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# The Method of Inversion in the Theory of Plates

*La méthode d'inversion dans la théorie des plaques*

*Anwendung der Inversionsmethode in der Plattentheorie*

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## 1. Introductory Remarks

It is often advantageous to use conformal mapping to solve two-dimensional problems in the theory of elasticity, the inversion type of transformation being comparatively simple, especially for certain types of boundary conditions. In addition to the fundamental work by J. H. MICHELL [2], papers by A. TIMPE [13], P. FILLUNGER [7] and R. SONNTAG [8] should be mentioned; together with those by one of the present authors [3, 4, 5], who generalized this method, giving solutions for some new problems. In the theory of plates, however, this method is not generally used. It is only in A. E. H. LOVE's monograph [6] that an example of the use of this type of conformal mapping is discussed as applied to the problem of a circular plate clamped at the periphery and loaded by a concentrated eccentric force.

In the present paper the basic relations of the application of the transformation of inversion to the theory of plates will first be discussed, and this will be followed by some solutions for circular plates with eccentric holes.

## 2. Basic Relations for the Transformation of Inversion

The middle surface of the plate will be assumed to be a plane of the complex variable. Let us map every point $z$ of this plane into a point $Z$ of a corresponding complex variable plane by means of the relation

$$Z = \frac{k^2}{\bar{z}}, \qquad (k > 0) \tag{2.1}$$

where $z$ and $\bar{z}$ are conjugate complex variables.

Putting: $z = r \cdot e^{i\varphi}, \qquad \bar{z} = r \cdot e^{-i\varphi}, \qquad Z = R \cdot e^{i\Phi},$

we have
$$R \cdot e^{i\Phi} = \frac{k^2}{r} e^{i\varphi},$$

that is
$$R = \frac{k^2}{r}, \qquad \Phi = \varphi. \tag{2.2}$$

It is evident that between the points of the $z$- and the $Z$-plane there is a one-valued correspondence, according to (2.2). Such a method of mapping is termed "inversion mapping" or mapping by means of reciprocal (inverted) radii. Assuming two systems of polar coordinates $(r, \varphi)$ and $(R, \Phi)$ having a common origin, which is also the centre of inversion, the geometrical properties of this transformation can easily be established. Thus, a region bounded by a circle of radius $r_k$, the distance of its centre from the origin being $h$ $(h > r_k)$, maps into a region bounded by a circle of radius $R_k$, the distance of the centre from the origin being $H$, where:

$$R_k = \frac{k^2}{h^2 - r_k{}^2} \cdot r_k, \qquad H = \frac{k^2}{h^2 - r_k{}^2} \cdot h. \tag{2.3}$$

In the particular case of circles passing through the origin $(h = r_k)$ these map into straight lines and vice versa; thus, the regions within these circles are represented on half-planes. If the centre of inversion lies within a circle $(h < r_k)$ its inner region maps into its outer region, in other words, into a plane having a circular hole. The circle of radius $r_k = k$ the centre of which coincides with the centre of inversion is termed the "inversion circle". The points of the periphery of this circle map into themselves. It should be added that after double inversion (2.2) we return to the initial system.

In the subsequent considerations the system $(R, \Phi)$ will be referred to as the original system $(O)$, while the system $(r, \varphi)$ will be referred to as the inverted system $(J)$.

Let the deflection of an arbitrary point of the plate in the (original) system $O$ be denoted by $W(R, \Phi)$. The deflection of the corresponding point in the (inverted) system $J$ will be assumed to be:

$$w = \frac{r^2}{k^2} \cdot W. \tag{2.4}$$

The correspondence between the stress functions in plane problems is of the same type. This type of correspondence is adopted in view of its fundamental property, which is as follows. If in the system $O$ the function $W(R, \Phi)$ satisfies the biharmonic equation $\nabla^4 W(R, \Phi) = 0$, the function

$$w(r, \varphi) = \frac{r^2}{k^2} W\left(\frac{k^2}{r}, \varphi\right)$$

in the system $J$ also satisfies that equation $(\nabla^4 w = 0)$. This property, which can easily be verified, will be extensively used in the paper.

### 3. Relations Between the Fields of Moments and Shearing Forces in the Two Systems

The equations of equilibrium for an element of the plate subjected to a continuously distributed transverse load $q(r, \varphi)$ are, in terms of polar coordinates, as follows:

$$\begin{aligned} \frac{\partial Q_\varphi}{\partial \varphi} + \frac{\partial}{\partial r}(r \cdot Q_r) &= -q \cdot r, \\ \frac{M_r - M_\varphi}{r} + \frac{\partial M_r}{\partial r} + \frac{\partial M_{r\varphi}}{r\,\partial \varphi} &= Q_r, \\ \frac{2M_{r\varphi}}{r} + \frac{\partial M_{r\varphi}}{\partial r} + \frac{\partial M_\varphi}{r\,\partial \varphi} &= Q_\varphi. \end{aligned} \tag{3.1}$$

The bending moments $M_r$ and $M_\varphi$, the twisting moment $M_{r\varphi}$, and the shearing forces $Q_r$ and $Q_\varphi$ are expressed by the equations:

$$\begin{aligned} M_r &= -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right], \\ M_\varphi &= -D\left[\nu\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right], \\ M_{r\varphi} &= -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \varphi}\right), \\ Q_r &= -D\frac{\partial(\nabla^2 w)}{\partial r} = -D\left[\frac{\partial^3 w}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} - \frac{2}{r^3}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial^3 w}{\partial r\,\partial \varphi^2}\right], \\ Q_\varphi &= -D\frac{1}{r}\frac{\partial(\nabla^2 w)}{\partial \varphi} = -D\left[\frac{1}{r^3}\frac{\partial^3 w}{\partial \varphi^3} + \frac{1}{r}\frac{\partial^3 w}{\partial r^2\,\partial \varphi} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r\,\partial \varphi}\right]; \end{aligned} \tag{3.2}$$

the notation $h$ represents the (uniform) thickness of the plate, and $D = \frac{E h^3}{12(1-\nu^2)}$ its flexural rigidity.

Analogous operations are performed on the function $W$ in the system $(R, \Phi)$. Using the relations (2.4) and (2.2), we establish the relations between the moment fields in the two systems in the form:

$$\begin{aligned} M_R &= \frac{r^2}{k^2}M_r - \frac{2}{k^2}D(1+\nu)\left(w - r\frac{\partial w}{\partial r}\right), \\ M_\Phi &= \frac{r^2}{k^2}M_\varphi - \frac{2}{k^2}D(1+\nu)\left(w - r\frac{\partial w}{\partial r}\right), \\ M_{R\Phi} &= -\frac{r^2}{k^2}M_{r\varphi}. \end{aligned} \tag{3.3}$$

The shearing forces $Q_R$ and $Q_\Phi$ are found from the equilibrium equations (3.1):

$$\begin{aligned} Q_R &= \frac{M_R - M_\Phi}{R} + \frac{\partial M_R}{\partial R} + \frac{\partial M_{R\Phi}}{R\,\partial \Phi} = \frac{r}{k^2}\left(M_R - M_\Phi - r\frac{\partial M_R}{\partial r} + \frac{\partial M_{R\Phi}}{\partial \varphi}\right), \\ Q_\Phi &= \frac{2M_{R\Phi}}{R} + \frac{\partial M_{R\Phi}}{\partial R} + \frac{\partial M_\Phi}{R\,\partial \Phi} = \frac{r}{k^2}\left(2M_{R\Phi} - r\frac{\partial M_{R\Phi}}{\partial r} + \frac{\partial M_\Phi}{\partial \Phi}\right). \end{aligned} \tag{3.4}$$

The corresponding quantities are shown in Fig. 1.

Before we proceed to the discussion of the properties of the moment fields which are related by the eqs. (3.3) let us recall that if the inversion is applied to plane problems, the stress components in the two corresponding systems are related by the equations:

$$\begin{aligned}\sigma_R &= \frac{r^2}{k^2}\sigma_r + \frac{2}{k^2}\left(f - r\frac{\partial f}{\partial r}\right),\\ \sigma_\Phi &= \frac{r^2}{k^2}\sigma_\varphi + \frac{2}{k^2}\left(f - r\frac{\partial f}{\partial r}\right),\\ \tau_{R\Phi} &= -\frac{r^2}{k^2}\tau_{r\varphi},\end{aligned} \tag{3.5}$$

where $f(r,\varphi)$ is the stress function in the inverted system $J$, related to the stress function $F(R,\Phi)$ in the original system $O$ by the equation:

$$f = \frac{r^2}{k^2}\cdot F. \tag{3.6}$$

We see that a close analogy exists between the corresponding stress fields in the plane problems (3.5), and the corresponding moment fields in the theory of plates (3.3). This results from the fact that bending and twisting moments are determined by the function $W$ (or $w$) in a manner analogous to that in which linear combinations of stresses in the plane problem are expressed by the stress function $F$ (or $f$). Thus, for instance, the linear combination:

$$\sigma_\Phi + \nu\sigma_R = \frac{\partial^2 F}{\partial R^2} + \nu\left(\frac{1}{R}\frac{\partial F}{\partial R} + \frac{1}{R^2}\frac{\partial^2 F}{\partial \Phi^2}\right)$$

corresponds to the bending moment $M_R$. The combination $\sigma_R + \nu\sigma_\Phi$ corresponds to the moment $M_\Phi$, and $\tau_{R\Phi}$ corresponds to the twisting moment $M_{R\Phi}$.

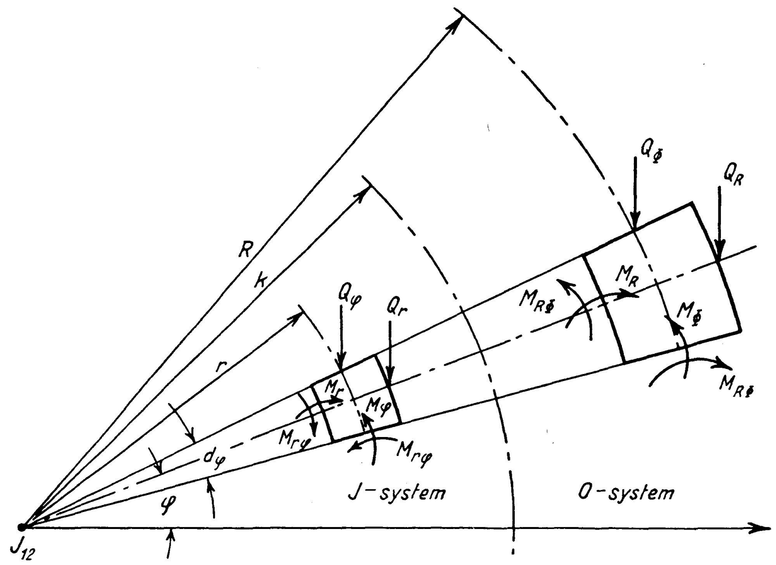


Fig. 1

The form of the eqs. (3.3) indicates that, for the relations between the moment fields in the systems $O$ and $J$, the orientation of corresponding linear elements is now of no importance; indeed, these equations can be written in the general form:

$$\begin{aligned} M_G &= \frac{r^2}{k^2} M_g + E = M'_G + E, \\ M_S &= -\frac{r^2}{k^2} M_s, \quad \text{where} \quad E = -\frac{2}{k^2} D(1+\nu)\left(w - r\frac{\partial w}{\partial r}\right); \end{aligned} \tag{3.7}$$

thus, we are free from reference to any particular system of coordinates (in particular its origin and orientation); it must always be borne in mind, however, that the symbols $M_G$, $M_S$ and $M_g$, $M_s$ denote the bending and twisting moments acting on the corresponding elements in the two systems $O$ and $J$, respectively.

The eqs. (3.7) have a simple physical meaning. If we assume that in one of the systems the bending moments $M_g \cdot ds$ and the twisting moments $M_s \cdot ds$ act on an element $ds$, the corresponding element in the other system $dS$ (where the linear element $ds$ is transformed into $dS = \frac{k^2}{r_2} ds$) will be subjected to the same bending and twisting moments; in other words:

$$\begin{aligned} M_g \cdot ds &= \quad M'_G \cdot dS, \\ M_s \cdot ds &= -M_S \cdot dS; \end{aligned}$$

the bending moment having to be complemented by the superposition of an additional moment $E$, which is the same in all directions. Thus, the trajectories of the principal moments in the system $O$ will map into analogous trajectories in the system $J$, and vice versa. It is evident that the character of the mapping depends solely on the choice of the centre of inversion, whereas the choice of the auxiliary system of coordinates should be determined by the possibility of a convenient establishment of the boundary conditions.

A similar physical meaning can be attributed to the eqs. (3.5) relating the corresponding stress fields in the plane problem. The state of stress in the corresponding system can be considered to result from the superposition of the state of stress $\frac{r^2}{k^2}\sigma_r$, whereby the forces acting on the corresponding elements are equal, and a hydrostatic tension $\frac{2}{k^2}\left(f - r\frac{\partial f}{\partial r}\right)$, variable with the coordinates of the point considered. Obviously, this analogy is well founded, the bending and twisting moments being the results of integrations, over the thickness of the plate, of normal and shear stresses acting parallel to the middle surface.

However, the analogy with the plane problem is somewhat less close if the boundary conditions are considered. In plane problems the contour free from shear stresses and subjected to a uniform normal tension in one system is transformed into a contour of the same properties in the other system. In plate problems a contour free from twisting moments preserves this property after transformation, whereas a contour subjected to constant bending moments is mapped into a contour subjected to bending moments which are, as a rule, variable. This entails a considerable difficulty in the solution of mixed boundary problems and also of those involving free edges, the shear forces in both systems being related in a relatively complicated manner.

A complete analogy, as regards boundary conditions, with the plane state of stress, exists only in the particular case of $\nu = 0$; it concerns, however, the "circumferential" bending moments. This is because the contour for which $M_\varphi$ = const. (with $M_{r\varphi} = 0$) is transformed, in the other system, into a contour for which we have also $M_\Phi$ = const. (and $M_{R\Phi} = 0$). This condition is, however, of little practical importance.

## 4. The Generalized Inversion

In the above considerations both coordinate systems had a common origin at the point $J_{12}$ (fig. 1), which, according to our assumption, was, at the same time, the centre of inversion. In the general case the centre of inversion does not coincide, however, with one particular point of the plate (its centre for instance); this may render cumbersome the establishment of the boundary conditions. These difficulties will be overcome by an appropriate choice of the origin of the auxiliary system of coordinates $(r, \varphi)$. For instance, if the original sytem $O$ maps in the inverted system into a circle, we choose the origin at the centre of this circle ($J_2$). The centre of inversion $J_1$ itself is independent of this choice and its position is determined by the required

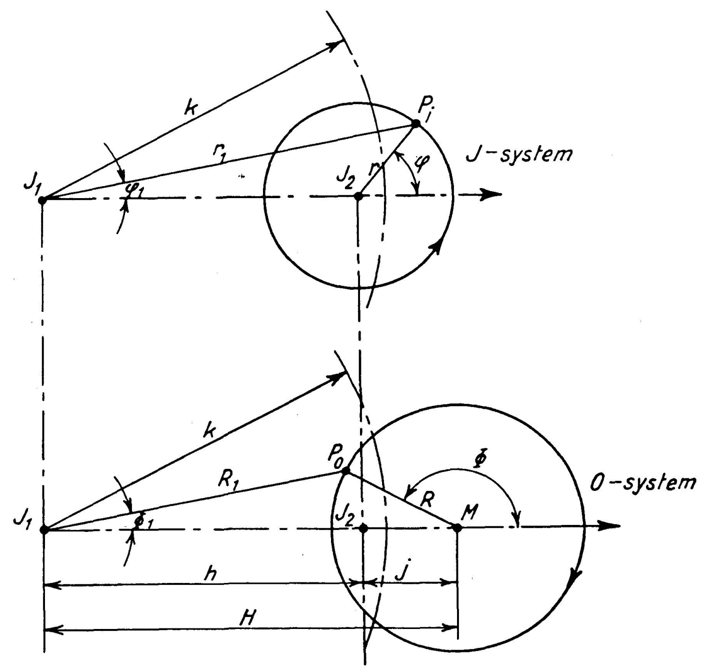


Fig. 2

mapping function. The distance between these two characteristic points is $\overline{J_1 J_2} = h$ (fig. 2).

Stating the problem in the above manner we think, for instance, of the problems, for which the solution can be sought by mapping the system examined into a concentric annular plate. This will be, in such a case, a two-parameter family of circular plates with eccentric holes (fig. 3a); a one-parameter family of semi-infinite plates bounded by straight lines with circular holes (fig. 3b); and a two-parameter family of infinite plates with two circular holes (fig. 3c). Each of these systems can be mapped by inversion into a doubly connected concentric region — an annulus ($r = a, b$). The location of the inversion pole is in each case different. In the first case $h \begin{cases} < a, \\ > b, \end{cases}$ in the second case $h = \begin{cases} a, \\ b, \end{cases}$ and in the third case $a < h < b$.

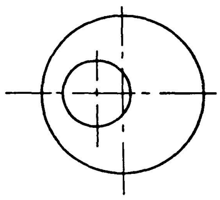
Fig. 3a

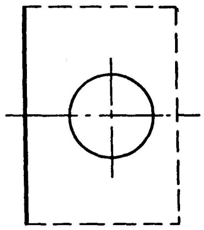
Fig. 3b

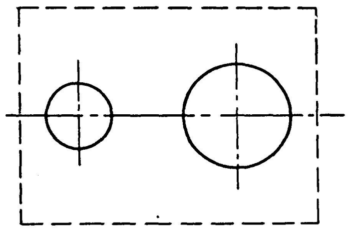
Fig. 3c

The mapping described above constitutes a generalization of the mapping discussed previously and can be represented in the complex plane by the relation:

$$Z = \frac{k^2}{\bar{z} + h} \qquad (k > 0). \tag{4.1}$$

The elements lying on the periphery of the circle of radius $r$ in the system $J$ correspond to elements lying on the periphery of the circle of radius $R$ in the system $O$. Covering the circular plate in the system $J$ with a net of concentric circles and radii passing through the origin $J_2$ we obtain, in the original system $O$, a curvilinear net composed of eccentric circles of variable radii $R$ and variable centres $M$, and a family of circles passing through the point $J_1$ and the point corresponding to $J_2$. It is natural, therefore, to associate the reference system with the variable point $M$ and to determine the state of stress in the coordinates $(R, \Phi)$ for each group of elements lying on the periphery of the circle of radius $R$. Thus, the stress field in the system $J$, described in the coordinates $(r, \varphi)$, corresponds to a stress field in the system $O$ described in the coordinates $(R, \Phi)$.

In order to find the relation between the corresponding moment fields we should proceed in a manner similar to that used previously. We express the bending moments in the system $O$:

$$M_R = -D\left[\frac{\partial^2 W}{\partial R^2} + \nu\left(\frac{1}{R^2}\frac{\partial^2 W}{\partial \Phi^2} + \frac{1}{R}\frac{\partial W}{\partial R}\right)\right],$$
$$M_\Phi = -D\left[\nu\frac{\partial^2 W}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 W}{\partial \Phi^2} + \frac{1}{R}\frac{\partial W}{\partial R}\right], \tag{4.2}$$
$$M_{R\Phi} = -D(1-\nu)\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial W}{\partial \Phi}\right),$$

where
$$W = \frac{1}{k^2}R_1^2 w = \frac{k^2}{r^2 + 2hr\cos\varphi + h^2}w.$$

The differential operations can be performed in the system $J$ writing:

$$\frac{\partial W}{\partial R} = \frac{\partial W}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial W}{\partial \varphi}\frac{\partial \varphi}{\partial R}; \qquad \frac{\partial^2 W}{\partial R^2} = \left(\frac{\partial r}{\partial R}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial R}\frac{\partial}{\partial \varphi}\right)\left(\frac{\partial r}{\partial R}\frac{\partial W}{\partial r} + \frac{\partial \varphi}{\partial r}\frac{\partial W}{\partial \varphi}\right);$$
$$\frac{\partial W}{\partial \Phi} = \frac{\partial W}{\partial r}\frac{\partial r}{\partial \Phi} + \frac{\partial W}{\partial \varphi}\frac{\partial \varphi}{\partial \Phi}; \qquad \frac{\partial^2 W}{\partial \Phi^2} = \left(\frac{\partial r}{\partial \Phi}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial}{\partial \varphi}\right)\left(\frac{\partial r}{\partial \Phi}\frac{\partial W}{\partial r} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial W}{\partial \varphi}\right).$$

The derivatives of the coordinates in the system $J$, with respect to the coordinates of the system $O$, can easily be determined by considering the geometrical relations. Introducing the relations:

$$R = \frac{k^2}{h^2 - r^2}r, \qquad H = \frac{k^2}{h^2 - r^2}h, \qquad (h > r),$$
$$R = \frac{k^2}{r^2 - h^2}r, \qquad H = \frac{k^2}{r^2 - h^2}h, \qquad (h < r),$$

and taking into account small increments $\Delta R$ and $\Delta\Phi$ of the coordinates $R$ and $\Phi$, respectively, we obtain, after determining the increments of the coordinates $r$ and $\varphi$ and passing to the limit, the relations:

$$\frac{\partial r}{\partial R} = \pm\frac{r_1^2}{k^2}, \qquad \frac{\partial r}{\partial \Phi} = 0,$$
$$\frac{\partial \varphi}{\partial R} = 0, \qquad \frac{\partial \varphi}{\partial \Phi} = -\frac{r_1^2}{h^2 - r^2}. \tag{4.3}$$

Performing the operations indicated, we have:

$$M_R = \frac{r_1^2}{k^2}M_r - \frac{2}{k^2}D(1+\nu)\cdot G,$$
$$M_\Phi = \frac{r_1^2}{k^2}M_\varphi - \frac{2}{k^2}D(1+\nu)\cdot G, \tag{4.4}$$
$$M_{R\Phi} = -\frac{r_1^2}{k^2}M_{r\varphi},$$

where
$$G = w - \frac{\partial w}{\partial r}(r + h\cos\varphi) + \frac{\partial w}{\partial \varphi}\frac{h}{r}\sin\varphi.$$

The shearing forces will be obtained, as before, from the conditions of equilibrium (when $h > r$: positive sign; $h < r$: negative sign):

$$
\begin{aligned}
Q_R &= \frac{M_R - M_\Phi}{R} + \frac{\partial M_R}{\partial R} + \frac{\partial M_{R\Phi}}{R\,\partial\Phi} = \\
&= \pm\left[\frac{h^2 - r^2}{k^2 r}(M_R - M_\Phi) + \frac{r_1^2}{k^2}\frac{\partial M_R}{\partial r} - \frac{r_1^2}{k^2 r}\frac{\partial M_{R\Phi}}{\partial\varphi}\right], \\
Q_\Phi &= \frac{2\,M_{R\Phi}}{R} + \frac{\partial M_{R\Phi}}{\partial R} + \frac{\partial M_\Phi}{R\,\partial\Phi} = \\
&= \pm\left[\frac{2\,(h^2 - r^2)}{k^2 r} M_{R\Phi} + \frac{r_1^2}{k^2}\frac{\partial M_{R\Phi}}{\partial r} - \frac{r_1^2}{k^2 r}\frac{\partial M_\Phi}{\partial\varphi}\right].
\end{aligned}
\tag{4.5}
$$

The equations obtained are similar to those in (3.3). Substituting $h=0$ we obtain the case considered before[1]). All the properties of the transformation described in the case of concentric inversion retain their validity and will not be discussed again.

In particular, the general equations (3.7) retain their validity independently of the orientation of the system of coordinates, provided that the function $E(r,\varphi)$ is replaced by the more general function $G(r,\varphi)$ which, for the substitution $h=0$, becomes $E(r,\varphi)$.

The advantage of the generalized inversion of essential meaning, for the cases considered here, consists in the possibility of direct establishment of actual boundary conditions. These conditions can be expressed for the system $O$ in a simple manner. The circles mapping in the system $J$ into concentric circles constitute, in the system $O$, a family of eccentric circles. This family includes the contours of the plate considered (e.g. in the form of an eccentric ring). It is sufficient, therefore, to substitute $r=$ const. to determine the required quantities on the edges of the plate in the system $O$.

Thus, the inverted system $J$, operating as an auxiliary figure, makes it possible to analyse the states of stress and strain in the original system $O$. If this system consists, for instance, of a circular plate with an eccentric hole, it can be reduced by the transformation of inversion to a simple system in the form of a plate bounded by concentric circles; all the required quantities concerning the states of stress and strain in the system $O$ are then related (by means of the quoted relations) to the coordinates $(r,\varphi)$ of the system $J$; thus, the boundary conditions are expressed in a simple manner.

Let us now examine the transformation of the external loads. Consider a transversal load $q_0(R,\Phi)$ in the system $O$; the basic differential equation of the plate problem has the form:

$$
\nabla^4 W = \frac{q_0}{D}. \tag{4.6}
$$

Applying the transformation (2.4) to the function $W$ we find at the same time $q_i(r,\varphi)$ for the auxiliary system $J$, in which the basic equation can be written as:

[1]) The discontinuity of $Q_R$ and $Q_\Phi$, when passing from values $h>r$ to values $h<r$, is caused by the discontinuous change of orientation of the coordinate system $(R, \Phi)$.

$$\nabla^4 w = \frac{q_i}{D}. \tag{4.6a}$$

The deflections $W$ and $w$ can be represented as a sum of integrals of the biharmonic equation and of particular integrals, that is:

$$w \doteq w_I + w_{II} \tag{4.7}$$

where

$$\nabla^4 w_I = 0, \qquad \nabla^4 w_{II} = \frac{q_i}{D}, \tag{4.7a}$$

and

$$W = \frac{R_1^2}{k^2} w_I + \frac{R_1^2}{k^2} w_{II} = W_I + W_{II} \tag{4.8}$$

where

$$\nabla^4 W_I = 0, \qquad \nabla^4 W_{II} = \frac{q_0}{D}. \tag{4.8a}$$

Since

$$\nabla^4 W_{II} = \left[\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \Phi^2}\right]\left[\frac{\partial^2 W_{II}}{\partial R^2} + \frac{1}{R}\frac{\partial W_{II}}{\partial R} + \frac{1}{R^2}\frac{\partial^2 W_{II}}{\partial \Phi^2}\right] =$$

$$= \frac{k^8}{R_1^6}\nabla^4 w_{II} = \frac{k^8}{R_1^6}\frac{q_i}{D},$$

we, therefore, obtain

$$q_i = \frac{R_1^6}{k^8} q_0 = \frac{1}{k^8}\frac{q_0}{(r^2 + 2hr\cos\varphi + h^2)^3}. \tag{4.9}$$

## 5. The Case $\nabla^4 W = 0$

The deflection of the plate is expressed by a biharmonic function in the system $O$ and in the system $J$ as well.

The general integral of the biharmonic equation can be expressed in the form:

$$w = f_1(z) + f_2(\bar{z}) + (x^2 + y^2)[f_3(z) + f_4(\bar{z})],$$

where $f_k\,(k = 1, 2, 3, 4)$ are symbols of harmonic functions, $\left.\begin{matrix}\bar{z} = x + iy \\ z = x - iy\end{matrix}\right\}$ being conjugate complex variables.

The character of the geometrical form of the systems considered implies the necessity of using polar coordinates $(r, \varphi)$. Any function satisfying the biharmonic eq. (4.7a) can be represented in the form

$$w = f_5 + r^2 f_6,$$

where $f_5$ and $f_6$ are (plane) harmonic functions. Thus, $w$ will be obtained as the sum of particular integrals:

$$w = w_s + w_{as}; \qquad \begin{aligned} w_s = {} & a_0 + b_0 \ln r + c_0 r^2 + d_0 r^2 \ln r + \\ & + (a_1 r + b_1 r^3 + c_1 r^{-1} + d_1 r \ln r) \cos \varphi + \\ & + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + c_n r^{-n} + d_n r^{-n+2}) \cos n\varphi, \\ w_{as} = {} & (\bar{a}_1 r + \bar{b}_1 r^3 + \bar{c}_1 r^{-1} + \bar{d}_1 r \ln r) \sin \varphi + \\ & + \sum_{n=2}^{\infty} (\bar{a}_n r^n + \bar{b}_n r^{n+2} + \bar{c}_n r^{-n} + \bar{d}_n r^{-n+2}) \sin n\varphi. \end{aligned} \tag{5.1}$$

The asymmetric function, $w_{as}$, will appear in that case only when the external load is not symmetric with respect to the axis $\Phi = 0$ (or $\varphi = 0$).

The moments $M_r$, $M_\varphi$, and $M_{r\varphi}$ in the system $J$ will be determined by using the eqs. (3.2) and the moments $M_R$, $M_\Phi$, and $M_{R\Phi}$ in the system $O$ by using the eqs. (4.4). We assume that $h = k = 1$, which by no means restricts the generality of the treatment and is equivalent to an appropriate choice of the unit of length.

For the symmetric terms of the function $w$ we have:

a) in the inverted system:

$$\begin{aligned} M_r = {} & -D\Big\{[-b_0 r^{-2}(1-\nu) + 2c_0(1+\nu) + 2d_0 \ln r\,(1+\nu) + d_0(3+\nu)] + \\ & + [2b_1 r(3+\nu) + 2c_1 r^{-3}(1-\nu) + d_1 r^{-1}(1+\nu)] \cos\varphi + \\ & + \sum_{n=2}^{\infty} \{n(n-1)(1-\nu) a_n r^{n-2} + [n^2(1-\nu) + n(3+\nu) + 2(1+\nu)] b_n r^n + \\ & + n(n+1)(1-\nu) c_n r^{-n-2} + [n^2(1-\nu) - n(3+\nu) + 2(1+\nu)] d_n r^{-n}\} \cos n\varphi\Big\}; \\ M_\varphi = {} & -D\Big\{[b_0 r^{-2}(1-\nu) + 2c_0(1+\nu) + 2d_0 \ln r\,(1+\nu) + d_0(1+3\nu)] + \\ & + [2b_1 r(1+3\nu) - 2c_1 r^{-3}(1-\nu) + d_1 r^{-1}(1+\nu)] \cos\varphi + \\ & + \sum_{n=2}^{\infty} \{-n(n-1)(1-\nu) a_n r^{n-2} + [-n^2(1-\nu) + n(1+3\nu) + (2+\nu)] b_n r^n + \\ & - n(n+1)(1-\nu) c_n r^{-n-2} + [-n^2(1-\nu) - n(1+3\nu) + 2(1+\nu)] d_n r^{-n}\} \cos n\varphi\Big\}; \\ M_{r\varphi} = {} & -D(1-\nu)\{(-2b_1 r + 2c_1 r^{-3} - d_1 r^{-1}) \sin\varphi + \\ & + \sum_{n=2}^{\infty} [-n(n-1) a_n r^{n-2} - n(n+1) b_n r^n + \\ & + n(n+1) c_n r^{-n-2} + n(n-1) d_n r^{-n}] \sin n\varphi\}; \end{aligned} \tag{5.2}$$

b) in the original system:

$$\begin{aligned} M_R = {} & -D[(a_0\alpha_0 + b_0\beta_0 + c_0\gamma_0 + d_0\delta_0 + a_1\alpha_1'' + b_1\beta_1'' + c_1\gamma_1'' + d_1\delta_1'') + \\ & + \sum_{n=1}^{\infty} (a_{n-1}\alpha'_{n-1} + a_n\alpha_n + a_{n+1}\alpha''_{n+1} + \\ & + b_{n-1}\beta'_{n-1} + b_n\beta_n + b_{n+1}\beta''_{n+1} + \\ & + c_{n-1}\gamma'_{n-1} + c_n\gamma_n + c_{n+1}\gamma''_{n+1} + \\ & + d_{n-1}\delta'_{n-1} + d_n\delta_n + d_{n+1}\delta''_{n+1}) \cos n\varphi]. \end{aligned} \tag{5.3}$$

The values of the coefficients appearing in the series (5.3) are:
for $n=0$:

$$\begin{aligned} &\alpha_0 = 2(1+\nu), \quad \beta_0 = 2\ln r(1+\nu)-(3+\nu)-r^{-2}(1-\nu),\\ &\gamma_0 = 2(1+\nu), \quad \delta_0 = 2\ln r(1+\nu)+r^2(1-\nu)+3+\nu \\ &\alpha_1'' = -2(1+\nu), \quad \beta_1'' = 2r^2(1-\nu), \quad \gamma_1'' = 2r^{-2}(1-\nu), \quad \delta_1'' = -2\ln r(1+\nu); \end{aligned} \tag{5.3a}$$

for $n=1$:

$$\begin{aligned} &\alpha_0' = 0, \quad \beta_0' = -4r^{-1}, \quad \gamma_0' = 0,\\ &\alpha_1 = 0, \quad \beta_1 = 2r^3(1-\nu)+2r(3+\nu), \quad \gamma_1 = 2r^{-1}(3+\nu)+2r^{-3}(1-\nu),\\ &\alpha_2'' = -2(1+3\nu)r, \quad \beta_2'' = 6(1-\nu)r^3, \quad \gamma_2'' = 6(1-\nu)r^{-1},\\ &\delta_0' = 4r,\\ &\delta_1 = -r(1+\nu)+r^{-1}(1+\nu),\\ &\delta_2'' = -2(1+3\nu)r; \end{aligned} \tag{5.3b}$$

for $n \geqq 2$:

$$\begin{aligned} &\alpha_{n-1}' = (n-1)(n-2)(1-\nu)r^{n-2}, \quad \beta_{n-1}' = (n-1)[n(1-\nu)+2(1+\nu)]r^n,\\ &\gamma_{n-1}' = (n-1)[n(1-\nu)+2(1+\nu)]r^{-n-2}, \quad \delta_{n-1}' = (n-1)(n-2)(1-\nu)r^{-n},\\ &\alpha_n = (n-1)[n(1-\nu)-2(1+\nu)]r^n+n(n-1)(1-\nu)r^{n-2},\\ &\beta_n = n(n+1)(1-\nu)r^{n+2}+(n+1)[n(1-\nu)+2(1+\nu)]r^n,\\ &\gamma_n = (n+1)[n(1-\nu)+2(1+\nu)]r^{-n}+n(n+1)(1-\nu)r^{-n-2},\\ &\delta_n = n(n-1)(1-\nu)r^{-n+2}+(n-1)[n(1-\nu)-2(1+\nu)]r^{-n},\\ &\alpha_{n+1}'' = (n+1)[n(1-\nu)-2(1+\nu)]r^n,\\ &\beta_{n+1}'' = (n+1)(n+2)(1-\nu)r^{n+2},\\ &\gamma_{n+1}'' = (n+1)(n+2)(1-\nu)r^{-n},\\ &\delta_{n+1}'' = (n+1)[n(1-\nu)-2(1+\nu)]r^{-n+2}. \end{aligned} \tag{5.3c}$$

$$\begin{aligned} M_\Phi = &-D[(a_0\zeta_0+b_0\eta_0+c_0\xi_0+d_0\kappa_0+a_1\zeta_1''+b_1\eta_1''+c_1\xi_1''+d_1\kappa_1'')+\\ &+\sum_{n=1}^{\infty}(a_{n-1}\zeta_{n-1}'+a_n\zeta_n+a_{n+1}\zeta_{n+1}''+b_{n-1}\eta_{n-1}'+b_n\eta_n+b_{n+1}\eta_{n+1}''+\\ &+c_{n-1}\xi_{n-1}'+c_n\xi_n+c_{n+1}\xi_{n+1}''+d_{n-1}\kappa_{n-1}'+d_n\kappa_n+d_{n+1}\kappa_{n+1}'')\cos n\varphi]. \end{aligned} \tag{5.4}$$

The values of the coefficients appearing in the series (5.4) are now equal to:
for $n=0$:

$$\begin{aligned} &\zeta_0 = 2(1+\nu), \quad \eta_0 = 2\ln r(1+\nu)-(1+3\nu)+r^{-2}(1-\nu),\\ &\xi_0 = 2(1+\nu), \quad \kappa_0 = 2\ln r(1+\nu)+(1+3\nu)-r^{-2}(1-\nu),\\ &\zeta_1'' = -2(1+\nu), \quad \eta_1'' = -2r^2(1-\nu), \quad \xi_1'' = -2r^{-2}(1-\nu), \quad \kappa_1'' = -2\ln r(1+\nu) \end{aligned} \tag{5.4a}$$

for $n=1$:

$$
\begin{aligned}
&\zeta_0' = 0, \quad \eta_0' = -4\nu r^{-1}, \quad \xi_0' = 0, \quad \kappa_0' = 4\nu r,\\
&\zeta_1 = 0, \quad \eta_1 = -2r^3(1-\nu) + 2r(1+3\nu), \quad \xi_1 = 2r^{-1}(1+3\nu) - 2r^{-3}(1-\nu),\\
&\qquad \kappa_1 = -r(1+\nu) + r^{-1}(1+\nu),\\
&\zeta_2'' = -2(3+\nu)r, \quad \eta_2'' = -6(1-\nu)r^3, \quad \xi_2'' = -6(1-\nu)r^{-1}, \quad \kappa_2'' = -2(3+\nu)r,
\end{aligned}
\tag{5.4b}
$$

for $n \geqq 2$:

$$
\begin{aligned}
&\zeta'_{n-1} = -(n-1)(n-2)(1-\nu)r^{n-2}, \quad \eta'_{n-1} = -(n-1)[n(1-\nu) - 2(1+\nu)]r^n,\\
&\xi'_{n-1} = -(n-1)[n(1-\nu) - 2(1+\nu)]r^{-n-2}, \quad \kappa'_{n-1} = -(n-1)(n-2)(1-\nu)r^{-n},\\
&\zeta_n = -(n-1)[n(1-\nu) - 2(1+\nu)]r^{-n} - n(n+1)(1-\nu)r^{-n-2},\\
&\kappa_n = -n(n-1)(1-\nu)r^{-n+2} - (n-1)[n(1-\nu) + 2(1+\nu)]r^{-n},\\
&\eta_n = -n(n+1)(1-\nu)r^{n+2} - (n+1)[n(1-\nu) - 2(1+\nu)]r^n,\\
&\xi_n = -(n+1)[n(1-\nu) - 2(1+\nu)]r^{-n} - n(n+1)(1-\nu)r^{-n-2},\\
&\zeta''_{n+1} = -(n+1)[n(1-\nu) + 2(1+\nu)]r^n, \quad \eta''_{n+1} = -(n+1)(n+2)(1-\nu)r^{n+2},\\
&\xi''_{n+1} = -(n+1)(n+2)(1-\nu)r^{-n}, \quad \kappa''_{n+1} = -(n+1)(n(1-\nu) + 2(1+\nu)]r^{-n+2}.
\end{aligned}
\tag{5.4c}
$$

The moment $M_{R\Phi}$ can be represented in a simpler form without having recourse to Fourier series:

$$
\begin{aligned}
M_{R\Phi} = -D(1-\nu)\{&-2b_1 \sin\varphi\,(r^3 + 2r^2\cos\varphi + r) - 2c_1 \sin\varphi\,(-r^{-1} - 2r^{-2}\cos\varphi - r^{-3}) -\\
&- d_1 \sin\varphi\,(r + 2\cos\varphi + r^{-1}) +\\
+ \sum_{n=2}^{\infty} \sin n\varphi\,[&-a_n n(n-1)(r^n + 2r^{n-1}\cos\varphi + r^{n-2}) -\\
&- b_n n(n+1)(r^{n+2} + 2r^{n+1}\cos\varphi + r^n) +\\
&+ c_n n(n+1)(r^{-n} + 2r^{-n-1}\cos\varphi + r^{-n-2}) +\\
&+ d_n n(n-1)(r^{-n+2} + 2r^{-n+1}\cos\varphi + r^{-n})]\}.
\end{aligned}
\tag{5.5}
$$

For the asymmetric function $w_{as}$ the moments $\overline{M}_r$, $\overline{M}_\varphi$, $\overline{M}_{r\varphi}$ can be obtained by replacing $\cos\varphi$ and $\cos n\varphi$ in eq. (5.2) by $\sin\varphi$ and $\sin n\varphi$, respectively, and vice versa, taking $M_{r\varphi}$ with the opposite sign.

On the other hand, we have for the asymmetric statical quantities $\overline{M}_R$, $\overline{M}_\Phi$, $\overline{M}_{R\Phi}$:

$$
\begin{aligned}
\overline{M}_R = -D \sum_{n=1}^{\infty} [(&\bar{a}_{n-1}\bar{\alpha}'_{n-1} + \bar{a}_n\bar{\alpha}_n + \bar{a}_{n+1}\bar{\alpha}''_{n+1} +\\
&+ \bar{b}_{n-1}\bar{\beta}'_{n-1} + \bar{b}_n\bar{\beta}_n + \bar{b}_{n+1}\bar{\beta}''_{n+1} +\\
&+ \bar{c}_{n-1}\bar{\gamma}'_{n-1} + \bar{c}_n\bar{\gamma}_n + \bar{c}_{n+1}\bar{\gamma}''_{n+1} +\\
&+ \bar{d}_{n-1}\bar{\delta}'_{n-1} + \bar{d}_n\bar{\delta}_n + \bar{d}_{n+1}\bar{\delta}''_{n+1})\sin n\varphi].
\end{aligned}
\tag{5.6}
$$

For this case the values of the coefficients are:

for $n=1$:

$$
\begin{aligned}
&\bar{\alpha}_0' = 0, && \bar{\beta}_0' = 0, && \bar{\gamma}_0' = 0,\\
&\bar{\alpha}_1 = 0, && \bar{\beta}_1 = 2r^3(1-\nu)+2r(3+\nu), && \bar{\gamma}_1 = 2r^{-1}(3+\nu)+2r^{-3}(1-\nu),\\
&\bar{\alpha}_2'' = -2(1+3\nu)r, && \bar{\beta}_2'' = 6(1-\nu)r^3, && \bar{\gamma}_2'' = 6(1-\nu)r^{-1},\\
&&& \bar{\delta}_0' = 0,\\
&&& \bar{\delta}_1 = -r(1+\nu)+r^{-1}(1+\nu),\\
&&& \bar{\delta}_2'' = -2(1+3\nu)r;
\end{aligned}
\tag{5.6a}
$$

for $n \geqq 2$ all coefficients are expressed by the relations already derived in (5.6).

$$
\begin{aligned}
\bar{M}_\Phi = -D \sum_{n=1}^{\infty} [(&\bar{a}_{n-1}\bar{\zeta}'_{n-1}+\bar{a}_n\bar{\zeta}_n+\bar{a}_{n+1}\bar{\zeta}''_{n+1}+\\
&+\bar{b}_{n-1}\bar{\eta}'_{n-1}+\bar{b}_n\bar{\eta}_n+\bar{b}_{n+1}\bar{\eta}''_{n+1}+\\
&+\bar{c}_{n-1}\bar{\xi}'_{n-1}+\bar{c}_n\bar{\xi}_n+\bar{c}_{n+1}\bar{\xi}''_{n+1}+\\
&+\bar{d}_{n-1}\bar{\kappa}'_{n-1}+\bar{d}_n\bar{\kappa}_n+\bar{d}_{n+1}\bar{\kappa}''_{n+1})\sin n\varphi].
\end{aligned}
\tag{5.7}
$$

For $n=1$:

$$
\begin{aligned}
&\bar{\zeta}_0' = 0, && \bar{\eta}_0' = 0,\\
&\bar{\zeta}_1 = 0, && \bar{\eta}_1 = -2r^3(1-\nu)+2r(1+3\nu),\\
&\bar{\zeta}_2'' = -2(3+\nu)r, && \bar{\eta}_2'' = -6(1-\nu)r^3,\\
&\bar{\xi}_0' = 0, && \bar{\kappa}_0' = 0,\\
&\bar{\xi}_1 = 2r^{-1}(1+3\nu)-2r^{-3}(1-\nu), && \bar{\kappa}_1 = -r(1+\nu)+r^{-1}(1+\nu),\\
&\bar{\xi}_2'' = -6(1-\nu)r^{-1}, && \bar{\kappa}_2'' = -2(3+\nu)r.
\end{aligned}
\tag{5.7a}
$$

Whereas for $n \geqq 2$ all coefficients are given by relations as represented in (5.4c).

The moment $\bar{M}_{R\Phi}$ is given by (5.5) when $\sin\varphi$ and $\sin n\varphi$ are replaced by $\cos\varphi$ and $\cos n\varphi$, respectively, and changing the sign.

The expressions for the shearing forces in the system $O$ are not derived here, their form being somewhat cumbersome.

## 6. Circular Plate with an Eccentric Hole

The problem of a circular plate with an eccentric hole has, so far, only been solved for the case where both edges are clamped. The solution of this case, using a curvilinear system of coordinates, is to be found in the book by Ya. S. Ufland [11], some particular cases being treated by N. W. Kudriavtzev [9], S. Woinowsky-Krieger [12] and Chin-Bing Ling [10]. In the present paper different cases of loads on such a plate will be considered for a wider class of problems, assuming one edge to be clamped and the other simply supported. Reference to the solutions of those cases which were treated by the above mentioned authors will allow a comparison to be made between the method of inversion and the approach to the problem by the use of curvilinear coordinates.

A plate with an eccentric hole can be characterized by two dimensionless parameters:

$$\rho = \frac{c}{d} \qquad (0 \leqq \rho \leqq 1) \qquad \text{(the ratio of radii)},$$

$$\zeta = \frac{e}{d-c} \qquad (0 \leqq \zeta \leqq 1) \qquad \text{(the eccentricity)}.$$

In addition, one of the absolute dimensions of the plate should be given. Since (according to the assumption) $h = k$, the value of $k$ can be determined, starting from the absolute dimensions $\bar{e}, \bar{c}, \bar{d}$ of the plate, provided that the eccentric plate will map, after transformation, on a concentric annulus in the system $J$:

$$\frac{k^2 \bar{j}_w}{\bar{j}_w^{\,2} - \bar{c}^2} = \frac{k^2 (\bar{j}_w + \bar{e})}{(\bar{j}_w + \bar{e})^2 - \bar{d}^2} = k.$$

Dividing the linear dimensions by $k$ we obtain the dimensionless qualities $e, c, d$ equivalent to the assumption of $k = 1$.

The concentric annulus into which the primary eccentric system is mapped, is determined by the two radii: interior $a$ and exterior $b$, which can be obtained from the data characterizing the eccentric system, using the relations:

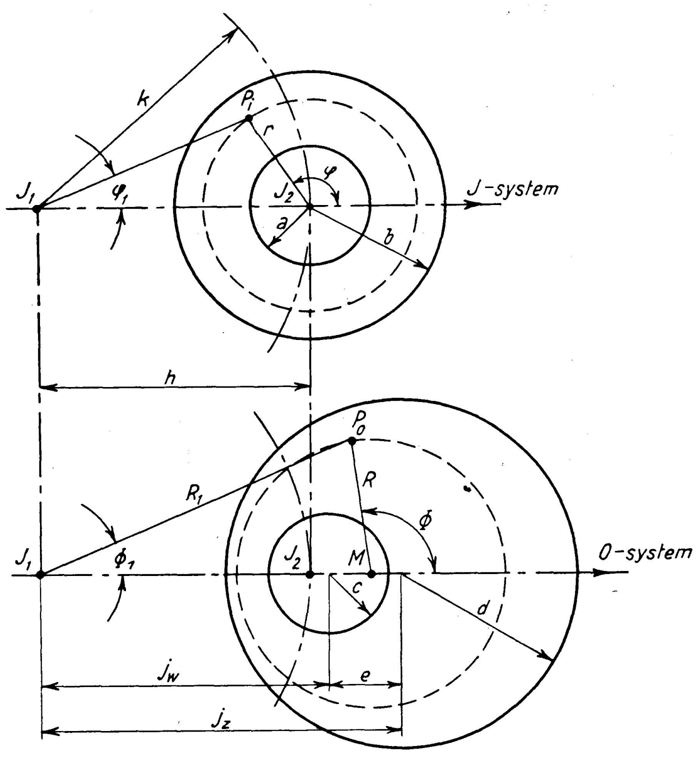


Fig. 4

$$
\begin{aligned}
a &= \frac{1}{2}\,\frac{1}{\rho\cdot\zeta}\left[(1+\rho)-\zeta^2(1-\rho)-\sqrt{S^4}\right],\\
b &= \frac{1}{2}\,\frac{1}{\zeta}\left[(1+\rho)-\zeta^2(1-\rho)-\sqrt{S^4}\right],
\end{aligned}
\tag{6.1}
$$

where
$$
S^4 = (1+\zeta+\eta)(1+\zeta-\eta)(1-\zeta+\eta)(1-\zeta-\eta),
$$
$$
\eta = \sqrt{\rho\,(\zeta^2-1)}.
$$

It can easily be seen that, after the transformation of inversion, a clamped edge in one system corresponds to a clamped edge in the other system. This follows from the following simple relations. For example, in the system $J$, let $w=0$ and $\frac{\partial w}{\partial r}=0$, for $r=b$.

In the system $O$ we have, for $r=b$

$$
W = R_1^2 w = 0,
$$

$$
\frac{\partial W}{\partial R} = \frac{\partial\left(\frac{1}{r_1^2}w\right)}{\partial r}\,r_1^2 = \frac{\partial w}{\partial r} - \frac{2\,(r+\cos\varphi)}{r_1^2}\,W = 0.
$$

In the case of a simply supported plate, however, we have, for instance, for $r=b$, when reasoning in an analogous manner,

$$
w = 0, \qquad W = 0; \qquad \frac{\partial W}{\partial R} = \frac{\partial w}{\partial r}.
$$

This means that a simply supported edge in one system transforms into a simply supported edge in the other system, the deflection angles being, at corresponding points, the same.

Let us now pass to the general solution for a given load and given boundary conditions. The particular integral $w_{II}$ (in the system $J$) of the biharmonic equation can be expressed in the coordinates $(r,\varphi)$ and represented in the form of a Fourier series:

$$
w_{II} = L_0(r) + \sum_{n=1}^{\infty} K_n(r)\sin n\varphi + \sum_{n=1}^{\infty} L_n(r)\cos n\varphi. \tag{6.2}
$$

The integral of the biharmonic equation $w_I$ can be assumed according to the expression (5.1).

Thus, the deflection of the plate will be represented in the form of a sum of two Fourier series

$$
w = w_I(r,\varphi) + w_{II}(r,\varphi). \tag{6.3}
$$

Let us consider two cases of edge support.

### *A. Both edges clamped*

In this case the boundary conditions (for $r=a$ and $r=b$) will be written:

$$w = w_I + w_{II} = 0,$$
$$\frac{\partial w}{\partial r} = \frac{\partial w_I}{\partial r} + \frac{\partial w_{II}}{\partial r} = 0. \tag{6.4}$$

Substituting here the expressions (5.1) and (6.2) we obtain, for every $n$, four equations for the coefficients of $\cos n\varphi$ and $\sin n\varphi$. The equations are:

For $n=0$:

$$\begin{aligned} a_0 + b_0 \ln b + c_0 b^2 + d_0 b^2 \ln b &= -L_0(b), \\ \frac{b_0}{b} + 2c_0 b + d_0(2\ln b + 1)b &= -L_0'(b), \\ a_0 + b_0 \ln a + c_0 a^2 + d_0 a^2 \ln a &= -L_0(a), \\ \frac{b_0}{a} + 2c_0 a + d_0(2\ln a + 1)a &= -L_0'(a). \end{aligned} \tag{6.5a}$$

For $n=1$:

$$\begin{aligned} a_1 b + b_1 b^3 + c_1 b^{-1} + d_1 b \ln b &= -L_1(b), \\ a_1 + 3b_1 b^2 + c_1 b^{-2} + d_1(\ln b + 1) &= -L_1'(b), \\ a_1 a + b_1 a^3 + c_1 a^{-1} + d_1 a \ln a &= -L_1(a), \\ a_1 + 3b_1 a^2 - c_1 a^{-2} + d_1(\ln a + 1) &= -L_1'(a). \end{aligned} \tag{6.5b}$$

For $n \geqq 2$:

$$\begin{aligned} a_n b^n + b_n b^{n+2} + c_n b^{-n} + d_n b^{-n+2} &= -L_n(b), \\ n a_n b^{n-1} + (n+2) b_n b^{n+1} - n c_n b^{-n-1} - (n-2) d_n b^{-n+1} &= -L_n'(b), \\ a_n a^n + b_n a^{n+2} + c_n a^{-n} + d_n a^{-n+2} &= -L_n(a), \\ n a_n a^{n-1} + (n+2) b_n a^{n+1} - n c_n a^{-n-1} - (n-2) d_n a^{-n+1} &= -L_n'(a). \end{aligned} \tag{6.5c}$$

The values of the coefficients $\bar{a}_n$, $\bar{b}_n$, $\bar{c}_n$, $\bar{d}_n$ for $n=1$ and $n \geqq 2$ will be obtained from the relations analogous to the above equations, replacing the coefficients $L_1$ and $L_n$ by $K_1$ and $K_n$, respectively.

If, therefore, for a given load, the deflection $w_{II}$ can be expressed in the form (6.2), the problem of the plate clamped along both edges is solved.

### *B. The case of one (e.g. exterior) edge clamped, the other (interior) being simply supported*

The boundary conditions can be written as:

$$\begin{aligned} &\text{for} \quad r = b \qquad w = 0, \qquad \frac{\partial w}{\partial r} = 0; \\ &\text{for} \quad r = a \qquad w = 0, \qquad M_R = 0. \end{aligned} \tag{6.6}$$

We see that the first three conditions are identical with those in the preceding case, the 3 constants $a_n, b_n, c_n$ can be therefore determined for each $n$ expressed as a function of the fourth $d_n$. We can write:

$$\begin{aligned} a_n &= \lambda_n' + \lambda_n'' d_n, \\ b_n &= \omega_n' + \omega_n'' d_n, \\ c_n &= \chi_n' + \chi_n'' d_n. \end{aligned} \tag{6.7}$$

We can determine the coefficients $\lambda_n$, $\omega_n$, $\chi_n$ from the three first equations of (6.5a), (6.5b), (6.5c).

The condition $M_R = 0$ can be written as follows (we confine our considerations to deflections determined by a symmetrie function $w_s$; for an asymmetric function analogous results can be obtained):

For $n=0$:

$$a_0 \alpha_0 + b_0 \beta_0 + c_0 \gamma_0 + d_0 \delta_0 + a_1 \alpha_1'' + b_1 \beta_1'' + c_1 \gamma_1'' + d_1 \delta_1'' = +M_0. \tag{6.8}$$

For $n \geqq 1$:

$$\begin{aligned} & a_{n-1} \alpha_{n-1}' + b_{n-1} \beta_{n-1}' + c_{n-1} \gamma_{n-1}' + d_{n-1} \delta_{n-1}' + \\ & + a_n \alpha_n + b_n \beta_n + c_n \gamma_n + d_n \delta_n + \\ & + a_{n+1} \alpha_{n+1}'' + b_{n+1} \beta_{n+1}'' + c_{n+1} \gamma_{n+1}'' + d_{n+1} \delta_{n+1}'' = +M_n. \end{aligned} \tag{6.9}$$

Here $M_0$, $M_n$ denote the Fourier coefficients for a moment calculated from $w_{II}$.

Substituting the relations (6.7) in (6.8) and (6.9) we obtain the following system of equations with the unknown coefficients $d_0, d_1, d_2, \ldots, d_n, \ldots$

$$\begin{aligned} & s_0{}^0 d_0 + s_0{}^1 d_1 = P_0, \\ & s_1{}^0 d_0 + s_1{}^1 d_1 + s_1{}^2 d_2 = P_1, \\ & \qquad s_2{}^1 d_1 + s_2{}^2 d_2 + s_2{}^3 d_3 = P_2, \\ & \qquad \cdots\cdots\cdots\cdots\cdots\cdots \\ & \qquad\qquad s_n^{n-1} d_{n-1} + s_n^n d_n + s_n^{n+1} d_{n+1} = P_n, \end{aligned} \tag{6.10}$$

where

$$\begin{aligned} s_n^{n-1} &= \alpha_{n-1}' \lambda_{n-1}'' + \beta_{n-1}' \omega_{n-1}'' + \gamma_{n-1}' \chi_{n-1}'' + \delta_{n-1}', \\ s_n^n &= \alpha_n \lambda_n'' + \beta_n \omega_n'' + \gamma_n \chi_n'' + \delta_n, \\ s_n^{n+1} &= \alpha_{n+1}'' \lambda_{n+1}'' + \beta_{n+1}'' \omega_{n+1}'' + \gamma_{n+1}'' \chi_{n+1}'' + \delta_{n+1}'', \\ P_n &= -(\alpha_{n-1}' \lambda_{n-1}' + \beta_{n-1}' \omega_{n-1}' + \gamma_{n-1}' \chi_{n-1}' + \alpha_n \lambda_n' + \beta_n \omega_n' \\ & \quad + \gamma_n \chi_n' + \alpha_{n+1}'' \lambda_{n+1}' + \beta_{n+1}'' \omega_{n+1}' + \gamma_{n+1}'' \chi_{n+1}' - M_n), \\ P_0 &= -(\alpha_0 \lambda_0' + \beta_0 \omega_0' + \gamma_0 \chi_0' + \alpha_1'' \lambda_1' + \beta_1'' \omega_1' + \gamma_1'' \chi_1' - M_0). \end{aligned} \tag{6.10a}$$

If the characteristic determinant of the system satisfies the Koch conditions (the necessary condition in our case being that $\sum_{i=0}^{\infty} |s_i^i - 1|$ and $\sum_{i=0}^{\infty} P_i^2$

be bounded) we can determine the values of the roots in an approximate manner by "cutting" from the infinite system of equations a determinant of the order $n$ and calculating the unknowns to the $n$-th order inclusively.

Let us now consider some different cases of loading.

1. For a circular plate, with an eccentric hole, clamped along both edges direct solutions can be found for those types of loads which, after transformation, lead, in the system $J$, to cases already known, for which the function $w$ can, therefore, be written directly.

Let us assume, as an example, the following type of loading of the plate in the system $O$

$$q_0 = \frac{q}{R_1^6} = q\,(r^2 + 2\,r\cos\varphi + 1)^3 \qquad (q = \text{const.}). \tag{6.11}$$

Then, in the system $J$ we obtain:

$$q_i = q = \text{const.}, \tag{6.12}$$

and the function $w$ takes the well-known form:

$$w = a_0 + b_0 \ln r + c_0\,r^2 + d_0\,r^2 \ln r + \frac{q\,r^4}{64\,D}. \tag{6.13}$$

The fulfilment of the boundary conditions leads to the coefficients given by the eqs. (6.5a), where

$$L_0 = \frac{q\,r^4}{64\,D}.$$

After transformation into the system $O$ we have

$$W = R_1^2\,w = \frac{1}{r^2 + 2\,r\cos\varphi + 1}\,w. \tag{6.14}$$

A. E. H. Love [6] proceeds in an analogous manner, mapping a plate loaded by a concentrated force at its centre into a plate loaded by an eccentric force.

2. As the next example let us consider a plate mapping into a concentric plate in the system $J$, clamped at the exterior edge and simply supported on the interior edge and loaded by constant moments $M_r = M$ along the latter.

Taking the function $w$ in the form

$$w = a_0 + b_0 \ln r + c_0\,r^2 + d_0\,r^2 \ln r, \tag{6.15}$$

the constants $a_0$, $b_0$, $c_0$, $d_0$ can be found from the eqs. (6.5a) and from the condition:

$$M_r = -D\left[-b_0\,a^2\,(1-\nu) + 2\,c_0\,(1+\nu) + d_0\,(2\ln a + 2\ln a\,\nu + 3 + \nu)\right] = M.$$

In the system $O$ the eccentric plate will have the edges supported in the same manner, the moments along the interior edge being:

$$M_R = -D\{2a_0(1+\nu)+b_0[2\ln r(1+\nu)-(3+\nu)-r^{-2}(1-\nu)]+ \\ +2c_0(1+\nu)+d_0[2\ln r(1+\nu)+r^2(1-\nu)+(3+\nu)]+ \\ +\left(-\tfrac{4}{r}b_0+4rd_0\right)\cos\varphi\}. \tag{6.16}$$

3. Consider a plate with an eccentric hole, subjected to a constant load $q_0$. The function $W_{II}$ in the system $O$ can be assumed in the form:

$$W_{II} = \frac{q_0 R_1^4}{64\cdot D}; \tag{6.17}$$

it obviously satisfies (in the system $(R, \Phi)$) the condition:

$$\nabla^4 W_{II} = \frac{q_0}{D}.$$

Transforming this into the system $(r, \varphi)$ we obtain:

$$w_{II} = r_1^2 W_{II} = \frac{q_0 R_1^2}{64D} = \frac{q_0}{64D}\frac{1}{r^2+2r\cos\varphi+1}. \tag{6.18}$$

The expression (6.16) can be represented in the form of a Fourier series

$$w_{II} = L_0 + \sum_{n=1}^{\infty} L_n \cos n\varphi, \tag{6.19a}$$

where

$$\left.\begin{aligned} L_0 &= k\frac{1}{1-r^2} \\ L_n &= (-1)^n k\frac{2r^n}{1-r^2} \end{aligned}\right\} \text{for } r<1 \qquad \left.\begin{aligned} L_0 &= k\frac{1}{r^2-1} \\ L_n &= (-1)^n k\frac{2}{r^2-1}\frac{1}{r^n} \end{aligned}\right\} \text{for } r>1 \tag{6.19b}$$

$$k = \frac{q_0}{64D}.$$

Substituting the coefficients $L_0$, $L_n$ obtained (for $r<1$) in the expressions (6.5a) and the following, we have definitely determined the function $w_I$. Thus, our problem is solved.

The special case of $a=0$ will furnish the solution for a plate without a hole clamped along the exterior edge and at one additional point.

4. A circular plate with an eccentric hole loaded by a concentrated force at any point.

In the case of a concentrated force the deflection surface of the plate can be represented in the form of a biharmonic function containing a singular term $W_P$:

$$W = W_I + W_P, \qquad \nabla^4 W = 0.$$

Let us assume that the plate is loaded by a concentrated force $P$ at the point $S_0$. The term $W_P$ in the system $O$ can be assumed in the form:

$$W_P = \frac{P}{8\pi D}\rho_0^2 \ln \rho_0 = \tag{6.20}$$
$$= \frac{P}{16\pi D}[R_1^2 - 2R_1 S_1 \cos(\Phi_1 - \Phi_{01}) + S_1^2] \ln [R_1^2 - 2R_1 S_1 \cos(\Phi_1 - \Phi_{01}) + S_1^2],$$

where $\rho_0$ denotes the distance of the point of application of the force from the point considered $T_0$ (fig. 5).

Transforming the function $W_P$, we obtain in the system $J$

$$w_p = \frac{P S_1^2}{16\pi D}[r^2 + s^2 - 2rs\cos(\varphi - \varphi_0)] \ln [r^2 + s^2 - 2rs\cos(\varphi - \varphi_0)] + \psi(r\varphi)$$
$$= \frac{P S_1^2}{8\pi D}\rho_i^2 \ln \rho_i + \psi(r, \varphi), \tag{6.21}$$

where $\psi(r, \varphi)$ is a biharmonic function (having no singularity) in the domain of the plate.

We see that the concentrated force $P$ at the point $S_0$ in the system $O$ corresponds to the concentrated force $P S_1^2$ at the point $S_i$ in the auxiliary system $J$.

The expression (6.21) can be represented, after rejecting the function $\psi(r, \varphi)$, in the form of a Fourier series

$$w_p(r, \varphi) = L_0 + \sum_{n=1}^{\infty} L_n \cos n\varphi + \sum_{n=1}^{\infty} K_n \sin n\varphi. \tag{6.22}$$

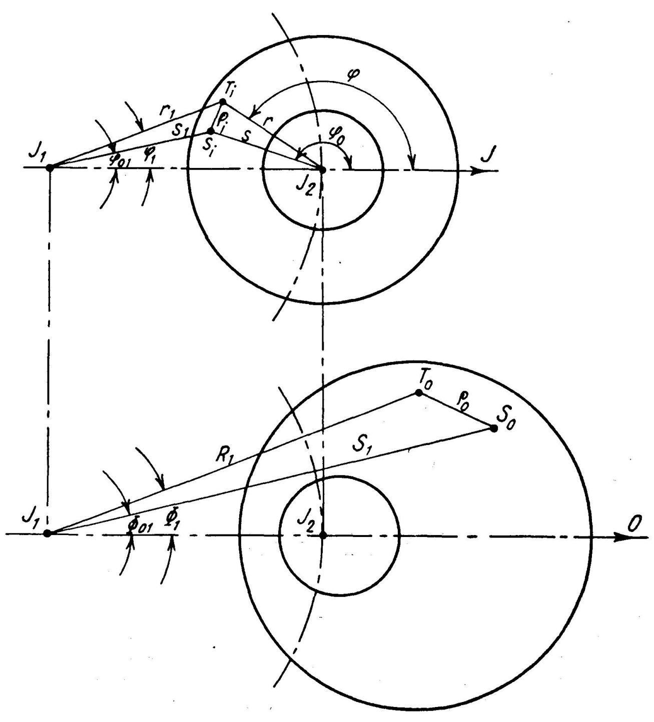


Fig. 5

The values of the coefficients are:

for
$$p = \frac{r}{s} < 1$$

$$\begin{aligned}
L_0 &= k\,[2 \ln s\,(r^2 + s^2) + 2\,r^2],\\
L_1 &= -k \left(\frac{r^3}{s} + 4\,r\,s \ln s + 2\,r\,s\right) \cos \varphi_0,\\
K_1 &= -k \left(\frac{r^3}{s} + 4\,r\,s \ln s + 2\,r\,s\right) \sin \varphi_0,\\
L_n &= k\,[r\,s\,(M_{n+1} + M_{n-1}) - (r^2 + s^2)\,M_n] \cos n\,\varphi_0,\\
K_n &= k\,[r\,s\,(M_{n+1} + M_{n-1}) - (r^2 + s^2)\,M_n] \sin n\,\varphi_0,\\
M_n &= \frac{2}{n}\left(\frac{r}{s}\right)^n, \qquad k = \frac{P\,S_1^{\,2}}{16\,\pi\,D}.
\end{aligned} \tag{6.22b}$$

For $p = \frac{r}{s} < 1$, on the other hand, we obtain $M_n = \frac{2}{n}\left(\frac{s}{r}\right)^n$. All the remaining quantities will be obtained replacing $r$ by $s$ and vice versa.

The solution for this case represents Green's function for the plate considered.

5. *A semi-infinite plate with a circular hole.* This problem is contained in the foregoing and constitutes its limit case. Indeed, if we choose the centre of inversion so that
$$h = b = 1,$$
the circle of radius $r = b$ in the system $J$ will map into a straight line in the system $O$. A semi-infinite plate with a circular hole maps, therefore, into a concentric plate in the system $J$.

The case of such a plate subjected to a concentrated force at any point can be solved by substituting $b = 1$ in the eqs. (6.5a) and the following.

Thus, Green's function is also found for this case.

However, we cannot pass to this limit in the case of a uniformly distributed load $q_0 = \text{const.}$, for the coefficients of the Fourier series given in (6.19b) tend to infinity, although the function $w_{II}$, determined by the eqs. (6.18), is bounded for every $\varphi$, except $\varphi = \pi$.

## Final Remarks

It is obvious that the above examples do not cover the whole class of problems that can be solved by using the method of inversion. As an example, plates mapping into rectangles, wedges or circular sectors in the system $J$ should be mentioned. It was found that many problems which can be described in bipolar coordinates can be generalized to new problems, not yet known, by transforming known solutions and adapting them to new systems obtained from the first by the inversion method.

The advantage of this method is the possibility of obtaining a simple physical interpretation concerning the relations between the two corresponding systems (which leads, for instance, to the possibility of direct transformation of trajectories of principal moments from one system to the other). In addition, there is a possibility of easy determination of particular integrals $W_{II}$ in problems connected with concrete systems of loads, the auxiliary system adopted of coordinates being a polar reference system. The facility of establishing the commonly encountered boundary conditions should be stressed.

Some other problems, for which solutions have been found by applying the method of inversion will be discussed separately. Such solutions refer, for instance, to the infinite plate with two circular holes and loaded by an isolated force.

Finally, using the same method, we shall investigate problems of the ultimate load-carrying capacity of plates of eccentric shapes considered in the present paper. These limit analysis problems will be treated as problems of limit equilibrium of the theory of plasticity, with the introduction of a suitably formulated yield criterion.

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**Summary**

In this paper the application of the method of inversion to the theory of plates is considered. This method is based on the following two operations:

1. The transformation of inversion of the plate; the plane of the complex variable is assumed to coincide with the middle plane of the plate; two corresponding systems, the original system $O$, and the inverted system $J$, are thus obtained.
2. The determination of correspondence between the functions $W$ and $w$ representing the deflections of the plates in the two systems mentioned above.

This correspondence between the functions $W$ and $w$ is assumed to be analogous to that between the stress functions in the inversion method as applied to plane problems of the theory of elasticity. The "generalized inversion", introduced by one of the authors in 1934 and 1935, is used. The relations between the fields of the bending and twisting moments in the two systems are derived. These exhibit close analogy with the relations of the plane problems. (This concerns, in particular, e.g., the trajectories of the principal moments etc.) This analogy ceases, however, to be valid if the boundary conditions are considered.

Next, some examples are presented. Solutions for a plate with a circular eccentric hole are derived for different loads and different boundary conditions (clamped or simply supported at the edges). The possibility of transition to limit cases is indicated (e.g. to the case of a semi-infinite plate with a circular hole, etc.).

The paper is intended to constitute a basis for further investigations concerning the ultimate load-carrying capacity of such plates examined as a problem of the theory of plasticity.

**Résumé**

Le présent mémoire est consacré à l'application de la méthode d'inversion à la théorie des plaques. Cette méthode est basée sur deux opérations fondamentales:

1. La représentation de la plaque par inversion, le plan moyen de la plaque étant choisi comme plan de la variable complexe; on obtient ainsi deux systèmes correspondants: le système original $O$ et le système inversé $J$.
2. La détermination de la correspondance entre les fonctions $W$ et $w$ exprimant la flèche de la plaque dans les deux systèmes.

La correspondance entre les fonctions $W$ et $w$ est introduite en analogie à celle entre les fonctions de tension dans la méthode d'inversion pour les problèmes plans de la théorie de l'élasticité. On part des relations de base établies, pour le cas de „l'inversion généralisée", par un des auteurs en 1934 et 1935. On deduit les relations entre les champs de moments de flexion et de torsion dans les deux systèmes en mettant en evidence une étroite analogie avec les relations dans les problèmes bidimensionnels. (Ceci concerne, en particulier, p. e. les trajectoires des moments principaux, etc.). Toutefois cette analogie n'est plus valable lorsqu'on passe aux problèmes aux limites.

On considère ensuite plusieurs exemples en présentant quelques solutions pour une plaque circulaire percée d'un trou excentré pour différentes charges et différentes conditions aux limites (plaque encastrée ou simplement appuyée sur son contour). On indique la possibilité de passage aux cas limites (p. e., pour une plaque semi-indéfinie percée d'un trou circulaire, etc).

Le présent travail constitue une base pour des recherches concernant la capacité portante (la charge limite) de telles plaques traitée comme problème de la théorie de la plasticité.

## Zusammenfassung

Die Arbeit behandelt die Anwendung der Inversionsmethode in der Plattentheorie. Diese Methode besteht aus zwei grundlegenden Operationen:

1. Einer geometrischen Inversionsabbildung der Platte, wobei die Ebene der komplexen Veränderlichen mit der Mittelfläche der Platte zusammenfällt; auf diese Art erhält man zwei einander zugeordnete Systeme: das Originalsystem $O$ und das invertierte System $J$.

2. Einer gegenseitigen Zuordnung (in den beiden Systemen) der Funktionen $W$ und $w$, die die Durchbiegung der Platte darstellen.

Diese gegenseitige Zuordnung der Funktionen $W$ und $w$ wurde in analoger Art wie die gegenseitige Zuordnung der Spannungsfunktionen bei Anwendung der Inversionsmethode in der Theorie der zweidimensionalen Probleme der Elastizitätstheorie vorgenommen.

Es wird hierbei von den Zusammenhängen der „verallgemeinerten Inversion" (angegeben von einem der Verfasser 1934 und 1935) Gebrauch gemacht.

Es werden die Zusammenhänge, welche zwischen den Feldern der Biege- und der Drillmomente in den beiden Systemen bestehen, abgeleitet. Diese sind durch eine weitgehende Analogie mit Zusammenhängen, die für zweidimensionale Probleme charakteristisch sind, gekennzeichnet. (Dies betrifft insbesondere z. B. die Trajektorien der Hauptmomente usw.) Diese Analogie bricht jedoch beim Übergang zu den Randbedingungen ab.

Weiterhin wurden einige Beispiele behandelt. Es wurden die Lösungen für eine Kreisplatte mit einer exzentrischen Öffnung, beim Auftreten von verschiedenen Typen von Belastungen, angegeben. Es werden dabei Platten mit verschiedenen Randbedingungen untersucht (Ränder eingespannt, Ränder frei gestützt). Es wurde auf Möglichkeiten betreffend den Übergang zu Grenzfällen hingewiesen (Platte in Gestalt einer Halbebene mit kreisförmigem Ausschnitt usw.).

Die Arbeit wird als Grundlage für weitere Untersuchungen über die Grenztragfähigkeit derartiger Gebilde, die als Probleme der Plastizitätstheorie behandelt werden, betrachtet.