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Autor: Eb, W.J. van der

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A New Method of Calculating Circular Cylindrical Shells

(Illustrated with one example of a calculation)

Nouvelle méthode pour le calcul des coques cylindriques

Eine neue Methode für die Berechnung zylindrischer Schalen

W. J. VAN DER EB, Civil Engineer, Rijswijk (Holland)

Introduction

It may be assumed to be a known fact that the state of membrane stresses in circular cylindrical shells, if these do *not* form tubes of completely cylindrical shape, is inadequate per se to produce an equilibrium with the external loads. For instance, in general, the membrane shearing stresses along the edge seem to be far from adequate in combination with the other edge reactions, to allow the application of an arbitrary edge member. First of all, a considerably higher edge shearing stress will always be required to make such application possible. Only in very exceptional cases such is possible with the proper membrane stresses, to build up with edge members of a very definite shape.

This additional edge shearing stress which takes the course $\tau_{xy} = \tau_r \left(\frac{1}{2} l - x\right)$ in free span shells, cannot develop per se in circular cylindrical shells with continuous curvature.

This can be proved as follows: Take a symmetrical barrel-vault shell which is preliminary approximated by a prismatic structure. The top descriptive is the symmetrical axis.

Moreover, and entirely irrespective of the shape, the following applies to prismatic structures for an nth line of intersection (the discs are all of the same height and thickness), if no exterior stresses are operative:

$$\tau_{n-1} + 4\,\tau_n + \tau_{n+1} = 0$$

or, expressed in calculus of finite differences:

$$\Delta^2 \tau_n + 6\,\tau_n = 0,$$

with the solution:

$$\tau_n = C_1(-0{,}2679)^n + C_2(-3{,}7321)^n.$$

If the counting is started from the symmetrical axis, the following conditions apply: $\tau_n = 0$, when $n = 0$, and $\tau_n = \tau_r$, when $n = p$, if a shearing stress is operative along the edge p.

We find that:

$$C_1 = -C_2 = \frac{\tau_r}{(-0{,}2679)^p - (-3{,}7321)^p}.$$

τ_n becomes:

$$\tau_n = \left[\frac{(-0{,}2679)^n}{(-0{,}2679)^p - (-3{,}7321)^p} - \frac{(-3{,}7321)^n}{(-0{,}2679)^p - (-3{,}7321)^p}\right]\tau_r.$$

If p becomes large, then the denominator in the above formulæ becomes immediately very large. The first term between the brackets very quickly approaches 0 and can be neglected in regard to the second; further, in the case of a large p-value, $(-0{,}2679)^p$ can also be neglected as far as $(-3{,}7321)^p$ is concerned.

So, finally:

$$\tau_n = -\frac{(-3{,}7321)^n}{(-3{,}7321)^p}\cdot \tau_r$$

or:

$$\tau_n = -\frac{\tau_r}{(-3{,}7321)^{p-n}}.$$

This formula indicates that τ_n calculated from the edge, approximates 0 very quickly. The first discs are affected to some extent; the central ones remain practically without load.

If $p = \sim$, i.e. when the limit is reached and the prismatic structure changes into a continuously curved shell, then it becomes clear that the shearing stress is arrested in the outermost edge fibres and at specific places causes strong folding-phenomena in the membrane.

Herewith it has been proved that an additional shearing stress cannot occur *per se* in a membrane with continuous curvature. If, however, radial loads can develop on the membrane, then this *is* possible. The edge longitudinal stresses and the transverse stresses (σ_x and σ_y) are likewise greatly influenced thereby.

In the following theory, a shell is provisionally assumed to consist of two major parts:

1. A membrane.

2. A series of closely adjacent circularly curved small rigid bars which are lying close to the membrane by means of numerous radially orientated short pendulums and are hinged to the membrane at the edge, in the simplest instance, without any edge members, as pictures below:

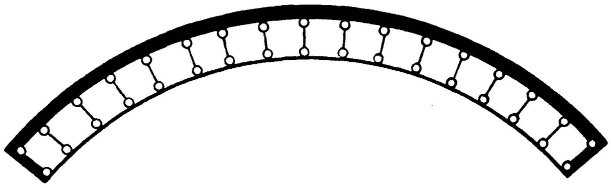

Fig. 1

What will happen now? Caused by the transverse load a state of membrane stresses is developed. Along the edges, shearing stresses and transverse loads develop i. a. These transverse load operate upon the rigid arches which are thereby pressed against the membrane. This gives rise to the additional radial load on the membrane (which accordingly acts at the same time upon the arches) and which again is necessary to bring about the corrective stresses in the membrane. This radial load is developing in such a manner that:

1. The tangential and radial reactions along the edge are 0.
2. The edge shearing stress must also be 0.
3. The deformations in membrane and arches are equal, for the membrane presses closely to the arches at all points.

Thus a system evolves which can indeed thoroughly resist the exterior loads.

Apart from the arches, we can also imagine that there exists a series of longitudinal small bars arranged closely next to one another along the shell, while these bars are supported at the two ends of the shell. In this case, a pressure also develops between the longitudinal bars and the transverse arches while the membrane is loaded by the algebraic sum of these two radial loads. This system must be taken into account in shorter shells. In long shells, this is not necessary in view of the fact that in this case, the radial load between the longitudinal bars and the transverse arches only arises to any important extent close to the supports and is furthermore so small that it can be neglected. In long shells with which we are mostly concerned, the influence of the longitudinal bars may therefore as a rule be disregarded.

Finally, one may also imagine that the longitudinal bars and the transverse arches are connected to one another in a manner resistant to torsion. This is the most accurate solution of the case and it is always possible to obtain it in its exact form whenever we are dealing with a shell which is supported on two end bases. In the case of continuous shells, a very close approximation is possible, though not theoretically exact, with the so-called "eigen" functions. This elaboration of the theory is introduced below. It seems to call for the calculation of one additional coefficient only and therefore does not involve any difficulties in itself. It is only useful, however, to apply it to shells that are definitely short, when the length of the shell is smaller than the width. *For, it appeared from many examples of calculations, the theory which calculates only with arches has a very wide field of application. In practical cases, one may almost exclusively use this theory.*

The unknown radial load $Z = f(\varphi x)$ is composed of

1. a number of symmetrical functions, and
2. a number of anti-symmetrical functions.

Both types of functions consist of a so-called linear part, and a perturbation part. The linear part does not appear to fulfil the condition that the deformations of membrane and arches are equal. For this purpose, a function is added which has been evolved in Fourier-series which we shall hereafter call the perturbation function; this latter function is determined by the condition that arches and membrane must undergo the same deformations. The linear functions are called so by the fact that they cause σ_x-stresses in the membrane which, if projected upon the plane of symmetry and perpendicularly thereto, take a linear course.

These functions are following here:

A. Symmetrical functions

1. Linear functions

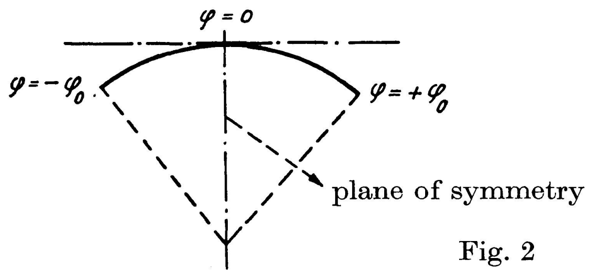

Fig. 2

$$Z = \left[\left\{\frac{\varphi_0(\cos\varphi - \cos\varphi_0) + \frac{1}{2}\sin\varphi_0(\varphi^2 - \varphi_0^2)}{\varphi_0 - \sin\varphi_0}\right\}\sigma_s - \right.$$
$$\left. - \left\{\frac{(\cos\varphi - \cos\varphi_0) + \frac{1}{2}(\varphi^2 - \varphi_0^2)}{\varphi_0 - \sin\varphi_0}\right\} t_s + p_0\right] \sin\frac{m\pi x}{l}.$$

2. Perturbation functions

$$Z = \Sigma p_{mn} \cdot \cos n\varphi \sin\frac{m\pi x}{l} \quad \text{wherein} \quad n = \left.\frac{k\pi}{2\varphi_0}\right|_{k=1\ 3,5,\ldots}$$

B. Anti-symmetrical functions

1. Linear functions

$$Z = \left[\left\{\frac{\varphi_0\sin\varphi - \varphi\sin\varphi_0}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\right\}\sigma_{ss} + \left\{\frac{\varphi^3 - \varphi_0^2\varphi}{2\varphi_0^2}\right\} t_{ss} + \frac{\varphi}{\varphi_0} q_0\right] \sin\frac{m\pi x}{l}.$$

2. Perturbation functions

$$Z = \Sigma q_{mn} \cdot \sin n \varphi \sin \frac{m \pi x}{l} \quad \text{where in} \quad n = \left.\frac{k \pi}{2 \varphi_0}\right|_{k=2,4,6,\ldots}$$

One can easily convince oneself by referring to the tables of formulæ on pages 117—122 that in the case of the symmetrical linear functions, the σ_s and t_s-functions projected upon the plane of symmetry yield σ_x-stresses that take a linear course, while of the anti-symmetrical linear functions the σ_{ss}-function does so if projected upon the plane which is perpendicular to the plane of symmetry, and the t_{ss}-function yields a σ_x which takes a linear course that is developed in the shell plane.

Furthermore, the p_0 and q_0-functions are required in order to bring about such tangential and radial reactions that a connection with arbitrary edge members is rendered possible. Of these functions, therefore, σ_s, t_s, p_0 or σ_{ss}, t_{ss} and q_0 are still unknown quantities which must be more closely determined by means of the edge conditions.

Furthermore, two other unknown quantities are added in a somewhat devious way, namely M_{0s} and M_{0ss}.

These are fixed end moments along the shell edges. Hence there is a total of eight unknown quantities which can be considered as integration constants.

We now revert to the condition by which the perturbation functions must be determined, namely that deformations of membrane and arches must be equal. It is simpler and requires less calculation to work with the moment planes. In the arches, certain moment planes occur. Let us now take a series of imaginary arches equal in number to the real arches which is forced to undergo the same deformations of the membrane; the moment planes of the real and those of the imaginary arches must of course be equal if they are to give equal deformations.

This method of calculation has the great advantage that the transverse moments can immediately be determined with certain calculated quantities which are necessary for establishing the final equations.

In the following:

$$\left.\begin{array}{l} M_u = \text{the moment plane in the real arches} \\ M_i = \text{the moment plane in the imaginary arches} \end{array}\right\} \begin{array}{l} \text{Hence } M_i \\ \text{must equal } M_u. \end{array}$$

Furthermore, the index l will refer to the linear functions, s to the symmetrical functions and ss to the anti-symmetrical functions.

Finally it is put that "eigen" functions can be applied in developing series in a longitudinal direction. These functions, however, have the great advantage that:

1. They are orthogonal.
2. They comply with the property that the fourth derivative and the function itself are of exactly the same form.

3. They themselves all fulfil their own edge conditions so that it does not make any difference whether we are dealing with shells based on two supports or on several supports, if one only calculates directly with the correct "eigen" function and "eigen" values.

In the derivation shown below, the calculation for the longitudinal direction is based for convenience upon the simplest "eigen" function, namely that for the beam upon two supports, i. e. the sine-curve. We are working here, then, with Fourier series in a longitudinal direction, which are in actual fact special "eigen" functions, namely those of the beam upon two supports. One can assume all these sine-functions to be replaced by any other arbitrary "eigen" function; this does not make any difference. Only here and there it does not work out exactly, since in the other "eigen" functions there is no longer any equality of form between integral or differential curves with relation to the original "eigen" function. A minor error is then made. Since these errors as a rule bear upon the influence of the shearing stresses and these are of considerably less influence than the longitudinal stresses, we need not be concerned too much with them. All other "eigen" functions can be applied without any difficulty.

To sum up, we may therefore say that the essence of the theory is as follows:

1. The avoidance of the usual solution of the differential equation of the 8th order by introducing linear σ_x-stresses to which are added perturbation stresses developed in Fourier series in transverse direction, which must ensure the condition is fulfilled of arches and membrane undergoing the same deformations.
2. That this latter condition is obtained by the formula:

$$M_i = M_u.$$

This latter formula is the basic condition of the theory given below.

A. The Membrane Stresses and Deformations

What will be the state of the membrane if $Z = z \cdot \sin \frac{m \pi x}{l}$, in which $z = f(\varphi)$. Hence, according to the differential equation of the state of membrane stresses, the following is valid:

$$n_\varphi = -r \cdot Z = -r \cdot z \cdot \sin \frac{m \pi x}{l}.$$

Therefore:

$$\frac{\partial n_x}{\partial x} + \frac{1}{r} \frac{\partial n_\varphi}{\partial \varphi} = 0.$$

So:

$$\frac{\partial n_{x\varphi}}{\partial x} = + \frac{\partial z}{\partial \varphi} \cdot \sin \frac{m \pi x}{l}.$$

Whereby:
$$n_{x\varphi} = -\frac{l}{m\pi}\cdot\frac{\partial z}{\partial\varphi}\cdot\cos\frac{m\pi x}{l} + C_1(\varphi).$$

For a beam with hinged edges, in the case of symmetrical loading the shearing force $=0$ when $x=\frac{1}{2}l$; accordingly, for the shell $n_{x\varphi}=0$. From this condition it follows that $C_1(\varphi)=0$.

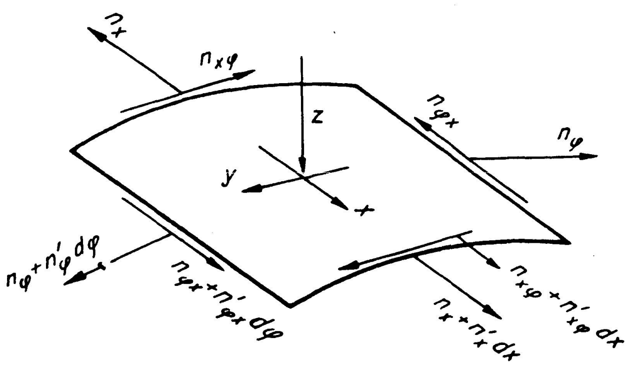

Fig. 3

Furthermore:
$$\frac{\partial n_x}{\partial x} + \frac{1}{r}\frac{\partial n_{x\varphi}}{\partial\varphi} = 0.$$

So:
$$\frac{\partial n_x}{\partial x} = +\frac{l}{m\pi r}\cdot\frac{\partial^2 z}{\partial\varphi^2}\cdot\cos\frac{m\pi x}{l}.$$

Whereby:
$$n_x = \frac{l^2}{m^2\pi^2 r}\cdot\frac{\partial^2 z}{\partial\varphi^2}\cdot\sin\frac{m\pi x}{l} + C_2(\varphi).$$

In view of the fact that in the case of a beam with hinged ends the moment $=0$ when $x=0$ and $x=l$, hence for the shell $n_x=0$, it appears therefore that $C_2(\varphi)=0$ as well.

Recapitulating, we obtain with $\lambda = \frac{m\pi r}{l}$;

$$z = f(\varphi); \qquad z' = \frac{\partial z}{\partial\varphi} = f'(\varphi); \qquad z'' = \frac{\partial^2 z}{\partial\varphi^2} = f''(\varphi);$$

$$n_\varphi = -r\cdot z\cdot\sin\frac{m\pi x}{l}, \tag{a}$$

$$n_{x\varphi} = -\frac{r}{\lambda}\cdot z'\cdot\cos\frac{m\pi x}{l}, \tag{b) (I}$$

$$n_x = +\frac{r}{\lambda^2}\cdot z''\cdot\sin\frac{m\pi x}{l}. \tag{c}$$

Regarding the deformations the following equations are valid:

$$\frac{\partial u}{\partial x} = \frac{1}{E\delta}(n_x - \nu n_\varphi)$$

or:
$$\frac{\partial u}{\partial x} = \frac{1}{E\delta}\left[\frac{l^2}{m^2\pi^2 r}\cdot z'' + \nu r\cdot z\right]\sin\frac{m\pi x}{l}$$

so:
$$u = -\frac{1}{E\delta}\left[\frac{l^3}{m^3\pi^3 r}\cdot z'' + \frac{\nu r l}{m\pi}\cdot z\right]\cos\frac{m\pi x}{l} + C_1(\varphi).$$

Considering when $x = \frac{1}{2}l$, u must equal 0 everywhere, it follows that $C_1(\varphi) = 0$, thus:
$$\frac{\partial u}{\partial \varphi} = -\frac{1}{E\delta}\left[\frac{l^3}{m^3\pi^3 r}\cdot z''' + \frac{\nu r l}{m\pi}\cdot z'\right]\cos\frac{m\pi x}{l}.$$

Further:
$$\frac{\partial u}{\partial \varphi} + r\frac{\partial v}{\partial x} = \frac{2r(1+\nu)}{E\delta}n_{x\varphi}.$$

Or:
$$\frac{\partial v}{\partial x} = -\frac{2(1+\nu)}{E\delta m\pi}\cdot z'\cdot\cos\frac{m\pi x}{l} + \frac{1}{E\delta}\left[\frac{l^3}{m^3\pi^3 r^2}\cdot z''' + \frac{\nu l}{m\pi}\cdot z'\right]\cos\frac{m\pi x}{l}$$

so:
$$v = \frac{1}{E\delta}\left[\frac{l^4}{m^4\pi^4 r^2}\cdot z''' - \frac{(2+\nu)l^2}{m^2\pi^2}\cdot z'\right]\sin\frac{m\pi x}{l} + C_2(\varphi).$$

Considering when $x = 0$ and $x = l$, v must be equal 0 everywhere; accordingly, $C_2(\varphi) = 0$ too, hence:
$$\frac{\partial v}{\partial \varphi} = \frac{1}{E\delta}\left[\frac{l^4}{m^4\pi^4 r^2}\cdot z'''' - \frac{(2+\nu)l^2}{m^2\pi^2}\cdot z''\right]\sin\frac{m\pi x}{l}.$$

Finally:
$$\frac{\partial v}{\partial \varphi} + w = \frac{r}{E\delta}(n_\varphi - \nu n_x)$$

or:
$$w = \frac{r}{E\delta}\left[-rz - \frac{\nu l^2}{m^2\pi^2 r}\cdot z''\right]\sin\frac{m\pi x}{l} - \frac{1}{E\delta}\left[\frac{l^4}{m^4\pi^4 r^2}\cdot z'''' - \frac{(2+\nu)l^2}{m^2\pi^2}\cdot z''\right]\sin\frac{m\pi x}{l}$$

so:
$$w = -\frac{1}{E\delta}\left[r^2 z - \frac{2l^2}{m^2\pi^2}\cdot z'' + \frac{l^4}{m^4\pi^4 r^2}\cdot z''''\right]\sin\frac{m\pi x}{l}.$$

Recapitulating, we obtain with
$$z''' = \frac{\partial^3 z}{\partial\varphi^3} = f'''(\varphi); \qquad z'''' = \frac{\partial^4 z}{\partial\varphi^4} = f''''(\varphi);$$

$$u = -\frac{r^2}{E\delta}\left[\frac{1}{\lambda^3}\cdot z'' + \frac{\nu}{\lambda}\cdot z\right]\cos\frac{m\pi x}{l}, \tag{a}$$

$$v = +\frac{r^2}{E\delta}\left[\frac{1}{\lambda^4}\cdot z''' - \frac{(2+\nu)}{\lambda^2}\cdot z'\right]\sin\frac{m\pi x}{l}, \tag{b) (II}$$

$$w = -\frac{r^2}{E\delta}\left[\frac{1}{\lambda^4}\cdot z'''' - \frac{2}{\lambda^2}\cdot z'' + z\right]\sin\frac{m\pi x}{l}. \tag{c}$$

In view of the fact that when equalling M_i to M_u, we are using exclusively cosine- and sine-functions, we derive the formulæ for M_i only for these two goniometrical functions.

Generally, the following is valid:
$$\frac{d^2 w}{d\varphi^2} + \left(1 - \frac{\nu m^2\pi^2 r^2}{l^2}\right)w = +\frac{M_i\cdot r^2(1-\nu^2)}{EI}$$

from which:

$$M_i = +\frac{EI}{(1-\nu^2)r^2}\left[\frac{d^2 w}{d\varphi^2} + \left(1 - \frac{\nu m^2 \pi^2 r^2}{l^2}\right) w\right],$$

which becomes with (IIc):

$$\begin{aligned} M_i = \frac{-\delta^2}{12(1-\nu^2)}\Bigg[&\frac{1}{\lambda^4}\{z''''''' + (1-\nu\lambda^2) z'''''\} - \frac{2}{\lambda^2}\{z'''' + (1-\nu\lambda^2) z''\} + \\ &+ \{z'' + (1-\nu\lambda^2) z\} \cdot \sin\frac{m\pi x}{l}. \end{aligned} \tag{III}$$

Substituting the following two z-functions in (III):

a) Symmetrical functions

$$z = p_{mn} \cos n\varphi, \quad \text{in which} \quad n = \frac{k\pi}{2\varphi_0} \quad (k = 1, 3, 5, \ldots \text{ etc.}).$$

b) Anti-symmetrical functions

$$z = q_{mn} \sin n\varphi, \quad \text{in which} \quad n = \frac{k\pi}{2\varphi_0} \quad (k = 2, 4, \ldots \text{ etc.}).$$

We find therefore that:

$$M_{ips} = \frac{\delta^2}{12(1-\nu^2)} \cdot \left[\frac{1}{\lambda^4} + \frac{2}{n^2\lambda^2} + \frac{1}{n^4}\right] n^4 (n^2 - 1 + \nu\lambda^2)\, p_{mn} \cos n\varphi \sin\frac{m\pi x}{l}$$

or:

$$M_{iqss} = \frac{\delta^2}{12(1-\nu^2)} \cdot \left[\frac{1}{\lambda^4} + \frac{2}{n^2\lambda^2} + \frac{1}{n^4}\right] n^4 (n^2 - 1 + \nu\lambda^2)\, q_{mn} \sin n\varphi \sin\frac{m\pi x}{l}.$$

For concrete shells, usually ν is taken $= 0$. The formulæ are then simplified to:

$$M_{ips} = A_m \cdot B_{mn} \cdot n^4 (n^2 - 1)\, p_{mn} \cdot \cos n\varphi \cdot \sin\frac{m\pi x}{l} \tag{a}$$

or:

$$M_{iqss} = A_m \cdot B_{mn} \cdot n^4 (n^2 - 1)\, q_{mn} \cdot \sin n\varphi \cdot \sin\frac{m\pi x}{l} \tag{b}$$

when:

$$\begin{aligned} A_m &= \frac{\delta^2}{12\lambda^4}, \\ B_{mn} &= 1 + 2\frac{\lambda^2}{n^2} + \frac{\lambda^4}{n^4}, \\ \lambda &= \frac{m\pi r}{l} \quad \text{and} \quad n = \frac{k\pi}{2\varphi_0}, \end{aligned} \tag{IV}$$

$k = 1, 3, 5 \ldots$ etc. or $2, 4, 6 \ldots$ etc.

B. The Arches

In this case the three following differential equations are valid:

$$1.\ \frac{dM_u}{d\varphi} = Q_u \cdot r; \qquad 2.\ \frac{dQ_u}{d\varphi} + n_\varphi + z \cdot r = 0; \qquad 3.\ \frac{dn_\varphi}{d\varphi} = Q_u.$$

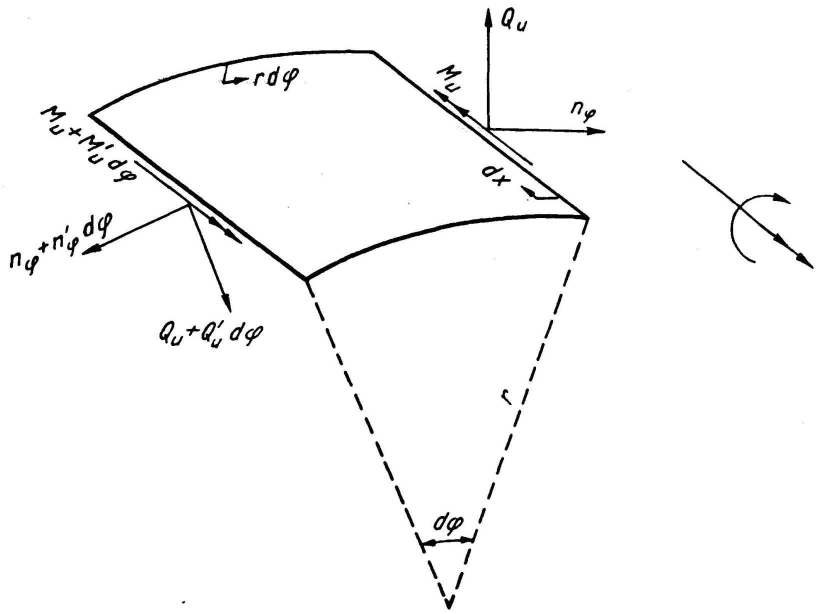

Fig. 4

Differentiating (1) twice and substituting this result in (3) as well as in (2) which has been differentiated once prior thereto, and multiplying by r, we obtain:

$$\frac{d^3 M_u}{d\varphi^3} + \frac{dM_u}{d\varphi} = -r^2 \frac{dz}{d\varphi}.$$

And after integrating once:

$$\frac{d^2 M_u}{d\varphi^2} + M_u = -z \cdot r^2. \qquad \text{(V)}$$

Regarding the arches, the following is valid: $z = -p_{mn} \cdot \cos n\varphi$, so the differential equation becomes:

$$\frac{d^2 M_u}{d\varphi^2} + M_u = r^2 \cdot p_{mn} \cdot \cos n\varphi \sin \frac{m\pi x}{l}$$

whereby:

$$M_u = \left[C_1 \cos\varphi + C_2 \sin\varphi - \frac{r^2}{n^2-1} \cdot p_{mn} \cdot \cos n\varphi \right] \sin \frac{m\pi x}{l}.$$

As $M_u = 0$ when $\varphi = +\varphi_0$ and $\varphi = -\varphi_0$, C_1 and C_2 are equal 0 when $k = 1, 3, 5 \ldots$ etc.

So:

$$M_{ups} = -\frac{r^2}{n^2-1} \cdot p_{mn} \cdot \cos n\varphi \cdot \sin \frac{m\pi x}{l} \qquad \text{(V a)}$$

and if $z = -q_{mn} \sin n\varphi$, we find analogously with $k = 2, 4, 6 \ldots$ etc.

$$M_{uqss} = -\frac{r^2}{n^2-1} \cdot q_{mn} \cdot \sin n\varphi \cdot \sin \frac{m\pi x}{l}. \tag{V b}$$

The radial reactions, considered in their relation to the edge members for which the $-$ sign is introduced, are found by employing the differential equation (1) for the shearing force, after substituting $\varphi = +\varphi_0$, and accordingly:

$$q_r = Q_u\Big|_{\varphi=\varphi_0} = -\frac{1}{r} \cdot \frac{d M_u}{d\varphi}\Big|_{\varphi=\varphi_0}. \tag{VI}$$

When $z = -p_{mn} \cdot \cos n\varphi$:

$$q_{rps} = -\frac{n r}{n^2-1} \cdot p_{mn} \cdot \sin \frac{k\pi}{2} \cdot \sin \frac{m\pi x}{l}\Big|_{k=1,3,5\ldots} \tag{VI a}$$

When $z = -q_{mn} \cdot \sin n\varphi$:

$$q_{rqss} = +\frac{n r}{n^2-1} \cdot q_{mn} \cdot \cos \frac{k\pi}{2} \cdot \sin \frac{m\pi x}{l}\Big|_{k=2,4,6\ldots} \tag{VI b}$$

This concludes the calculation for the arches. It should further be remarked that the tangential reactions are caused by the membrane. These quantities can be determined directly from n_φ when $\varphi = +\varphi_0$. When $z = p_{mn} \cdot \cos n\varphi$, we find:

$$q_{tps} = R_t = -r \cdot p_{mn} \cdot \cos \frac{k\pi}{2} \cdot \sin \frac{m\pi x}{l}\Big|_{k=1,3,5\ldots} = 0$$

and when: $z = q_{mn} \cdot \sin n\varphi$:

$$q_{tqss} = R_t = -r \cdot q_{mn} \cdot \sin \frac{k\pi}{2} \cdot \sin \frac{m\pi x}{l}\Big|_{k=2,4,6\ldots} = 0.$$

C. The Linear Functions

We now come to the linear functions. As described in the introduction, the following is valid for the symmetrical linear function:

$$Z = \left[\left\{\frac{\varphi_0 (\cos\varphi - \cos\varphi_0) + \frac{1}{2}\sin\varphi_0 (\varphi^2 - \varphi_0^2)}{\varphi_0 - \sin\varphi_0}\right\}\sigma_s - \right.$$
$$\left. - \left\{\frac{(\cos\varphi - \cos\varphi_0) + \frac{1}{2}(\varphi^2 - \varphi_0^2)}{\varphi_0 - \sin\varphi_0}\right\} t_s + p_0\right] \sin \frac{m\pi x}{l}.$$

and for the anti-symmetrical function:

$$Z = \left[\left\{\frac{\varphi_0 \sin\varphi - \varphi\sin\varphi_0}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\right\}\sigma_{ss} + \left\{\frac{\varphi^3 - \varphi_0^2\varphi}{2\varphi_0^2}\right\} t_{ss} + \frac{\varphi}{\varphi_0} q_0\right] \sin \frac{m\pi x}{l}.$$

From this it can be seen that the general form for the symmetrical function as well as for the term of σ_s and t_s is: $z = A\cos\varphi + B\varphi^2 + C$, and for the anti-symmetrical function: $z = D\sin\varphi + E\varphi^3 + F\varphi$.

We solve the differential equation for the arches with the above general forms for the perturbation functions. Thus the following is valid for the symmetrical functions:

$$\frac{d^2 M_{uls}}{d\varphi^2} + M_{uls} = [A\cos\varphi + B\varphi^2 + C]\, r^2 \cdot \sin\frac{m\pi x}{l}.$$

The complete solution of which is as follows:

$$M_{uls} = C_1\cos\varphi + C_2\sin\varphi + [\tfrac{1}{2}A\,\varphi\sin\varphi + B(\varphi^2-2) + C]\, r^2\cdot\sin\frac{m\pi x}{l}.$$

When $\varphi = \pm\varphi_0$, M_u must equal 0 from which it follows that:

$$C_1 = \left[-A\frac{\varphi_0\sin\varphi_0}{2\cos\varphi_0} - B\frac{(\varphi_0^2-2)}{\cos\varphi_0} - C\frac{1}{\cos\varphi_0}\right] r^2\cdot\sin\frac{m\pi x}{l}.$$

$C_2 = 0$, so that:

$$M_{uls} = r^2\left[A\left\{\frac{\cos\varphi_0\cdot\varphi\sin\varphi - \cos\varphi\cdot\varphi_0\sin\varphi_0}{2\cos\varphi_0}\right\} + \right.$$
$$\left. + B\left\{\frac{(\varphi^2-2)\cos\varphi_0 - (\varphi_0^2-2)\cos\varphi}{\cos\varphi_0}\right\} + C\left\{\frac{\cos\varphi_0-\cos\varphi}{\cos\varphi_0}\right\}\right]\cdot\sin\frac{m\pi x}{l} \qquad \text{(VII a)}$$

hence:

$$q_{rls} = Q_{uls} = -\frac{1}{r}\frac{d M_{uls}}{d\varphi},$$

$$q_{rls} = r\left[-A\left\{\frac{\cos\varphi_0(\sin\varphi + \varphi\cos\varphi) + \varphi_0\sin\varphi_0\cdot\sin\varphi}{2\cos\varphi_0}\right\} - \right.$$
$$\left. - B\left\{\frac{2\varphi\cos\varphi_0 + (\varphi_0^2-2)\sin\varphi}{\cos\varphi_0}\right\} - C\left\{\frac{\sin\varphi}{\cos\varphi_0}\right\}\right]\cdot\sin\frac{m\pi x}{l}. \qquad \text{(VIII a)}$$

If we proceed analogously, we obtain the following for the anti-symmetrical functions:

$$\frac{d^2 M_{ulss}}{d\varphi^2} + M_{ulss} = [D\sin\varphi + E\varphi^3 + F\varphi]\, r^2\cdot\sin\frac{m\pi x}{l},$$

with the complete solution:

$$M_{ulss} = C_3\cos\varphi + C_4\sin\varphi - [\tfrac{1}{2}D\,\varphi\cos\varphi - E(\varphi^3-6\varphi) - F\varphi]\, r^2\cdot\sin\frac{m\pi x}{l},$$

when $\varphi = \pm\varphi_0$, M_u must equal 0, or:

$$C_4 = \left[+\tfrac{1}{2}D\frac{\varphi_0\cos\varphi_0}{\sin\varphi_0} - E\frac{\varphi_0^3-6\varphi_0}{\sin\varphi_0} - F\frac{\varphi_0}{\sin\varphi_0}\right] r^2\cdot\sin\frac{m\pi x}{l}.$$

$C_3 = 0$, so that:

$$M_{ulss} = r^2\left[D\left\{\frac{\varphi_0\cos\varphi_0\cdot\sin\varphi - \varphi\cos\varphi\cdot\sin\varphi_0}{2\sin\varphi_0}\right\} + \right.$$
$$\left. + E\left\{\frac{(\varphi^3-6\varphi)\sin\varphi_0 - (\varphi_0^3-6\varphi_0)\sin\varphi}{\sin\varphi_0}\right\} + F\left\{\frac{\varphi\sin\varphi_0 - \varphi_0\sin\varphi}{\sin\varphi_0}\right\}\right]\cdot\sin\frac{m\pi x}{l} \qquad \text{(VII b)}$$

further:
$$q_{rlss} = Q_{ulss} = -\frac{1}{r}\frac{d M_{ulss}}{d\varphi},$$

$$q_{rlss} = r\left[-D\left\{\frac{\varphi_0\cos\varphi_0\cdot\cos\varphi-\sin\varphi_0(\cos\varphi-\varphi\sin\varphi)}{2\sin\varphi_0}\right\}-\right. \qquad \text{(VIII b)}$$
$$\left.-E\left\{\frac{(3\varphi^2-6)\sin\varphi_0-(\varphi_0^3-6\varphi_0)\cos\varphi}{\sin\varphi_0}\right\}-F\left\{\frac{\sin\varphi_0-\varphi_0\cos\varphi}{\sin\varphi_0}\right\}\right]\sin\frac{m\pi x}{l}.$$

Considering the fact that the condition $M_i = M_u$ will be expressed in a Fourier series with the course $\sim$, we must develop M_u in a Fourier series. We can do this direct; it is simpler, however, to develop Z in a Fourier series.

According to the theory of the eigen-functions (e.g. for symmetrical functions)
$$z = \Sigma C_{mn}\cos n\varphi|_{k=1,3,5\ldots}$$
then:
$$C_{mn} = \frac{1}{\varphi_0}\int_{-\varphi_0}^{+\varphi_0}[A\cos\varphi + B\varphi^2 + C]\cos n\varphi\, d\varphi =$$
$$= \frac{A}{\varphi_0}\left[-\frac{1}{n^2-1}\sin\varphi\cos n\varphi + \frac{n}{n^2-1}\cos\varphi\sin n\varphi\right]_{-\varphi_0}^{+\varphi_0} +$$
$$+\frac{B}{\varphi_0}\left[\frac{\varphi^2\sin n\varphi}{n} + \frac{2\varphi\cos n\varphi}{n^2} - \frac{2}{n^3}\sin n\varphi\right]_{-\varphi_0}^{+\varphi_0} + \frac{C}{\varphi_0}\left[\frac{\sin n\varphi}{n}\right]_{-\varphi_0}^{+\varphi_0} =$$
$$= \frac{2}{\varphi_0}\left[A\cdot\frac{n}{n^2-1}\cdot\cos\varphi_0 + B\left(\frac{\varphi_0^2}{n}-\frac{2}{n^3}\right) + C\cdot\frac{1}{n}\right]\sin\frac{k\pi}{2},$$
since: $\cos n\varphi_0 = \cos\frac{k\pi}{2} = 0$ when $k = 1, 3, 5\ldots$

while: $\sin n\varphi_0 = \sin\frac{k\pi}{2} = \pm 1.$

Analogously, for the anti-symmetrical functions:
$$z = \Sigma C_{mn}\sin n\varphi|_{k=2,4\ldots}$$
Hence it follows that:
$$C_{mn} = \frac{1}{\varphi_0}\int_{-\varphi_0}^{+\varphi_0}[D\sin\varphi + E\varphi^3 + F\varphi]\sin n\varphi\, d\varphi =$$
$$= \frac{D}{\varphi_0}\left[\frac{1}{n^2-1}\sin n\varphi\cos\varphi - \frac{n}{n^2-1}\cos n\varphi\sin\varphi\right]_{-\varphi_0}^{+\varphi_0} +$$
$$+\frac{E}{\varphi_0}\left[-\frac{\varphi^3\cos n\varphi}{n} + \frac{3\varphi^2\sin n\varphi}{n^2} + \frac{6\varphi\cos n\varphi}{n^3}\right]_{-\varphi_0}^{+\varphi_0} +$$
$$+\frac{F}{\varphi_0}\left[-\frac{\varphi\cos n\varphi}{n} + \frac{\sin n\varphi}{n^2}\right]_{-\varphi_0}^{+\varphi_0} =$$
$$= -\frac{2}{\varphi_0}\left[D\cdot\frac{n}{n^2-1}\cdot\sin\varphi_0 - E\cdot\left(\frac{6\varphi_0}{n^3}-\frac{\varphi_0^3}{n}\right) + F\cdot\frac{\varphi_0}{n}\right]\cos\frac{k\pi}{2}$$

in which, analogously with the above:

$$\sin n\varphi_0 = \sin\frac{k\pi}{2} = 0 \quad \text{when} \quad k = 2, 4\ldots$$

$$\cos n\varphi_0 = \cos\frac{k\pi}{2} = \pm 1.$$

With formulæ (V a) and (V b) we find for the symmetrical functions:

$$M_{uls} = \sum -\frac{2r^2\sin\frac{k\pi}{2}}{(n^2-1)\varphi_0}\left[A\cdot\frac{n\cos\varphi_0}{n^2-1} + B\cdot\left(\frac{\varphi_0^2}{n} - \frac{2}{n^3}\right) + C\cdot\frac{1}{n}\right]\cos n\varphi\sin\frac{m\pi x}{l} \quad \text{(IX a)}$$

and for the anti-symmetrical functions:

$$M_{ulss} = \sum +\frac{2r^2\cos\frac{k\pi}{2}}{(n^2-1)\varphi_0}\left[D\cdot\frac{n\sin\varphi_0}{n^2-1} - E\cdot\left(\frac{6\varphi_0}{n^3} - \frac{\varphi_0^3}{n}\right) + F\cdot\frac{\varphi_0}{n}\right]\sin n\varphi\sin\frac{m\pi x}{l}. \quad \text{(IX b)}$$

In the above, the following must be substituted:

1. For the symmetrical functions:

a) The σ_s term

$$A = \frac{\varphi_0}{\varphi_0 - \sin\varphi_0}\sigma_s; \qquad B = \frac{\frac{1}{2}\sin\varphi_0}{\varphi_0 - \sin\varphi_0}\sigma_s$$

$$\text{and} \quad C = -\frac{\varphi_0\cos\varphi_0 + \frac{1}{2}\varphi_0^2\sin\varphi_0}{\varphi_0 - \sin\varphi_0}\sigma_s.$$

b) The t_s term (X a)

$$A = -\frac{1}{\varphi_0 - \sin\varphi_0}t_s; \qquad B = -\frac{\frac{1}{2}}{\varphi_0 - \sin\varphi_0}t_s$$

$$\text{and} \quad C = +\frac{\cos\varphi_0 + \frac{1}{2}\varphi_0^2}{\varphi_0 - \sin\varphi_0}t_s.$$

c) The p_0 term

$$A = 0; \qquad B = 0; \quad \text{and} \quad C = p_0.$$

2. For the anti-symmetrical functions:

a) The σ_{ss} term

$$D = \frac{\varphi_0}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\sigma_{ss}; \qquad E = 0 \quad \text{and} \quad F = -\frac{\sin\varphi_0}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\sigma_{ss}.$$

b) The t_{ss} term (X b)

$$D = 0; \qquad E = \frac{1}{2\varphi_0^2}t_{ss} \quad \text{and} \quad F = -\tfrac{1}{2}t_{ss}.$$

c) The q_0 term

$$D = 0; \qquad E = 0 \quad \text{and} \quad F = \frac{1}{\varphi_0}q_0.$$

Before going on to the equation to determine the various perturbation functions, which is: $M_i = M_u$, as has been explained in the beginning, first the question of M_i as a result of the linear functions must be dealt with. These appear to be only very small and constitute a quantity which can be disregarded in the process of determining the perturbation functions as a function of the unknown quantities σ_s, t_s, p_0 or σ_{ss}, t_{ss} and q_0. For, we can determine M_i with the aid of formula (III).

When $\nu=0$, the following applies:

$$M_i = \frac{-\delta^2}{12}\left[\frac{1}{\lambda^4}(z''' + z'''') - \frac{2}{\lambda^2}(z'''' + z'') + (z'' + z)\right]\cdot \sin\frac{m\pi x}{l}.$$

If we carry out the calculation and divide the formula into a variable and a constant part, we find e.g. for the linear symmetrical functions:

$$M_{ils} = \frac{-\delta^2}{12}\left[\left\{\frac{\frac{1}{2}(\varphi^2-\varphi_0^2)}{\varphi_0-\sin\varphi_0}(\sin\varphi_0\cdot\sigma_s - t_s)\right\} + \right.$$
$$\left. + \left\{-\frac{\cos\varphi_0}{\varphi_0-\sin\varphi_0}(\varphi_0\sigma_s - t_s) - \frac{\left(\frac{2}{\lambda^2}-1\right)}{\varphi_0-\sin\varphi_0}(\sin\varphi_0\cdot\sigma_s - t_s) + p_0\right\}\right]\sin\frac{m\pi x}{l}.$$

The constant part simply requires the presence in the M_u-plane of 2 fixed end moments of the same magnitude which together yield the same constant moment plane, so that the condition $M_i = M_u$ is satisfied in this respect.

For determining p_{mn} therefore, only the variable part matters. This is as follows:

$$M_{ils} = -\frac{\delta^2}{12}\left[\left\{\frac{\frac{1}{2}(\varphi^2-\varphi_0^2)}{\varphi_0-\sin\varphi_0}\right\}(\sin\varphi_0\cdot\sigma_s - t_s)\right]\cdot\sin\frac{m\pi x}{l}$$

or, developed in Fourier series:

$$M_{ils} = +\frac{\delta^2}{12}\sum\frac{1}{n^3}\cdot\frac{2\sin\frac{k\pi}{2}}{\varphi_0(\varphi_0-\sin\varphi_0)}\cdot(\sin\varphi_0\cdot\sigma_s - t_s)\cdot\cos n\varphi\cdot\sin\frac{m\pi x}{l}.$$

Analogously, we find the following for the linear anti-symmetrical functions if we split these into a curvilinear variable and a linear variable part, which latter part is cancelled out in the condition $M_i = M_u$ against an identical moment plane of the M_u-plane:

$$M_{ilss} = \frac{-\delta^2}{12}\left[\left\{\frac{\varphi^3-\varphi\varphi_0^2}{2\varphi_0^2}\cdot t_{ss}\right\} - \right.$$
$$\left. -\varphi\left\{\frac{\sin\varphi_0}{\varphi_0\cos\varphi_0-\sin\varphi_0}\cdot\sigma_{ss} + \frac{3\left(\frac{2}{\lambda^2}-1\right)}{\varphi_0^2}\cdot t_{ss} - \frac{1}{\varphi_0}q_0\right\}\right]\sin\frac{m\pi x}{l}.$$

For determinig q_{mn}, only the following remains therefore:

$$M_{ilss} = \frac{-\delta^2}{12}\left[\left\{\frac{\varphi^3-\varphi\varphi_0^2}{2\varphi_0^2}\right\}t_{ss}\right]\cdot\sin\frac{m\pi x}{l}.$$

or, developed in Fourier series:

$$M_{ilss} = -\frac{\delta^2}{12}\sum\frac{1}{n^3}\cdot\frac{6\cos\frac{k\pi}{2}}{\varphi_0^2}\cdot t_{ss}\cdot\sin n\varphi\cdot\sin\frac{m\pi x}{l}.$$

Prior to establishing the general condition $M_i = M_u$, we first present a tabulated survey of the various formulæ which will appear to be necessary for the further calculation, divided according to symmetrical and anti-symmetrical functions. Starting from $z = f(\varphi)$, all these formulæ can be easily determined with the aid of the basic formulæ established for this purpose, namely (I), (II), (III), (VII a and b), (VIII a and b), (IX a and b) and (X a and b), while the perturbation functions have likewise been included here.

The indices have been changed for this purpose. Thus, e.g.:

$$M_{us} = M_{ups} + M_{uls}$$
$$M_{uss} = M_{uqss} + M_{ulss}$$

etc., which is a simplification of these indices. The index s now exclusively refers to the symmetrical functions and the index ss to the anti-symmetrical functions. Furthermore, the index 0 has been introduced. This is applied to all edge quantities, and in particular to that edge of the shell when $\varphi = +\varphi_0$. In the theory of the northlight shells are definitely asymmetrical, the index b will be used in addition; this latter refers to alle edge quantities of the shell edge when $\varphi = -\varphi_0$.

Furthermore, p_{mn} is replaced by P_{mn}, and q_{mn} by Q_{mn}. The significance of this will be made clear below.

The signs for the edge quantities have been changed in such a manner that they are valid for the edge beams. Here, a downward reaction whether vertical or oblique, is counted as positive, since it acts as a load upon the edge beam; further, deflections directed downward, whether vertical or oblique, are likewise counted as positive.

The state of membrane stresses deriving from the shell weight

This is the state of membrane stresses deriving from the dead weight, excluding the edge perturbations. These formulæ can be found in all text books on shell constructions.

$$n_{\varphi e} = -\sum \frac{2(1-\cos m\pi)}{m\pi} \cdot g r \cdot \cos\varphi \sin \frac{m\pi x}{l}.$$

$$n_{x\varphi e} = +\sum \frac{4(1-\cos m\pi)}{m\pi} \cdot \frac{g r}{\lambda} \cdot \sin\varphi \cos \frac{m\pi x}{l}.$$

$$n_{xe} = -\sum \frac{4(1-\cos m\pi)}{m\pi} \cdot \frac{g r}{\lambda^2} \cdot \cos\varphi \sin \frac{m\pi x}{l}.$$

$$w_e = \sum \frac{4(1-\cos m\pi)}{m\pi} \cdot \frac{g r^2}{E\lambda^4\delta} \cdot (1 + 2\lambda^2 + \tfrac{1}{2}\lambda^4) \cos\varphi \sin \frac{m\pi x}{l}.$$

$$v_e = \sum \frac{4(1-\cos m\pi)}{m\pi} \cdot \frac{g r^2}{E\lambda^4\delta} \cdot (1 + 2\lambda^2) \sin\varphi \sin \frac{m\pi x}{l}.$$

g = the weight of the shell per cm² of shell surface.

Applied formulas	Symmetrical functions
	$$z_s = \left[\left\{\frac{\varphi_0(\cos\varphi - \cos\varphi_0) + \frac{1}{2}\sin\varphi_0(\varphi^2 - {\varphi_0}^2)}{\varphi_0 - \sin\varphi_0}\right\}\sigma_s - \left\{\frac{(\cos\varphi - \cos\varphi_0) + \frac{1}{2}(\varphi^2 - {\varphi_0}^2)}{\varphi_0 - \sin\varphi_0}\right\}t_s + p_0 + \sum P_{mn}\cos n\varphi\right]$$ $$n = \left.\frac{k\pi}{2\varphi_0}\right\|_{k=1,3,5\ldots}$$ $$\lambda = \frac{m\pi r}{l}$$ $$P_{mn} = p_{mn} + \frac{4\sin\frac{k\pi}{2}}{A_m \cdot B_{mn} \cdot k\pi \cdot n^4(n^2-1)} \cdot M_{0s}$$
(I a)	$$n_{\varphi s} = -r\left[\left\{\frac{\varphi_0(\cos\varphi - \cos\varphi_0) + \frac{1}{2}\sin\varphi_0(\varphi^2 - {\varphi_0}^2)}{\varphi_0 - \sin\varphi_0}\right\}\sigma_s - \left\{\frac{(\cos\varphi - \cos\varphi_0) + \frac{1}{2}(\varphi^2 - {\varphi_0}^2)}{\varphi_0 - \sin\varphi_0}\right\}t_s + p_0 + \sum P_{mn}\cos n\varphi\right]\sin\frac{m\pi x}{l}$$
(I b)	$$n_{x\varphi s} = +\frac{r}{\lambda}\left[\left\{\frac{\varphi_0\sin\varphi - \varphi\sin\varphi_0}{\varphi_0 - \sin\varphi_0}\right\}\sigma_s - \left\{\frac{\sin\varphi - \varphi}{\varphi_0 - \sin\varphi_0}\right\}t_s + \sum n \cdot P_{mn}\sin n\varphi\right]\cdot\cos\frac{m\pi x}{l}$$
(I c)	$$n_{xs} = -\frac{r}{\lambda^2}\left[\left\{\frac{\varphi_0\cos\varphi - \sin\varphi_0}{\varphi_0 - \sin\varphi_0}\right\}\sigma_s - \left\{\frac{\cos\varphi - 1}{\varphi_0 - \sin\varphi_0}\right\}t_s + \sum n^2 \cdot P_{mn}\cos n\varphi\right]\cdot\sin\frac{m\pi x}{l}$$
(II b)	$$v_s = +\frac{r^2}{E\lambda^4\delta}\left[\left\{\frac{\varphi_0\sin\varphi}{\varphi_0 - \sin\varphi_0}\right\}\sigma_s - \left\{\frac{\sin\varphi}{\varphi_0 - \sin\varphi_0}\right\}t_s + 2\lambda^2\left[\left\{\frac{\varphi_0\sin\varphi - \varphi\sin\varphi_0}{\varphi_0 - \sin\varphi_0}\right\}\sigma_s - \left\{\frac{\sin\varphi - \varphi}{\varphi_0 - \sin\varphi_0}\right\}t_s\right] + \sum\left(1 + 2\frac{\lambda^2}{n^2}\right)n^3 \cdot P_{mn}\sin n\varphi\right]\cdot\sin\frac{m\pi x}{l}$$

(II c)

$$w_s = -\frac{r^2}{E\lambda^4\delta}\Bigg[\Bigg[\left\{\frac{\varphi_0\cos\varphi}{\varphi_0-\sin\varphi_0}\right\}\sigma_s - \left\{\frac{\cos\varphi}{\varphi_0-\sin\varphi_0}\right\}t_s + 2\lambda^2\left[\left\{\frac{\varphi_0\cos\varphi-\sin\varphi_0}{\varphi_0-\sin\varphi_0}\right\}\sigma_s - \left\{\frac{\cos\varphi-1}{\varphi_0-\sin\varphi_0}\right\}t_s\right]+$$

$$+\lambda^4\left[\left\{\frac{\varphi_0(\cos\varphi-\cos\varphi_0)+\frac{1}{2}\sin\varphi_0(\varphi^2-\varphi_0{}^2)}{\varphi_0-\sin\varphi_0}\right\}\sigma_s - \left\{\frac{(\cos\varphi-\cos\varphi_0)+\frac{1}{2}(\varphi^2-\varphi_0{}^2)}{\varphi_0-\sin\varphi_0}\right\}t_s + p_0\right]+$$

$$+\sum\left(1+2\frac{\lambda^2}{n^2}+\frac{\lambda^4}{n^4}\right)n^4\cdot P_{mn}\cos n\varphi\Bigg]\cdot\sin\frac{m\pi x}{l}$$

(III)

$$M_{is} = \left[-\frac{\delta^2}{12}\left\{\frac{\frac{1}{2}(\varphi^2-\varphi_0{}^2)}{\varphi_0-\sin\varphi_0}\cdot(\sin\varphi_0\cdot\sigma_s-t_s)\right\}+\sum A_m\cdot B_{mn}\cdot n^4(n^2-1)\cdot p_{mn}\cos n\varphi+\overline{M}_{0s}\right]\cdot\sin\frac{m\pi x}{l}$$

$$\text{waarin: } \overline{M}_{0s} = -\frac{\delta^2}{12}\left[-\frac{\cos\varphi_0}{\varphi_0-\sin\varphi_0}(\varphi_0\sigma_s-t_s)+\frac{1-\frac{2}{\lambda^2}}{\varphi_0-\sin\varphi_0}(\sin\varphi_0\cdot\sigma_s-t_s)+p_0\right]\cdot\sin\frac{m\pi x}{l}$$

(VII a) (X a) (V a)

$$M_{us} = r^2\Bigg[\Bigg\{\frac{\varphi_0(\cos\varphi_0\cdot\varphi\sin\varphi-\varphi_0\sin\varphi_0\cdot\cos\varphi)+\sin\varphi_0(\cos\varphi_0\cdot\varphi^2-\varphi_0{}^2\cdot\cos\varphi)-}{2\cos\varphi_0(\varphi_0-\sin\varphi_0)}$$

$$\frac{-(\varphi_0{}^2\sin\varphi_0+2\sin\varphi_0+2\varphi_0\cos\varphi_0)(\cos\varphi_0-\cos\varphi)}{2\cos\varphi_0(\varphi_0-\sin\varphi_0)}\Bigg\}\sigma_s - \Bigg\{\frac{(\cos\varphi_0\cdot\varphi\sin\varphi-\varphi_0\sin\varphi_0\cdot\cos\varphi)+}{2\cos\varphi_0(\varphi_0-\sin\varphi_0)}$$

$$\frac{+(\cos\varphi_0\cdot\varphi^2-\varphi_0{}^2\cdot\cos\varphi)-(\varphi_0{}^2+2+2\cos\varphi_0)(\cos\varphi_0-\cos\varphi)}{2\cos\varphi_0(\varphi_0-\sin\varphi_0)}\Bigg\}t_s+\left\{\frac{\cos\varphi_0-\cos\varphi}{\cos\varphi_0}\right\}p_0+$$

$$+\frac{M_{0s}+\overline{M}_{0s}}{r^2}-\sum\frac{1}{n^2-1}\cdot P_{mn}\cos n\varphi\Bigg]\cdot\sin\frac{m\pi x}{l}$$

Developed in Fourier-series

(III)

$$M_{is} = \Bigg[\frac{\delta^2}{12}\sum\sin\frac{k\pi}{2}\left[+\left\{\frac{\frac{2}{n^3}\sin\varphi_0}{\varphi_0(\varphi_0-\sin\varphi_0)}\right\}\sigma_s-\left\{\frac{\frac{2}{n^3}}{\varphi_0(\varphi_0-\sin\varphi_0)}\right\}t_s\right]+$$

$$+\sum A_m\cdot B_{mn}\cdot n^4(n^2-1)\cdot p_{mn}\Bigg]\cdot\cos n\varphi\sin\frac{m\pi x}{l}$$

(IXa) (Xa) (Va)	$M_{us} = r^2 \sum \left[\sin\frac{k\pi}{2} \left[-\left\{ \frac{\left(\frac{2\cos\varphi_0}{n(n^2-1)^2}\right)\varphi_0 - \left(\frac{2}{n^3(n^2-1)}\right)\sin\varphi_0}{\varphi_0(\varphi_0-\sin\varphi_0)} \right\}\sigma_s + \left\{ \frac{\left(\frac{2\cos\varphi_0}{n(n^2-1)^2}\right) - \left(\frac{2}{n^3(n^2-1)}\right)}{\varphi_0(\varphi_0-\sin\varphi_0)} \right\} t_s - \left\{ \frac{2}{n(n^2-1)} \cdot \frac{1}{\varphi_0} \right\} \cdot p_0 \right] - \frac{1}{n^2-1} \cdot P_{mn} \right] \cdot \cos n\varphi \cdot \sin\frac{m\pi x}{l}$
	Edge quantities
$q_{0ts} = -n_{\varphi s}\vert_{\varphi=\varphi_0}$	$q_{0ts} = r \cdot p_0 \cdot \sin\frac{m\pi x}{l}$
$q_{0rs} = -\frac{1}{r} \cdot \frac{dM_{us}}{d\varphi}\Big\vert_{\varphi=\varphi_0}$	$q_{0rs} = r\left[-\left\{ \frac{\varphi_0 \sin\varphi_0 \cos\varphi_0 + {\varphi_0}^2 - 2\sin^2\varphi_0}{2\cos\varphi_0(\varphi_0-\sin\varphi_0)} \right\}\sigma_s + \left\{ \frac{\varphi_0 + 2\varphi_0\cos\varphi_0 - 2\sin\varphi_0 - \sin\varphi_0\cos\varphi_0}{2\cos\varphi_0(\varphi_0-\sin\varphi_0)} \right\} t_s - \mathrm{tg}\,\varphi_0 \cdot p_0 - \sum \frac{n\sin\frac{k\pi}{2}}{n^2-1} \cdot P_{mn} \right] \cdot \sin\frac{m\pi x}{l}$
$T_{0s} = n_{x\varphi s}\vert_{\varphi=\varphi_0}$	$T_{0s} = +\frac{r}{\lambda}\left[t_s + \sum n \sin\frac{k\pi}{2} \cdot P_{mn} \right] \cdot \cos\frac{m\pi x}{l}$
$\sigma_{0s} = \frac{n_{xs}}{\delta}\Big\vert_{\varphi=\varphi_0}$	$\sigma_{0s} = -\frac{r}{\lambda^2\delta}\left[\left\{ \frac{\varphi_0\cos\varphi_0 - \sin\varphi_0}{\varphi_0 - \sin\varphi_0} \right\}\sigma_s - \left\{ \frac{\cos\varphi_0 - 1}{\varphi_0 - \sin\varphi_0} \right\} t_s \right] \cdot \sin\frac{m\pi x}{l}$
$v_{0s} = v_s\vert_{\varphi=\varphi_0}$	$v_{0s} = \frac{r^2}{E\lambda^4\delta}\left[\left\{ \frac{\varphi_0\sin\varphi_0}{\varphi_0 - \sin\varphi_0} \right\}\sigma_s - \left\{ \frac{\sin\varphi_0}{\varphi_0 - \sin\varphi_0} \right\} t_s + 2\lambda^2 \cdot t_s + \sum \left(1 + 2\frac{\lambda^2}{n^2}\right) n^3 \sin\frac{k\pi}{2} \cdot P_{mn} \right] \cdot \sin\frac{m\pi x}{l}$
$w_{0s} = -w_s\vert_{\varphi=\varphi_0}$	$w_{0s} = \frac{r^2}{E\lambda^4\delta}\left[\left\{ \frac{\varphi_0\cos\varphi_0}{\varphi_0 - \sin\varphi_0} \right\}\sigma_s - \left\{ \frac{\cos\varphi_0}{\varphi_0 - \sin\varphi_0} \right\} t_s + 2\lambda^2\left[\left\{ \frac{\varphi_0\cos\varphi_0 - \sin\varphi_0}{\varphi_0 - \sin\varphi_0} \right\}\sigma_s - \left\{ \frac{\cos\varphi_0 - 1}{\varphi_0 - \sin\varphi_0} \right\} t_s \right] + \lambda^4 \cdot p_0 \right] \cdot \sin\frac{m\pi x}{l}$

Applied formulas	Anti-metrical functions

$$n = \frac{k\pi}{2\varphi_0}\bigg|_{k=2,4,6\ldots}$$

$$\lambda = \frac{m\pi r}{l}$$

$$Q_{mn} = q_{mn} - \frac{4\cos\frac{k\pi}{2}}{A_m \cdot B_{mn} \cdot k\pi \cdot n^4 (n^2-1)} \cdot M_{0ss}$$

$$z_{ss} = \left[\left\{\frac{\varphi_0 \sin\varphi - \varphi\sin\varphi_0}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\right\}\sigma_{ss} + \left\{\frac{\varphi^3 - \varphi\cdot\varphi_0^2}{2\varphi_0^2}\right\}t_{ss} + \right.$$
$$\left. + \frac{\varphi}{\varphi_0}\cdot q_0 + \sum Q_{mn}\sin n\varphi\right]$$

(Ia)
$$n_{\varphi ss} = -r\left[\left\{\frac{\varphi_0\sin\varphi - \varphi\sin\varphi_0}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\right\}\sigma_{ss} + \left\{\frac{\varphi^3-\varphi\cdot\varphi_0^2}{2\varphi_0^2}\right\}t_{ss} + \frac{\varphi}{\varphi_0}\cdot q_0 + \sum Q_{mn}\sin n\varphi\right]\cdot\sin\frac{m\pi x}{l}$$

(Ib)
$$n_{x\varphi ss} = -\frac{r}{\lambda}\left[\left\{\frac{\varphi_0\cos\varphi - \sin\varphi_0}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\right\}\sigma_{ss} + \left\{\frac{3\varphi^2-\varphi_0^2}{2\varphi_0^2}\right\}t_{ss} + \frac{1}{\varphi_0}q_0 + \sum n\,Q_{mn}\cos n\varphi\right]\cdot\cos\frac{m\pi x}{l}$$

(Ic)
$$n_{xss} = -\frac{r}{\lambda^2}\left[\left\{\frac{\varphi_0\sin\varphi}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\right\}\sigma_{ss} - \left\{\frac{3\varphi}{\varphi_0^2}\right\}t_{ss} + \sum n^2 Q_{mn}\sin n\varphi\right]\cdot\sin\frac{m\pi x}{l}$$

(IIb)
$$v_{ss} = \frac{r^2}{E\lambda^4\delta}\left[-\left\{\frac{\varphi_0\cos\varphi}{\varphi_0\cos\varphi_0-\sin\varphi_0}\right\}\sigma_{ss} + \left\{\frac{3}{\varphi_0^2}\right\}t_{ss} - 2\lambda^2\left[\left\{\frac{\varphi_0\cos\varphi-\sin\varphi_0}{\varphi_0\cos\varphi_0-\sin\varphi_0}\right\}\sigma_{ss} + \right.\right.$$
$$\left.\left. + \left\{\frac{3\varphi^2-\varphi_0^2}{2\varphi_0^2}\right\}t_{ss} + q_0\right] - \sum\left(1+2\frac{\lambda^2}{n^2}\right)n^3 Q_{mn}\cos n\varphi\right]\cdot\sin\frac{m\pi x}{l}$$

(IIc)
$$w_{ss} = -\frac{r^2}{E\lambda^4\delta}\left[\left[\left\{\frac{\varphi_0\sin\varphi}{\varphi_0\cos\varphi_0-\sin\varphi_0}\right\}\sigma_{ss} + 2\lambda^2\left[\left\{\frac{\varphi_0\sin\varphi}{\varphi_0\cos\varphi_0-\sin\varphi_0}\right\}\sigma_{ss} - \left\{\frac{3\varphi}{\varphi_0^2}\right\}t_{ss}\right] + \right.\right.$$
$$+\lambda^4\left[\left\{\frac{\varphi_0\sin\varphi-\varphi\sin\varphi_0}{\varphi_0\cos\varphi_0-\sin\varphi_0}\right\}\sigma_{ss} + \left\{\frac{\varphi^3-\varphi\cdot\varphi_0^2}{2\varphi_0^2}\right\}t_{ss} + \frac{\varphi}{\varphi_0}\cdot q_0\right] +$$
$$\left. + \sum\left(1+2\frac{\lambda^2}{n^2}+\frac{\lambda^4}{n^4}\right)n^4\cdot Q_{mn}\sin n\varphi\right]\cdot\sin\frac{m\pi x}{l}$$

(III)

$$M_{iss} = \left[-\frac{\delta^2}{12} \left\{ \frac{\varphi^3 - \varphi_0{}^2 \cdot \varphi}{2\,\varphi_0{}^2} \right\} t_{ss} + \sum A_m \cdot B_{mn} \cdot n^4 (n^2 - 1) \cdot q_{mn} \cos n\,\varphi + \frac{\varphi}{\varphi_0} \overline{M}_{0ss} \right] \cdot \sin \frac{m\,\pi\,x}{l}$$

$$\text{waarin: } \overline{M}_{0ss} = -\frac{\delta^2}{12} \left[-\varphi_0 \left\{ \frac{\sin \varphi_0}{\varphi_0 \cos \varphi_0 - \sin \varphi_0} \sigma_{ss} + \frac{3}{\varphi_0{}^2} \left(\frac{2}{\lambda^2} - 1 \right) t_{ss} - \frac{1}{\varphi_0} \cdot q_0 \right\} \right] \cdot \sin \frac{m\,\pi\,x}{l}$$

(VIIb)
(Xb)
(Vb)

$$M_{uss} = r^2 \left[\left\{ \frac{2 \sin \varphi_0 (\varphi_0 \sin \varphi - \varphi \sin \varphi_0) + \varphi_0 (\varphi_0 \cos \varphi_0 \cdot \sin \varphi - \varphi \cos \varphi \cdot \sin \varphi_0)}{2 \sin \varphi_0 (\varphi_0 \cos \varphi_0 - \sin \varphi_0)} \right\} \sigma_{ss} + \right.$$

$$+ \left\{ \frac{\sin \varphi_0 (\varphi^3 - \varphi_0{}^2 \cdot \varphi) + 6 (\varphi_0 \sin \varphi - \sin \varphi_0 \cdot \varphi)}{2\,\varphi_0{}^2 \sin \varphi_0} \right\} t_{ss} + \left\{ \frac{\sin \varphi_0 \cdot \varphi - \varphi_0 \sin \varphi}{\varphi_0 \sin \varphi_0} \right\} q_0 +$$

$$\left. + \frac{\varphi}{\varphi_0} \left(\frac{M_{0ss} + \overline{M}_{0ss}}{r^2} \right) - \sum \frac{1}{n^2 - 1} Q_{mn} \sin n\,\varphi \right] \cdot \sin \frac{m\,\pi\,x}{l}$$

Developed in Fourier-series

(III)

$$M_{iss} = \left[\frac{\delta^2}{12} \sum \cos \frac{k\,\pi}{2} \left[- \left\{ \frac{\frac{6}{n^3}}{\varphi_0{}^2} \right\} t_{ss} \right] + \sum A_m \cdot B_{mn} \cdot n^4 (n^2 - 1) \cdot q_{mn} \right] \cdot \sin n\,\varphi \sin \frac{m\,\pi\,x}{l}$$

(IXb)
(Xb)
(Vb)

$$M_{uss} = r^2 \sum \left[\cos \frac{k\,\pi}{2} \left[\left\{ \frac{2 \sin \varphi_0}{\varphi_0 \cos \varphi_0 - \sin \varphi_0} \cdot \frac{1}{n (n^2 - 1)^2} \right\} \sigma_{ss} - \left\{ \frac{6}{\varphi_0{}^2} \cdot \frac{1}{n^3 (n^2 - 1)} \right\} t_{ss} + \right. \right.$$

$$\left. \left. + \left\{ \frac{2}{n (n^2 - 1)} \cdot \frac{1}{\varphi_0} \right\} q_0 \right] - \sum \frac{1}{n^2 - 1} Q_{mn} \right] \cdot \sin n\,\varphi \cdot \sin \frac{m\,\pi\,x}{l}$$

	Edge quantities
$q_{0tss} = -n_{\varphi ss}\big\vert_{\varphi=\varphi_0}$	$q_{0tss} = r\cdot q_0\cdot\sin\frac{m\,\pi\,x}{l}$
$q_{0rss} = = -\frac{1}{r}\cdot\frac{d\,M_{uss}}{\alpha\,\varphi}\Big\vert_{\varphi=\varphi_0}$	$q_{0rss} = r\Big[-\Big\{\frac{{\varphi_0}^2-2\sin^2\varphi_0+\varphi_0\sin\varphi_0\cos\varphi_0}{2\sin\varphi_0\,(\varphi_0\cos\varphi_0-\sin\varphi_0)}\Big\}\sigma_{ss}-\Big\{\frac{2\,{\varphi_0}^2\sin\varphi_0+6\,(\varphi_0\cos\varphi_0-\sin\varphi_0)}{2\,{\varphi_0}^2\sin\varphi_0}\Big\}t_{ss}-$ $-\Big\{\frac{\sin\varphi_0-\varphi_0\cos\varphi_0}{\varphi_0\sin\varphi_0}\Big\}q_0+\sum\frac{n\cos\frac{k\,\pi}{2}}{n^2-1}\cdot Q_{mn}\Big]\cdot\sin\frac{m\,\pi\,x}{l}$
$T_{0ss} = n_{x\varphi ss}\big\vert_{\varphi=\varphi_0}$	$T_{0ss} = -\frac{r}{\lambda}\Big[\sigma_{ss}+t_{ss}+\Big\{\frac{1}{\varphi_0}\Big\}q_0+\sum n\cos\frac{k\,\pi}{2}\cdot Q_{mn}\Big]\cdot\cos\frac{m\,\pi\,x}{l}$
$\sigma_{0ss} = \frac{n_{xss}}{\delta}\Big\vert_{\varphi=\varphi_0}$	$\sigma_{0ss} = -\frac{r}{\lambda^2\,\delta}\Big[\Big\{\frac{\varphi_0\sin\varphi_0}{\varphi_0\cos\varphi_0-\sin\varphi_0}\Big\}\sigma_{ss}-\Big\{\frac{3}{\varphi_0}\Big\}t_{ss}\Big]\cdot\sin\frac{m\,\pi\,x}{l}$
$v_{0ss} = v_{ss}\big\vert_{\varphi=\varphi_0}$	$v_{0ss} = \frac{r^2}{E\,\lambda^4\,\delta}\Big[-\Big\{\frac{\varphi_0\cos\varphi_0}{\varphi_0\cos\varphi_0-\sin\varphi_0}\Big\}\sigma_{ss}+\Big\{\frac{3}{{\varphi_0}^2}\Big\}t_{ss}-$ $-2\,\lambda^2\,[\sigma_{ss}+t_{ss}+q_0]-\sum\Big(1+2\,\frac{\lambda^2}{n^2}\Big)\,n^3\cos\frac{k\,\pi}{2}\cdot Q_{mn}\Big]\cdot\sin\frac{m\,\pi\,x}{l}$
$w_{0ss} = -w_{ss}\big\vert_{\varphi=\varphi_0}$	$w_{0ss} = \frac{r^2}{E\,\lambda^4\,\delta}\Big[\Big\{\frac{\varphi_0\sin\varphi_0}{\varphi_0\cos\varphi_0-\sin\varphi_0}\Big\}\sigma_{ss}+2\,\lambda^2\Big[\Big\{\frac{\varphi_0\sin\varphi_0}{\varphi_0\cos\varphi_0-\sin\varphi_0}\Big\}\sigma_{ss}-\Big\{\frac{3}{\varphi_0}\Big\}t_{ss}\Big]+\lambda^4\cdot q_0\Big]\cdot\sin\frac{m\,\pi\,x}{l}$

D. Establishing of the Condition $M_i = M_u$

We now come to the condition with the aid of which the perturbation functions can be more precisely determined. If we confine ourselves at first to the symmetrical functions, the condition is as follows: $M_{is} = M_{us}$, and it can be established with the aid of the above without any further difficulty. For an arbitrary harmonic function m in a longitudinal direction, after division of both parts of the equation by $\sin \frac{m\pi x}{l}$, the following is therefore valid:

$$\sum \left[A_m \cdot B_{mn} \cdot n^4 (n^2-1) \cdot p_{mn} + \frac{\delta^2}{12} \cdot \frac{1}{n^3} \cdot \frac{2 \sin \frac{k\pi}{2}}{\varphi_0 (\varphi_0 - \sin \varphi_0)} \cdot (\sin \varphi_0 \cdot \sigma_s - t_s) \right] \cdot \cos n \varphi =$$

$$= r^2 \sum \left[\sin \frac{k\pi}{2} \left[- \left\{ \frac{\left(\frac{2 \cos \varphi}{n (n^2-1)^2}\right) \varphi_0 - \left(\frac{2}{n^3 (n^2-1)}\right) \sin \varphi_0}{\varphi_0 (\varphi_0 - \sin \varphi_0)} \right\} \sigma_s + \right.\right.$$

$$\left.\left. + \left\{ \frac{\left(\frac{2 \cos \varphi}{n (n^2-1)^2}\right) - \left(\frac{2}{n^3 (n^2-1)}\right)}{\varphi_0 (\varphi_0 - \sin \varphi_0)} \right\} t_s - \left\{ \frac{2}{n (n^2-1)} \cdot \frac{1}{\varphi_0} \right\} p_0 \right] - \frac{1}{n^2-1} \cdot p_{mn} \right] \cos n \varphi .$$

In this equation, M_{ils}, developed in Fourier series, has still to be taken into account. It is furthermore clear that this summation condition has been exactly satisfied if the above condition (without the Σ symbol) has been satisfied as regards each part of the harmonic function.

Therefore, after elimination of $\cos n\varphi$ and after division of both parts of the equation by r^2, we can establish the following for each part of the harmonic function:

$$\left\{ \frac{A_m \cdot B_{mn}}{r^2} \cdot n^4 (n^2-1) + \frac{1}{n^2-1} \right\} \cdot p_{mn} = \tag{XI a}$$

$$= \sin \frac{k\pi}{2} \left[- \left\{ \frac{1}{n (n^2-1)^2} \cdot \frac{2 \varphi_0 \cos \varphi_0}{\varphi_0 (\varphi_0 - \sin \varphi_0)} - \frac{1}{n^3 (n^2-1)} \cdot \frac{2 \sin \varphi_0}{\varphi_0 (\varphi_0 - \sin \varphi_0)} + \right.\right.$$

$$\left. + \frac{\delta^2}{12 r^2} \cdot \frac{1}{n^3} \cdot \frac{2 \sin \varphi_0}{\varphi_0 (\varphi_0 - \sin \varphi_0)} \right\} \sigma_s + \left\{ \frac{1}{n (n^2-1)^2} \cdot \frac{2 \cos \varphi_0}{\varphi_0 (\varphi_0 - \sin \varphi_0)} - \frac{1}{n^3 (n^2-1)} \cdot \right.$$

$$\left.\left. \cdot \frac{2}{\varphi_0 (\varphi_0 - \sin \varphi_0)} + \frac{\delta^2}{12 r^2} \cdot \frac{1}{n^3} \cdot \frac{2}{\varphi_0 (\varphi_0 - \sin \varphi_0)} \right\} t_s - \left\{ \frac{1}{n (n^2-1)} \cdot \frac{2}{\varphi_0} \right\} p_0 \right]$$

from which p_{mn} can be resolved directly and expressed in σ_s, t_s and p_0.

Analogously, we find, with the aid of the condition $M_{iss} = M_{uss}$, the following for the anti-symmetrical functions:

$$\left\{ \frac{A_m \cdot B_{mn}}{r^2} \cdot n^4 (n^2-1) + \frac{1}{n^2-1} \right\} q_{mn} = \tag{XI b}$$

$$= \cos \frac{k\pi}{2} \left[+ \left\{ \frac{1}{n (n^2-1)^2} \cdot \frac{2 \sin \varphi_0}{\varphi_0 \cos \varphi_0 - \sin \varphi_0} \right\} \sigma_{ss} - \left\{ \frac{1}{n^3 (n^2-1)} \cdot \frac{6}{\varphi_0^2} - \frac{\delta^2}{12 r^2} \cdot \right.\right.$$

$$\left.\left. \cdot \frac{1}{n^3} \cdot \frac{6}{\varphi_0^2} \right\} t_{ss} + \left\{ \frac{1}{n (n^2-1)} \cdot \frac{2}{\varphi_0} \right\} q_0 \right]$$

from which q_{mn} can be directly expressed in σ_{ss}, t_{ss} and q_0. It is evident that in (XIa) as well as in (XIb), the expressions which have the coefficient $\frac{\Delta^2}{12\,r^2}$, constitute a negligible influence as regards the others; for, the coefficient referred to varies in practice between the values $3\cdot10^{-6}$ and $7\cdot10^{-6}$. The average error lies at approximately $\frac{1}{2}$ per mille when one simply eliminates the expressions.

Where p_{mn} can now be expressed in σ_s, t_s and p_0, or q_{mn} in σ_{ss}, t_{ss} and q_0, all quantities at the shell edge can also be expressed in these same magnitudes by substitution. In the edge equations, only σ_s, t_s, p_0, or σ_{ss}, t_{ss}, q_0 occur. These quantities can therefore be determined from the above. Once they are known, the stress lines can be directly established with them.

E. Taking into Account of Longitudinal and Torque Rigidities

These can still be computed in a rather simple manner. The complete equilibrium equations are as follows:

$$r\frac{\partial n_x}{\partial x}+\frac{\partial n_x}{\partial \varphi}=0, \tag{a}$$

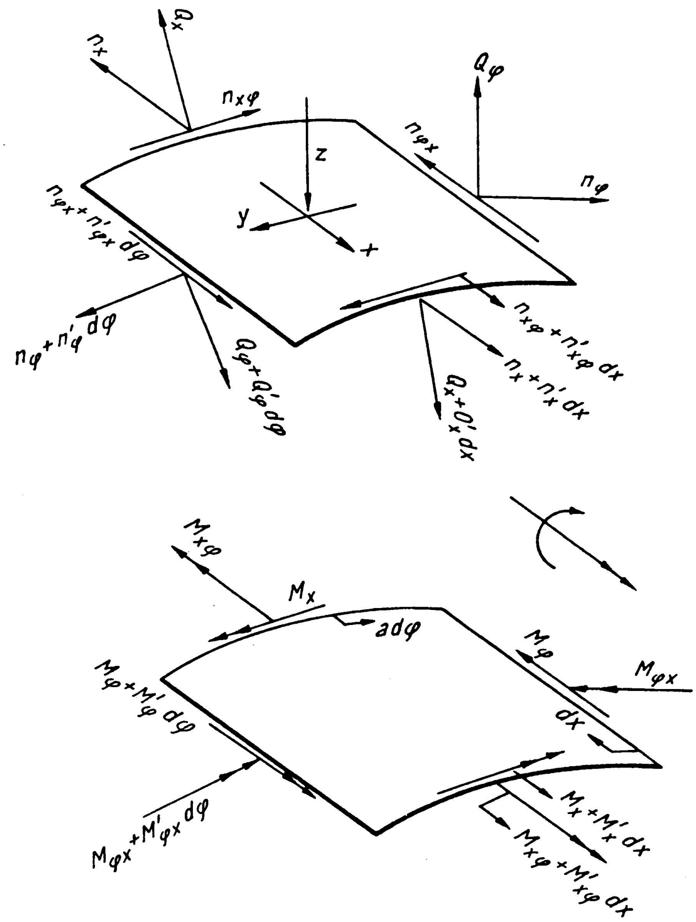

Fig. 5

$$\frac{\partial n_\varphi}{\partial \varphi} + r \frac{\partial n_{x\varphi}}{\partial x} - Q_\varphi = 0, \tag{b}$$

$$r \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\varphi}{\partial \varphi} + n_\varphi + q r = 0, \tag{c}$$

$$-r \frac{\partial M_{x\varphi}}{\partial x} - \frac{\partial M_\varphi}{\partial \varphi} + r Q_\varphi = 0, \tag{d}$$

$$\frac{\partial M_{x\varphi}}{\partial \varphi} + r \frac{\partial M_x}{\partial x} - r Q_x = 0. \tag{e}$$

Equations (b), (c) and (d) have already been applied when $M_{x\varphi}$, Q_x as well as the expression $r \frac{\partial n_{x\varphi}}{\partial x}$ were disregarded. Hereby it was found that:

$$\frac{d^2 M_\varphi}{d \varphi^2} + M_\varphi = -r^2 q \sin \frac{m \pi x}{l}$$

if q represents the shearing load which is exclusively taken up by the arches.

Further, the following applies:

$$M_\varphi = \frac{E I}{r^2} \left[\frac{d^2 w}{d \varphi^2} + w \right]$$

which, substituted in the above equation, yields:

$$\frac{d^4 w}{d \varphi^4} + 2 \frac{d^2 w}{d \varphi^2} + w = + \frac{r^4}{E I} \cdot q \sin \frac{m \pi x}{l}. \tag{1}$$

In this expression, the sign for q has been reversed, since the loads are operative upon the arches inversely to those operative upon the membrane.

Let us now investigate what will be the situation, if M_x and $M_{x\varphi}$ are not disregarded:

The differentiation of (c) to φ results in:

$$r \frac{\partial^2 Q_x}{\partial x \, \partial \varphi} + \frac{\partial^2 Q_\varphi}{\partial \varphi^2} + \frac{\partial n_\varphi}{\partial \varphi} = -r \frac{\partial q}{\partial \varphi}. \tag{2}$$

(d) differentiated twice to φ results in:

$$r \frac{\partial^2 Q_\varphi}{\partial \varphi^2} = \frac{\partial^3 M_\varphi}{\partial \varphi^3} + r \frac{\partial^3 M_{x\varphi}}{\partial x \, \partial \varphi^2}. \tag{3}$$

(e) differentiated twice to x and φ respectively, results in:

$$r \frac{\partial^2 Q_x}{\partial x \, \partial \varphi} = r \frac{\partial^3 M_x}{\partial x^2 \, \partial \varphi} + \frac{\partial^3 M_{x\varphi}}{\partial x \, \partial \varphi^2}. \tag{4}$$

(3) and (4) substituted in (2), after multiplication with r, results in:

$$r^2 \frac{\partial^3 M_x}{\partial x^2 \, \partial \varphi} + 2 r \frac{\partial^3 M_{x\varphi}}{\partial x \, \partial \varphi^2} + \frac{\partial^3 M_\varphi}{\partial \varphi^3} + \frac{\partial M_\varphi}{\partial \varphi} + r \frac{M_{x\varphi}}{\partial x} = -r^2 \frac{\partial q}{\partial \varphi}. \tag{5}$$

Disregarding the very negligible influence of v, we can write:

$$\left.\begin{aligned} M_x &= E\,I\frac{\partial^2 w}{\partial x^2} \\ M_{x\varphi} &= \frac{E\,I}{r}\left[\frac{\partial^2 w}{\partial x\,\partial\varphi}+\frac{\partial v}{\partial x}\right] \\ M_\varphi &= \frac{E\,I}{r^2}\left[\frac{\partial^2 w}{\partial\varphi^2}+\frac{\partial v}{\partial\varphi}\right] \end{aligned}\right\} \text{ while } \frac{\partial v}{\partial\varphi} = w. \tag{6}$$

Substituting (6) in (5), we obtain:

$$r^2\frac{\partial^6 w}{\partial x^4\,\partial\varphi^2}+2\frac{\partial^6 w}{\partial x^2\,\partial\varphi^4}+\frac{1}{r^2}\left[\frac{\partial^6 w}{\partial\varphi^6}+2\frac{\partial^4 w}{\partial\varphi^4}+\frac{\partial^2 w}{\partial\varphi^2}\right]+3\frac{\partial^4 w}{\partial x^2\,\partial\varphi^2}+\frac{\partial^2 w}{\partial x^2} = -\frac{r^2}{E\,I}\cdot\frac{\partial^2 q}{\partial\varphi^2}$$

multiplied by r^2 and computing the load inversely to that operative upon the membrane, we obtain:

$$\left[\frac{\partial^6 w}{\partial\varphi^6}+2\frac{\partial^4 w}{\partial\varphi^4}+\frac{\partial^2 w}{\partial\varphi^2}\right]+r^4\frac{\partial^6 w}{\partial x^4\,\partial\varphi^2}+2\,r^2\frac{\partial^6 w}{\partial x^2\,\partial\varphi^4}+3\,r^2\frac{\partial^4 w}{\partial x^2\,\partial\varphi^2}+r^2\frac{\partial^2 w}{\partial x^2} =$$
$$= +\frac{r^4}{E\,I}\cdot\frac{\partial^2 q}{\partial\varphi^2}.$$

When $w = \dot{W}\sin\frac{m\,\pi\,x}{l}$ and $q = q_0\sin\frac{m\,\pi\,x}{l}$, the differential equation, after elimination of $\sin\frac{m\,\pi\,x}{l}$, becomes:

$$\left[\frac{d^6 W}{d\,\varphi^6}+2\frac{d^4 W}{d\,\varphi^4}+\frac{d^2 W}{d\,\varphi^2}\right]+\frac{m^4\,\pi^4\,r^4}{l^4}\cdot\frac{d^2 W}{d\,\varphi^2}-2\frac{m^2\,\pi^2\,r^2}{l^2}\cdot\frac{d^4 W}{d\,\varphi^4}-3\frac{m^2\,\pi^2\,r^2}{l^2}\cdot\frac{d^2 W}{d\,\varphi^2}-$$
$$-\frac{m^2\,\pi^2\,r^2}{l^2}W = +\frac{r^4}{E\,I}\cdot\frac{\partial^2 q_0}{\partial\,\varphi^2}.$$

Here, the expression between the brackets actually represents the influence of the arches which can be easily understood if (1) is differentiated twice in succession to φ.

The total shearing load q_0 is composed of three different influences q_{01}, q_{02} and q_{03} as follows:

1. Operative upon the arches

$$\frac{d^2 q_{01}}{d\,\varphi^2} = \frac{E\,I}{r^4}\left[\frac{d^6 W}{d\,\varphi^6}+2\frac{d^4 W}{d\,\varphi^4}+\frac{d^2 W}{d\,\varphi^2}\right].$$

2. Resulting from the longitudinal force

$$\frac{d^2 q_{02}}{d\,\varphi^2} = \frac{E\,I}{r^4}\left[\frac{m^4\,\pi^4\,r^4}{l^4}\cdot\frac{d^2 W}{d\,\varphi^2}\right].$$

3. Resulting from the torque

$$\frac{d^2 q_{03}}{d\varphi^2} = -\frac{EI}{r^4}\left[\frac{m^2\pi^2 r^2}{l^2}\left(2\frac{d^4 W}{d\varphi^4}+3\frac{d^2 W}{d\varphi^2}+W\right)\right].$$

The part which is operative upon the arches is therefore $\frac{q_{01}}{q_0}$ times the total transversal load, that is to say that of the total shearing load which is operative between membrane and slab, only the part $\frac{q_{01}}{q_0}$ is operative upon those arches which are actually in this slab. These arches constitute the transversal bearing of the curved slab.

In the equation $M_i = M_u$, we must therefore introduce the coefficient $\frac{q_{01}}{q_0}$ for M_u. Considering the fact that in the establishing of the equation $M_i = M_u$, sine- and cosine-functions are applied and that it hereby appears that all quantities always take a course according to the same functions, we can start, for instance, from $W = w_0 \cos n\varphi$ and $W = w_0 \sin n\varphi$ next to $q_0 = Q_0 \cos n\varphi$ and $q_0 = Q_0 \sin n\varphi$.

The coefficient in both instances is as follows:

$$\Theta_{mn} = \frac{q_{01}}{q_0} = \frac{n^4 - 2n^2 + 1}{n^4 - 2n^2 + 1 + \frac{m^2\pi^2 r^2}{l^2}\left(2n^2 - 3 + \frac{1}{n^2}\right) + \frac{m^4\pi^4 r^4}{l^4}}.$$

or also:

$$\Theta_{mn} = \frac{1}{1 + \frac{\frac{m^2\pi^2 r^2}{l^2}\left(2n^2 - 3 + \frac{1}{n^2}\right) + \frac{m^4\pi^4 r^4}{l^4}}{n^4 - 2n^2 + 1}}.$$

When $\lambda = \frac{m\pi r}{l}$, while $n = \frac{k\pi}{2\varphi_0}$ by disregarding the negligible influence of $+\frac{1}{n^2}$, this coefficient finally reads:

$$\Theta_{mn} = \frac{1}{1 + \frac{\lambda^4 + \lambda^2(2n^2 - 3)}{n^4 - 2n^2 + 1}}.$$

We can now establish equations (XI a) and (XI b) for the general case, where longitudinal force and torque are not disregarded, where, however, the very negligible influences arising from $\frac{\delta^2}{12 r^2}$ have indeed been disregarded. Thus we get the following for the symmetrical functions:

$$A_m \cdot B_{mn} \cdot n^4 (n^2 - 1) \cdot p_{mn} = \Theta_{mn} \cdot r^2 \left[\sin\frac{k\pi}{2}\left[-\left\{\frac{1}{n(n^2-1)^2}\cdot\frac{2\varphi_0 \cos\varphi_0}{\varphi_0(\varphi_0 - \sin\varphi_0)} - \right.\right.\right.$$
$$\left. - \frac{1}{n^3(n^2-1)}\cdot\frac{2\sin\varphi_0}{\varphi_0(\varphi_0 - \sin\varphi_0)}\right\}\sigma_s + \left\{\frac{1}{n(n^2-1)^2}\cdot\frac{2\cos\varphi_0}{\varphi_0(\varphi_0 - \sin\varphi_0)} - \right.$$
$$\left.\left.\left. - \frac{1}{n^3(n^2-1)}\cdot\frac{2}{\varphi_0(\varphi_0 - \sin\varphi_0)}\right\} t_s - \left\{\frac{1}{n(n^2-1)}\cdot\frac{2}{\varphi_0}\right\} p_0\right] - \frac{1}{n^2-1}\cdot p_{mn}\right].$$

or:

$$\left\{\frac{A_m \cdot B_{mn} \cdot n^4 (n^2-1)}{r^2 \cdot \Theta_{mn}} + \frac{1}{n^2-1}\right\} p_{mn} = \sin\frac{k\pi}{2}\left[-\left\{\frac{1}{n(n^2-1)^2} \cdot \frac{2\varphi_0 \cos\varphi_0}{\varphi_0(\varphi_0-\sin\varphi_0)} - \right.\right.$$
$$\left. - \frac{1}{n^3(n^2-1)} \cdot \frac{2\sin\varphi_0}{\varphi_0(\varphi_0-\sin\varphi_0)}\right\}\sigma_s + \left\{\frac{1}{n(n^2-1)^2} \cdot \frac{2\cos\varphi_0}{\varphi_0(\varphi_0-\sin\varphi_0)} - \right. \tag{XIIa}$$
$$\left.\left. - \frac{1}{n^3(n^2-1)} \cdot \frac{2}{\varphi_0(\varphi_0-\sin\varphi_0)}\right\} t_s - \left\{\frac{1}{n(n^2-1)} \cdot \frac{2}{\varphi_0}\right\} p_0\right]$$

from which p_{mn} can therefore be directly expressed in σ_s, t_s and p_0. Analogously, we find for the anti-symmetrical functions:

$$\left\{\frac{A_m \cdot B_{mn} \cdot n^4 (n^2-1)}{r^2 \cdot \Theta_{mn}} + \frac{1}{n^2-1}\right\} q_{mn} = \cos\frac{k\pi}{2}\left[+\left\{\frac{1}{n(n^2-1)^2} \cdot \right.\right.$$
$$\left.\left. \cdot \frac{2\sin\varphi_0}{\varphi_0\cos\varphi_0 - \sin\varphi_0}\right\}\sigma_{ss} - \left\{\frac{1}{n^3(n^2-1)} \cdot \frac{6}{\varphi_0^2}\right\} t_{ss} + \left\{\frac{1}{n(n^2-1)} \cdot \frac{2}{\varphi_0}\right\} q_0\right] \tag{XIIb}$$

from which q_{mn} can again be directly expressed in σ_{ss}, t_{ss} and q_0.

The perturbation functions determined in this way are practically exact. Only extremely minor factors are allowed to be disregarded in this instance. As far as concrete calculations are concerned, they can in any event be considered as completely inessential.

We can now also contract the coefficient $\frac{A_m \cdot B_{mn}}{r^2 \cdot \theta_{mn}}$ into one coefficient. If we call it Φ_{mn}, then we get:

$$\Phi_{mn} = \frac{\delta^2}{12 r^2 \lambda^4}\left(1 + 2\frac{\lambda^2}{n^2} + \frac{\lambda^4}{n^4}\right)\left\{1 + \frac{\lambda^4 + \lambda^2(2n^2-3)}{n^4 - 2n^2 + 1}\right\}. \tag{XIIc}$$

Here: $m = 1, 3, 5 \ldots \qquad n = \frac{k\pi}{2\varphi_0},$

$k = 1, 2, 3, 4, 5 \ldots \qquad \lambda = \frac{m\pi r}{l}.$

When taking into account the longitudinal force and the torque, one must consider that the formulæ for q_{0r}, given on pages 119 and 122 are no longer valid. In this case, the formulæ for q_{0r} become considerably more complicated.

F. The Influences of the Fixed End Moments Along the Shell Edges

So far, we have actually all the time carried out calculations of shells which are hinged along their edges (long hinges). Now, the fourth unknown is introduced which consists of a moment along these shell edges. In the symmetrical functions, we are concerned with two equally large edge moments M_{0s} of the same sign, both of which bring about a constant moment plane in the arches, while in the anti-symmetrical functions, we have equally large moments M_{0ss}, however with the opposite sign, which bring about a moment plane with a linear course. In a longitudinal direction, they all take again the course according to $\sin \frac{m\pi x}{l}$. Restricting ourselves firstly to the symmetrical functions, if we have an intermediate pressure between arches and membrane which is as follows:

$$Z = \sum M_{0s} \cdot \frac{4 \sin \frac{k\pi}{2}}{A_m \cdot B_{mn} \cdot k\pi \cdot n^4 (n^2 - 1)} \cdot \cos n\varphi \cdot \sin \frac{m\pi x}{l}.$$

This intermediate pressure brings about a M_{is} of the following magnitude:

$$M_{is} = \sum A_m \cdot B_{mn} \cdot n^4 (n^2 - 1) \cdot M_{0s} \cdot \frac{4 \sin \frac{k\pi}{2}}{A_m \cdot B_{mn} \cdot k\pi \cdot n^4 (n^2 - 1)} \cdot \cos n\varphi \cdot \sin \frac{m\pi x}{l}$$

or:

$$M_{is} = \sum \frac{4 \sin \frac{k\pi}{2}}{k\pi} \cdot M_{0s} \cdot \cos n\varphi \cdot \sin \frac{m\pi x}{l}.$$

This represents a constant moment plane developed in Fourier series, namely:

$$M_{is} = M_{0s} \cdot \sin \frac{m\pi x}{l}.$$

This moment plane brings about an equally large moment plane in the arches, caused by the fixed end edge moments $M_{0s} \cdot \sin \frac{m\pi x}{l}$ at both shell edges. Hence these bring about a moment plane M_{us}, of the following magnitude:

$$M_{us_1} = M_{0s} \cdot \sin \frac{m\pi x}{l}.$$

However, the intermediate pressure in the arches brings about another moment plane which can easily be determined with the aid of the above, namely:

$$M_{us_2} = \sum r^2 M_{0s} \cdot \frac{-4 \sin \frac{k\pi}{2}}{A_m \cdot B_{mn} \cdot k\pi \cdot n^4 (n^2 - 1)^2} \cdot \cos n\varphi \cdot \sin \frac{m\pi x}{l}.$$

In the equation $M_i = M_u$, $M_{is} = M_{0s} \cdot \sin \frac{m\pi x}{l}$ must be added to M_i, and

$$M_{us} = M_{0s} \sin \frac{m\pi x}{l} - \sum r^2 M_{0s} \cdot \frac{4 \sin \frac{k\pi}{2}}{A_m \cdot B_{mn} \cdot k\pi \cdot n^4 (n^2 - 1)^2} \cdot \cos n\varphi \sin \frac{m\pi x}{l}$$

to M_u.

It is evident that the expressions $M_{0s} \sin \frac{m\pi x}{l}$ are cancelling each other out on either side of the equation. Therefore, only the second expression of M_{us} remains. After having been divided by $\sin \frac{m\pi x}{l}$ the following must be added to each part of the harmonic function:

$$-r^2 M_{0s} \cdot \frac{4 \sin \frac{k\pi}{2}}{A_m \cdot B_{mn} \cdot k\pi \cdot n^4 (n^2 - 1)^2}.$$

After dividing both parts of the equation by r^2, the following has to be added finally to the right-hand side of the equation (XII a):

$$-\frac{4\sin\frac{k\pi}{2}}{A_m\cdot B_{mn}\cdot k\pi\cdot n^4(n^2-1)^2}\cdot M_{0s}.$$

Entirely analogously, we obtain in the case of the anti-symmetrical functions the addition as below:

$$+\frac{4\cos\frac{k\pi}{2}}{A_m\cdot B_{mn}\cdot k\pi\cdot n^4(n^2-1)^2}\cdot M_{0ss}.$$

As a matter of fact, this causes the moment plane $M_{iss}=\frac{\varphi}{\varphi_0}\cdot M_{0ss}\cdot\sin\frac{m\pi x}{l}$, and is obtained from the intermediate pressure:

$$Z=\sum M_{0ss}\cdot\frac{-4\cos\frac{k\pi}{2}}{A_m\cdot B_{mn}\cdot k\pi\cdot n^4(n^2-1)}\cdot\sin n\varphi\cdot\sin\frac{m\pi x}{l}.$$

Obviously, the moment plane M_{iss} pertaining to this expression is:

$$M_{iss}=\sum\frac{-4\cos\frac{k\pi}{2}}{k\pi}\cdot M_{0ss}\cdot\sin n\varphi\cdot\sin\frac{m\pi x}{l}.$$

Equation (XII a) in its complete form is as follows:

$$\left[\Phi_{mn}\cdot n^4(n^2-1)+\frac{1}{n^2-1}\right]\cdot p_{mn}=\sin\frac{k\pi}{2}\left[-\left\{\frac{1}{n(n^2-1)^2}\,\frac{2\varphi_0\cos\varphi_0}{\varphi_0(\varphi_0-\sin\varphi_0)}-\right.\right.$$
$$\left.-\frac{1}{n^3(n^2-1)}\cdot\frac{2\sin\varphi_0}{\varphi_0(\varphi_0-\sin\varphi_0)}\right\}\sigma_s+\left\{\frac{1}{n(n^2-1)^2}\cdot\frac{2\cos\varphi_0}{\varphi_0(\varphi_0-\sin\varphi_0)}-\frac{1}{n^3(n^2-1)}\cdot\right.$$
$$\left.\left.\cdot\frac{2}{\varphi_0(\varphi_0-\sin\varphi_0)}\right\}t_s-\left\{\frac{1}{n(n^2-1)}\cdot\frac{2}{\varphi_0}\right\}p_0-\frac{4}{A_m\cdot B_{mn}\cdot k\pi\cdot n^4(n^2-1)^2}\cdot M_{0s}\right]$$

and likewise equation (XII b):

$$\left[\Phi_{mn}\cdot n^4(n^2-1)+\frac{1}{n^2-1}\right]\cdot q_{mn}=\cos\frac{k\pi}{2}\left[+\left\{\frac{1}{n(n^2-1)^2}\cdot\frac{2\sin\varphi_0}{\varphi_0\cos\varphi_0-\sin\varphi_0}\right\}\sigma_{ss}-\right.$$
$$\left.-\left\{\frac{1}{n^3(n^2-1)}\cdot\frac{6}{\varphi_0^2}\right\}t_{ss}+\left\{\frac{1}{n(n^2-1)}\cdot\frac{2}{\varphi_0}\right\}q_0+\frac{4}{A_m\cdot B_{mn}\cdot k\pi\cdot n^4(n^2-1)^2}\cdot M_{0ss}\right].$$

Furthermore, the total perturbation function as quoted below is present in the membrane:

$$P_{mn}=p_{mn}+\frac{4\sin\frac{k\pi}{2}}{A_m\cdot B_{mn}\cdot k\pi\cdot n^4(n^2-1)}\cdot M_{0s}$$

or:

$$Q_{mn}=q_{mn}-\frac{4\cos\frac{k\pi}{2}}{A_m\cdot B_{mn}\cdot k\pi\cdot n^4(n^2-1)}\cdot M_{0ss}.$$

respectively.

This completes in actual fact the theory, and we can now proceed to the practical application which consists determining the edge conditions, with which the unknown quantities can be determined. Prior to that, we want to ask ourselves, however, upon which differential equation the theory is actually based.

G. More Detailed Theoretical Considerations

Let us restrict ourselves first of all to the instance where longitudinal and torque bearing are disregarded.

The differential equation for the moment plane of the arches, included the inverse sign for the load, is as follows:

$$\frac{d^2 M_u}{d\varphi^2} + M_u = q r^2 \tag{1}$$

while M_i is determined from:

$$\frac{d^2 w}{d\varphi^2} + w = \frac{M_i r^2}{EI}$$

so

$$M_i = +\frac{EI}{r^2}\left[\frac{d^2 w}{d\varphi^2} + w\right] \tag{2}$$

however, the condition: $\qquad M_i = M_u \qquad (3)$

must prevail.

Hence, the following also applies:

$$M_u = +\frac{EI}{r^2}\left[\frac{d^2 w}{d\varphi^2} + w\right]. \tag{4}$$

(4) substituted in (1) results in:

$$\frac{d^4 w}{d\varphi^4} + 2\frac{d^2 w}{d\varphi^2} + w = +\frac{q r^4}{EI} \tag{5}$$

which with $q = z \cdot \sin\frac{m\pi x}{l}$ when $z = f(\varphi)$ gives: (6)

$$\frac{d^4 w}{d\varphi^4} + 2\frac{d^2 w}{d\varphi^2} + w = +\frac{r^4}{EI}\cdot z \cdot \sin\frac{m\pi x}{l}. \tag{7}$$

We found in formula (IIc) that when (6) applies:

$$w = -\frac{r^2}{E\delta}\left[\frac{1}{\lambda^4} z'''' - \frac{2}{\lambda^2} z'' + z\right] \sin\frac{m\pi x}{l} \tag{8}$$

in which:

$$\lambda = \frac{m\pi r}{l}. \tag{9}$$

Substitution of (8) in (5) finally results in:

$$\overset{''''}{z}'''' + 2(1-\lambda^2)\overset{'''}{z}''' + (1-4\lambda^2+\lambda^4) z'''' - 2(\lambda^2-\lambda^4) z'' + \lambda^4\left(1+\frac{12 r^2}{\delta^2}\right) z = 0. \tag{10}$$

This differential equation is practically identical with that of Finsterwalder. One can still correct the minor difference, however, this does not make any sense at all, in view of the fact that an error of only approximately 0,015 per mille is involved in this instance.

In the second instance, where the longitudinal and torque bearing are taken into account, the same derivations are valid up to (5). However, in this case a diminished bearing is computed for the arches, since a part is transmitted by the longitudinal force and the torque so that, in other words, q is reduced as regards the arches. This reduction amounts to:

$$q_{02}+q_{03} = \frac{E\,I}{r^4}\left[\lambda^4 w - 2\lambda^2 \frac{d^2 w}{d\varphi^2} - 3\lambda^2 w\right]\sin\frac{m\pi x}{l}. \tag{11}$$

This is the result, after the very negligible influence of $\lambda^2 w$ has been disregarded and after integration twice successively of the term to φ.

Consequently, (7), in changed form, is as follows:

$$\frac{d^4 w}{d\varphi^4} + 2\frac{d^2 w}{d\varphi^2} + w = \frac{r^4}{E\,I}\left[z\sin\frac{m\pi x}{l} - \frac{E\,I}{r^4}\left\{\lambda^4 w - 2\lambda^2\frac{d^2 w}{d\varphi^2} - 3\lambda^2 w\right\}\right] \tag{12}$$

or:

$$\frac{d^4 w}{d\varphi^4} + 2\frac{d^2 w}{d\varphi^2} + w + \lambda^4 w - 2\lambda^2\frac{d^2 w}{d\varphi^2} - 3\lambda^2 w = \frac{r^4}{E\,I} z\sin\frac{m\pi x}{l} \tag{13}$$

contracted:

$$\frac{d^4 w}{d\varphi^4} + 2(1-\lambda^2)\frac{d^2 w}{d\varphi^2} + (1 - 3\lambda^2 + \lambda^4)\,w = \frac{r^4}{E\,I} z\sin\frac{m\pi x}{l}. \tag{14}$$

Again, (8) applies:

$$w = -\frac{r^2}{E\,\delta}\left[\frac{1}{\lambda^4}z'''' - \frac{2}{\lambda^2}z'' + z\right]\sin\frac{m\pi x}{l}. \tag{15}$$

(15), substituted in (14), gives:

$$\begin{aligned} &\overset{''''}{z''''} - (4\lambda^2 - 2)\,\overset{'''}{z'''} + (6\lambda^4 - 7\lambda^2 + 1)\,z'''' - (4\lambda^6 - 8\lambda^4 + 2\lambda^2)\,z'' + \\ &\quad + \left\{\lambda^8 - 3\lambda^6 + \lambda^4\left(1 + \frac{12\,r^2}{\delta^2}\right)\right\} z = 0. \end{aligned} \tag{16}$$

In this instance, too, very minor factors have been neglected; proportionately, they are of about the same order as in the preceding instance. They might be introduced without any difficulty, but this is quite superfluous.

H. The Practical Application

Let us confine ourselves at first to the barrel-vault shells. These are symmetrical, so that the anti-symmetrical functions are cancelled out as a matter of course. Furthermore, all edge quantities in vertical and horizontal direction are necessary, while the so-called state of membrane stresses deriving from the dead weight must be added, in order to obtain the shell weight. The weight of the shell construction per cm² of shell surface is provisionally put at 1, and is only later introduced in true magnitude. We quote:

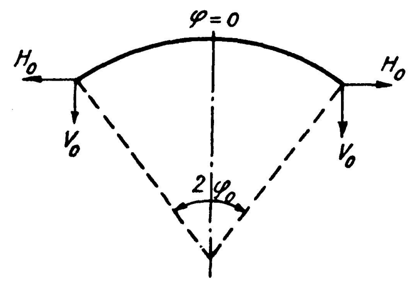

Fig. 6

$$\frac{l}{m\pi} T_0 = T_{0s} + \frac{8l}{m^2\pi^2} \sin\varphi_0,$$

$$\frac{l^2}{m^2\pi^2 r\delta} \sigma_0 = \sigma_{0s} - \frac{8l^2}{m^3\pi^3 r\delta} \cos\varphi_0,$$

$$r V_0 = q_{0ts} \sin\varphi_0 + q_{0rs} \cos\varphi_0 + \frac{4r}{m\pi} \sin\varphi_0 \cos\varphi_0 \quad \text{(positively directed downward)},$$

$$r H_0 = q_{0ts} \cos\varphi_0 - q_{0rs} \sin\varphi_0 + \frac{4r}{m\pi} \cos^2\varphi_0 \quad \text{(positively directed outward)}$$

$$\frac{l^4}{m^4\pi^4 E r^2 \delta} \cdot \Delta_{v0} = w_{0s} \cos\varphi_0 + v_{0s} \sin\varphi_0 + [1 + 2\lambda^2 + \tfrac{1}{2}\lambda^4 \cos^2\varphi_0] \frac{8l^4}{m^5\pi^5 E r^2 \delta}$$

(positively directed downward)

$$\frac{l^4}{m^4\pi^4 E r^2 \delta} \cdot \Delta_{h0} = -w_{0s} \sin\varphi_0 + v_{0s} \cos\varphi_0 - \tfrac{1}{2}\lambda^4 \sin\varphi_0 \cos\varphi_0 \frac{8l^4}{m^5\pi^5 E r^2 \delta}$$

(positively directed outward).

(The substitution $\lambda = \frac{m\pi r}{l}$ has been abandoned purposely in some instances.)

However, a better and very precise method of determining the edge displacements is that in which the horizontal displacement is calculated by means of the static moment of the transversal moment plane in the arches in regard to the horizontal line running through the ends of the arches and contracted along the axis of the arch. When this Δ_{h0} is known, then:

$$\frac{l^4}{m^4\pi^4 E r^2 \delta} \Delta_{v0} = \frac{w_{0s}}{\cos\varphi_0} + \operatorname{tg}\varphi_0 \cdot \Delta_{h0} + [1 + 2\lambda^2 + \tfrac{1}{2}\lambda^4 \cos^2\varphi_0] \frac{8l^4}{m^5\pi^5 E r^2 \delta}.$$

The simplest calculation of the static moment is effected by means of M_{us} which has been resolved in a Fourier series.

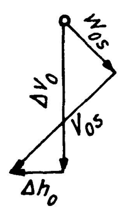

Fig. 7

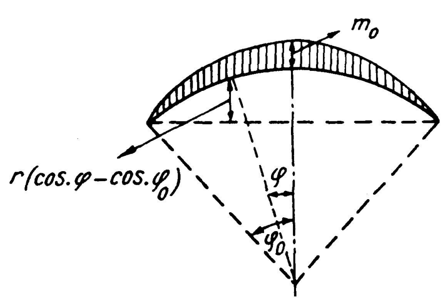

Fig. 8

Obviously, in general $M_{us} = m_0 \cos n\varphi$. Then, the following is valid for this harmonic function:

$$[\Delta_{h0}] = \frac{12}{E\,\delta^3} \int_0^{\varphi_0} m_0\, r^2 (\cos\varphi - \cos\varphi_0) \cos n\,\varphi \cdot d\,\varphi$$

or:

$$[\Delta_{h0}] = \frac{12\,r^2}{E\,\delta^3} \cdot \frac{\cos\varphi_0 \cdot \sin\frac{k\,\pi}{2}}{n\,(n^2-1)} \cdot m_0,$$

which formula can also be easily applied to the entire moment plane.

$$\begin{aligned}
[\Delta_{h0}] = {} & \frac{12\,r^4}{E\,\delta^3} \cos\varphi_0 \Bigg[\sum \frac{\sin^2\frac{k\,\pi}{2}}{n\,(n^2-1)} \Bigg[- \left\{\frac{1}{n\,(n^2-1)^2} \cdot \frac{2\,\varphi_0 \cos\varphi_0}{\varphi_0\,(\varphi_0 - \sin\varphi_0)} - \frac{1}{n^3\,(n^2-1)} \cdot \right.\\
& \left. \cdot \frac{2 \sin\varphi_0}{\varphi_0\,(\varphi_0 - \sin\varphi_0)} \right\} \sigma_s + \left\{ \frac{1}{n\,(n^2-1)^2} \cdot \frac{2\cos\varphi_0}{\varphi_0\,(\varphi_0 - \sin\varphi_0)} - \frac{1}{n^3\,(n^2-1)} \cdot \right. \qquad \text{(XIII)}\\
& \left. \cdot \frac{2}{\varphi_0\,(\varphi_0 - \sin\varphi_0)} \right\} t_s - \frac{1}{n\,(n^2-1)} \cdot \left.\frac{2}{\varphi_0}\right\} p_0 \Bigg] + \{\mathrm{tg}\,\varphi_0 - \varphi_0\} \cdot \frac{M_{0s}}{r^2} - \sum \sin\frac{k\,\pi}{2} \cdot \\
& \cdot \frac{1}{n\,(n^2-1)^2} \cdot P_{mn} \Bigg] - \tfrac{1}{2}\lambda^4 \sin\varphi_0 \cos\varphi_0 \frac{8\,l^4}{m^5\,\pi^5\,E\,r^2\,\delta} \cdot
\end{aligned}$$

Analogously, the angle of rotation at the shell edge is found by contracting the volume of the moment plane along the axis of the arch.

$$\psi_0 = \left.\frac{d\,w}{d\,\varphi}\right|_{\varphi=\varphi_0} = \frac{12}{E\,\delta^3} \int_0^{\varphi_0} m_0\, r \cos n\,\varphi\, d\varphi = \frac{12\,r}{E\,\delta^3} \cdot \frac{\sin\frac{k\,\pi}{2}}{n} \cdot m_0 .$$

Hence, the following applies:

$$\begin{aligned}
\psi_0 = {} & \frac{12\,r^3}{E\,\delta^3} \Bigg[\sum \frac{\sin^2\frac{k\,\pi}{2}}{n} \Bigg[- \left\{\frac{1}{n\,(n^2-1)^2} \cdot \frac{2\,\varphi_0 \cos\varphi_0}{\varphi_0\,(\varphi_0 - \sin\varphi_0)} - \frac{1}{n^3\,(n^2-1)} \cdot \right.\\
& \left. \cdot \frac{2 \sin\varphi_0}{\varphi_0\,(\varphi_0 - \sin\varphi_0)} \right\} \sigma_s + \left\{ \frac{1}{n\,(n^2-1)^2} \cdot \frac{2\cos\varphi_0}{\varphi_0\,(\varphi_0 - \sin\varphi_0)} - \frac{1}{n^3\,(n^2-1)} \cdot \right. \qquad \text{(XIV)}\\
& \left. \cdot \frac{2}{\varphi_0\,(\varphi_0 - \sin\varphi_0)} \right\} t_s - \left\{ \frac{1}{n\,(n^2-1)} \cdot \frac{2}{\varphi_0} \right\} p_0 \Bigg] + \varphi_0 \frac{M_{0s}}{r^2} - \sum \sin\frac{k\,\pi}{2} \cdot \frac{1}{n\,(n^2-1)} \cdot P_{mn} \Bigg] .
\end{aligned}$$

In the case of an inner shell, e.g., the following conditions apply:

$\Delta_{h0} = 0$ next to σ_x shell edge $= \sigma_x$ edge beam.
$\psi_0 = 0$,, ,, Δ_{v0} ,, ,, $= \Delta_{v0}$,, ,,

From these four conditions, σ_s, t_s, p_0 and M_{0s} can be resolved, whence all stress- and moment-lines are known. In order to determine the latter two conditions, one must consider that on the edge beam V_0, shell reactions T_0

as well as the shell weight, amounting to q_e per cm are acting, as sketched below.

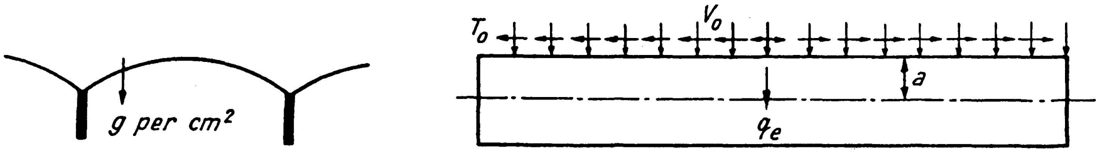

Fig. 9

a = position of neutral axis in relation to upper edge.

The conditions can now easily be established. If the development in Fourier series in a longitudinal direction is also applied in this case, then one finds the following for the first condition:

$$\frac{g\,l^2}{m^2\,\pi^2\,r\,\delta}\sigma_0 = \frac{l^2}{m^2\,\pi^2\,I}\left[\left(\frac{I}{F}+a^2\right)g\,T_0 - a\,g\,r\,V_0 - \frac{4\,a}{m\,\pi}\cdot q_e\right]$$

or:

$$\sigma_0 = \frac{\delta\,r^2}{I}\left[\left(\frac{\frac{I}{F}+a^2}{r}\right)T_0 - a\,V_0 - \frac{4\,a}{m\,\pi}\cdot\frac{q_e}{g\,r}\right] \tag{A}$$

and for the second:

$$\frac{g\,l^4}{m^4\,\pi^4\,E\,r^2\,\delta}\cdot\varDelta_{v0} = \frac{l^4}{m^4\,\pi^4\,E\,I}\left[g\,r\,V_0 - a\,g\,T_0 + \frac{4}{m\,\pi}q_e\right]$$

or:

$$\varDelta_{v0} = \frac{\delta\,r^3}{I}\left[V_0 - \frac{a}{r}T_0 + \frac{4}{m\,\pi}\cdot\frac{q_e}{g\,r}\right] \tag{B}$$

whereupon according to (XIII) the following is determined:

$$\varDelta_{h0} = 0 \tag{C}$$

and according to (XIV) the following is determined:

$$\psi_0 = 0 \tag{D}$$

In these formulæ the expressions signify therefore (see above):

$$T_0 = \frac{m\,\pi}{l}T_{0s} + \frac{8}{m\,\pi}\cdot\sin\varphi_0,$$

$$V_0 = \frac{1}{r}\left[q_{0ts}\sin\varphi_0 + q_{0rs}\cos\varphi_0\right] + \frac{4}{m\,\pi}\sin\varphi_0\cos\varphi_0,$$

$$\sigma_0 = \frac{m^2\,\pi^2\,r\,\delta}{l^2}\sigma_{0s} - \frac{8}{m\,\pi}\cos\varphi_0.$$

Considering the fact that in this instance the condition has been introduced, that $\varDelta_{h0}=0$, the following applies:

$$\varDelta_{v0} = \frac{m^4\,\pi^4\,E\,r^2\delta}{l^4}\cdot\frac{w_{0s}}{\cos\varphi_0} + \left[1 + 2\lambda^2 + \tfrac{1}{2}\lambda^4\cos^2\varphi_0\right]\frac{8}{m\,\pi}.$$

One can see from these formulæ that when a given φ_0-value is applied, all shells which have the same φ_0-value can be calculated with the aid of a table which can be practically reduced to the four final formulas (A), (B), (C) and (D), with the exception of some coefficients once and for all. It is of no importance which r, δ and l proportions these shells have, provided that φ_0 is identical. The same applies to the course of the stress- and moment-lines. This is indeed a great advantage of the method developed above.

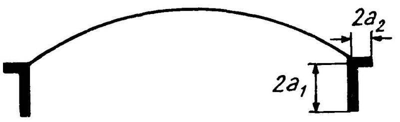

Fig. 10

q_e = total weight of edge beam.

Let us now follow the case sketched here, that is to say that of a freely suspended shell with two small horizontal beams at the edge of the shell. The quantities concerning the vertical beam are quoted with index 1 and those of the horizontal beam with index 2. First of all, the moment at fixed ends is known directly (in this instance, the torque rigidity has been disregarded), that is to say, it is: $M_{0s} = a_2 q_{e2}$.

The increase in shearing force, T_0 is split into two parts, that is to say T_{01} and T_{02}; hence, the following applies:

$$T_0 = T_{01} + T_{02}.$$

The longitudinal stresses of both beams must be identical at their point of meeting, and then the following applies:

$$\frac{1}{I_1}\left[\left(\frac{\frac{I_1}{F_1}+a_1^2}{r}\right) T_{01} - a_1 \cdot V_0 - \frac{4\,a_1}{m\,\pi} \cdot \frac{q_e}{g\,r}\right] = \\ = \frac{1}{I_2}\left[\left(\frac{\frac{I_2}{F_2}+a_2^2}{r}\right)(T_0 - T_{01}) - a_2 \cdot H_0\right] \tag{A}$$

Furthermore:

$$\sigma_0 = \frac{\delta\,r^2}{I_1}\left[\left(\frac{\frac{I_1}{F_1}+a_1^2}{r}\right) T_{01} - a_1 V_0 - \frac{4\,a_1}{m\,\pi} \cdot \frac{q_e}{g\,r}\right], \tag{B}$$

$$\Delta_{v0} = \frac{\delta\,r^3}{I_1}\left[V_0 - \frac{a_1}{r} T_{01} + \frac{4}{m\,\pi} \cdot \frac{q_e}{g\,r}\right] \tag{C}$$

while furthermore, the following has to be added:

$$\Delta_{h0} = \frac{\delta\,r^3}{I_2}\left[H_0 - \frac{a_2}{r}(T_0 - T_{01})\right]. \tag{D}$$

From these 4 equations, T_{01}, σ_s, t_s and p_0 can be resolved.

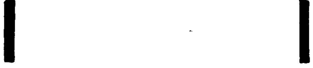
Fig. 11

Finally, we come to a third type which can only be used for smaller longitudinal spans. If the horizontal- and torque-rigidities of the vertical beam are disregarded as a matter of convenience, then the following is valid:

$$\begin{aligned} M_{0s} &= 0 \\ H_0 &= 0 \end{aligned} \qquad \text{(A)}$$

$$\sigma_0 = \frac{\delta r^2}{I}\left[\left(\frac{\frac{I}{F}+a^2}{r}\right) T_0 - a V_0 - \frac{4a}{m\pi}\cdot\frac{q_e}{g r}\right], \qquad \text{(B)}$$

$$\Delta_{v0} = \frac{\delta r^3}{I}\left[V_0 - \frac{a}{r} T_0 + \frac{4}{m\pi}\cdot\frac{q_e}{g r}\right] \qquad \text{(C)}$$

from which the three unknowns σ_s, t_s and p_0 can be computed.

This concludes the chapter on the three main types of symmetrical barrel-vault shells and we are now going to deal with the northlight shell types which are asymmetrical and therefore call for more elaborate calculations.

We are concerned here only with that type of northlight shell where the tangent at the top of the circular cross section runs horizontally.

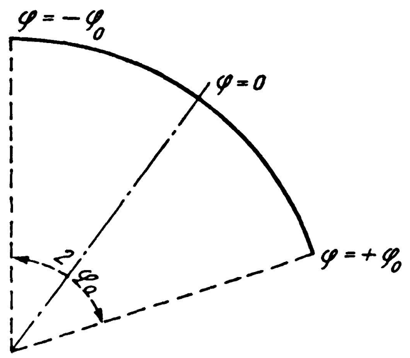

Fig. 12

As stated earlier on, all edge quantities of the top edge will have the index b, that is to say when $\varphi_0 = -\varphi_0$, and of the gutter edge the index 0, that is to say when $\varphi = +\varphi_0$.

In the first place, the following is valid:

$$\frac{l}{m\pi} T_b = -T_{0s} + T_{0ss},$$

$$\frac{l}{m\pi} T_0 = T_{0s} + T_{0ss} + \frac{8l}{m^2\pi^2}\sin 2\varphi_0,$$

$$\frac{l^2}{m^2\pi^2 r\delta}\sigma_b = \sigma_{0s} - \sigma_{0ss} - \frac{8l^2}{m^3\pi^3 r\delta},$$

$$\frac{l^2}{m^2\pi^2 r\delta}\sigma_0 = \sigma_{0s} + \sigma_{0ss} - \frac{8l^2}{m^3\pi^3 r\delta}\cos 2\varphi_0,$$

$$r V_b = q_{0rs} - q_{0rss},$$

$$r V_0 = (q_{0ts} + q_{0tss})\sin 2\varphi_0 + (q_{0rs} + q_{0rss})\cos 2\varphi_0 + \\ + \frac{4r}{m\pi}\sin 2\varphi_0 \cos 2\varphi_0,$$

$$r\,H_b = q_{0ts} - q_{0tss} + \frac{4\,r}{m\,\pi},$$

$$r\,H_0 = -(q_{0ts} + q_{0tss})\cos 2\,\varphi_0 + (q_{0rs} + q_{0rss})\sin 2\,\varphi_0 - \\ - \frac{4\,r}{m\,\pi}\cos^2 2\,\varphi_0,$$

$$\frac{l^4}{m^4\,\pi^4\,E\,r^2\,\delta}\,\Delta_{vb} = w_{0s} - w_{0ss} + (1 + 2\,\lambda^2 + \tfrac{1}{2}\,\lambda^4)\,\frac{8\,l^4}{m^5\,\pi^5\,E\,r^2\,\delta},$$

$$\frac{l^4}{m^4\,\pi^4\,E\,r^2\,\delta}\,\Delta_{v0} = (w_{0s} + w_{0ss})\cos 2\varphi_0 + (v_{0s} + v_{0ss})\sin 2\,\varphi_0 + \\ + [1 + 2\,\lambda^2 + \tfrac{1}{2}\,\lambda^4\cos^2 2\,\varphi_0]\,\frac{8\,l^4}{m^5\,\pi^5\,E\,r^2\,\delta},$$

$$\frac{l^4}{m^4\,\pi^4\,E\,r^2\,\delta}\,\Delta_{h0} = (w_{0s} + w_{0ss})\sin 2\,\varphi_0 - (v_{0s} + v_{0ss})\cos 2\,\varphi_0 + \\ + \tfrac{1}{2}\,\lambda^4\sin 2\,\varphi_0\cos 2\,\varphi_0\,\frac{8\,l^4}{m^5\,\pi^5\,E\,r^2\,\delta},$$

$$M_b = M_{0s} - M_{0ss},$$
$$M_0 = M_{0s} + M_{0ss}.$$

If the shell does not have a thickened top, the conditions as below are valid:

$$T_b = 0, \tag{A}$$
$$H_b = 0, \tag{B}$$
$$\Delta_{vb} = \Delta_{v0}, \tag{C}$$
$$M_b = 0. \tag{D}$$

This latter applies when for the sake of convenience the torque rigidity of the edge beams are assumed to be $\sim$.

The introduction of this simplification is admissible in every respect as has appeared from different numerical examples. Further, it is assumed that the top edge of each preceding shell is supported by frame trusses placed on the following gutter edge, as follows:

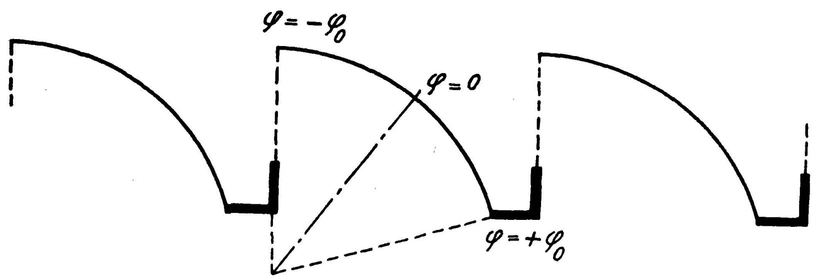

Fig. 13

This latter condition is very important since it renders a considerably more favourable distribution of stress possible than if this were not the case.

The frame planes are assumed to be vertical. Those that are placed obliquely will be dealt with later on. If the edge beams are considered to be a framework construction, which supposition is a safe one to make, then the following edge conditions apply:

$$\sigma_0 = \frac{\delta r^2}{b h_2^2 \left(1+\frac{h_1}{h_2}\right)} \left[\left(\frac{3h_1+4h_2}{r}\right)\cdot T_0 + \left(6+3\frac{h_1}{h_2}\right)\cdot H_0 - 3\frac{h_2}{h_1}\left(V_b+V_0+\frac{4}{m\pi}\cdot\frac{q_e}{gr}\right)\right], \tag{E}$$

$$\Delta_{vb} = \frac{3\delta r^3}{b h_1^3 \left(1+\frac{h_2}{h_1}\right)} \left[-\frac{h_1}{r}\cdot T_0 - 3\frac{h_1}{h_2}\cdot H_0 + \left(4+\frac{h_2}{h_1}\right)\left(V_b+V_0+\frac{4}{m\pi}\cdot\frac{q_e}{gr}\right)\right], \tag{F}$$

$$\Delta_{h0} = \frac{3\delta r^3}{b h_2^3 \left(1+\frac{h_1}{h_2}\right)} \left[\left(\frac{h_1+2h_2}{r}\right)\cdot T_0 + \left(4+\frac{h_1}{h_2}\right)\cdot H_0 - 3\frac{h_2}{h_1}\left(V_b+V_0+\frac{4}{m\pi}\cdot\frac{q_e}{gr}\right)\right], \tag{G}$$

$$\left.\frac{\partial w}{\partial \varphi}\right|_{\varphi=+\varphi_0} = 0. \tag{H}$$

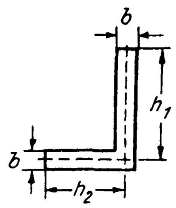

Fig. 14

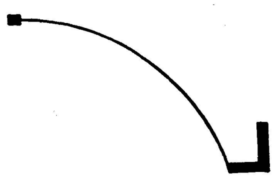

Fig. 15

If a thickened top edge is involved, then the following replaces condition (A):

$$-\frac{r\delta}{F_b}\cdot T_b = \sigma_b \tag{Aa}$$

where F_b represents the cross-section of the top-edge thickening. If the rigidity of the top-edge thickening in a horizontal direction is neglected, then (B) remains valid per se. This simplified supposition can be considered as admissible without any objections.

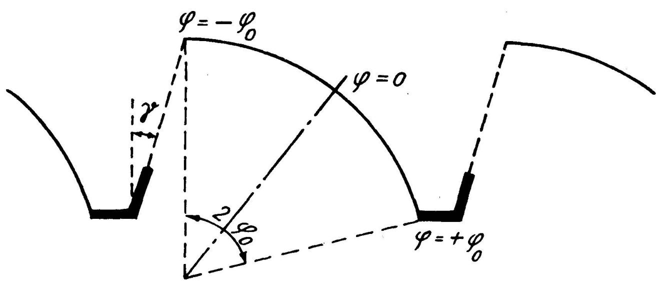

Fig. 16

If the frame planes are oblique, at an angle γ with the perpendicular, then the following applies instead of (B):

$$-H_b + V_b\cdot\operatorname{tg}\gamma = 0 \tag{Ba}$$

The equations (E), (F), and (G), are, however, as follows:

$$\sigma_0 = \frac{\delta r^2}{b\,h_2^2\left(1+\frac{h_1}{h_2}\right)}\left[\left(\frac{3h_1+4h_2}{r}\right)\cdot T_0 + \right.$$
$$\left. + \left(6+3\frac{h_1}{h_2}\right)\cdot \overline{H}_0 - 3\frac{h_2}{h_1}\left\{\overline{(V_b+V_0)} + \frac{4}{m\pi}\cdot\frac{q_e}{g\,r}\right\}\right], \tag{Ea}$$

$$\Delta_{vb} = \frac{3\,\delta r^3}{b\,h_1^3\left(1+\frac{h_2}{h_1}\right)\cos\gamma}\left[-\frac{h_1}{r}\cdot T_0 - 3\frac{h_1}{h_2}\cdot\overline{H}_0 + \left(4+\frac{h_2}{h_1}\right)\left\{\overline{(V_b+V_0)} + \frac{4}{m\pi}\cdot\frac{q_e}{g\,r}\right\}\right] -$$
$$- \frac{3\,\delta r^3\cdot \operatorname{tg}\gamma}{b\,h_2^3\left(1+\frac{h_1}{h_2}\right)}\left[\left(\frac{h_1+2h_2}{r}\right)\cdot T_0 + \left(4+\frac{h_1}{h_2}\right)\cdot\overline{H}_0 - 3\frac{h_2}{h_1}\left\{\overline{(V_b+V_0)} + \frac{4}{m\pi}\cdot\frac{q_e}{g\,r}\right\}\right], \tag{Fa}$$

$$\Delta_{h0} = \frac{3\,\delta r^3}{b\,h_2^3\left(1+\frac{h_1}{h_2}\right)}\left[\left(\frac{h_1+2h_2}{r}\right)\cdot T_0 + \left(4+\frac{h_1}{h_2}\right)\cdot\overline{H}_0 - 3\frac{h_2}{h_1}\left\{\overline{(V_b+V_0)} + \frac{4}{m\pi}\cdot\frac{q_e}{g\,r}\right\}\right]. \tag{Ga}$$

Here, $\overline{H}_0 = H_0 - V_0 \operatorname{tg}\gamma$ and: $\overline{(V_b+V_0)} = \dfrac{V_b+V_0}{\cos\gamma}$.

For the sake of convenience, it is again assumed that:

$$\left.\frac{\partial w}{\partial\varphi}\right|_{\varphi=+\varphi_0} = 0.$$

The torque influences can, of course, also be taken into account without any difficulty. Supposing the angle torsion is η (η is very small), then the following replaces equation (C):

$$\Delta_{v0} = \Delta_{vb} + \eta\,h_2, \tag{Ca}$$

while:

$$\left.\frac{\partial w}{\partial\varphi}\right|_{\varphi=+\varphi_0} = \eta.$$

η can be determined from the torque moment with regard to the neutral axis of the gutter beam which can be easily determined with the aid of V_0, V_b and H_0. Let us, however, repeat again that the most simplified supposition gives results which differ only very little from the exact ones. One must consider that the torque rigidity of the gutter edges is considerably enlarged by the frame trusses which are fastened to the top edge of the shell. These trusses have a large moment arm, so that minor shearing forces act rather as a check on the upper shell edge as far as the angle torsion of these edges is concerned. The taking into account of all these minor influences calls for a

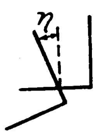

Fig. 17

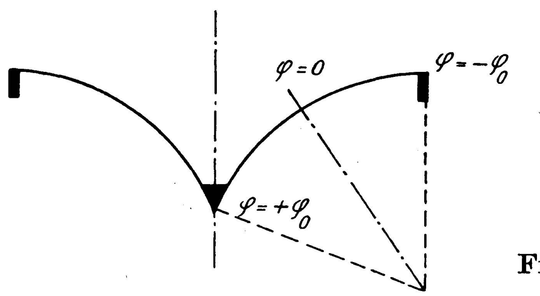

Fig. 18

considerably more complicated calculation, with the result that the tabulated elaboration cannot be carried out so far, and is therefore unnecessary. The influence upon the main forces is subsequently only very negligible (n_x, $n_{x\varphi}$, and n_φ).

We now come to another asymmetrical type of shell, i. e. the butterfly shell.

It is necessary in this case to place solid edge members upon the top edge of the shell, that is to say, to the underside of the shell. This is the most favourable position of the top edge member. If displaced toward the upper edge, its position becomes increasingly unfavourable, and it is therefore not advised to do so.

The edge conditions for this type are as follows:

$$\Delta_{h0} = 0, \tag{A}$$

$$V_0 + \frac{4}{m\pi} \cdot \frac{q_{e0}}{gr} = 0, \tag{B}$$

$$\frac{r\delta}{F_0} \cdot T_0 = \sigma_0, \tag{C}$$

$$\left.\frac{\partial w}{\partial \varphi}\right|_{\varphi = +\varphi_0} = 0. \tag{D}$$

Further, when neglecting the horizontal rigidity of the top edge beam:

$$H_b = 0, \tag{E}$$

$$\left.\frac{\partial w}{\partial \varphi}\right|_{\varphi = -\varphi_0} = 0, \tag{F}$$

$$\sigma_b = \frac{\delta r^2}{I_b}\left[-\left(\frac{\frac{I_b}{F_b} + a^2}{r}\right) T_b - a\, V_b - \frac{4a}{m\pi} \cdot \frac{q_{eb}}{gr}\right], \tag{G}$$

$$\Delta_{vb} = \frac{\delta r^3}{I_b}\left[V_b + \frac{a}{r} T_b + \frac{4}{m\pi} \cdot \frac{q_{eb}}{gr}\right]. \tag{H}$$

This concludes the discussion of the most important forms of this asymmetrical type.

The reader may have noticed that, although the method of determining the perturbation functions including the longitudinal and torque rigidities have been discussed in Chapter E, the formulæ which are quoted in the part on practical applications are nonetheless based upon the "arches theory". The reason is that its field of application is very wide. It is even so that the usual shell constructions in utility structures fall entirely within this group where one may neglect the influences of longitudinal and torque rigidities without any consideration and thus save a considerable amount of unnecessary calculations.

If longitudinal and torque rigidities must, however, be taken into account, then almost all formulæ remain in force, with the exception of those of the radial reaction q_{0r}.

This can easily be seen, since the edge quantities σ_0, T_0, q_{0t}, v_0 and w_0 have all been determined with the aid of the membrane formulæ, whereby the influence of longitudinal and torque rigidities has also been taken into account by means of establishing the perturbation functions.

The case of the radial reaction is different. This is determined by means of the moment plane which is developing in the transverse arches. Considering that this moment plane has been changed as regards the arches theory, subsequent to taking into account longitudinal and torque rigidities, the transverse reaction q_{0r} is naturally changed, too. The procedure for determining it is rather laborious and has not been included in this publication.

Example of a calculation

We now come to an example of a calculation and for this purpose, the well-known "Markthalle Budapest" that is to say, its inner shell, has been chosen. Let us state at once that of the perturbation functions, only the first harmonic function has been determined, i. e. for $k=1$.

The higher harmonic functions decrease so rapidly in magnitude that it is quite unnecessary to take them into account. Furthermore, we shall show in what manner a table is drawn up.

In the case of the "Markthalle", the following applies:

$$\varphi_0 = 0{,}61872, \text{ hence:}$$
$$\sin\varphi_0 = 0{,}57999 \qquad n_1 = 2{,}538783823$$
$$\cos\varphi_0 = 0{,}81462.$$

Herewith we determine the equation, see formula (XIIa) page 128, in order to determine the perturbation function p_{11}. After figures have been substituted to 9 decimals, this is, if M_i is neglected, i. e. if $M_i=0$ as follows:

$$[226{,}2218\,\Phi_{11} + 0{,}183\,640\,452]\,p_{11} = -0{,}015\,539\,523\,\sigma_s - 0{,}033\,516\,134\,t_s - \\ -0{,}233\,818\,144\,p_0 - \frac{1033{,}58\,000\cdot 10^{-6}}{\Phi_{11}}\cdot\frac{M_0}{r^2} \qquad (1)$$

if: $$\Phi_{11} = \frac{1}{1168{,}909\,m^4}\cdot\frac{l^4}{r^4}\cdot\frac{\delta^2}{r^4}(1 + 0{,}310\,297\,696\,\lambda^2 + 0{,}024\,071\,165\,\lambda^4) \qquad (2)$$

whereby: Θ_{11} has been put as 1, that is to say, the influence of longitudinal and torque rigidities has been neglected.

Also: $$P_{11} = p_{11} + \frac{5628{,}28\,063\cdot 10^{-6}}{\Phi_{11}}\cdot\frac{M_0}{r^2}. \qquad (3)$$

It is evident that if r, l and δ are known, it is possible, with the aid of these formulæ, to express p_{11} and P_{11} in σ_s, t_s, p_0 and $\frac{M_0}{r^2}$.

Conditions (C) and (D), page 135, can now also be established directly with the aid of formulæ (XIII) and (XIV), page 134. Hereby, $k = 1$, 3 and 5 are taken into account whereby the results are practically exact because of the extremely rapid convergence. Translated into numerical values, these conditions are as follows:

$$\begin{aligned} &-0{,}000\,915\,238\,\sigma_s - 0{,}001\,976\,012\,t_s - 0{,}013\,792\,279\,p_0 + \\ &+0{,}075\,968\,314\,\frac{M_0}{r^2} - 0{,}010\,821\,021\,P_{11} = 0 \end{aligned} \tag{4}$$

and:

$$\begin{aligned} &-0{,}006\,090\,189\,\sigma_s - 0{,}013\,279\,702\,t_s - 0{,}093\,201\,219\,p_0 + \\ &+0{,}618\,720\,000\,\frac{M_0}{r^2} - 0{,}072\,333\,423\,P_{11} = 0. \end{aligned} \tag{5}$$

Furthermore, we are able to determine in 6 decimals the following with the aid of the formulæ on page 135 for V_0 and T_0 and the table of formulæ, page 119.

$$V_0 = -0{,}030\,534\,\sigma_s - 0{,}073\,433\,t_s - 0{,}379\,795\,P_{11} + \frac{0{,}601\,569}{m} \tag{6}$$

and:

$$T_0 = t_s + 2{,}538\,784\,P_{11} + \frac{1{,}476\,933}{m}. \tag{7}$$

We can now write the conditions (A) and (B) with the aid of the formulæ for σ_0 and Δ_{v0}, page 135; and the table of formulæ; these, calculated in 6 decimals, are as follows:

$$\begin{aligned} &1{,}961\,629\,\sigma_s - 4{,}786\,703\,t_s - \frac{2{,}074\,414}{m} = \\ &= \frac{\delta r^2}{I}\left[\left(\frac{I}{F r} + \frac{a^2}{r}\right) T_0 - a\,V_0 - \frac{1{,}27\,324}{m}\cdot\frac{a\,q_e}{g\,r}\right] \end{aligned} \tag{8}$$

and:

$$\begin{aligned} &\{15{,}975\,213 - 4{,}815\,905\,\lambda^2\}\,\sigma_s - \{25{,}819\,778 - 11{,}751\,419\,\lambda^2\}\,t_s + 1{,}227\,566\,\lambda^4\cdot \\ &\cdot p_0 + \frac{2{,}54\,648}{m}\cdot(1 + 2\lambda^2 + \tfrac{1}{2}\lambda^4) = \frac{\delta r^3}{I}\left[-\frac{a}{r}\,T_0 + V_0 + \frac{1{,}27\,324}{m}\cdot\frac{q_e}{g r}\right]. \end{aligned} \tag{9}$$

Substituting (6) and (7) in (8) and (9) gives, in 6 decimals:

$$\begin{aligned} &\left\{1{,}961\,629 - 0{,}030\,534\,\frac{a\,\delta\,r^2}{I}\right\}\sigma_s - \left\{4{,}786\,703 + \frac{\delta r}{I}\left(\frac{I}{F} + a^2\right) + 0{,}073\,433\,\frac{a\,\delta\,r^2}{I}\right\} t_s - \\ &-\left\{2{,}538\,784\,\frac{\delta r}{I}\left(\frac{I}{F} + a^2\right) + 0{,}379\,795\,\frac{a\,\delta\,r^2}{I}\right\} P_{11} - \left\{\frac{2{,}074\,414}{m} + \frac{1{,}476\,933}{m}\cdot\right. \\ &\left.\cdot\frac{\delta r}{I}\left(\frac{I}{F} + a^2\right) - \frac{0{,}601\,569}{m}\cdot\frac{a\,\delta\,r^2}{I} - \frac{1{,}27\,324}{m}\cdot\frac{a\,\delta\,r^2}{I}\cdot\frac{q_e}{g r}\right\} = 0 \end{aligned} \tag{10}$$

and:

$$\left\{15{,}975210-4{,}815905\lambda^2+0{,}030534\frac{\delta r^3}{I}\right\}\sigma_s-\left\{25{,}819780-11{,}751419\lambda^2-\right.$$
$$\left.-\frac{a\delta r^2}{I}-0{,}073433\cdot\frac{\delta r^3}{I}\right\}t_s+1{,}227566\lambda^4\cdot p_0+\left\{2{,}538784\frac{a\delta r^2}{I}+\right.$$
$$\left.+0{,}379795\frac{\delta r^3}{I}\right\}P_{11}+\left\{\frac{2{,}54648}{m}\cdot(1+2\lambda^2+\tfrac{1}{2}\lambda^4)+\frac{1{,}476933}{m}\cdot\frac{a\delta r^2}{I}-\right. \tag{11}$$
$$\left.-\frac{0{,}601569}{m}\cdot\frac{\delta r^3}{I}-\frac{1{,}27324}{m}\cdot\frac{\delta r^3}{I}\cdot\frac{q_e}{gr}\right\}=0.$$

Herewith, practically everything is known and after eliminating P_{11} the unknown quantities σ_s, t_s, p_0 and M_0 can be resolved by numerical substitution with the aid of the four given equations (4), (5), (10) and (11).

Another method of working out, however, is possible. From (4) and (5) p_0 and $\frac{M_0}{r^2}$ can be expressed in σ_s, t_s and P_{11}.

We find that:

$$p_0=-0{,}071300059\,\sigma_s-0{,}147095163\,t_s-0{,}825844962\,P_{11} \tag{12}$$

and:

$$\frac{M_0}{r^2}=-0{,}000897115\,\sigma_s-0{,}000694573\,t_s-0{,}007493428\,P_{11}. \tag{13}$$

Substitution of (12) and (13) in (1), and a further application of (3), renders it possible to express P_{11} in σ_s and t_s. We find that:

$$P_{11}=-\left[\frac{10{,}519\cdot10^{-6}}{226{,}2218\,\Phi_{11}+83{,}847\cdot10^{-6}}\right]\sigma_s-\left[\frac{6{,}974\cdot10^{-6}}{226{,}2218\,\Phi_{11}+83{,}847\cdot10^{-6}}\right]t_s. \tag{14}$$

After substituting (12) in (11), and (14) in (10) and (11), σ_s and t_s can be expressed simply and explicitly in numerical and edge beam magnitudes. Thus we obtain the following table which is generally valid for symmetrical shells.

*

Table for $\varphi_0=0{,}61872$ valid for symmetrical barrel vault shells

$$\lambda=3{,}14159\cdot m\cdot\frac{r}{l}$$

$$\Phi_{11}=\frac{1}{1168{,}909\,m^4}\cdot\frac{l^4}{r^4}\cdot\frac{\delta^2}{r^2}(1+0{,}310297696\lambda^2+0{,}024071165\lambda^4)$$

$$R_0=\frac{10{,}519\cdot10^{-6}}{226{,}222\,\Phi_{11}+83{,}847\cdot10^{-6}}\qquad S_0=\frac{6{,}974\cdot10^{-6}}{226{,}222\,\Phi_{11}+83{,}847\cdot10^{-6}}$$

$$H_0=2{,}538784\cdot\frac{\delta r}{I}\left(\frac{I}{F}+a^2\right)+0{,}379795\cdot\frac{a\delta r^2}{I}$$

$$K_0 = 2{,}538\,784\cdot\frac{a\,\delta\,r^2}{I}+0{,}379\,795\frac{\delta\,r^3}{I}-1{,}013\,779\,\lambda^4$$

$$L_0 = 0{,}601\,569+1{,}27\,324\frac{q_e}{g\,r}$$

$$R_1 = 1{,}961\,629-0{,}030\,534\frac{a\,\delta\,r^2}{I}+R_0\cdot H_0$$

$$S_1 = 4{,}786\,703+\frac{\delta\,r}{I}\left(\frac{I}{F}+a^2\right)+0{,}073\,433\cdot\frac{a\,\delta\,r^2}{I}+S_0\cdot H_0$$

$$R_2 = 15{,}975\,210-4{,}816\,059\,\lambda^2-0{,}087\,526\,\lambda^4+0{,}030\,534\cdot\frac{\delta\,r^3}{I}-R_0\cdot K_0$$

$$S_2 = 25{,}819\,780-11{,}751\,419\,\lambda^2-0{,}180\,569\,\lambda^4-0{,}073\,433\cdot\frac{\delta\,r^3}{I}-\frac{a\,\delta\,r^2}{I}-S_0\cdot K_0$$

$$L_1 = \frac{1}{m}\left\{2{,}074\,414+1{,}476\,933\frac{\delta\,r}{I}\left(\frac{I}{F}+a^2\right)-\frac{a\,\delta\,r^2}{I}\cdot L_0\right\}$$

$$L_2 = \frac{1}{m}\left\{2{,}54\,648\,(1+2\,\lambda^2+\tfrac{1}{2}\,\lambda^4)+1{,}476\,933\cdot\frac{a\,\delta\,r^2}{I}-\frac{\delta\,r^3}{I}\cdot L_0\right\}$$

Then:

$$\sigma_s = -\frac{L_2\,S_1+L_1\,S_2}{R_2\,S_1-R_1\,S_2} \quad \text{and} \quad t_s = -\frac{L_2\,R_1+L_1\,R_2}{R_2\,S_1-R_1\,S_2}$$

$$P_{11} = -R_0\cdot\sigma_s-S_0\cdot t_s$$

$$p_0 = -0{,}071\,300\,\sigma_s-0{,}147\,095\,t_s-0{,}825\,845\,P_{11}$$

$$\frac{M_0}{r^2} = -0{,}000\,897\,\sigma_s-0{,}000\,695\,t_s-0{,}007\,493\,P_{11}$$

$\varphi=0$	$\sigma_x =$	$\frac{g\,l^2}{m^2\pi^2 r\,\delta}$	$[-1{,}000\,\sigma_s$	$-0{,}000\,t_s$	$-6{,}445\,P_{11}$	$-2{,}546\cdot\frac{1}{m}]$
$^1/_6\,\varphi_0$	,, =	,,	$[-0{,}915$,,	$-0{,}137$,,	$-6{,}226$,,	$-2{,}533\cdot$,,]
$^2/_6\,\varphi_0$	,, =	,,	$[-0{,}662$,,	$-0{,}547$,,	$-5{,}581$,,	$-2{,}493\cdot$,,]
$^3/_6\,\varphi_0$	,, =	,,	$[-0{,}241$,,	$-1{,}225$,,	$-4{,}557$,,	$-2{,}426\cdot$,,]
$^4/_6\,\varphi_0$	,, =	,,	$[+0{,}340$,,	$-2{,}165$,,	$-3{,}223$,,	$-2{,}333\cdot$,,]
$^5/_6\,\varphi_0$	,, =	,,	$[+1{,}076$,,	$-3{,}357$,,	$-1{,}669$,,	$-2{,}215\cdot$,,]
φ_0	,, =	,,	$[+1{,}962$,,	$-4{,}786$,,	$-0{,}000$,,	$-2{,}074\cdot$,,]

$\varphi=0$	$\tau_{xy} =$	$\frac{g\,l}{m\,\pi\,\delta}$	$[+0{,}000\,\sigma_s$	$+0{,}000\,t_s$	$+0{,}000\,P_{11}$	$+0{,}000\cdot\frac{1}{m}]$
$^1/_6\,\varphi_0$	,, =	,,	$[+0{,}100$,,	$+0{,}005$,,	$+0{,}658$,,	$+0{,}262\cdot$,,]
$^2/_6\,\varphi_0$	,, =	,,	$[+0{,}183$,,	$+0{,}038$,,	$+1{,}269$,,	$+0{,}521\cdot$,,]
$^3/_6\,\varphi_0$	,, =	,,	$[+0{,}231$,,	$+0{,}127$,,	$+1{,}795$,,	$+0{,}775\cdot$,,]
$^4/_6\,\varphi_0$	,, =	,,	$[+0{,}228$,,	$+0{,}299$,,	$+2{,}199$,,	$+1{,}021\cdot$,,]
$^5/_6\,\varphi_0$	,, =	,,	$[+0{,}155$,,	$+0{,}582$,,	$+2{,}452$,,	$+1{,}256\cdot$,,]
φ_0	,, =	,,	$[+0{,}000$,,	$+1{,}000$,,	$+2{,}539$,,	$+1{,}477\cdot$,,]

$\varphi=0$	$\sigma_y =$	$\frac{gr}{\delta}$	$\left[-0{,}095\,\sigma_s - 0{,}156\,t_s - 1{,}000\;p_0\; - 1{,}000\,P_{11} - 1{,}273\cdot\frac{1}{m}\right]$
$^1/_6\,\varphi_0$	,, =	,,	$[-0{,}090$,, $-0{,}155$,, $-1{,}000$,, $-0{,}966$,, $-1{,}267\cdot$,,$]$
$^2/_6\,\varphi_0$	,, =	,,	$[-0{,}075$,, $-0{,}153$,, $-1{,}000$,, $-0{,}866$,, $-1{,}247\cdot$,,$]$
$^3/_6\,\varphi_0$	,, =	,,	$[-0{,}053$,, $-0{,}145$,, $-1{,}000$,, $-0{,}707$,, $-1{,}213\cdot$,,$]$
$^4/_6\,\varphi_0$	,, =	,,	$[-0{,}029$,, $-0{,}125$,, $-1{,}000$,, $-0{,}500$,, $-1{,}167\cdot$,,$]$
$^5/_6\,\varphi_0$	,, =	,,	$[-0{,}003$,, $-0{,}091$,, $-1{,}000$,, $-0{,}259$,, $-1{,}108\cdot$,,$]$
φ_0	,, =	,,	$[-0{,}000$,, $-0{,}000$,, $-1{,}000$,, $-0{,}000$,, $-1{,}037\cdot$,,$]$

$\varphi=0$	$M_\varphi =$	$g r^2\cdot 10^{-6}$	$[-15454\,\sigma_s - 33465\,t_s - 227566\,p_0 - 183641\,P_{11}] + g\cdot M_0$
$^1/_6\,\varphi_0$	,, =	,,	$[-14806$,, -32569 ,, -221048 ,, -177397 ,, $]+$,,
$^2/_6\,\varphi_0$	,, =	,,	$[-13139$,, -29538 ,, -201554 ,, -159033 ,, $]+$,,
$^3/_6\,\varphi_0$	,, =	,,	$[-10566$,, -24522 ,, -169294 ,, -129834 ,, $]+$,,
$^4/_6\,\varphi_0$	,, =	,,	$[-7345$,, -17662 ,, -124610 ,, -91820 ,, $]+$,,
$^5/_6\,\varphi_0$	,, =	,,	$[-3648$,, -9453 ,, -67983 ,, -47563 ,, $]+$,,
φ_0	,, =	,,	$[\;0$,, -0 ,, -0 ,, -0 ,, $]+$,,

Calculation

$r = 1000$ cm $\qquad I = 199\cdot 10^5$ cm^4

$l = 4100$ cm $\qquad F = 3960$ cm^2

$\delta = 6$ cm $\qquad a = 154$ cm

$g = 0{,}0253$ kg/cm^2 $\qquad q_e = 6{,}66$ kg/cm^1

$m = 1$

dus:
$$\left\{\begin{aligned} \frac{q_e}{gr} &= 0{,}263\,241 \\ \frac{\delta r}{I}\left(\frac{I}{F}+a^2\right) &= 8{,}665\,704 \\ \frac{a\,\delta\,r^2}{I} &= 46{,}432\,161 \\ \frac{\delta\,r^3}{I} &= 301{,}507\,536 \end{aligned}\right.$$

With this, the following can be computed:

$\lambda = 0{,}766241$

$\Phi_{11} = 10{,}360482\cdot 10^{-6} \qquad \lambda^2 = 0{,}587126 \qquad \lambda^4 = 0{,}344717.$

Hence: $R_0 = 0{,}004333 \qquad S_0 = 0{,}002873$

$H_0 = 39{,}635055 \qquad K_0 = 232{,}042815 \qquad L_0 = 0{,}936738$

and: $R_1 = 0{,}715612 \qquad S_1 = 16{,}975924 \qquad L_1 = -28{,}621692$

$R_2 = +21{,}318171 \qquad S_2 = -50{,}381741 \qquad L_2 = -210{,}427257$

So: $\sigma_s = 5{,}352910 \qquad t_s = 1{,}911661 \qquad P_{11} = -0{,}028687$

$p_0 = -0{,}639168 \qquad M_0 = -5916$

and also the course of stresses and moments:

$$\frac{g\,l^2}{m^2\pi^2 r\,\delta} = 7{,}18187 \qquad g r^2 = 25300$$

$$\frac{g\,l}{m\,\pi\,\delta} = 5{,}50305 \qquad \frac{gr}{\delta} = 4{,}217$$

		$\sigma_x =$
$\varphi = 0$	$-\ \ 5{,}352 - 0{,}000 + 0{,}185 - 2{,}546 = -7{,}713$	$-55{,}4$
$^1/_6\,\varphi_0$	$-\ \ 4{,}898 - 0{,}262 + 0{,}179 - 2{,}533 = -7{,}514$	$-53{,}9$
$^2/_6\,\varphi_0$	$-\ \ 3{,}544 - 1{,}046 + 0{,}160 - 2{,}493 = -6{,}923$	$-49{,}7$
$^3/_6\,\varphi_0$	$-\ \ 1{,}290 - 2{,}342 + 0{,}131 - 2{,}426 = -5{,}927$	$-42{,}6$
$^4/_6\,\varphi_0$	$+\ \ 1{,}830 - 4{,}139 + 0{,}092 - 2{,}333 = -4{,}560$	$-32{,}7$
$^5/_6\,\varphi_0$	$+\ \ 5{,}760 - 6{,}417 + 0{,}048 - 2{,}215 = -2{,}824$	$-20{,}3$
φ_0	$+10{,}502 - 9{,}149 + 0{,}000 - 2{,}074 = -0{,}721$	$-\ \ 5{,}2$

		$\tau_{xy} =$
$\varphi = 0$	$+0{,}000 + 0{,}000 - 0{,}000 + 0{,}000 = \ \ 0{,}000$	$+\ \ 0{,}0$
$^1/_6\,\varphi_0$	$+0{,}535 + 0{,}010 - 0{,}019 + 0{,}262 = +0{,}788$	$+\ \ 4{,}4$
$^2/_6\,\varphi_0$	$+0{,}980 + 0{,}072 - 0{,}036 + 0{,}521 = +1{,}537$	$+\ \ 8{,}5$
$^3/_6\,\varphi_0$	$+1{,}237 + 0{,}243 - 0{,}051 + 0{,}775 = +2{,}204$	$+12{,}1$
$^4/_6\,\varphi_0$	$+1{,}220 + 0{,}572 - 0{,}063 + 1{,}021 = +2{,}750$	$+15{,}1$
$^5/_6\,\varphi_0$	$+0{,}830 + 1{,}113 - 0{,}070 + 1{,}256 = +3{,}129$	$+17{,}2$
φ_0	$+0{,}000 + 1{,}912 - 0{,}073 + 1{,}477 = +3{,}316$	$+18{,}2$

		$M_\varphi =$
$\varphi = 0$	$-82723 - 63974 + 145453 + 5268 = +4024 - 5916 = -1892$	$-\ \ 48$
$^1/_6\,\varphi_0$	$-79255 - 62261 + 141287 + 5089 = +4860 - 5916 = -1056$	$-\ \ 27$
$^2/_6\,\varphi_0$	$-70332 - 56467 + 128827 + 4562 = +6590 - 5916 = +\ \ 674$	$+\ \ 17$
$^3/_6\,\varphi_0$	$-56559 - 46878 + 108207 + 3725 = +8495 - 5916 = +2579$	$+\ \ 65$
$^4/_6\,\varphi_0$	$-39317 - 33764 + \ \ 79647 + 2634 = +9200 - 5916 = +3284$	$+\ \ 83$
$^5/_6\,\varphi_0$	$-19527 - 18071 + \ \ 43421 + 1364 = +7187 - 5916 = +1271$	$+\ \ 32$
φ_0	$-0 \quad -0 \quad +0 \quad +0 \quad = \quad -5916 = -5916$	-149

		$\sigma_y =$
$\varphi = 0$	$-0{,}508 - 0{,}298 + 0{,}639 + 0{,}028 - 1{,}273 = -1{,}412$	$-6{,}0$
$^1/_6\,\varphi_0$	$-0{,}482 - 0{,}296 + 0{,}639 + 0{,}028 - 1{,}267 = -1{,}378$	$-5{,}8$
$^2/_6\,\varphi_0$	$-0{,}401 - 0{,}292 + 0{,}639 + 0{,}025 - 1{,}247 = -1{,}276$	$-5{,}4$
$^3/_6\,\varphi_0$	$-0{,}284 - 0{,}277 + 0{,}639 + 0{,}020 - 1{,}213 = -1{,}115$	$-4{,}7$
$^4/_6\,\varphi_0$	$-0{,}155 - 0{,}239 + 0{,}639 + 0{,}014 - 1{,}167 = -0{,}908$	$-3{,}8$
$^5/_6\,\varphi_0$	$-0{,}016 - 0{,}174 + 0{,}639 + 0{,}007 - 1{,}108 = -0{,}652$	$-2{,}7$
φ_0	$-0 \quad -0 \quad +0{,}639 + 0 \quad -1{,}037 = -0{,}398$	$-1{,}7$

Note: This theory has more extensively been published in the Dutch language in a publication by the C. U. R. (Committee for Research) together with comprehensive and suitable tables. With the help of these the calculation of cylindrical shells may quickly be accomplished.

Summary

The method described in this paper is entirely different from those usually published in the literature. By means of an artifice, it proved possible to avoid the standard mathematical solution of the fundamental differential equation.

Starting with linear σ_x stresses, a part of the σ_x stresses due to disturbance proved necessary in order to satisfy the fundamental differential equation, governed by the condition $M_i = M_u$. The method is generally applicable and may be carried out to any required degree of accuracy.

It is evident that the formulas derived are particularly suitable for drawing up tables. It is even possible to give detailed results (see the worked example). In the Netherlands, tables are published which are elaborated to such an extent that, for example, it proved possible to reduce the number of end equations from 8 to 4 in the case of a Northlight Shell and from 4 to 2 for symmetrical shell construction. It is obvious that by this method the work involved in the calculation of a shell is considerably reduced.

Résumé

La méthode ici décrite diffère essentiellement de celles qui sont généralement mentionnées: à l'aide d'un artifice, il est possible d'éviter la résolution mathématique habituelle de l'équation différentielle de base. En partant des contraintes linéaires σ_x, un élément de perturbation des contraintes σ_x doit intervenir pour satisfaire l'équation différentielle de base, suivant la condition $M_i = M_u$. Cette méthode est universelle et peut être appliquée avec tout degré voulu de précision. Il apparaît manifestement que les formules dérivées se prêtent fort bien à l'établissement de tableaux. Il est même possible de fournir des résultats explicites (voir exemple de calculs).

Aux Pays-Bas, la publication de tables a été poussée à un point tel qu'il est par exemple possible de réduire le nombre des équations finales de 8 à 4 dans le cas d'une coque Nordlicht et de 4 à 2 dans le cas des coques symétriques. Il est évident que le calcul des coques est ainsi beaucoup simplifié.

Zusammenfassung

Die in diesem Aufsatz beschriebene Methode unterscheidet sich vollständig von der gewöhnlich verwendeten. Durch Anwendung eines Kunstgriffes wird es möglich, die übliche mathematische Lösung der Grunddifferentialgleichung zu vermeiden. Ausgehend von den linearen Spannungen σ_x wird das Störglied der σ_x-Spannungen zur Erfüllung der Grunddifferentialgleichung gebracht, die in der Bedingung $M_i = M_u$ liegt. Die Methode ist generell anwendbar und mit jedem Genauigkeitsgrad durchzuführen; wie gezeigt wird, sind die abgeleiteten Formeln sehr praktisch zur Verarbeitung in Tabellen. Es ist sogar möglich, explizite Formeln anzugeben (siehe Rechnungsbeispiel).

In Holland sind Tabellen veröffentlicht, die soweit ausgearbeitet sind, daß es z. B. möglich ist, die Anzahl der Endgleichungen von 8 auf 4 im Falle einer Nordlicht-Schale zu reduzieren, von 4 auf 2 bei symmetrischen Schalen. Es ist klar, daß auf diese Weise die Berechnung einer Schale stark vereinfacht wird.