gamma_B x}(\cos\gamma_B x + \sin\gamma_B x),$$

in which $$\gamma_A = \left(\frac{k_A}{4\,E\,I}\right)^{1/4}, \qquad \gamma_B = \left(\frac{k_B}{4\,E\,I}\right)^{1/4}.$$

Denoting the exterior loads on a beam by p_A and the interior loads by p_B and making the same approximation as for case Ia one has:

$$\begin{aligned} p_A &= k_A\, y_A\, s = \frac{P\,\gamma_A\, s}{2} e^{-\gamma_A x} (\cos\gamma_A x + \sin\gamma_A x), \\ p_B &= k_B\, y_B\, s = \frac{P\,\gamma_B\, s}{2} e^{-\gamma_B x} (\cos\gamma_B x + \sin\gamma_B x), \end{aligned} \tag{h}$$

in which

$$\gamma_A = \left(\frac{k_A}{4\,E\,I}\right)^{1/4} = \left\{\frac{3}{2}\,\frac{I_c}{I\,s}\,\frac{1}{a'\,g^2\left(4\frac{a'^2}{g^2}+12\frac{a'}{g}+6\frac{a'c}{g^2}+\frac{3c}{g}+3\right)}\right\}^{1/4},$$

$$\gamma_B = \left(\frac{k_B}{4\,E\,I}\right)^{1/4} = \left\{\frac{3}{2}\,\frac{I_c}{I\,s}\,\frac{1}{g^3\left(4\frac{a'^3}{g^3}+12\frac{a'^2}{g_2}+9\frac{a'c}{g^2}+9\frac{a'}{g}+\frac{3c}{g}+2\right)}\right\}^{1/4}.$$

N/l having been replaced by $1/s$.

In the expressions for γ_A and γ_B if g is taken as 5 feet and c, the distance center to center of inner rails, as 9 feet, γ_B is seen be a constant multiplied by γ_A, the constant depending only on the value of a'. If one writes $\gamma_B = \bar{\alpha}\,\gamma_A$

$$\bar{\alpha} = \left\{\frac{4\frac{a'^3}{g^3}+12\frac{a'^2}{g^2}+6\frac{a'^2c}{g^3}+3\frac{a'}{g}+3\frac{a'c}{g^2}}{4\frac{a'^3}{g^3}+12\frac{a'^2}{g^2}+9\frac{a'c}{g^2}+9\frac{a'}{g}+\frac{3c}{g}+2}\right\}^{1/4}. \tag{i}$$

Expressions (h) may therefore be written as follows omitting the subscript A in γ_A:

$$\begin{aligned} p_A &= \frac{P\,\gamma\, s}{2} e^{-\gamma x} (\cos\gamma x + \sin\gamma x), \\ p_B &= \frac{P\,\bar{\alpha}\,\gamma\, s}{2} e^{-\gamma x} (\cos\bar{\alpha}\,\gamma x + \sin\bar{\alpha}\,\gamma x). \end{aligned} \tag{j}$$

The maximum bending moment in the transverse beam due to the axle load of $2\,P$, thus becomes:

$$M = \frac{P\,s\,\gamma}{2}\{a'\,e^{-\gamma x}(\cos\gamma x + \sin\gamma x) + \bar{\alpha}\,(a'+g)\,e^{-\bar{\alpha}\gamma x}(\cos\bar{\alpha}\,\gamma x + \sin\bar{\alpha}\,\gamma x)\}. \tag{k}$$

Bending Moment due to a Train of Loads

The standard loading suggested for a single track bridge will be used here also. Consider the maximum effect of the three heaviest drivers, each of $2\,P$, placed symmetrically over a cross beam, the axle spacing being z. The maximum bending moment in the transverse beam for this loading is:

$$M = P\,s\,\gamma\{a'\,[\tfrac{1}{2} + e^{-\gamma z}(\cos\gamma z + \sin\gamma z)] + \bar{\alpha}\,(a'+g)\tfrac{1}{2} + e^{-\bar{\alpha}\gamma z}(\cos\bar{\alpha}\,\gamma z + \sin\bar{\alpha}\,\gamma z)]\}.$$

This may be simplified by writing:

$$M = P s \gamma [a' \beta_{\gamma z} + \overline{\alpha} (a' + g) \beta_{\overline{\alpha}\gamma z}], \tag{l}$$

in which

$$\beta_{\gamma z} = \tfrac{1}{2} + e^{-\gamma z} (\cos \gamma z + \sin \gamma z),$$
$$\beta_{\overline{\alpha}\gamma z} = \tfrac{1}{2} + e^{-\overline{\alpha}\gamma z} (\cos \overline{\alpha} \gamma z + \sin \overline{\alpha} \gamma z).$$

The values of β may be obtained from table 4 given in the Appendix. Table 2 gives the values of γ for values of I_c/I_s and a' commonly met with, and table 3 furnishes values of $\overline{\alpha}$ for corresponding values of a'. The numerical values of g and c have been taken as 5 ft. and 9 ft. respectively in the preparation of the tables.

It is found that for practical instances, the value of $\overline{\alpha}$ differs but little from unity, so that equation (1) may be approximately represented by:

$$M = P s \gamma (2 a' + g) \beta_{\gamma z}. \tag{m}$$

The Lateral Distribution of Loads

As stated in the Introduction it is assumed that the transverse "diaphragms" — whether these consist of ties alone or ties and actual diaphragms — have a continuous distribution. Furthermore the ties do not separate from the girders under deflection. All the longitudinal girders are assumed to be identical. Three cases are investigated: bridges with four, five and six longitudinal girders.

Case IIa. Four Longitudinal Girders

Consider a transverse section of the bridge (fig. 8). The wheel loads W cause external loads W_1, W_2, W_3 and W_4 to act on the longitudinal girders. Assume an arbitrary set of deflections $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$, positive downward, from their positions under no load and positive (clockwise) rotations of all longitudinal members. The unknowns are these deflections and rotations, which may be taken as:

$$\left.\begin{aligned} y_r &= \sum_{n=1}^{\infty} a_{nr} \sin \frac{n \pi x}{l} \\ \Theta_r &= \sum_{n=1}^{\infty} b_{nr} \sin \frac{n \pi x}{l} \end{aligned}\right\} \quad r = 1, 2. \tag{a}$$

Due to symmetry only beams 1 and 2 need be considered. Equations (a) satisfy the boundary conditions of simple support.

Let the external loads on the beams be represented by

$$W_r = \sum_{n=1}^{\infty} C_{nr} \sin \frac{n \pi x}{l} \quad r = 1, 2. \tag{b}$$

The transverse diaphragms being closely spaced, their flexural rigidity per unit length of bridge is $\frac{N E_c I_c}{l}$ where $E_c I_c$ is the flexural rigidity of each diaphragm, N the total number of diaphragms and 1 the length of the bridge.

Considering the portion of the diaphragm between any two longitudinal members, the following slope-deflection equations may be written:

$$\left.\begin{aligned} M_{12} &= \frac{N E_c I_c}{l a}\left[4\,\Theta_1 + 2\,\Theta_2 - 6\,\frac{(y_2 - y_1)}{a}\right], \\ M_{21} &= \frac{N E_c I_c}{l a}\left[2\,\Theta_1 + 4\,\Theta_2 - 6\,\frac{(y_2 - y_1)}{a}\right], \\ M_{23} &= \frac{N E_c I_c}{l a}(2\,\Theta_3), \\ M_{32} &= -M_{23}. \end{aligned}\right\} \quad \text{(c)}$$

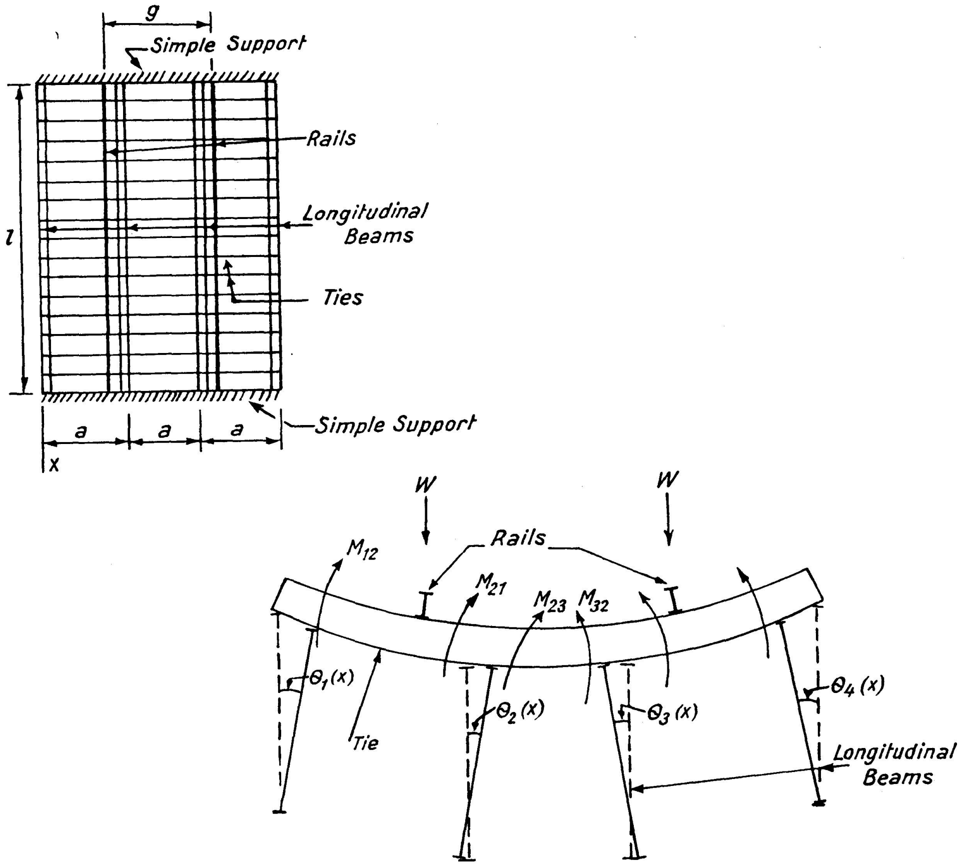

Fig. 8.

If the torsional stiffness of the beams is neglected it follows that:

$$M_{12} = M_{21} + M_{23} = 0,$$

or

$$\left.\begin{aligned} 4\,\Theta_1 + 2\,\Theta_2 - 6\,\frac{(y_2 - y_1)}{a} &= 0, \\ 2\,\Theta_1 + 6\,\Theta_2 - 6\,\frac{(y_2 - y_1)}{a} &= 0. \end{aligned}\right\} \quad \text{(d)}$$

Substituting the series (a) into equation (c) and solving for the b_{nr} in terms of the a_{nr} one finds:

$$b_{n1} = \frac{6}{5}\frac{(a_{n2} - a_{n1})}{a},$$
$$b_{n2} = \frac{3}{5}\frac{(a_{n2} - a_{n1})}{a}. \tag{e}$$

Denoting the downward shears on the beams 1 and 2 at the diaphragms by V_1 and V_2, one finds:

$$V_1 = -\frac{(M_{12} + M_{21})}{a},$$
$$V_2 = \frac{(M_{12} + M_{21})}{a},$$

i.e.

$$E I \frac{d^4 y_1}{d x^4} = -\frac{N E_c I_c}{l a^2}\left[6\Theta_1 + 6\Theta_2 - 12\frac{(y_2 - y_1)}{a}\right] + \text{External Load on Beam 1},$$
$$E I \frac{d^4 y_2}{d x^4} = \frac{N E_c I_c}{l a^2}\left[6\Theta_1 + 6\Theta_2 - 12\frac{(y_2 - y_1)}{a}\right] + \text{External Load on Beam 2}.$$

Substituting the series (a) and (b) in the above one obtains the following equations:

$$\frac{n^4 \pi^4}{l^4} a_{n1} = -\frac{6 N E_c I_c}{5 l a^3 E I}(a_{n1} - a_{n2}) + \frac{C_{n1}}{E I},$$
$$\frac{n^4 \pi^4}{l^4} a_{n2} = \frac{6 N E_c I_c}{5 l a^3 E I}(a_{n1} - a_{n2}) + \frac{C_{n2}}{E I}.$$

Denoting $\frac{N E_c I_c l^4}{l E I \pi^4 a^3}$ by λ, a dimensionless parameter, the following equations result:

$$a_{n1}\left(n^4 + \frac{6}{5}\lambda\right) - \frac{6}{5}\lambda a_{n2} = \frac{C_{n1} l^4}{\pi^4 E I},$$
$$a_{n2}\left(n^4 + \frac{6}{5}\lambda\right) - \frac{6}{5}\lambda a_{n1} = \frac{C_{n2} l^4}{\pi^4 E I}.$$

Solving which

$$a_{n1} = \frac{1}{E I \frac{n^4 \pi^4}{l^4}} \frac{C_{n1}\left(n^4 + \frac{6}{5}\lambda\right) + C_{n2}\frac{6}{5}\lambda}{\left(n^4 + \frac{12}{5}\lambda\right)},$$

$$a_{n2} = \frac{1}{E I \frac{n^4 \pi^4}{l^4}} \frac{C_{n2}\left(n^4 + \frac{6}{5}\lambda\right) + C_{n1}\frac{6}{5}\lambda}{\left(n^4 + \frac{12}{5}\lambda\right)}.$$

The bending moments are given by:

$$M_1(x) = -E I \frac{d^2 y_1}{d x^2} = \sum_{n=1}^{\infty} \frac{C_{n1}\left(n^4 + \frac{6\lambda}{5}\right) + C_{n1}\frac{6\lambda}{5}}{\frac{n^2 \pi^2}{l^2}\left(n^4 + \frac{12\lambda}{5}\right)} \sin\frac{n \pi x}{l},$$

$$M_2(x) = -EI\frac{d^2 y_2}{dx^2} = \sum_{n=1}^{\infty} \frac{C_{n2}\left(n^4+\frac{6\lambda}{5}\right)+C_{n1}\frac{6\lambda}{5}}{\frac{n^2\pi^2}{l^2}\left(n^4+\frac{12\lambda}{5}\right)} \sin\frac{n\pi x}{l}.$$

Let $\overline{M}_1(x)$ be the bending moment in beam 1 due to $\lambda = 0$.

Then
$$\overline{M}_1(x) = \sum_{n=1}^{\infty} \frac{C_{n1}}{\frac{n^2\pi^2}{l^2}} \sin\frac{n\pi x}{l}.$$

Hence
$$M_1(x) - \overline{M}_1(x) = \frac{6}{5}\lambda \sum_{n=1}^{\infty} \frac{(C_{n2}-C_{n1})}{\frac{n^2\pi^2}{l^2}\left(n^4+\frac{12}{5}\lambda\right)} \sin\frac{n\pi x}{l}. \qquad \text{(f)}$$

If the load W be a concentrated load at $x=\xi$, then

$$C_{n1} = \frac{2W_1}{l}\sin\frac{n\pi\xi}{l}.$$

Whence
$$\frac{M_1(x)}{l} = \frac{12}{5}\lambda\frac{(W_2-W_1)}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(n^4+\frac{12\lambda}{5}\right)} + \frac{\overline{M}_1(x)}{l}. \qquad \text{(g)}$$

A similar expression may be written for $\frac{M_2(x)}{l}$.

Case IIb. Five Longitudinal Beams

This case is schematically represented in fig. 9. A solution similar in every respect to that in case IIa yields the following expressions for the bending moments:

$$\begin{aligned}
\frac{M_1-\overline{M}_1}{l} = {} & 24\frac{(W_2-W_1)}{\pi^2}\lambda\sum_{n=1}^{\infty}\frac{\left(1+\frac{9\lambda}{n^4}\right)\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(7n^4+204\lambda+\frac{180\lambda^2}{n^4}\right)} \\
& -36\frac{(W_3-W_2)}{\pi^2}\lambda\sum_{n=1}^{\infty}\frac{\left(1-\frac{2\lambda}{n^4}\right)\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(7n^4+204\lambda+\frac{180\lambda^2}{n^4}\right)}, \\
\frac{M_2-\overline{M}_2}{l} = {} & 12\frac{(W_1-W_2)}{\pi^2}\lambda\sum_{n=1}^{\infty}\frac{\left(5+\frac{12\lambda}{n^4}\right)\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(7n^4+204\lambda+\frac{180\lambda^2}{n^4}\right)} \\
& +12\frac{(W_3-W_2)}{\pi^2}\lambda\sum_{n=1}^{\infty}\frac{\left(11+\frac{6\lambda}{n^4}\right)\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(7n^4+204\lambda+\frac{180\lambda^2}{n^4}\right)},
\end{aligned} \qquad \text{(h)}$$

where the same notation as in case IIa has been used.

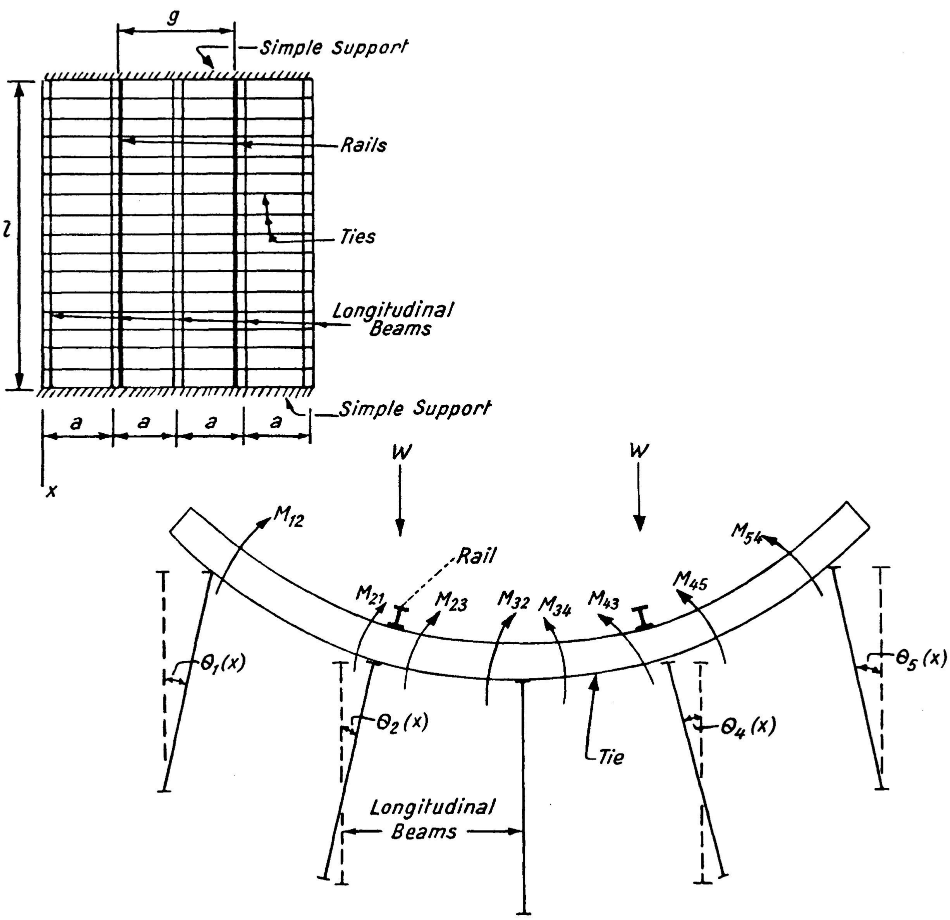

Fig. 9.

Case IIc. Six Longitudinal Beams

The definition sketch for this is shown in fig. 10. The wheel loads W cause external loads W_1, W_2, W_3, W_4, W_5, W_6 to act on the longitudinal girders. Again proceeding as in Case IIa the following expressions are obtained for the bending moments:

$$\frac{M_1-\overline{M}_1}{l} = \frac{12}{\pi^2}(W_2-W_1)\lambda\sum_{n=1}^{\infty}\frac{\left(5+\frac{12\lambda}{n^4}\right)\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(19n^4+264\lambda+\frac{108\lambda^2}{n^4}\right)}$$

$$-\frac{72}{\pi^2}(W_3-W_2)\lambda\sum_{n=1}^{\infty}\frac{\left(1-\frac{\lambda}{n^4}\right)\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(19n^4+264\lambda+\frac{108\lambda^2}{n^4}\right)},$$

$$\frac{M_2-\overline{M}_2}{l} = \frac{12}{\pi^2}(W_1-W_2)\lambda\sum_{n=1}^{\infty}\frac{\left(11+\frac{6\lambda}{n^4}\right)\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(19n^4+264\lambda+\frac{108\lambda^2}{n^4}\right)}$$

$$+\frac{12}{\pi^2}(W_3-W_2)\lambda\sum_{n=1}^{\infty}\frac{\left(17+\frac{\lambda}{n^4}\right)\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}}{n^2\left(19n^4+264\lambda+\frac{108\lambda^2}{n^4}\right)}.$$

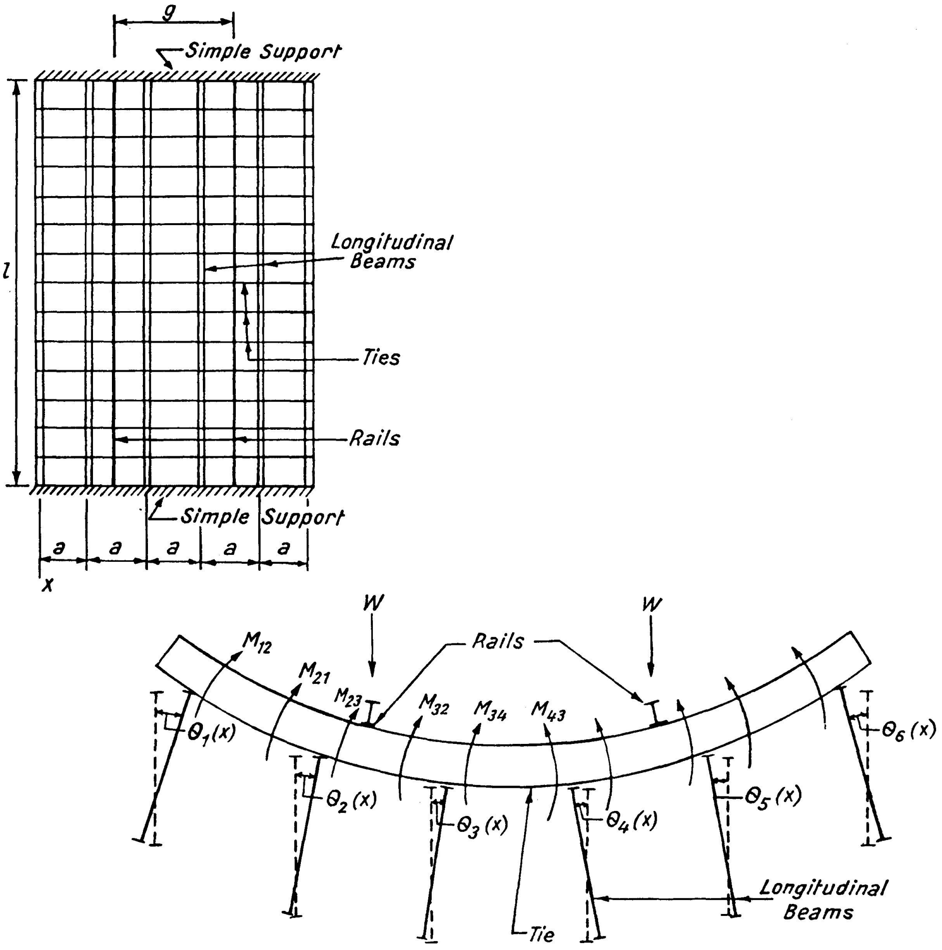

Fig. 10.

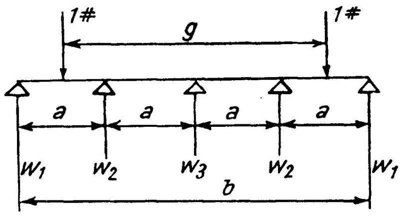

Fig. 11.

Tables for Lateral Distribution

An examination of the equations for lateral distribution shows that they may be put in the general form:

$$\frac{M_r(x)}{l} = K_1 f_1(x) + K_2 f_2(x) + K_3 F(x),$$

where the $f(x)$ denote sums of infinite series, $F(x)$ denotes the statical moment divided by l at any point due to a unit wheel load and the $K's$ are parameters depending on the total width of the bridge. It is clear that the values of $f(x)$

and K may be tabulated for various values of λ and ξ and different widths of bridge. This has been done in reference [9]. The tables are not included here owing to limitations of space. By the use of these tables it is possible to find the bending moments in any beam of a four, five or six beam bridge. By using the reciprocal theorem these tables may be further used for drawing the curve of maximum moments in any beam, in these three types of bridges, due to a train of loads.

Conclusions and General Remarks

It is realized that the analysis presented should logically be extended to cover an arbitrary number of longitudinal diaphragms (Case I) and an arbitrary number of longitudinal beams (Case II), in order to be sufficiently comprehensive. For these cases, the simplest picture is given by the "gridwork". But although such a solution is mathematically feasible [9], the results are not of sufficiently simple form to be attractive to the designer.

As a matter of fact it may be argued that even the use of the tables is too laborious for the design office. Even simpler formulas can, of course, be derived if experimental results are available from a fairly large sample of actual bridges that have been tested, so that the formulas may be modified appropriately to take account of field conditions (such as effect of ballast, floor plate, spacing of ties, etc.). In the absence of adequate information of this kind it is believed that a greater simplicity at the expense of accuracy is not warranted.

Acknowledgement

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References

1. M. HETENYI, "Beams on Elastic Foundations". University of Michigan Press, 1946.
2. CH. MASSONNET[1]), «Contribution au calcul des ponts à poutres multiples». Ann. des Travaux Publics de Belgique, 1950.
3. CH. MASSONNET, "Method of Calculation for Bridges with Several Longitudinal Beams taking into Account Their Torsional Resistance". Publications of the International Assoc. Bridge and Struct. Engg. 10, 1950.
4. PIPPARD and WAELE, "The Loading of Interconnected Crossgirders". Journal of the Inst. of C. E., Nov. 1938, Vol. 10, p. 97.
5. F. LEONHARDT, «Die vereinfachte Berechnung zweiseitig gelagerter Trägerroste». Bautechnik, 1938, p. 535.

[1]) A full bibliography may be found at the end of this article.

6. KEIICHI HAYASHI[2]), «Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen». Julius Springer, Berlin, 1926.
7. NATHAN M. NEWMARK, CHESTER P. SIESS and ROBERT R. PENMAN, "Studies of Slab and Beam Highway Bridges". Bulletin 363 of the University of Illinois, 1946.
8. N. M. NEWMARK, "A Distribution Procedure for the Analysis of Slabs Continuous over Flexible Beams". Bulletin 304 of the University of Illinois, 1938.
9. THEIN WAH, "The Longitudinal and Lateral Distribution of Loads in Open-Deck Railway Bridges". Dissertation, University of Illinois, 1953.

Appendix

Table 1

Type I a

$\frac{I_c}{I_s}$	Values of γ							
	a'							
	3	4	5	6	7	8	9	10
1	0.2985	0.2527	0.2213	0.1982	0.1802	0.1658	0.1539	0.1439
2	0.3549	0.3005	0.2632	0.2357	0.2144	0.1972	0.1830	0.1705
3	0.3928	0.3325	0.2913	0.2609	0.2372	0.2182	0.2025	0.1894
4	0.4221	0.3573	0.3130	0.2803	0.2549	0.2345	0.2177	0.2035
5	0.4463	0.3778	0.3310	0.2964	0.2695	0.2480	0.2302	0.2152
6	0.4671	0.3955	0.3464	0.3102	0.2821	0.2595	0.2409	0.2252
7	0.4855	0.4110	0.3600	0.3224	0.2932	0.2697	0.2504	0.2340
8	0.5020	0.4249	0.3722	0.3333	0.3031	0.2789	0.2588	0.2420
9	0.5170	0.4377	0.3834	0.3433	0.3122	0.2872	0.2666	0.2492
10	0.5308	0.4493	0.3936	0.3525	0.3205	0.2949	0.2737	0.2559
12	0.5555	0.4703	0.4120	0.3689	0.3355	0.3086	0.2865	0.2678
14	0.5773	0.4887	0.4281	0.3834	0.3487	0.3207	0.2977	0.2783
16	0.5969	0.5053	0.4427	0.3964	0.3605	0.3316	0.3078	0.2878
18	0.6148	0.5204	0.4559	0.4082	0.3713	0.3415	0.3170	0.2963
20	0.6312	0.5343	0.4681	0.4191	0.3812	0.3507	0.3255	0.3043
25	0.6674	0.5650	0.4949	0.4432	0.4031	0.3708	0.3442	0.3217
30	0.6985	0.5913	0.5180	0.4638	0.4218	0.3881	0.3602	0.3367
35	0.7260	0.6146	0.5384	0.4821	0.4384	0.4033	0.3744	0.3500
40	0.7506	0.6354	0.5566	0.4984	0.4533	0.4170	0.3871	0.3618
45	0.7730	0.6544	0.5733	0.5133	0.4668	0.4295	0.3986	0.3726
50	0.7937	0.6719	0.5886	0.5270	0.4793	0.4409	0.4093	0.3826
60	0.8307	0.7032	0.6160	0.5516	0.5017	0.4615	0.4284	0.4004
70	0.8633	0.7309	0.6402	0.5733	0.5214	0.4796	0.4452	0.4162
80	0.8926	0.7557	0.6619	0.5928	0.5391	0.4959	0.4603	0.4303
90	0.9193	0.7782	0.6817	0.6105	0.5552	0.5107	0.4740	0.4432
100	0.9439	0.7990	0.6999	0.6268	0.5700	0.5244	0.4867	0.4550

[2]) Used in the preparation of tables.

Tables 1, 2, 3 and 4 are for finding the longitudinal distribution of loads in single track and double track bridges, types I a and I b respectively. These types are shown in figs. 1 and 2.

To find the maximum bending moment in any transverse beam in type I a, obtain first the values of $I_c/I s$, which must be known or assumed. Then from

Table 2

Type I b

$\frac{I_c}{I s}$	Values of γ							
	a'							
	3	4	5	6	7	8	9	10
1	0.1708	0.1505	0.1359	0.1246	0.1155	0.1080	0.1017	0.0962
2	0.2030	0.1790	0.1616	0.1481	0.1374	0.1284	0.1209	0.1144
3	0.2247	0.1981	0.1788	0.1640	0.1520	0.1421	0.1338	0.1266
4	0.2415	0.2129	0.1922	0.1762	0.1633	0.1527	0.1438	0.1361
5	0.2554	0.2251	0.2032	0.1863	0.1727	0.1615	0.1520	0.1439
6	0.2673	0.2356	0.2127	0.1950	0.1808	0.1690	0.1591	0.1506
7	0.2778	0.2449	0.2210	0.2026	0.1879	0.1757	0.1654	0.1565
8	0.2872	0.2532	0.2285	0.2095	0.1943	0.1816	0.1710	0.1618
9	0.2958	0.2608	0.2354	0.2158	0.2000	0.1871	0.1761	0.1667
10	0.3037	0.2677	0.2416	0.2215	0.2054	0.1921	0.1808	0.1711
12	0.3178	0.2802	0.2529	0.2319	0.2150	0.2010	0.1892	0.1791
14	0.3303	0.2912	0.2628	0.2410	0.2234	0.2089	0.1967	0.1861
16	0.3415	0.3011	0.2718	0.2492	0.2310	0.2160	0.2033	0.1924
18	0.3517	0.3101	0.2799	0.2566	0.2379	0.2224	0.2094	0.1982
20	0.3611	0.3184	0.2873	0.2634	0.2442	0.2284	0.2150	0.2035
25	0.3818	0.3366	0.3038	0.2786	0.2583	0.2415	0.2273	0.2152
30	0.3996	0.3523	0.3180	0.2916	0.2703	0.2528	0.2379	0.2252
35	0.4154	0.3662	0.3305	0.3030	0.2809	0.2627	0.2473	0.2340
40	0.4294	0.3786	0.3417	0.3133	0.2905	0.2716	0.2557	0.2420
45	0.4423	0.3899	0.3519	0.3227	0.2991	0.2797	0.2633	0.2492
50	0.4541	0.4003	0.3613	0.3313	0.3071	0.2872	0.2703	0.2559
60	0.4753	0.4190	0.3782	0.3467	0.3214	0.3006	0.2829	0.2678
70	0.4939	0.4355	0.3930	0.3603	0.3341	0.3124	0.2941	0.2783
80	0.5107	0.4502	0.4064	0.3726	0.3454	0.3230	0.3040	0.2878
90	0.5260	0.4637	0.4185	0.3837	0.3557	0.3326	0.3131	0.2964
100	0.5400	0.4761	0.4297	0.3940	0.3652	0.3415	0.3215	0.3043

Table 3

Type I b

a'	3	4	5	6	7	8	9	10
$\bar{\alpha}$	0.8585	0.8897	0.9225	0.9475	0.9668	0.9820	0.9942	1.0040

Table 4

x	β_x	x	β_x	x	β_x	x	β_x
0.51	1.3173	0.91	1.0648	1.31	0.8303	1.71	0.6540
0.52	1.3113	0.92	1.0584	1.32	0.8251	1.72	0.6505
0.53	1.3054	0.93	1.0521	1.33	0.8199	1.73	0.6470
0.54	1.2994	0.94	1.0459	1.34	0.8148	1.74	0.6435
0.55	1.2934	0.95	1.0396	1.35	0.8098	1.75	0.6400
0.56	1.2873	0.96	1.0333	1.36	0.8047	1.76	0.6365
0.57	1.2813	0.97	1.0270	1.37	0.7997	1.77	0.6332
0.58	1.2752	0.98	1.0207	1.38	0.7948	1.78	0.6299
0.59	1.2690	0.99	1.0145	1.39	0.7898	1.79	0.6266
0.60	1.2628	1.00	1.0083	1.40	0.7849	1.80	0.6234
0.61	1.2566	1.01	1.0021	1.41	0.7801	1.81	0.6202
0.62	1.2503	1.02	0.9960	1.42	0.7753	1.82	0.6170
0.63	1.2442	1.03	0.9899	1.43	0.7705	1.83	0.6138
0.64	1.2379	1.04	0.9839	1.44	0.7658	1.84	0.6108
0.65	1.2315	1.05	0.9778	1.45	0.7611	1.85	0.6078
0.66	1.2252	1.06	0.9716	1.46	0.7565	1.86	0.6048
0.67	1.2189	1.07	0.9656	1.47	0.7519	1.87	0.6018
0.68	1.2126	1.08	0.9596	1.48	0.7474	1.88	0.5989
0.69	1.2062	1.09	0.9536	1.49	0.7429	1.89	0.5960
0.70	1.1997	1.10	0.9476	1.50	0.7384	1.90	0.5932
0.71	1.1933	1.11	0.9416	1.51	0.7339	1.91	0.5904
0.72	1.1869	1.12	0.9356	1.52	0.7295	1.92	0.5876
0.73	1.1805	1.13	0.9298	1.52	0.7252	1.93	0.5849
0.74	1.1741	1.14	0.9240	1.54	0.7209	1.94	0.5822
0.75	1.1676	1.15	0.9183	1.55	0.7166	1.95	0.5795
0.76	1.1611	1.16	0.9126	1.56	0.7125	1.96	0.5769
0.77	1.1547	1.17	0.9069	1.57	0.7082	1.97	0.5743
0.78	1.1483	1.18	0.9012	1.58	0.7041	1.98	0.5717
0.79	1.1418	1.19	0.8955	1.59	0.7000	1.99	0.5692
0.80	1.1353	1.20	0.8898	1.60	0.6960	2.00	0.5667
0.81	1.1289	1.21	0.8842	1.61	0.6919	2.01	0.5643
0.82	1.1225	1.22	0.8786	1.62	0.6879	2.02	0.5619
0.83	1.1160	1.23	0.8731	1.63	0.6840	2.03	0.5595
0.84	1.1096	1.24	0.8677	1.64	0.6801	2.04	0.5571
0.85	1.1032	1.25	0.8623	1.65	0.6763	2.05	0.5549
0.86	1.0968	1.26	0.8569	1.66	0.6725	2.06	0.5526
0.87	1.0904	1.27	0.8515	1.67	0.6686	2.07	0.5504
0.88	1.0840	1.28	0.8462	1.68	0.6648	2.08	0.5482
0.89	1.0776	1.29	0.8408	1.69	0.6612	2.09	0.5460
0.90	1.0712	1.30	0.8355	1.70	0.6576	2.10	0.5438

Table 4 (Contd.)

x	β_x	x	β_x	x	β_x	x	β_x
2.11	0.5417	2.51	0.4824	2.91	0.4594	3.31	0.4579
2.12	0.5397	2.52	0.4815	2.92	0.4591	3.32	0.4580
2.13	0.5377	2.53	0.4805	2.93	0.4589	3.33	0.4581
2.14	0.5357	2.54	0.4796	2.94	0.4587	3.34	0.4582
2.15	0.5337	2.55	0.4787	2.95	0.4585	3.35	0.4583
2.16	0.5317	2.56	0.4779	2.96	0.4583	3.36	0.4585
2.17	0.5288	2.57	0.4772	2.97	0.4581	3.37	0.4587
2.18	0.5280	2.58	0.4763	2.98	0.4580	3.38	0.4589
2.19	0.5262	2.59	0.4754	2.99	0.4579	3.39	0.4591
2.20	0.5244	2.60	0.4746	3.00	0.4578	3.40	0.4592
2.21	0.5226	2.61	0.4739	3.01	0.4577	3.41	0.4594
2.22	0.5208	2.62	0.4731	3.02	0.4576	3.42	0.4596
2.23	0.5191	2.63	0.4724	3.03	0.4575	3.43	0.4597
2.24	0.5174	2.64	0.4717	3.04	0.4574	3.44	0.4599
2.25	0.5157	2.65	0.4711	3.05	0.4573	3.45	0.4601
2.26	0.5141	2.66	0.4704	3.06	0.4572	3.46	0.4603
2.27	0.5125	2.67	0.4698	3.07	0.4571	3.47	0.4603
2.28	0.5110	2.68	0.4692	3.08	0.4570	3.48	0.4608
2.29	0.5095	2.69	0.4686	3.09	0.4569	3.49	0.4610
2.30	0.5080	2.70	0.4680	3.10	0.4569	3.50	0.4612
2.31	0.5065	2.71	0.4674	3.11	0.4569	3.51	0.4614
2.32	0.5050	2.72	0.4669	3.12	0.4568	3.52	0.4616
2.33	0.5036	2.73	0.4663	3.13	0.4568	3.53	0.4618
2.34	0.5022	2.74	0.4658	3.14	0.4568	3.54	0.4620
2.35	0.5008	2.75	0.4653	3.15	0.4568	3.55	0.4622
2.36	0.4995	2.76	0.4648	3.16	0.4568	3.56	0.4624
2.37	0.4982	2.77	0.4644	3.17	0.4568	3.57	0.4627
2.38	0.4969	2.78	0.4639	3.18	0.4569	3.58	0.4629
2.39	0.4956	2.79	0.4635	3.19	0.4569	3.59	0.4632
2.40	0.4944	2.80	0.4631	3.20	0.4569	3.60	0.4634
2.41	0.4932	2.81	0.4627	3.21	0.4570	3.61	0.4637
2.42	0.4920	2.82	0.4623	3.22	0.4570	3.62	0.4639
2.43	0.4908	2.83	0.4619	3.23	0.4571	3.63	0.4641
2.44	0.4897	2.84	0.4615	3.24	0.4572	3.64	0.4644
2.45	0.4886	2.85	0.4612	3.25	0.4573	3.65	0.4646
2.46	0.4875	2.86	0.4609	3.26	0.4574	3.66	0.4649
2.47	0.4865	2.87	0.4606	3.27	0.4575	3.67	0.4652
2.48	0.4854	2.88	0.4603	3.28	0.4576	3.68	0.4654
2.49	0.4844	2.89	0.4600	3.29	0.4577	3.69	0.4657
2.50	0.4834	2.90	0.4597	3.30	0.4578	3.70	0.4659

Table 4 (Contd.)

x	β_x	x	β_x	x	β_x	x	β_x
3.71	0.4662	3.91	0.4717	4.22	0.4801	4.62	0.4893
3.72	0.4664	3.92	0.4720	4.24	0.4806	4.64	0.4897
3.73	0.4667	3.93	0.4722	4.26	0.4807	4.66	0.4901
3.74	0.4670	3.94	0.4725	4.28	0.4817	4.68	0.4904
3.75	0.4673	3.95	0.4728	4.30	0.4821	4.70	0.4908
3.76	0.4676	3.96	0.4731	4.32	0.4826	4.72	0.4912
3.77	0.4678	3.97	0.4733	4.34	0.4831	4.74	0.4915
3.78	0.4681	3.98	0.4736	4.36	0.4836	4.76	0.4919
3.79	0.4684	3.99	0.4739	4.38	0.4841	4.78	0.4922
3.80	0.4686	4.00	0.4742	4.40	0.4845	4.80	0.4925
3.81	0.4689	4.02	0.4747	4.42	0.4850	4.82	0.4928
3.82	0.4692	4.04	0.4753	4.44	0.4855	4.84	0.4932
3.83	0.4695	4.06	0.4758	4.46	0.4859	4.86	0.4935
3.84	0.4697	4.08	0.4764	4.48	0.4864	4.88	0.4938
3.85	0.4700	4.10	0.4769	4.50	0.4868	4.90	0.4941
3.86	0.4703	4.12	0.4776	4.52	0.4872	4.92	0.4944
3.87	0.4706	4.14	0.4780	4.54	0.4877	4.94	0.4947
3.88	0.4708	4.16	0.4785	4.56	0.4881	4.96	0.4949
3.89	0.4711	4.18	0.4791	4.58	0.4885	4.98	0.4952
3.90	0.4714	4.20	0.4796	4.60	0.4889	5.00	0.4954

table 1 obtain the corresponding value of γ. Choose from the expected train load on the bridge the three largest wheel loads P with their axle spacings z. (These three loads must be all equal and equally spaced to yield exact values for the result but a slight discrepancy in the value of the loads or axle spacing will not seriously affect the result.) Obtain from table 4 the value of β_{yz}. Then the maximum moment in a transverse beam is given by $M = P a' s \gamma \beta_{yz}$.

To find the maximum bending moment in any transverse beam in type Ib, obtain first the value of $I_c/I\,s$ for the bridge, which must be known or assumed. Then from table 2 obtain the corresponding value of γ, and from table 3 obtain the value of $\bar{\alpha}$. As before choose the largest three loads P from the expected load, the loads P being z feet apart. From table 4 obtain the values of β_{yz} and $\beta_{\bar{\alpha}yz}$. The maximum bending moment in any transverse is given by

$$M = P s \gamma \{a' \beta_{yz} + \bar{\alpha}\,(a' + g)\,\beta_{\bar{\alpha}yz}\}.$$

Summary

The question of the distribution of train loads in the direction parallel to traffic ("longitudinal" distribution) and in the direction perpendicular to traffic ("lateral" distribution) arises in the design of certain types of railroad bridges. The problems involved in this question are analyzed mathematically for open-deck bridges and formulas derived for the longitudinal and lateral distribution of loads. Approximate solutions are also devised wherever feasible in order to furnish simple formulas which may be of use in design. Some of the formulas are translated into tables given in the Appendix.

Résumé

Dans l'étude des projets de ponts de chemin de fer, peut se poser la question de la répartition des charges mobiles en direction parallèle au sens du trafic (répartition longitudinale) et en direction perpendiculaire à la précédente (répartition transversale). Les problèmes corrélatifs ont été étudiés mathématiquement et des formules ont été établies pour exprimer ces deux répartitions. L'auteur indique des solutions d'approximation dans tous les cas où il a été possible d'établir des formules simples, susceptibles d'une utilisation pratique dans l'étude des projets. Quelques-unes de ces formules ont fait l'objet d'une tabulation qui est présentée en annexe.

Zusammenfassung

Beim Entwurf von Eisenbahnbrücken kann sich die Frage der Verteilung der beweglichen Lasten in Richtung parallel zum Verkehr («Längsverteilung») und in Richtung senkrecht zum Verkehr («Querverteilung») stellen. Die diese Frage einschließenden Probleme werden mathematisch untersucht, und es werden Formeln für die Längs- und Querverteilung der Lasten abgeleitet. Näherungslösungen werden angegeben, wo immer es möglich war, einfache Formeln aufzustellen, die beim Projektieren von Nutzen sein können. Einige dieser Formeln wurden in Tabellen umgewandelt und im Anhang veröffentlicht.