Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 16 (1956)

Artikel: Generalized elastic theory for pile groups

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DOI: https://doi.org/10.5169/seals-15055

Generalized Elastic Theory for Pile Groups

Généralisation de la théorie de l'élasticité pour des groupes de pieux

Verallgemeinerte Elastizitätstheorie für Pfahlgruppen

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Origin and Development of Theory

In 1902 Professor PER GULLANDER of Chalmers Institute of Technology applied the theories for statically indeterminate elastic structures to plane pile groups. The method was developed by himself and by the Gothenburg engineers FELLENIUS, EKWALL and T. HULTIN, and by WESTERGAARD:

GULLANDER P., Bidrag till teorien för grundpålningar (Contribution to theory of pile foundations), Tekn. Tidskrift 1902, p. V 51.

— Grafostatisk metod för beräkning av grundpålningar (Graphostatic method for the calculation of pile foundations), Tekn. T. 1911, p. V 55.

— Till teorien för grundpålningar (On the theory of pile foundations), Tekn. T. 1912, p. V 40.

— Teori för grundpålningar (Theory of pile foundations), Stockholm 1914.

FELLENIUS W., Om beräkning av kajbyggnader o. d. (On the calculation of quay walls etc.), Tekn. T. 1910, p. V 29.

EKWALL R., Beräkning av grundpålningar (Calculation of pile foundations), Tekn. T. 1911, p. V 79.

HULTIN T., Om beräkning av grundpålningar (On the calculation of pile foundations), Industritidningen Norden 1911, p. 370.

WESTERGAARD H. M., The resistance of a group of piles, J. Western Soc. Eng. Dec. 1917.

Professors OSTENFELD and NØKKENTVED in Copenhagen further systematized the theory. Results were summarized and the calculation somewhat systematized by the writer: The pile slopes were expressed as direction cosines and a few simplifications given.

Three-dimensional pile-groups were analysed by Nøkkentved, and, with matrix algebra, by the writer:

Ostenfeld A., Beregning av Paleverker (Calculation of pile foundations), Dansk Tekn. T. 1921, No. 1.

Nøkkentved Chr., Beregning av Paleverker (Calculation of pile foundations), Copenhagen 1924.

Asplund S. O., Beräkning och anordning av plana pålgrupper (Calculation and arrangement of plane pile-groups), Tekn. T. 1945, p. 1414.

— A study of three-dimensional pile-groups, AIPC Memoires (Zürich) 1947, p. 1.

Transversal Forces in the Piles

The pile group theories generally assume that the piles have hinges at both ends and that the surrounding medium (the earth) does not offer any side resistance. Some of the authors cited also discuss bending and transversal forces in the piles and fixed pile heads.

Agatz reviews the practical advantages of different assumptions and concludes that no considerable economy is won in the calculation of plane pile-groups by assumptions other than hinged pile ends. (Agatz A., Der Kampf des Ingenieurs gegen Erde und Wasser im Grundbau (The engineer's battle against earth and water in foundation construction), Berlin 1936, p. 193.) This assumption is on the side of extra safety. Therefore many design specifications demand that transversal forces and moments in the piles be neglected in the calculation of the axial forces of the piles.

Nevertheless, in for instance building foundations, only vertical piles are driven in spite of the fact that a pile foundation with only vertical piles, without side resistances, is unstable for horizontal loads. The obvious side stiffness of such pile foundations proves that side resistances generally exist. Many engineers consider it to be over-conservative always to neglect all side-resistances when their consideration with small safe values would mean a material economic gain.

Jampel and Hrennikoff consider each pile as a beam on elastic foundation:

Jampel S., An analysis of pile-groups, Thesis, London University Library, London 1947.

— An analysis of groups of piles, Concrete and Constructional Engineering (London), July 1949.

Hrennikoff A., Analysis of pile foundations with batter piles, ASCE Transactions 1950, p. 351.

Gullander, Hultin, Nøkkentved and Jampel take account of transversal force in a hinged pile by introducing at the pile head a fictitious pile, perpendicular to the real pile. Jampel accounts for moments and shears in a fixed pile head by introducing two fictitious piles at different heights. The methods of analysis of these authors is however not very systematic nor adapted to practical, numerical treatment.

The present paper includes the effect of an elastic side resistance from the earth upon piles with fixed or hinged pile heads. The paper aims at rationalizing the calculations by a systematic arrangement of formulas and forms for calculation.

Elastic Theory

Hooke's law is assumed to be valid. Varying and practically indeterminable properties of earth and piles would appear to make results of other assumptions just as uncertain. It therefore seems to be unwarranted to abandon a usable elastic theory in favor of a complicated and tedious theory even if the latter is founded upon "more realistic" assumptions of plastic earths and piles. Also, the transition from plastic to elastic treatment is principally "on the safe side" with respect to the risk of failure.

The real forces are assumed to cause small displacements relative to the dimensions of the pile foundation. A pile driven into the ground can then be assigned three sping constants or "pile stiffnesses" (force causing a unit displacement): one axial, $S_n = EA/L$, for axial forces, one transversal, S_t, for transversal forces, and one rotational spring constant, S_r, for moment. The spring constant, S_v, for torsion about the axis of the pile may become active in three-dimensional pile groups.

Axial Pile Stiffness

An axial load F_n upon the pile head, fig. 1, shortens a point-loaded foundation pile of length L, average cross sectional area A and elastic modulus E, by

$$U_n = F_n/S_n, \quad S_n = s\,EA/L, \quad s = 1. \tag{1}$$

In a friction pile the axial force decreases from F_n to zero at a point. Hrennikoff (l.c. p. 2) for this case writes $s=2$, but it is evident that the surrounding earth also can yield elastically at the same time as the pile. Tests by Jampel (l.c. p. 2) shows that $s=3$ to 1.5 for friction piles. For piles driven through silt or clay to point loading against fine sand he found $s=2$ to 1, and, if the pile point instead had been driven in gravel, $s=1.3$ to 0.6. When more accurate values are required, the axial pile stiffness S_n should be determined for piles driven upon the building site and by measuring their elastic deformation during a test loading.

Displacements of and External Forces Upon a Pile

The displacement of an encased pile head is measured at the end B' of a fictitious, stiff lever arm $A'\,B'$, fig. 1, connected to the pile head and extending to the depth $A'B' = b$. The end B of this stiff lever arm displaces BB', with U_n along the pile axis $A\,1''$ and with U_t perpendicularly to the pile axis, along $A\,2''$. The arm rotates by U_r in a positive direction (from the $1''$ to the $2''$ axis).

Forces acting upon the pile head can be reduced to an equivalent force system F_n, F_t, F_r, fig. 1, applied at the lower end B or B' of the stiff lever arm.

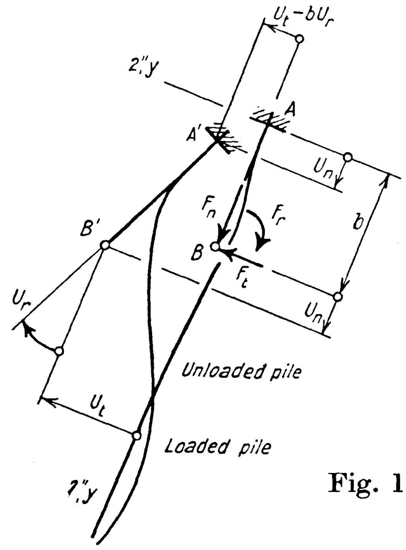

Fig. 1

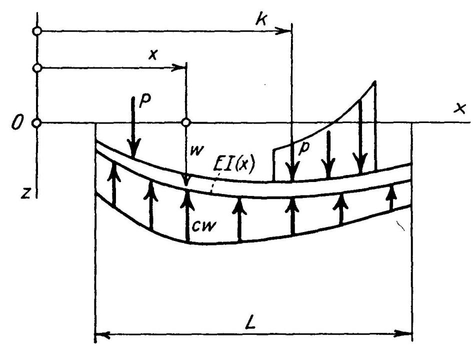

Fig. 2

Beams Supported by Continuous Springs

Consider a horizontal beam, fig. 2, with constant or variable stiffness $EI(x)$. The beam is continuously supported by springs yielding, according to Winkler's assumption, an upward reaction of cw per unit length at a beam element depression of w. (WINKLER E., Die Lehre von der Elastizität und Festigkeit (The theory of elasticity and strength), Prague 1867, p. 182.)

The beam is loaded from above by vertical loads P, p in its vertical plane of symmetry. The beam will be deflected by small distances w. Elementary theory yields

$$M = -EIw'', \qquad M'' = -(p - cw)$$
$$(EIw'')'' + cw = p \tag{1}$$

the fundamental differential equation for an elastic beam supported by continuous springs.

Solution of the Differential Equation for Constant EI and c

For constant EI and c, (1) assumes the form

$$EIw^{(4)} + cw = p. \tag{2}$$

For beam elements where the load $p = 0$ the homogeneous equation holds

$$E\,I\,w^{(4)} + c\,w = 0 \tag{3}$$

Use of the transformed variable $X = x/2b$, $b = \sqrt[4]{E\,I/4c}$, where $2b$ is usually called the characteristic length, changes (3) to

$$\frac{d^4 w}{d X^4} + 4w = 0. \tag{4}$$

A general integral of (4) is

$$w = e^{X}(A\cos X + B\sin X) + e^{-X}(C\cos X + D\sin X) \tag{5}$$

which is easily verified by differentiation and substitution in (3).

Semi-Infinite Beam

The semi-infinite beam, fig. 3, is loaded at its left end at $x=0$ with a moment M_0 and a transversal force T_0. The only other forces acting on the beam are the spring-reactions $-c\,w$, caused by the deflections w of the beam.

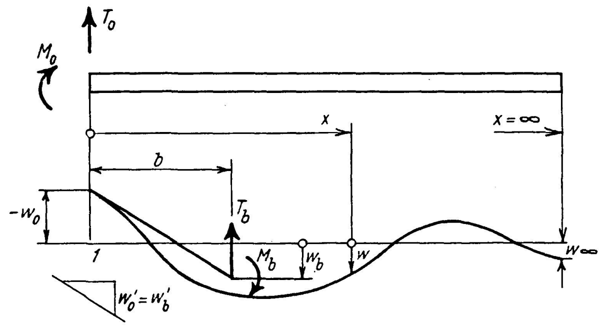

Fig. 3

The loads at $x=0$ are instead reduced to T_b, M_b acting at the end $x=b$ of an entirely stiff arm, connected at $x=0$.

When x or X increases to infinity, w approaches 0. Hence in

$$A = B = 0 \tag{5}$$

$$\begin{array}{lll}
w & = & e^{-X}(+C\cos X + D\sin X) \\
d\,w/d\,X = 2b\,w' & = & e^{-X}((-C+D)\cos X + (-C-D)\sin X) \\
(2b)^2 w'' & = & 2e^{-X}(-D\cos X + C\sin X) \\
(2b)^3 w^{(3)} & = & 2e^{-X}((+C+D)\cos X + (-C+D)\sin X) \\
(2b)^4 w^{(4)} & = & 4e^{-X}(-C\cos X - D\sin X) = -4w
\end{array} \tag{6}$$

and for $X=0$

$$\begin{array}{rcl}
w_0 & = & C \\
2b\,w_0' & = & -C+D \\
-2b^2 M_0/E\,I & = & -D \\
-4b^3 T_0/E\,I & = & C+D
\end{array} \tag{7}$$

By formation of $\frac{1}{2}C + \frac{1}{2}D$ and $-\frac{1}{2}C + \frac{1}{2}D$,

$$\begin{array}{rcl}
w_0 + b\,w_0' & = & -T_0 \cdot 2b^3/E\,I \\
b\,w_0' & = & (M_0 + T_0 b)\cdot 2b^2/E\,I
\end{array}$$

hence

$$w_b = -T_b/S,\ w_b' = M_b/b^2 S,\ S = EI/2b^3 = 2bc \tag{8}$$

A vertical force acting upon the lever end consequently moves the lever end vertically without rotation. A moment upon the lever end rotates it without translation.

The integration constants

$$C = w_0 = w_b - b\,w_b' \text{ and } D = w_0 + 2b\,w_0' = w_b + b\,w_b'$$

yield, substituted in (6)

$$\begin{aligned} w &= e^{-X}((w_b - b\,w_b')\cos X + (w_b + b\,w_b')\sin X)\\ w' &= e^{-X}(w_b'\cos X + (w_b/b)\sin X)\\ M &= e^{-X}((-T_b b + M_b)\cos X + (T_b b + M_b)\sin X)\\ T &= e^{-X}(T_b\cos X + (-M_b/b)\sin X)\end{aligned} \tag{9}$$

Application to an Encased Pile with Side Resistance

The solution for beams upon spring supports, fig. 1, yields when

$$\begin{aligned} b^4 &= EI/4c, \quad S = EI/2b^3 = 2bc\\ F_t &= S\,U_t = S_t\,U_t\\ F_r &= S\,b^2\,U_r = S_r\,U_r\end{aligned} \tag{10}$$

Thus a transversal force F_t at the lever end B causes a transversal movement of B by F_t/S but no rotation while a moment F_r rotates the lever end with $F_r/b^2 S$ without translation.

Hinged Piles with Side Resistance

An encased wooden pile head yields substantially perpendicularly to the fibres. It is therefore more correct to treat a wooden pile head as hinged if it is encased, say, to a smaller length than its diameter.

A hinge at $x=0$, $F_t b + F_r = 0$, moves the pile head by

$$U_t - b\,U_r = F_t/S + b\,F_t b/b^2 S = F_t/\tfrac{1}{2}S \tag{11}$$

Therefore a transversal force F_t displaces a hinged pile head transversally by $F_t/\frac{1}{2}S$. The length $b = (EI/4c)^{1/4}$ here loses its meaning of a lever arm.

Numerical Values of the Pile Stiffnesses and Test Results

Hrennikoff (l. c. p. 2) distinguishes three kinds of earth with the following elastic foundation moduli k in t/m³: very loose 20, loose 200, medium 2000. Whether these figures can be used for side resistances in the same way as for vertical loads, only soil mechanical tests and considerations can determine. They can evidently be increased because of the relatively small side surfaces of the piles and they can be decreased on account of the nearness of the horizontal stresses to the free upper surface of the earth.

A sheet piling around the pile pier could be treated more or less as a pile without axial pile stiffness.

An oak pile of diameter 0.33 m slightly below the soil surface, $EA = 130\ \mathrm{t/cm^2}\ (\pi/4)\ 33^2\ \mathrm{cm^2} = 110\,000\ \mathrm{t}$, length $L = 12$ m, $S_n = 9200$ t/m, $EI = EAi^2 = 110\,000\ \mathrm{t}\ (0.33\ \mathrm{m}/4)^2 = 750\ \mathrm{tm^2}$ is driven in medium earth, $k = 2000\ \mathrm{t/m^3}$, $c = 2000 \cdot 0.33 = 660\ \mathrm{t/m^2}$.

A circular pile cross section can reasonably be given a somewhat lower elastic side resistance c than a square pile cross section, but because of the fourth root in (8) such a correction has smaller influence compared with other unavoidable errors. Equations (8), (9), (10) yield

$$b^4 = EI/4c = 750\ \mathrm{tm^2}/(4 \cdot 660\ \mathrm{t/m^2}) = (0.73\ \mathrm{m})^4$$

$$S_r = EI/2b = 750\ \mathrm{tm^2}/2 \cdot 0.73\ \mathrm{m} = 515\ \mathrm{tm}, \quad S_t = S_r/b^2 = 970\ \mathrm{t/m}$$

More reliable values of b, S_t, S_r can be found by test loads upon single piles driven on the building site. Pile stiffnesses obtained by this method should however be reduced, if they are applied to pile-groups where many piles are situated near and behind each other.

An oak pile, $E = 130\ \mathrm{t/cm^2}$, 0.33 m in diameter slightly below the earth surface was driven in sand comparable to medium earth, and loaded by 4.5 t horizontally, 0.35 m above the earth surface, fig. 4. (Feagin L. B., Lateral pile loading tests, ASCE Trans. 1937, p. 247.)

At the load level Feagin measured a horizontal movement of 1.3 cm. The load 4.5 t in this example would have displaced the end of the stiff lever arm by $P_t/S_t = 4.5\ \mathrm{t}/970\ \mathrm{t/m} = 0.46$ cm and rotated by $P_r/S_r = -4.5\ \mathrm{t}.$ $(0.35 + 0.73)\ \mathrm{m}/515\ \mathrm{tm} = -9.6 \cdot 10^{-3}$ rad. This displaces the point of application of the load by $0.46 + 108 \cdot 9.6 \cdot 10^{-3} = 1.49$ cm, in fairly good agreement with Feagin's observed value 1.3 cm.

If the same pile figure, fig. 5, had been driven in loose earth $c = 200\ \mathrm{t/m^3}$, the pile constants $b = 1.30$ m, $S_r = 290$ tm, $S_t = 170$ t/m would have resulted,

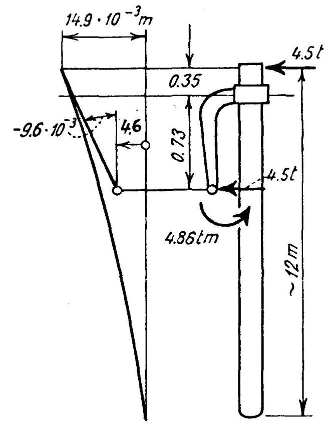

Fig. 4

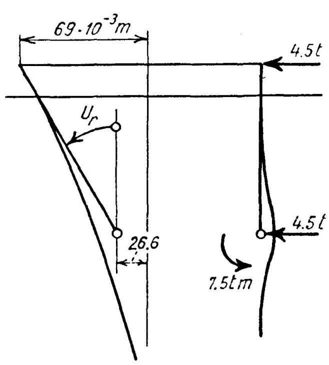

Fig. 5

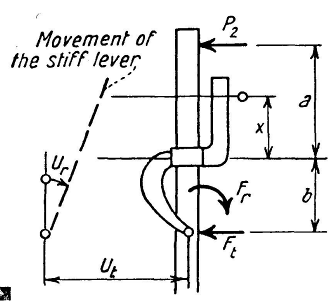

Fig. 6

and the transversal load 4.5 t of Feagin would have been displaced by $(4.5/170 + 4.5\,(0.35 + 1.30)^2/290)\,m = 6.9$ cm.

A 30 cm square pile $E = 210$ t/cm² with $EA = 210 \cdot 900 = 189\,000$ t and $L = 12$ m has $S_n = 189\,000$ t/12 m = 15 700 t/m, $EI = 189\,000$ t $\cdot\, 0.3^2$ m²/12 = 1420 tm². Medium earth $k = 2000$ t/m² yields according to (4), (2) $c = 660$ t/m², $b^4 = 1420$ tm²/(4·660) t/m² = (0.88 m)⁴, $S_r = 1420$ tm²/2.0,88 m = 810 tm, $S_t = 810/0{,}88^2 = 1050$ t/m.

Test Load to Determine *b*, *S*, *EI*, and *c*

Concrete piles 0.25·0.25 m in cross section reinforced by four deformed bars of diameter 25 mm (for $E = 250\,000$ kg/cm² (= at or kg/cm²), a value $EI = 1180$ tm² is calculated) were driven in clay containing about 65 volume per cent of water, weighing 1.53 t/m³ and having a shear strength (roughly) of 3 t/m². The "strength numbers" ("hållfasthetstal") H_1 and H_3 were determined by cone tests, giving $H_3 \approx 128$ (undisturbed clay) and $H_1 \approx 10$ (remolded clay). Hence the sensitivity $S_t \approx 11$ (Jakobson, Bernt, The landslide at Surte on the Göta River, Royal Swedish Geotech. Inst., Proc. No. 5, 1952, p. 44).

A test load P_2 was applied by the author horizontally at height a above the earth surface, fig. 6, 7. Horizontal movements v were measured, fig. 8, at the height x above the earth on a stiff lever arm, rigidly connected to the pile at the soil surface. Fig. 6 yields

$$\begin{aligned} F_t &= S\,U_t = P_2 \\ F_r &= S\,b^2\,U_r = -P_2\,(a+b) \\ v &= U_t - (b+x)\,U_r \\ S\,b^2 v &= P_2\,(b^2 + (b+x)\,(a+b)) \end{aligned}$$

At the levels x_1, x_2 were measured v_1, v_2

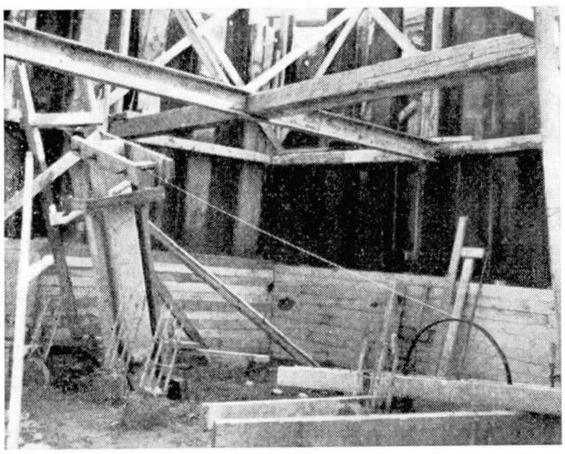

Fig. 7

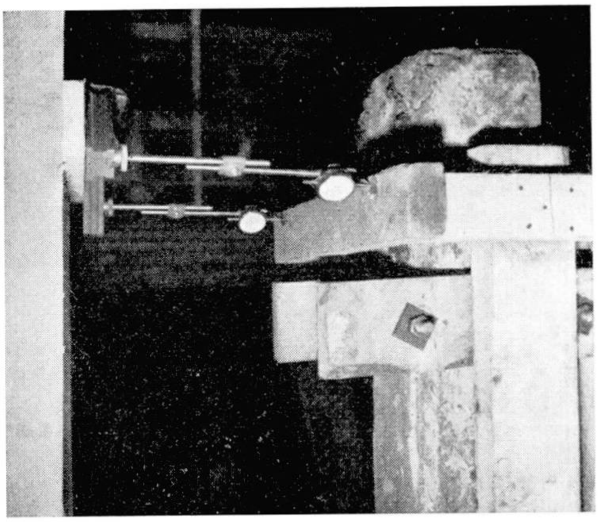

Fig. 8

$$\frac{v_1}{v_2} = \frac{b^2 + (b + x_1)(a + b)}{b^2 + (b + x_2)(a + b)}.$$

Hence

$$b^2 + \tfrac{1}{2}(a - Q)\,b = \tfrac{1}{2}a\,Q$$
$$Q = (x_2 v_1 - x_1 v_2)/(v_2 - v_1)$$

Measured and calculated values for two different test loads 1, 2 are entered in table 1.

Table 1

	Dim.	1	2		Dim.	1	2
a	10^{-3} m	1956	1595	$v_2 - v_1$	10^{-3} m	0.30	0.26
P_2	10^{-3} t	49.5	50	Q	10^{-3} m	1300	1290
x_2	10^{-3} m	1805	1805	b	10^{-3} m	976	941
v_2	10^{-3} m	0.73	0.63	$S b^2 = S_r$	tm	625	622
x_1	10^{-3} m	528	528	EI	tm²	1217	1172
v_1	10^{-3} m	0.43	0.37	c	t/m²	338	374
$x_2 v_1 - x_1 v_2$	10^{-3} m²	0.39	0.34	$S = S_t$	t/m	655	700

In this case the two different test loads are seen to have resulted in quite equal values of b, S_r, EI, c, and S_t. The values of EI were found to agree rather closely with those calculated above for $E = 250\,000$ at. The foundation modulus $c = 350$ t/m² thus found for $0.25 \cdot 0.25$ m piles corresponds to a clay that is rather loose.

Joint Action of Piles

Each pile is localized in an orthogonal coordinate frame 012, fig. 9a, by its direction cosines q_1, q_2 of the pile axis and by the projections q_4, q_5 upon, and perpendicularly to the pile axis, of the distance from the origin 0 to the end B of the stiff lever arm of the pile. Positive values of q_4 and q_5 are noted when the projections are situated as in fig. 9a.

All pile heads are encased in a pile pier, here treated as perfectly rigid. When the pile pier is loaded, a small translation u_1, u_2 takes place in the directions 01, 02, and a small rotation u_6 about 0. The end of the stiff lever arm of each pile translates axially U_n along $B\,1''$, transversally U_t along $B\,2''$ and rotates U_r around the origin 0, fig. 9b.

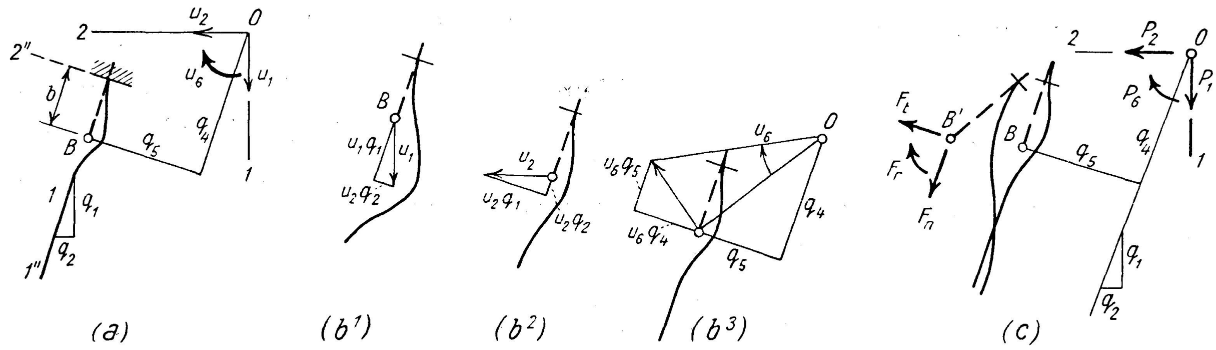

Fig. 9

In this paper, for clearness, matrix notation of the simplest kind will be used. Transposed matrices will be denoted by an asterisk.

Fig. 9b yields

$$\begin{bmatrix} U_n \\ U_t \\ U_r \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & -q_5 \\ -q_2 & q_1 & q_4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_6 \end{bmatrix}, \quad U = q\,u \tag{12}$$

and the pile reacts upon the pier with elastic forces

$$F = \begin{bmatrix} F_n \\ F_t \\ F_r \end{bmatrix} = \begin{bmatrix} S_n\,U_n \\ S_t\,U_t \\ S_r\,U_r \end{bmatrix} = S\,q\,u, \quad S = \begin{bmatrix} S_n & 0 & 0 \\ 0 & S_t & 0 \\ 0 & 0 & S_r \end{bmatrix} \tag{13}$$

In addition to such forces from all piles, the pile pier, fig. 9c, is acted upon by loads P_1, P_2, P_6 (for instance V, H, M). The equilibrium of the pile pier demands, fig. 9c

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_6 \end{bmatrix} = \Sigma \begin{bmatrix} q_1 & -q_2 & 0 \\ q_2 & q_1 & 0 \\ -q_5 & q_4 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_t \\ F_r \end{bmatrix} = \Sigma q^* F = \Sigma q^* S\,q\,u = R\,u \tag{14}$$

The sum is extended over all piles.

The symmetric matrix $R = \Sigma q^* S\,q$ has the elements (which can be calculated by elementary matrix multiplication)

$$R = \begin{bmatrix} \Sigma(S_n q_1^2 + S_t q_2^2) & \Sigma(S_n - S_t)\,q_1 q_2 & \Sigma(-S_n q_5 q_1 - S_t q_4 q_2) \\ R_{12} & \Sigma(S_n q_2^2 + S_t q_1^2) & \Sigma(-S_n q_5 q_2 + S_t q_4 q_1) \\ R_{13} & R_{23} & \Sigma(S_n q_5^2 + S_t q_4^2 + S_r) \end{bmatrix} \tag{15}$$

The movements of the pile pier are obtained from (14):

$$u = R^{-1} P \tag{16}$$

by forming the matrix R^{-1} inverse to R, or by carrying through the corresponding algebraic solution. The movements u are thus expressed as linear functions of the acting loads P. These movements u can be substituted in (13), yielding all pile forces

$$F = S\,q\,R^{-1} P. \tag{17}$$

Coordinate Transformation to the Principal Axes of the Pile Group

This solution of the pile group problem is theoretically simple but numerically laborious. It is also difficult to comprehend: as a matter of fact it conceals from the designer the influence of fundamental factors upon the pile forces.

A particular coordinate frame $G\,1'\,2'$ is therefore sought for which the relation between loads P' acting on the origin G and the resulting pile pier movements is reduced to diagonal form.

The coordinate frame $G\,1'\,2'$ is determined by the angle g_a, having direction cosines g_1, g_2 between the 01- and $G\,1'$-axes, and by the projections g_4,

g_5, fig. 10. It is further localized in such a way that a load P_1', P_2' along any of the coordinate axes $G\,1'$, $G\,2'$ only translates the pile pier in the direction of the same axis and so that a moment load upon the pile pier only rotates the pier around G.

$$P_1' = P_n' = G_n u_1', \qquad P_2' = P_t' = G_t u_2', \qquad P_6' = P_r' = G_r u_6'. \tag{18}$$

Because of its practical importance the origin G is given a special name, the pile-group center. Forces P' and movements u' reduced to the pile-group center G can be transformed to 0 in analogy with the transformation of F and U of the separate piles to 0 according to (15):

$$\begin{aligned} &R_{11} = G_n g_1^2 + G_t g_2^2, \qquad R_{22} = G_n g_2^2 + G_t g_1^2, \qquad R_{11} + R_{22} = G_n + G_t \\ &R_{12} = (G_n - G_t)\, g_1 g_2, \qquad R_{12}/g_1 g_2 = G_n - G_t \end{aligned}$$

$$G_n;\; G_t = \tfrac{1}{2}\,(R_{11} + R_{22} \pm R_{12}/g_1 g_2) \tag{19}$$

$$\frac{2\,R_{12}}{R_{11} - R_{22}} = \frac{2\,(G_n - G_t)\, g_1 g_2}{(G_n - G_t)\,(g_1^2 - g_2^2)} = \frac{2\, g_1 g_2}{1 - 2\, g_2^2} = \tan 2\, g_a. \tag{20}$$

The axis rotation g_a is obtained from (20) and the axial and transverse stiffnesses G_n, G_t of the pile group from (19).

Further, cf. (15),

$$\begin{aligned} \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} &= \begin{bmatrix} g_1 & -g_2 \\ g_2 & g_1 \end{bmatrix} \begin{bmatrix} -G_n\, g_5 \\ G_t\, g_4 \end{bmatrix} \\ \begin{bmatrix} -G_n\, g_5 \\ G_t\, g_4 \end{bmatrix} &= \begin{bmatrix} g_1 & g_2 \\ -g_2 & g_1 \end{bmatrix} \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} \end{aligned} \tag{21}$$

Alternatively, R_{13r}, and R_{23r} can be evaluated in a coordinate frame $01'\,2'$ rotated (by g_a) where $g_{1r} = 1$ and $g_{2r} = 0$. Then (21) is simplified to

$$-\, G_n g_5 = R_{13r}, \qquad G_t g_4 = R_{23r}. \tag{22}$$

This localizes the pile-group center G. Finally, the moment stiffness G_r of the pile-group is calculated according to (15)

$$G_r = R'_{33} = \Sigma\, S_n q_5'^2 + \Sigma\, S_t q_4'^2 + \Sigma\, S_r. \tag{23}$$

Minimum Property of the Pile-Group Center

Properties of the pile-group which are invariant under coordinate transformations are expressed by G_n, G_t, G_r. Differentiating g_2^2 with respect to g_a yields

$$\frac{d\, g_2^2}{d\, g_a} = 2\, g_2 \frac{d \sin g_a}{d\, g_a} = 2\, g_2 \cos g_a = 2\, g_1 g_2 = 2\, R_{12}/(G_n - G_t)$$

but R_{12} is assumed to vanish when 01 takes the direction $G\,1'$. Then g_2^2 and $R_{22} = G_t + (G_n - G_t)\, g_2^2$ will be minima according to (15), and $R_{11} = G_n - (G_n - G_t)\, g_2^2$ will be a maximum.

Partial derivatives of $R_{33} = G_n g_5^2 + G_t g_4^2 + G_r$ with respect to g_5 and g_4 are

$$\partial R_{33}/\partial g_5 = 2\,G_n g_5, \qquad \partial R_{33}/\partial g_4 = 2\,G_t g_4 .$$

These two are according to (21) zero when 0 coincides with G, because R_{13} and R_{23} are then assumed to vanish. Thus R_{33} takes on its minimum value when 0 is located in G.

Location of the Pile-Group Center

If a pile-group is symmetric, the pile-group center G is located upon its axis of symmetry. Furthermore, the principal axis $G\,1'$ coincides with the axis of symmetry. This saves the calculation of the coordinate rotation g_a.

For hinged or long piles driven in very loose soil, $S_t = S_r = 0$, the following rules also hold: In the pile-group, fig. 12, consisting of two or several sub-groups of parallel piles arranged so that the lines of symmetry of these sub-groups intersect in one point, this intersection will be the pile-group center. Denoting by q_5 the perpendicular distance to the pile axes from a variable point 0, $R_{33} = \Sigma\, S_n q_5^2$ for each subgroup is a minimum and constant along its axis of symmetry. For all subgroups $R_{33} = \Sigma\, S_n q_5^2$ will be smallest at the point of intersection of the axes of symmetry of all subgroups.

For piles transferring transversal loads $\Sigma\,(S_n q_5^2 + S_t q_4^2 + S_r)$ obviously is a minimum and the pile-group center is situated somewhere below the point where $\Sigma\, S_n q_5'^2$ is a minimum.

If the pile-group center G can be localized by such simple considerations, the origin 0 in the first coordinate frame should at once be taken there in order to save special calculations for its transfer to G.

Pile Forces

In the new coordinate frame $G\,1'\,2'$ determined in this way, the direction cosines q_1', q_2' and lever arms q_4', q_5' for each pile are calculated. Then according to (13), (18)

$$F = S\,q'\,u' = S\,q'\,R'^{-1}\,P' \tag{24}$$

$$\begin{bmatrix} F_n/S_n \\ F_t/S_t \\ F_r/S_r \end{bmatrix} = \begin{bmatrix} q_1' & q_2' & -q_5' \\ -q_2' & q_1' & q_4' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_n'/G_n \\ P_t'/G_t \\ P_r'/G_r \end{bmatrix} \tag{25}$$

that is

$$\begin{aligned} F_n &= P_n'\,q_1' \cdot S_n/G_n + P_t'\,q_2' \cdot S_n/G_t + P_r'\,(-q_5') \cdot S_n/G_r \\ F_t &= P_n'\,(-q_2') \cdot S_t/G_n + P_t'\,q_1' \cdot S_t/G_t + P_r'\,q_4' \cdot S_t/G_r \\ F_r &= P_r' \cdot S_r/G_r \end{aligned}$$

Analogies of theoretical interest can be drawn between the force distribution in a plane pile group and the Navier formulas for stress in a compressed and bent beam. The direct practical use of these analogies is, however, small.

Numerical Calculation of Pile Groups

The numerical calculations for a pile-group begin with the choice of a simple coordinate frame 012 with origin 0 somewhere near the point where $\Sigma(S_n q_5^2 + S_t q_4^2 + S_r)$ is estimated to be a minimum. The 01-axis could be placed in the average main direction of the piles, or vertically. The calculation of the pile-group, fig. 11, is carried through in a form, table 2.

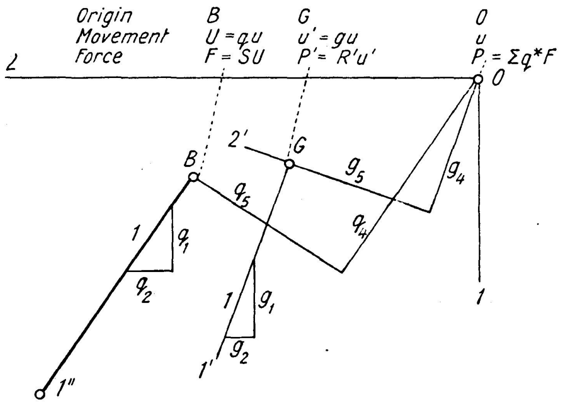

Fig. 10

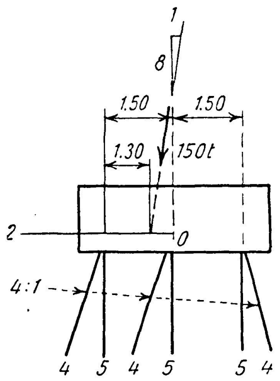

Fig. 11

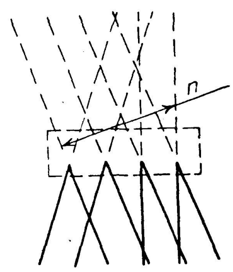

Fig. 12

Simplified Numerical Calculations

Uncertainties about the properties of the earth and about the driving and elasticity of the piles justify all simplifications of the calculation which introduce only small, harmless errors. Piles are, for instance, not driven at a slope larger than one to four or one to three from the vertical. Then

$$q_1 > \left[1 + \left(\frac{1}{3}\right)^2\right]^{-\frac{1}{2}} > 1 - \frac{1}{18} = 0.95.$$

Thus q_1' and q_1 are approximately equal to one and $q_2' = \sin q_a'$, $q_2 = \sin q_a$ to the arcs q_a', q_a. This somewhat simplifies the calculations of table 2.

When no piles have side resistances, $S_t = S_r = 0$, 12 lines in the form, table 2, will remain blank.

Successive Approximations for Side Resistances

A pile group with S_t and $S_r = 0$ can first be calculated under the assumption $S_t = S_r = 0$. The resulting movements u_0' of the pile pier can be used to calculate F_{n1}, F_{t1}, F_{r1} for each pile when these are assumed to transfer side forces. This first approximation obviously can be successively improved.

Results of Numerical Pile-Group Calculations

K. Magnusson (The influence of side resistances and encasement of piles in the calculation of plane pile-groups according to the theory of elasticity,

Table 2

Σ_{piles}	Dim., multip.	Pile, Mark No. of each	a 4	b 5	c 4	d 5	e 5	f 4
(1) 49410	1830 t/m	Axial stiffness S_n	1	1	1	1	1	1
(2) 5400	200 t/m	Transv. stiffn. S_t	1	1	1	1	1	1
(4) 1436	53,2 tm	Rot. stiffness S_r	1	1	1	1	1	1
	10^{-3} rad	Pile batter q_a	244	0	244	0	0	−244
	10^{-3}	$\cos q_a$ q_1	970	1000	970	1000	1000	970
	10^{-3}	$\sin q_a$ q_2	244	0	244	0	0	−244
(5) 1170	1630 t/m	$(S_n - S_t)\,q_2^2$	0,06	0	0,06	0	0	0,06
$R_{12} = 1560$	1630 t/m	$(S_n - S_t)\,q_1 q_2$	0,24	0	0,24	0	0	−0,24
	10^{-3} rad	$q_a - g_a = q_a'$	207	−37	207	−37	−37	−281
	10^{-3}	q_1'	978	1000	978	1000	1000	961
	10^{-3}	q_2'	207	−37	207	−37	−37	−281
	m	Lever arm b	0,52	0,52	0,52	0,52	0,52	0,52
	m	Arm q_5	1,46	1,50	0	0	−1,50	−1,46
	m	Arm q_4	0,88	0,52	0,52	0,52	0,52	0,88
	1830 t	$S_n q_5$	1,46	1,50	0	0	−1,50	−1,46
(6) 220	1830 t	$S_n q_5 q_1'$	1,43	1,50	0	0	−1,50	−1,40
(7) 5200	1830 t	$S_n q_5 q_2'$	0,30	−0,05	0	0	0,05	0,41
	200 t	$S_t q_4$	0,88	0,52	0,52	0,52	0,52	0,88
(8) 3340	200 t	$S_t q_4 q_1'$	0,86	0,52	0,51	0,52	0,52	0,85
(9) −28	200 t	$S_t q_4 q_2'$	0,18	−0,02	0,11	−0,02	−0,02	−0,25
	m	$q_5 + g_4 q_2' - g_5 q_1 = q_5'$	1,40	1,51	−0,06	0,01	−1,49	−1,38
	m	$q_4 - g_4 q_1' - g_5 q_2 = q_4'$	1,17	0,82	0,81	0,82	0,82	1,16
(10) 69430	1830 tm	$S_n q_5'^2$	1,96	2,28	0	0	2,22	1,90
(11) 4710	200 tm	$S_t q_4'^2$	1,37	0,67	0,66	0,67	0,67	1,35
	10^{-3}	$q_1' \cdot S_n/G_n = F_n/P_n'$	37,0	37,7	37,0	37,7	37,7	36,3
	10^{-3}	$q_2' \cdot S_n/G_t = F_n/P_t'$	60,0	−10,7	60,0	−10,7	−10,7	−81,5
	10^{-3}	$-q_5' \cdot S_n/G_r = F_n/P_r'$	−33,8	−36,5	1,5	−0,2	3,6	33,3
	10^{-3}	$-q_2' \cdot S_t/G_n = F_t/P_n'$	−0,85	0,15	−0,85	0,15	0,15	1,16
	10^{-3}	$q_1' \cdot S_t/G_t = F_t/P_t'$	31,0	31,8	31,0	31,8	31,8	30,6
	10^{-3}	$q_4' \cdot S_t/G_r = F_t/P_r'$	3,1	2,2	2,1	2,2	2,2	3,1
	10^{-3}	$S_r/G_r = F_r/P_r'$	0,70	0,70	0,70	0,70	0,70	0,70

$R_{11} = (1) - (5) = \underline{48{,}2 \cdot 10^3}$ $\quad R_{22} = (2) + (5) = \underline{6{,}6 \cdot 10^3}$

$\operatorname{tg} 2g_a = 2\,R_{12}/(R_{11} - R_{22}) = 3{,}12\,/\,41{,}6 \quad = \underline{0{,}075}$

$g_a = \underline{37 \cdot 10^{-3}}$ $\quad g_1 g_2 = \underline{1{,}000 \cdot 37 \cdot 10^{-3}} \quad = \underline{37 \cdot 10^3}$

$R_{13\,r} = -(6) - (9) = \underline{-192} = -G_n g_5$

$R_{23\,r} = -(7) + (8) = \underline{-1860} = G_t g_4$

$G_r = (10) + (11) + (4) = \underline{75{,}6 \cdot 10^3}$

$G_n;\ G_t = \tfrac{1}{2}((1) + (2) \pm R_{12}/g_1 g_2) =$

$\underline{48{,}5 \cdot 10^3}$, $\underline{6{,}3 \cdot 10^3}$

$g_5 = \underline{0{,}004}$

$g_4 = \underline{-0{,}295}$

Publ. 52:2. Department of Bridge Engineering, Chalmers University of Technology, Gothenburg 1952), calculated under various assumptions the pile-group, fig. 11. This consists of 27 wooden piles of diameter 0.2 m, $EA = 70 \cdot 314 = 22000$ t, $EI = 22000 \cdot 0.05^2 = 55$ tm^2, and a length of 12 m, $S_n = 22000/12 = 1830$ t/m, driven to point bearing through earth with an elasticity of 0, 100, 1000, and 10000 t/m^3. The last-mentioned kind of soil yields

$$b = \sqrt[4]{55\,\text{t/m}^2 / 4 \cdot 0.2\,\text{m} \cdot 10000\,\text{t/m}^3} = 0.29\ \text{m}$$

$$S_r = 55/2 \cdot 0.29 = 95\,\text{t/m}, \quad S_t = 95/0.29^2 = 1130\,\text{t/m}.$$

Hinged piles and zero soil elasticity were first calculated, then fixed pile ends and soil elasticities 0, 100, 1000, and 10000 t/m^3. The resulting largest axial pile-forces gradually decreased by 45 per cent. The smallest axial pile forces increased. Thus the transversal forces of the piles were equalized by increasing side resistances and fixation of the pile heads in the pile pier.

For soil elasticity 1000 t/m^3, $c = 200$ t/m^2, the transversal forces F_2 were between 0.05 and 0.47 t, far below permissible values for shear in wood. Then all piles resisted by their transversal forces 45 per cent of the total horizontal load upon the pile pier. The bending moments in the pile heads varied between 0.11 and 0.20 tm. This caused bending stresses in the pile head of less than 21 at.

Encasement and side resistances strongly decreased the side movements of the pile pier, namely from 13 mm to 2 mm. The vertical movements changed very little. The small rotation of the pile pier reversed from $0.4 \cdot 10^{-3}$ to $-0.3 \cdot 10^{-3}$ rad.

Square 0.25 m concrete piles in the pile group fig. 11, received almost the same axial forces as the wooden piles. In fixed pile heads bending stresses less than 10 at developed. The side-movement of the pile-pier was less than for wooden piles, since the concrete piles were stiffer.

The simplified method mentioned gave results within 5 per cent for axial forces and within 10 per cent for transverse forces, compared to the results according to the more exact method. The side movements differed by only 1 per cent.

Economic Lay-Out of Piles in a Pile-Group

The difficulty in the design of a pile-group lies not in the calculation of the pile forces. It has here been shown that this calculation is simple and direct. On the other hand it is difficult to locate the piles in a pile-group so that the total cost of the pile pier and the piles will be the smallest possible.

The situation is complicated by the fact that a pile-group must as a rule withstand two or more different cases of loading P instead of only one. Generally it is sufficient to consider only two cases of loading, namely those whose resultant has the largest and the smallest slope P_2/P_1. The forces of rotation resulting from the term P_r'/G_r' are often quite large. Therefore, the pile-group center should be located near the resultant of the resultant of the two extreme loadings just mentioned. The forces of rotation P_r'/G_r' for the two cases of loading are then equal and of opposite sign. The pile forces can be reduced by such artifices; it is nevertheless advisable to make sure that they do not become larger for any intermediate case of loading.

The forces of rotation in the piles increase, if the "broom", fig. 12, formed by the pile axes and their elongations is given a small "neck" n. The pile-group center is situated in this neck for hinged piles without side resistances where $\Sigma S_n q_5^2 = \min$, and for piles with side resistances that are somewhat lower, where $R_{33} = \Sigma(S_n q_5^2 + S_t q_4^2 + S_r) = \min$. To make the forces of rotation P_r'/G_r' as small as possible, the location of the pile axes in this neck should be as far as possible from the pile-group center. In other words, the minimum of R_{33} which locates the pile-group center should be as large as possible.

It is often specified that no piles may be in tension. When the piles have no side resistances pull is avoided by driving a sufficient number of battered piles in a larger slope in both directions than that of the both load resultants just mentioned. If this condition cannot be satisfied by natural distribution of piles, it may be necessary to change the acting loads, for instance by increasing the weight of the pile pier.

For piles which are entirely or partly free in water or air the buckling safety must be established. Hj. Granholm (On the elastic stability of piles surrounded by a supporting medium, Diss., Stockholm 1929) has shown that piles as a rule will not buckle if they are entirely surrounded by clay which may be quite soft. In very soft clay slender piles may sometimes buckle.

Apart from such considerations there are practical requirements that the piles must be driven at specified minimum distances, that battered piles must not interfere with other piles in the earth, that the pile-group can be driven at all, and that the pile foundation should admit of a simple construction. Neither should the point of view be overlooked that the pile-group should be so designed that it will be simple to calculate.

An economical pile-group designed to resist appreciable side forces will in the main contain many battered piles in both directions driven at the largest practical slope and so that the minimum $\Sigma S_n q_5^2$ (more exactly, of R_{33}) is as large as possible.

One additional design consideration for pile-groups should finally be proposed. Creep in the earth and the pile-heads makes it sound practise to regard a pile-group as hinged and without side resistances when it is loaded by the permanent load of the foundation (gravity, no traffic load, normal water level, mean temperature), and to consider the pile heads as fixed in the pier and the side resistances as fully active for all load variations of short duration, such as traffic, wind, and changes from normal water level and mean temperature.

Three-Dimensional Pile-Groups

In a suitable, orthogonal coordinate frame 0123, immovable with respect to the subsoil, the direction of each pile is given by the downward unit vector $[q_1, q_2, q_3]^* = q$ along its axis: the components of q are the direction cosines of the pile axis. Fictitious stiff lever arms of length $b = \sqrt[4]{EI/4c}$ are attached to the pile head and to the pile-pier. Their lower ends B are located by the vectors $0B = [q_4, q_5, q_6]^* = q_0$, fig. 13a.

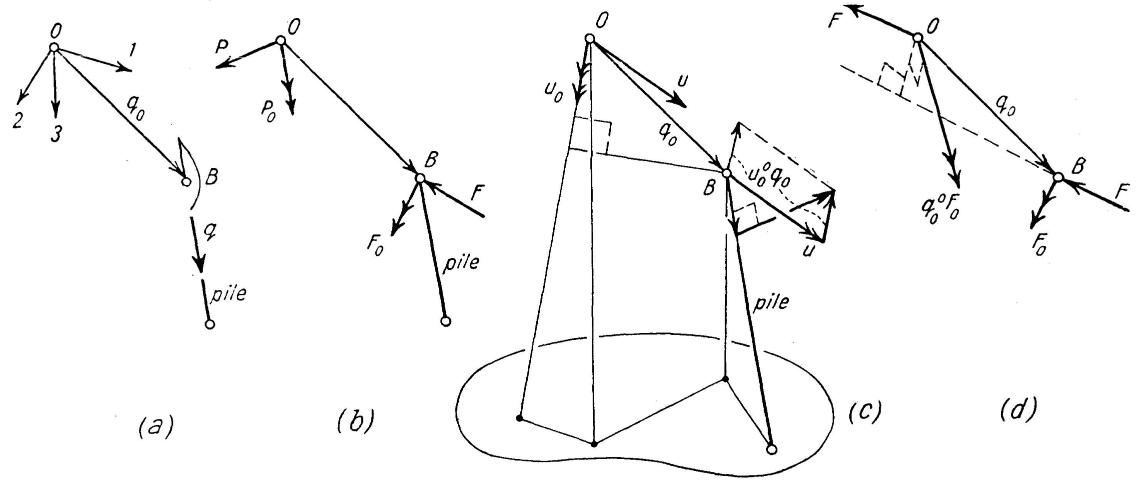

Fig. 13

The pile heads are encased in a stiff pile pier. External loads acting upon the pile pier are reduced to a force P at the origin 0 and a moment P_0, fig. 13b. They produce forces $-F$, $-F_0$ in the piles. The resulting pile deformations displace the pile pier. The small displacement of the pile pier can be reduced to a rotation u_0 about 0 and a translation u, fig. 13c. The translation u of the pile pier displaces the stiff lever ends B by distances u. The rotation u_0 of the pile pier about 0 displaces B perpendicularly to u_0 and q_0 by the vector product $u_0{}^0 q_0$: the end B of a stiff lever arm is thus totally displaced by

$$u_B = u + u_0{}^0 q_0 = u - q_0{}^0 u_0. \tag{26}$$

Further, the pile at the end B of the stiff lever arm and the pile head rotates through the angle $u_{0B} = u_0$.

Here the superscript 0 denotes an antisymmetric matrix containing the components of the vector in question, for instance

$$x^0 = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad x^0 y = \begin{bmatrix} -x_3 y_2 + x_2 y_3 \\ x_3 y_1 - x_1 y_3 \\ -x_2 y_1 + x_1 y_2 \end{bmatrix} \tag{27}$$

The matrix product $x^0 y$ thus yields a column matrix containing the components of the vector product of x and y.

The displacement u_B and the rotation u_{0B} can be resolved in components parallel to and normal to the pile axis:

$$\begin{aligned} u_B &= q q^* u_B + (u_B - q q^* u_B) \\ u_{0B} &= q q^* u_{0B} + (u_{0B} - q q^* u_{0B}) \end{aligned} \tag{28}$$

(The projection of u_B upon the pile axis q is the scalar product $q^* u_B$.) The pile thus reacts upon the pile pier with a force F through B and with a moment F_0, fig. 13d,

$$\begin{aligned} F &= -S_n q q^* u_B - S_t (u_B - q q^* u_B) \\ F_0 &= -S_v q q^* u_{0B} - S_r (u_{0B} - q q^* u_{0B}) \end{aligned} \tag{29}$$

The spring constants S_n, S_t, S_r, S_v are the axial, transversal, rotational bending, and torsional stiffnesses of the pile. The torsional stiffness S_v of the pile about its own axis can as a rule be neglected. Still, S_v may become active in foundation shafts. It is included here for the sake of completeness.

The load P, P_0 and the pile reactions F, F_0 maintain the pile-pier in equilibrium. This requires that all forces have the vector sum 0 and the moment sum 0 about any point, for instance the origin 0

$$\begin{aligned} P + \Sigma F &= 0 \\ P_0 + \Sigma q_0{}^0 F + \Sigma F_0 &= 0. \end{aligned} \tag{30}$$

The sums include all piles. Introducing F, F_0 from (29) and writing $S_D = S_n - S_t$ and $S_d = S_v - S_r$, one obtains

$$\begin{aligned} P &= \Sigma S_D \; q q^* u_B + \Sigma S_t \; u_B \\ P_0 &= \Sigma S_D q_0{}^0 q q^* u_B + \Sigma S_t q_0{}^0 u_B + \\ &\quad + \Sigma S_d \; q q^* u_{0B} + \Sigma S_r \; u_{0B} \end{aligned} \tag{31}$$

Substitution of u_B from (26) and $u_{0B} = u_0$ yields

$$\begin{aligned} P &= R_{11} u + R_{14} u_0 \\ P_0 &= R_{41} u + R_{44} u_0 \end{aligned} \tag{32}$$

$$\begin{aligned} R_{11} &= \Sigma S_D \; q q^* + \Sigma S_t \\ R_{14} &= -\Sigma S_D \; q q^* q_0{}^0 - \Sigma S_t q_0{}^0 \\ R_{41} &= \Sigma S_D q_0{}^0 q q^* + \Sigma S_t q_0{}^0 = R_{14}^* \\ R_{44} &= -\Sigma S_D q_0{}^0 q q^* q_0{}^0 - \Sigma S_t q_0{}^0 q_0{}^0 + \Sigma S_d q q^* + \Sigma S_r. \end{aligned} \tag{33}$$

The elements of the matrices R can be calculated directly from the stiffnesses S and positions q, q_0 in 0123 of the piles.

The system of equations (32) can be solved by inverting one $6\cdot 6$-matrix, but the numerical work is more manageable if $3\cdot 3$-matrices are employed: to this end multiply (32a, b) by $R_{41} R_{11}^{-1}$ and $R_{14} R_{44}^{-1}$:

$$\begin{aligned} R_{41} R_{11}^{-1} P &= + R_{41} u + R_{41} R_{11}^{-1} R_{14} u_0 \\ R_{14} R_{44}^{-1} P_0 &= R_{14} R_{44}^{-1} R_{41} u + R_{14} u_0 \end{aligned} \tag{34}$$

and subtract (34b, a) from (32a, b)

$$\begin{aligned} P - R_{14} R_{44}^{-1} P_0 &= (R_{11} - R_{14} R_{44}^{-1} R_{41}) u = R_{110} u \\ -R_{41} R_{11}^{-1} P + P_0 &= (R_{44} - R_{41} R_{11}^{-1} R_{14}) u_0 = R_{440}' u_0 \end{aligned} \tag{35}$$

and multiply by R_{110}^{-1}, R_{440}^{-1}

$$\begin{aligned} u &= \quad R_{110}^{-1} \qquad \cdot P - R_{110}^{-1} R_{14} R_{44}^{-1} P_0 \\ u_0 &= - R_{440}^{-1} R_{41} R_{11}^{-1} P + R_{440}^{-1} \qquad \cdot P_0 . \end{aligned} \tag{36}$$

With these formulas the displacements u, u_0 of the pile pier can be calculated from its load P, P_0. Finally the components F, F_0 of each pile force can be obtained by substitution of u, u_0 in (26), (29).

Numerical Calculations

The recommended sequence of numerical calculations is given by the formulas (33), (35), (36). Pile notation and stiffnesses S_n, S_t, S_r, S_v, S_D, S_d are entered in columns for each pile. Pile batter q_1, q_2, q_3 and positions q_4, q_5, q_6 of the stiff lever arm end B are denoted in a suitable coordinate frame 0123. In the same table the components of the vector product $q_0{}^0 q\,(=(-q^* q_0{}^0)^*)$ are calculated. From the table are summed the elements R_{11}, R_{14}, $(R_{41} = R_{14}^*)$, R_{44} of the primary matrix terms on the right side of (33). The adjoint matrices, determinants, and reciprocal matrices of R_{11} and R_{44} are calculated. $R_{14} R_{44}^{-1}$ and $R_{41} R_{11}^{-1}$ are multiplied, likewise $R_{14} R_{44}^{-1} R_{41}$ and $R_{41} R_{11}^{-1} R_{14}$. Subtraction from R_{11} and R_{44} yields R_{110} and R_{440}. Their adjoint matrices, determinants, and reciprocal matrices R_{110}^{-1} and R_{440}^{-1} are calculated. These are postmultiplied with $R_{14} R_{44}^{-1}$ and $R_{41} R_{11}^{-1}$, just calculated. This completes the solution according to (36).

Pile-Group Center

A pile-group center can also be established for pile groups in space. In the case $S_t = S_r = S_v = 0$ for instance, the pile-group center has the property that the sum of the products of S_n and the square of the perpendicular distance to each pile is a minimum. At the pile-group center the matrices R_{14} and R_{41} are equal and symmetric. This simplifies the calculations somewhat, but as a rule does not offset the calculations needed to locate the pile-group center.

When it is possible to find the pile-group center by some simple minimum consideration, as for plane pile-groups, the origin of the coordinate frame should be located there. If not, the origin should be taken somewhere near the middle of the neck of the broom-shaped structure of the pile axes. The coordinate frame can then be considered as "suitable". Direct calculation according to (36) without subtleties seems to be the easiest way to reach the goal.

Numerical Example

For all piles in the symmetric three-dimensional pile-group, fig. 14, all S_n are equal and all other S are zero. The stiff lever arms b are zero (hinged piles).

Because all S_t and S_r are zero, B can be chosen as any point of the pile axis. The coordinate frame 0123 is located according to the figure. In table 3 pile positions and data for the matrix elements are given.

Table 3

Pile	Mult.	a	b	c	d	e	f	g	h	i
$S_n =$	$S_n \cdot$	1	1	1	1	1	1	1	1	1
q_1	10^{-2}	–24		24				–24		24
q_2	10^{-2}		24						–24	
q_3	10^{-2}	97	97	97	100	100	100	97	97	97
q_4	e				–1		1			
q_5	e	1		1				–1		–1
q_6	e									
$(q_0{}^0 q)_1$	$10^{-2}e$	97		97				–97		–97
$(q_0{}^0 q)_2$	$10^{-2}e$				100		–100			
$(q_0{}^0 q)_3$	$10^{-2}e$	24		–24				–24		24

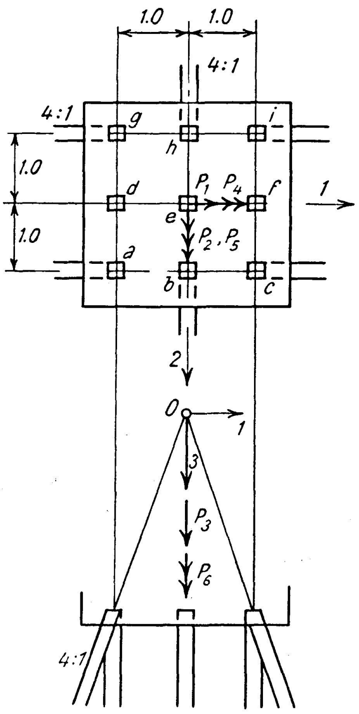

Fig. 14

$$R_{11} = \Sigma S_n \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} [q_1, q_2, q_3] = S_n \begin{bmatrix} 0.23 & 0 & 0 \\ 0 & 0.115 & 0 \\ 0 & 0 & 8.64 \end{bmatrix}$$

$$R_{41} = \Sigma S_n \begin{bmatrix} (q_0{}^0 q)_1 \\ (q_0{}^0 q)_2 \\ (q_0{}^0 q)_3 \end{bmatrix} [q_1, q_2, q_3] = e\, S_n \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{44} = \Sigma S_n \begin{bmatrix} (q_0{}^0 q)_1 \\ (q_0{}^0 q)_2 \\ (q_0{}^0 q)_3 \end{bmatrix} \begin{bmatrix} (q_0{}^0 q)_1 \\ (q_0{}^0 q)_2 \\ (q_0{}^0 q)_3 \end{bmatrix}^* =$$

$$= e^2 S_n \begin{bmatrix} 3.76 & 0 & 0 \\ 0 & 2.00 & 0 \\ 0 & 0 & 0.23 \end{bmatrix}$$

$$R_{11}^{-1} = \frac{R_{11}^a}{|R_{11}|} = \frac{1}{0.228\, S_n{}^3} S_n{}^2 \begin{bmatrix} 0.995 & 0 & 0 \\ 0 & 1.99 & 0 \\ 0 & 0 & 0.026 \end{bmatrix}$$

$$R_{44}^{-1} = \frac{R_{44}^a}{|R_{44}|} =$$

$$= \frac{1}{1.728\, e^6 S_n{}^3} e^4 S_n{}^2 \begin{bmatrix} 0.46 & 0 & 0 \\ 0 & 0.865 & 0 \\ 0 & 0 & 7.52 \end{bmatrix}$$

$$R_{14} R_{44}^{-1} = 0, \quad R_{41} R_{11}^{-1} = 0, \quad R_{14} R_{44}^{-1} R_{41} = 0, \quad R_{41} R_{11}^{-1} R_{14} = 0$$

$$R_{110} = R_{11}, \quad R_{440} = R_{44}, \quad R_{110}^{-1} R_{14} R_{44}^{-1} = 0, \quad R_{440}^{-1} R_{41} R_{11}^{-1} = 0$$

(36) assumes the form

$$u = \frac{1}{0.228\,S_n}\begin{bmatrix} 0.995 & 0 & 0 \\ 0 & 1.99 & 0 \\ 0 & 0 & 0.026 \end{bmatrix} P = \frac{1}{S_n}\begin{bmatrix} 4.37\,P_1 \\ 8.73\,P_2 \\ 0.11\,P_3 \end{bmatrix}$$

$$u_0 = \frac{1}{1.728\,e^2\,S_n}\begin{bmatrix} 0.46 & 0 & 0 \\ 0 & 0.865 & 0 \\ 0 & 0 & 7.52 \end{bmatrix} P_0 = \frac{1}{e^2\,S_n}\begin{bmatrix} 0.27\,P_4 \\ 0.50\,P_5 \\ 4.35\,P_6 \end{bmatrix}.$$

The pile forces are given by (29), (26)

$$F = q\,S_n\,q^*\,(u - q_0{}^0\,u_0)$$

$$= q\left(\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}^* \cdot \begin{bmatrix} 4.37\,P_1 \\ 8.73\,P_2 \\ 0.11\,P_3 \end{bmatrix} + \begin{bmatrix} (q_0{}^0 q)_1 \\ (q_0{}^0 q)_2 \\ (q_0{}^0 q)_3 \end{bmatrix}^* \cdot \begin{bmatrix} 0.27\,P_4 \\ 0.50\,P_5 \\ 4.35\,P_6 \end{bmatrix}\right)$$

The axial force in the pile a is, for instance,

$$F_a = \begin{bmatrix} -0.24 \\ 0 \\ 0.97 \end{bmatrix}^* \cdot \begin{bmatrix} 4.37\,P_1 \\ 8.73\,P_2 \\ 0.11\,P_3 \end{bmatrix} + \begin{bmatrix} 0.97 \\ 0 \\ 0.24 \end{bmatrix}^* \cdot \begin{bmatrix} 0.27\,P_4 \\ 0.50\,P_5 \\ 4.35\,P_6 \end{bmatrix}$$

$$= -1.05\,P_1 + 0.11\,P_3 + 0.26\,P_4 + 1.04\,P_6.$$

A numerical example of an unsymmetrical pile-group without side-resistances is completely calculated in S. O. ASPLUND, Three-dimensional pile-groups, AIPC Memoires 1947, p. 1.

Summary

The calculation of beams on continuous spring supports is facilitated by the device of a stiff lever arm, fastened to the beam end and extending to its "elastic center" where a transversal force produces a transverse displacement but no rotation and where a moment produces a rotation but no displacement. Longitudinal, transverse and rotational stiffnesses referred to this center, are defined. Tests for the actual evaluation of these stiffnesses and of the length of the lever arm are designed and measurements and results reported.

A matrix method for the force and displacement analysis of pile groups with side resistances is developed up to a workable formulaire for making the numerical computations. A simplified method and the question of the design of pile groups are discussed.

Finally the method is extended to the analysis in three dimensions of pile groups with side resistances.

Résumé

Le calcul des poutres portant sur des appuis élastiques continus est facilité par l'artifice que constitue l'introduction d'un bras de levier rigide, fixé à

l'extrémité de la poutre et s'étendant jusqu'à son ,,centre élastique'', endroit auquel un effort transversal produit un déplacement transversal sans rotation et un moment produit une rotation sans déplacement proprement dit. Les rigidités longitudinale, transversale et rotationnelle sont définies par rapport à ce centre. L'auteur rend compte d'essais pour l'évaluation de ces rigidités et de la longueur du bras de levier, ainsi que des résultats des mesures.

Il développe une méthode matricielle pour l'étude des efforts et des déplacements des groupes de pieux avec résistance latérales, mettant au point un formulaire pour les calculs numériques. Il discute une méthode simplifiée et l'étude du projet des groupes de pieux.

Enfin, il étend cette méthode à l'analyse tridimensionnelle des groupes de pieux avec résistances latérales.

Zusammenfassung

Die Berechnung von kontinuierlichen Trägern auf elastischen Auflagern wird erleichtert durch den Kunstgriff der Einführung eines steifen Hebelarmes, der am Trägerende befestigt wird und bis zu einem ,,elastischen Schwerpunkt'' reicht, wo eine Querkraft nur eine Querverschiebung, aber keine Drehung verursacht, und ein Drehmoment nur eine Drehung, aber keine Verschiebung erzeugt. Längs-, Quer- und Rotationssteifigkeit in bezug auf diesen Schwerpunkt werden definiert. Proben für die Abschätzung dieser Steifigkeit und der Länge des Hebelarmes werden beschrieben und über Spannungen und Ergebnisse berichtet. Es wird eine Matrizenmethode für die Kraft- und Verschiebungsberechnung von Pfahlgruppen mit seitlichen Widerständen entwickelt und zu einer praktisch verwendbaren Formel für die numerischen Berechnungen verarbeitet. Eine vereinfachte Methode und die Frage des Entwurfes von Pfahlgruppen werden diskutiert.

Schließlich wird die Methode erweitert auf die Berechnung dreidimensionaler Pfahlgruppen mit seitlichen Widerständen.