Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 12 (1952)

Artikel: Statics of Vierendeel girder

Autor: El-Demirdash, I.A.

DOI: https://doi.org/10.5169/seals-12318

Statics of the Vierendeel Girder

Statik des Vierendeel-Trägers

Statique de la poutre Vierendeel

Prof. Dr. I. A. EL DEMIRDASH, Dr. sc. techn. (E.T.H. Zürich). Professor of Bridges and Theory of Structures, Faculty of Engineering, Fouad I University, Giza

Introduction

The Vierendeel girder, fig. 1a, is a framed structure with a high degree of redundancy. In this respect, it is similar to a truss with rigid joints, fig. 1c. An exact treatment of the problem necessitates the solution of 3 m equations of elasticity, where m is the number of closed panels in the system. Such a procedure involves tedious mathematical calculations and requires a good deal of time. For this reason, the designer is ready to welcome any assumptions, provided that they lead to fairly good results.

In the case of a truss, the general practice is to neglect the effect of rigid connections entirely, and to assume all members to be hinged at their ends, fig. 1d. The structure is thus transformed into a perfect truss, which is stiff enough to carry the external loads. Here, the stiffness of the structure does not require rigid joints.

Obviously, the assumption of a hinged truss renders the solution of the problem very simple. Loads acting at the panel points produce axial forces in the different members. Results thus obtained coincide fairly well with the axial forces produced in a truss with rigid joints, at least for simple triangular

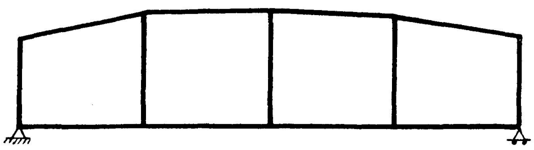


Fig. 1a. Vierendeel Girder

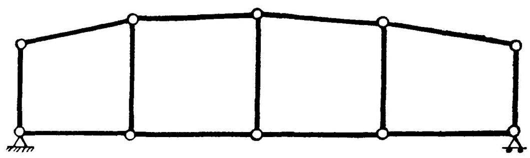


Fig. 1b. Deficient Hinged System

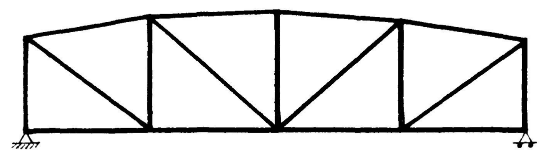


Fig. 1c. Truss with Rigid Joints

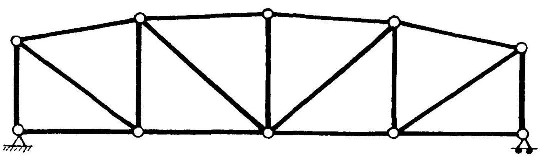


Fig. 1d. Truss with Hinged Joints

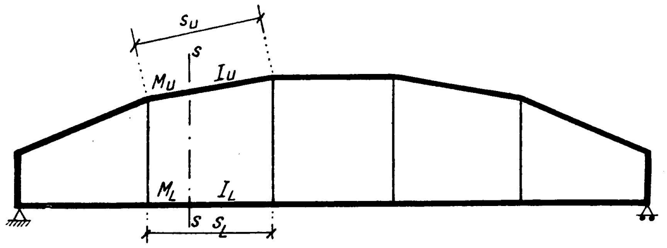
systems. Further, the end moments which exist in the latter cause bending stresses of about $^1/_3$ the normal stresses due to the axial forces. It is understood, that these are accounted for in deciding the working stresses.

In short, the hinged truss can be used, in routine calculations, as a fairly good approximation for the actual case of rigid connections. It gives almost correct axial forces and joint displacements. Consequently, it is suitable also as a main system for an accurate calculation of the rigid truss.

Such an approximation, however, is not possible in the case of a Vierendeel girder. The absence of the diagonal members renders the system with hinged connections, fig. 1b, deficient, and thus incapable of resisting the external loads. Here, rigid joints are necessary for the stiffness of the structure. Further, the end moments produce bending stresses in the different members which cannot be neglected. In this respect, the Vierendeel girder differs from the rigid truss.

Approximate Methods

For the sake of an approximate solution of the Vierendeel girder, the relatively slender verticals may be assumed hinged at their ends to the chord members, fig. 2. In this way, the number of redundants is reduced to $m+2$, where m is the number of closed panels in the system.



M_U and M_L = Bending Moments.
I_U and I_L = Moments of Inertia.
S_U and S_L = Lengths

Fig. 2. System with Hinged Verticals

Further, by neglecting the effect of normal forces on deformations, the vertical displacements of the upper chord joints will be equal to those of the corresponding lower chord joints. If, now, a constant ratio of stiffness $\frac{I_U \cdot S_L}{I_L \cdot S_U} = c$ is maintained between the two chords of the Vierendeel girder, the moments produced in the two chords, under vertical loads acting in the panel points, will bear the same ratio to one another, i. e. $\frac{M_U}{M_L} = c$. Further, by adopting the same chord stiffness throughout, the moments of the upper and lower chords will be equal.

In this way, the problem is so simplified that with vertical loads in the panel points, it can be refered to an once indeterminate system. By introducing a hinge at one end, and a movable bearing at the other, fig. 3, the elastic line due to $H = \pm 1\,t$ gives the influence line of the thrust H, which is the only redundant value.

The internal forces in the upper and lower chords at any vertical section $s-s$ give two equal and opposite forces R_U and R_L which act at a point intermediate between the two chords such that its distance from the two chords is in direct ratio to their relative stiffness, i. e. $\frac{y_U}{y_L}=c$. Consequently, the R-polygon gives the bending moment diagrams of the two chords due to $H = \pm 1\,t$, and hence the required elastic line of the thrust H can be obtained.

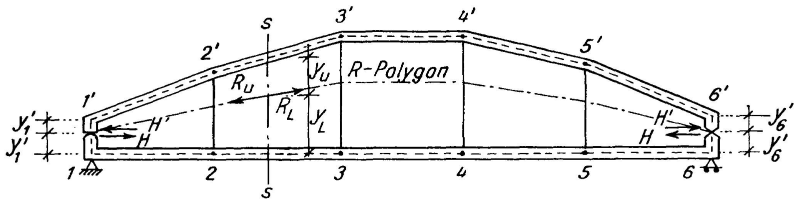


Fig. 3. Case of Loading $H = \pm 1$ on Main System

Another simplification of the problem is made in case of one chord being very stiff, while the other relatively weak. Here the Vierendeel girder is referred to as a tied arch, fig. 4a, or as a bow-string girder, fig. 4b, according to whether the top or bottom chord is stiff. The end connections of the verticals, as well as those of the non-stiff chord, are assumed to be hinged. The two simplified systems are once statically indeterminate.

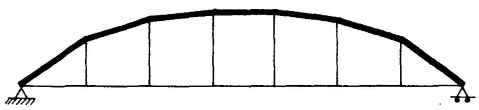


Fig. 4a. Arch with a Tie

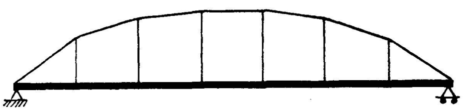


Fig. 4b. Bow String Girder

It is also possible to render the redundant Vierendeel girder statically determinate by introducing hinges in the mid-points of the members, fig. 5. Such a hinged system is easily calculated. It has been accepted as a fair approximation in the calculation of battened compression members, and even in the preliminary design of Vierendeel girders. In the case of equal chord stiffness, the moments obtained by this assumption are not very far from the true values of the indeterminate system. The discrepencies are limited to the

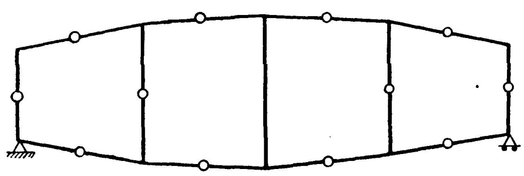


Fig. 5. System with Hinges in the Mid-Points

panels which lie in the vicinity of the loading. Besides, bending moment diagrams of the two systems are more or less similar. This means that the elastic behaviour of the hinged system, as well as its joint displacements do not differ much from those of the actual girder.

Exact Methods

The general rule is to refer the redundant Vierendeel girder to a statically determinate main system. This is usually done by cutting one of the chord members in every panel, fig. 6. The so-formed main system is a simply supported beam. The effect of the redundant values is here limited to one panel only. This is an advantage of the main system. However, the statical behaviour of the Vierendeel girder is far from being similar to that of the simple beam.

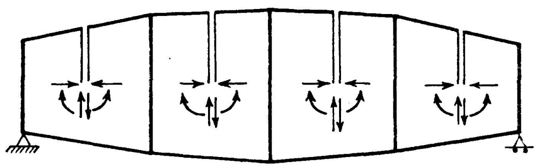


Fig. 6. The Simple Beam as Main System

On the other hand, if hinges are introduced in the mid-points of the different members, fig. 5, a more suitable main system is supplied. This main system behaves more or less similarly to the redundant Vierendeel girder. In the latter, the bending moments produced at the ends of the members are much bigger than those which occur at the mid-points. Consequently, the redundant moments at the introduced hinges will be mere corrections. Their

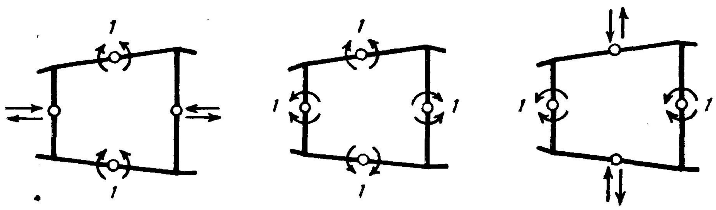


Fig. 7. Moment Groups as Virtual Cases of Loading

values, and subsequently their effect upon the system, will be less than those in other main systems. Further, if suitable groups of moments instead of single couples are introduced as virtual cases of loading, fig. 7, the effect of such groups can be restricted to the individual panels where they occur. In this way, the number of unknowns in each equation of elasticity will be reduced, and the exact solution of the problem is simplified.

In short, the statically determinate main system, which is formed by introducing hinges in the mid-points of the members, is suitable as an approximate method, and also as a main system for an exact calculation of the actual case. As will be seen later, the same hinged system adapts itself very well for the solution of the problem by successive approximations.

Further, in the case of a symmetrical Vierendeel girder, it is advisable to split up the loads into symmetrical and oppositely symmetrical half loadings. In this way, the number of unknowns and equations will be halved. Finally, in most cases, the coefficients which appear in the elastic equations can be simplified by neglecting the effect of the normal forces on the deformations of the structure. For slender members, this effect is relatively small.

Method of Elastic Weights

Similar to a continuous beam, which can be split up into a number of simple spans, the Vierendeel girder may also be divided into successive closed panels. The bending moment diagrams of these closed panels possess certain properties which lead to the determination of certain fixed poles which are similar to the fixed points of the continuous beam.

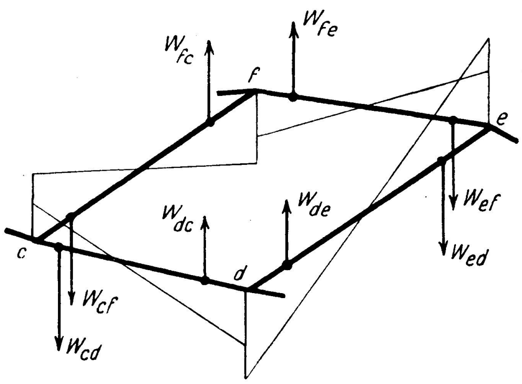


Fig. 8a. Equilibrium of the Elastic Weights in a Closed Panel

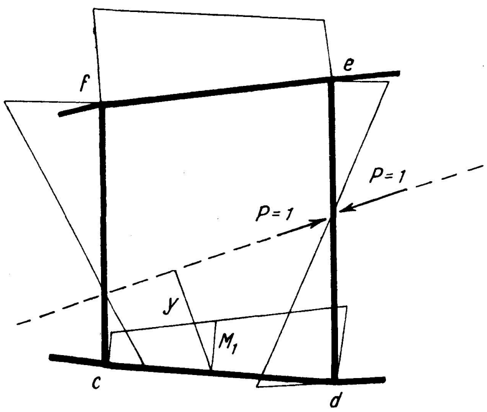


Fig. 8b. The Moment Diagram of the "Free Main System"

Neglecting the effect of normal and shearing forces, elastic weights are given simply by the $\frac{M}{EI}$ diagrams of the different members. Areas and moments of these diagrams give relative slopes and displacements of the structure. It is thus easy to understand why the elastic weights in a closed panel with rigid joints must be in equilibrium, fig. 8a. Here, the elastic weights are assumed to act along the centre lines of the different members, and in a direction perpendicular to the plane of the closed panel. Positive values may be assumed pointing towards the reader, while negative values point away from him.

In order to prove this statement, the closed panel is referred to a statically determinate main system by cutting one of its sides, fig. 8b. Two equal and opposite forces $P = \pm 1$ are applied at the severed ends. This virtual loading represents a system of external loads in equilibrium, which are applied to a just-stiff part of the structure.

Since no relative displacement of the two ends can take place in the indeterminate structure; $\int \frac{M_1 \cdot M \cdot ds}{E I} = 0$, where M_1 = bending moment due to $P = \pm 1$ in the main system, and M = bending moment in the indeterminate structure. But $M_1 = y$ and $\frac{M}{E I} \cdot ds = d W$, hence $\int y \cdot d W = 0$. In other words, statical moment of the elastic weights about the line of action of the virtual loads is zero. This condition is true for any position of the severed ends as well as any direction of the virtual loads P. Hence, elastic weights of the closed panel are in equilibrium.

The fore-going analysis shows that it is not at all necessary to refer the indeterminate structure to one and the same main system throughout the calculation. On the contrary, the main system may be altered by changing the position of any introduced hinges or severed ends. By means of this "free main system", it is possible to limit the integrations to few members, and thus to simplify the equations of elasticity.

Further, in the case of an externally indeterminate problem, supports may be removed altogether, and reactions introduced as external forces. The structure thus becomes a deformed free body with definite boundary conditions. Any stiff part of this elastic body can be used as a "free main system". However, the virtual loads and couples applied to this part of the structure must be in equilibrium. The elastic equations are formed by equating external and internal work in such a way as to satisfy the boundary conditions of the indeterminate structure. In this manner, every equation will contain a few number of redundant terms, and the mathematical computation will be simpler.

The fact that the elastic weights in a closed panel must be in equilibrium can be utilised in solving the Vierendeel girder. First a single closed panel, fig. 9a, is considered. The indeterminate system is then referred to a convenient main system. The bending moments of the closed panel are given by the expression: $M = M_0 + M_r$, where: M_0 = the bending moments in the main system due to the external loads, fig. 9b, and M_r = the bending moments due to the redundant values, fig. 9c. Superposition of the corresponding values should satisfy the equilibrium, as well as the elastic conditions of the closed frame.

Now the M_0-diagram depends on the external loading and on the type of main system. On the other hand, the M_r-diagram has a standard form, fig. 9c. Further, the pairs of equal and opposite redundant values produce no external reactions. The corresponding internal forces at any vertical section will be in equilibrium. In other words, the difference between the upper and lower chord moments produced by the redundant values in a vertical section through the frame will be proportional to the height of the frame at this section. Consequently, the standard shape of the M_r-diagram will depend on the values of the end moments M_1 and M_2, and the difference m_1. These three values represent the 3 unknowns of the indeterminate system.

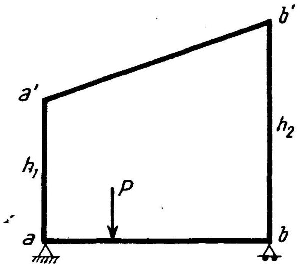


Fig. 9a
Single Closed Panel

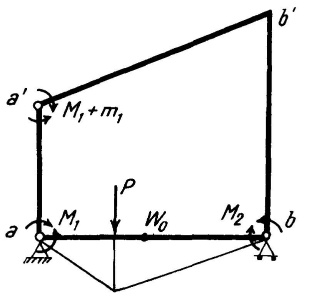


Fig. 9b
M_0-Diagram and W_0-Elastic Weights

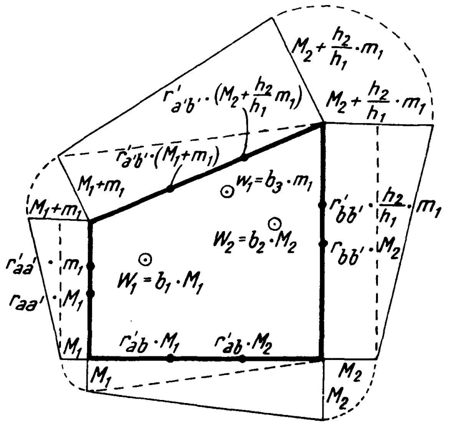


Fig. 9c
M_r-Diagrams and W_r-Elastic Weights

If now the geometrical shape and cross sections of the closed frame are known, the elastic weights $\frac{M}{EI}ds$ acting along the elemental length ds of every member can be expressed as function of M_1, M_2 and m_1. These are combined into the resultant elastic weights $r'_{ab}\cdot M_1$, $r'_{ab}\cdot M_2$, etc. shown in fig. 9c. The coefficients r' and r depend of course on the dimensions of the different members.

Further, all the elastic weights which depend on the 3 unknown moments M_1, M_2, and m_1 can be added together giving the 3 common resultants: $W_1 = b_1\cdot M_1$, $W_2 = b_2\cdot M_2$, and $w_1 = b_3\cdot m_1$. The coefficients b_1, b_2, and b_3 depend entirely on the dimensions of the frame, and not on the external loading. The same applies also to the positions of the 3 resultant elastic weights, which are termed the "elastic poles" of the closed frame.

Since all the joints of the frame are rigid, all the elastic weights must be in equilibrium. In other words, the 3 resultant elastic weights W_1, W_2 and w_1 due to the redundant values should keep the resultant elastic weight W_0, due to external loads, in equilibrium. Consequently, if the frame is considered to be a slab supported at its 3 elastic poles and loaded by the elastic weight W_0, the corresponding reactions will be simply W_1, W_2, and w_1. Hence the unknown moments M_1, M_2, and m_1 are obtained.

A similar procedure can be applied to the Vierendeel girder. It must be remembered, however, that successive panels of the Vierendeel girder have common verticals, so that elastic weights along these verticals will depend on the upper and lower chord moments on either side of each vertical. In this way, the elastic weights in an end panel involve 5, while those in every intermediate panel involve 7 unknown moments. The solution of the problem, however, can be simplified in the following manner.

Starting at the left hand panel I, fig. 10a, its elastic weights must be in equilibrium. Hence, the moments m_2 and M_3 can be expressed in terms of m_1, M_2 and $W_1{}^0$. Going over to panel II, the elastic weights in m_1 and M_2 are expressed in terms of m_2, M_3 and $W_1{}^0$. The condition of equilibrium of all elastic weights in this panel gives m_3 and M_5 as functions of m_2, M_4, $W_1{}^0$ and $W_2{}^0$.

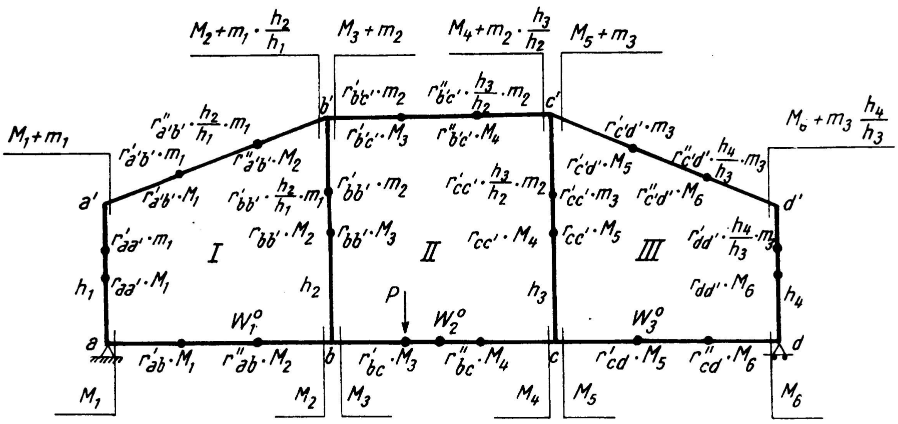


Fig. 10a. The Elastic Weights of the Vierendeel Girder

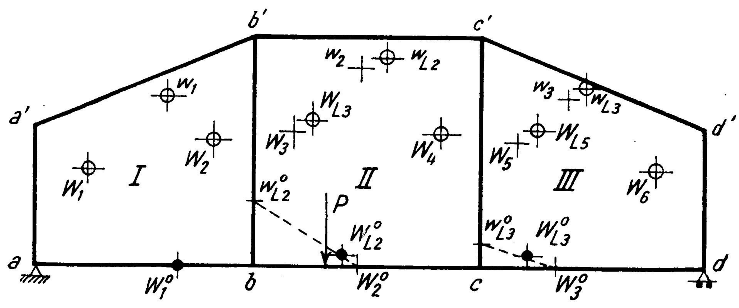


Fig. 10b. The corresponding Elastic Poles

Similarly for panel III, elastic weights in m_2 and M_4 are expressed in terms of m_3, M_5, $W_1{}^0$ and $W_2{}^0$. Equations of equilibrium of all elastic weights in this panel supply the values of the unknown moments m_3, M_5 and M_6. If the steps are retraced back to panel II, and finally to panel I, the rest of the moments can be obtained. This successive solution of the Vierendeel girder from left to right and then back to left replaces the solution of the equations of elasticity as a matrix.

The problem may also be solved by using the elastic poles. These poles are first determined by neglecting the effect of the adjacent panels, fig. 10b. Starting at panel I, it may be considered as a slab supported at its 3 poles w_1, W_1 and W_2. Assuming elastic weights $r'_{bb'} \cdot m_2$ and $r_{bb'} \cdot M_3$ to be external loads, and leaving out for the time being the elastic weight $W_1{}^0$, the relation between m_1, M_2 and m_2, M_3 may be found.

Going over to panel II, and expressing the elastic weights $r'_{bb'} \cdot \frac{h_2}{h_1} \cdot m_1$ and $r_{bb'} \cdot M_2$ in terms of m_2 and M_3, these elastic weights are added to w_2 and W_3 respectively in order to obtain the new elastic poles w_{L2} and W_{L3}. The index L means that the effect of the left hand panel is included. Panel II is then considered as a slab supported at the two shifted poles w_{L2} and W_{L3} and at W_4. Assuming the elastic weights $r'_{cc'} \cdot m_3$ and $r_{cc'} \cdot M_5$ to be external loading, and leaving out the effect of the elastic weight $W_2{}^0$, the relation between m_3, M_5 and m_2, M_4 may be found.

Similarly for panel III, elastic weights $r'_{cc'} \cdot \frac{h_3}{h_2} \cdot m_2$ and $r_{cc'} \cdot M_4$ are expressed in terms of m_3 and M_5. These elastic weights are included in the resultants by shifting w_3 and W_5 to w_{L3} and W_{L5} respectively. Needless to say, leaving out the effect of the elastic weights W^0, the positions of the new elastic poles are made independent of the external loading. These positions are thus valid for all cases.

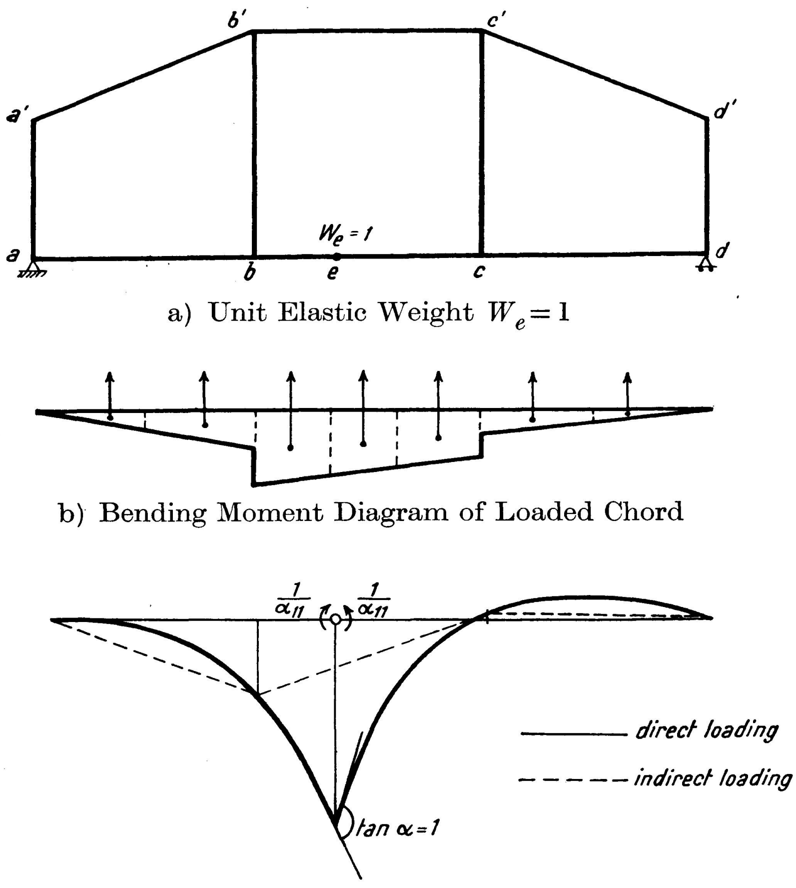
In order to determine the effect of the elastic weights W^0 for a certain case of loading, panel I is again considered first with $W_1{}^0$ acting. The corresponding values of m_1 and M_2 are determined as functions of $W_1{}^0$. Going over to panel II, elastic weights $r'_{bb'} \cdot \frac{h_2}{h_1} \cdot m_1$ and $r_{bb'} \cdot M_2$ are calculated in terms of $W_1{}^0$, and added into a resultant w^0_{L2} on the common vertical bb'. Further, w^0_{L2} and $W_2{}^0$ are combined into a new elastic weight W^0_{L2}.

Panel II is then considered as a slab supported at the poles w_{L2}, W_{L3} and W_4, and loaded by W^0_{L2}. The values of m_2 and M_4 are found in terms of W^0_{L2}. Hence, elastic weights $r'_{cc'} \cdot \frac{h_3}{h_2} \cdot m_2$ and $r_{cc'} \cdot M_4$ are calculated. These are expressed in terms of W^0_{L2} and give together an additional elastic weight w^0_{L3} along the vertical cc'.

Finally, for panel III, w^0_{L3} and $W_3{}^0$ are combined into the resultant elastic weight W^0_{L3}. Hence, m_3, M_5 and M_6 are determined. Returning back to panel II, and introducing the values of m_3 and M_5, m_2, M_3 and M_4 are found. Finally, for panel I, the values of m_2 and M_3 are introduced, and the remaining moments m_1, M_1, and M_2 are computed.

The method of the elastic weights is applicable to direct as well as indirect loading. Besides, the effect of variable moments of inertia and haunched ends, if any, is included in the coefficients r. These coefficients are of course much simpler in cases where the moment of inertia is constant over the whole length of each member. This method is also suited for drawing the influence lines. Referring to fig. 11, the influence line of the moment M_e is simply the elastic line of the loaded chord for a unit elastic weight $W_e = 1$ at e. The structure is assumed to be subject to such a loading as would produce a unit elastic weight at e. The corresponding values of W_r are obtained from the conditions of equilibrium of the elastic weights in every closed panel. They supply bending moment diagrams of the different members. Finally, deflections are found graphically by drawing the funicular polygon of the elastic weights, or analytically by virtual work.

Fig. 11



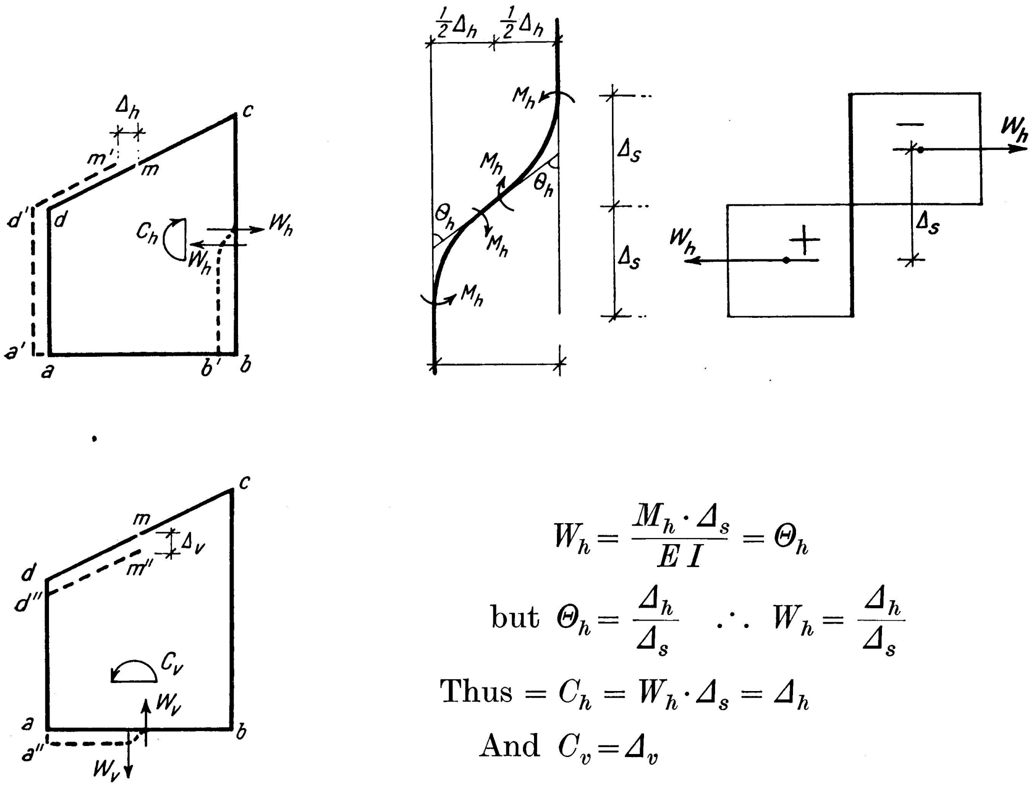
a) Unit Elastic Weight $W_e = 1$

b) Bending Moment Diagram of Loaded Chord

c) Influence Line of M_e

This method complies with the general rule of introducing a hinge at e and applying two equal and opposite couples $M = \pm 1$ on either side of the hinge. The elastic line of the loaded chord due to this virtual loading gives, to the scale of the relative slope α_{11}, the required influence line. In other words, the influence line of M_e can be obtained directly by assuming two virtual couples $M = \pm \frac{1}{\alpha_{11}}$ at the hinge e. Such a loading, however, produces unit relative slope at e, and is thus equivalent to the action of a unit elastic weight $W_e = 1$.

If the influence lines are completed between the panel points by the curved centre lines of the deflected members, the effect of direct loads can be ascertained. On the other hand, if the ordinates under the panel-points are connected by straight lines, the effect of indirect loads only is obtained. It is interesting to remark here that the moments produced by direct and indirect loads are nearly of the same order. Consequently, the effect of joint displacements and rotations must be included in the computation both for direct as well as for indirect loading. In this respect, the Vierendeel girder differs from a truss with rigid joints. The effect of direct loading in the latter case is much more pronounced. The corresponding end moments can be determined by assuming the joints to rotate while in their original positions, i. e. without any displacement. Such an assumption would be wrong for the Vierendeel girder.



$$W_h = \frac{M_h \cdot \Delta_s}{EI} = \Theta_h$$

$$\text{but } \Theta_h = \frac{\Delta_h}{\Delta_s} \quad \therefore \quad W_h = \frac{\Delta_h}{\Delta_s}$$

$$\text{Thus} = C_h = W_h \cdot \Delta_s = \Delta_h$$

$$\text{And } C_v = \Delta_v$$

Fig. 12. The Elastic Couples C_h and C_v

In all the fore-going investigations, the effect of the longitudinal deformations due to normal forces has not been included in the solution of the statically indeterminate rigid frames. These deformations produce no changes of slope in the statically determinate main system, which is formed by cutting the closed panel. The members are displaced parallel to themselves, and a gap is formed at the section where the frame is cut. In order to include the effect of the normal forces in the method of elastic weights and elastic poles, it is necessary to replace these axial forces by another system of loading, which gives the same gap as the normal forces without change of slope, and which is of the same nature as the elastic weights. These conditions are fulfilled by the elastic couples $C_h = \varDelta_h$, and $C_v = \varDelta_v$, fig. 12, where $\varDelta_h$ and $\varDelta_v$ are the horizontal and vertical components of the gap.

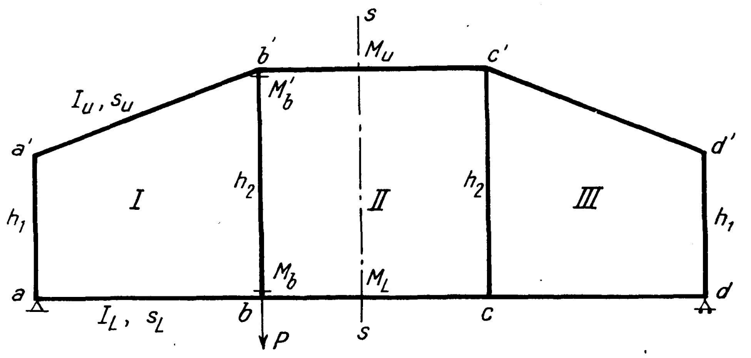
Remembering that in the closed panel the gap and the change of slope due to the bending moments and axial forces are equal to zero, the elastic weights and the introduced elastic couples must be in equilibrium, which condition supplies the redundant moments of the closed panel. The introduced elastic couples are subdivided into separate elastic couples corresponding to the redundant moments and to the external loading respectively. The first group of elastic couples shift the elastic poles horizontally and vertically, while the latter group shift the elastic weight W_0 due to the external loading. The corresponding coefficients remain unaltered. The shifted elastic poles give the new points of support of the closed panel.

The idea involved in the elastic couples can be utilised in determining the influence lines of the shearing forces. For example, the influence line of Q_e, fig. 11, is the elastic line of the loaded chord due to a unit relative displacement at e in the transverse direction without any change in slope. Such a sliding, however, can be produced by a unit elastic couple acting at e. Consequently, the influence line of the shearing force Q_e is simply the elastic line of the loaded chord due to a unit elastic couple at e.

Method of Elastic Couples

This method is applicable to the special case of a Vierendeel girder with equal chord stiffness subject to indirect panel-point loading. Neglecting the effect of normal forces, the deflections of the upper and lower chords will be equal. Consequently, the corresponding moments in both chords will be the same, fig. 13a. Further, judging by the stresses produced in the flanges of each chord, the corresponding elastic weights will be equal and opposite. These can be combined into "elastic couples", which are represented by vectors in the plane of the structure. The verticals give also similar elastic couples.

If, now, the Vierendeel girder is referred to the main system shown in fig. 13b, the moments M^0 due to the external loads will give the elastic couples C^0. Similarly, moments M_r produced by the redundant values will give elastic couples C_r. However, the redundant values being pairs of equal and opposite moments, or forces, produce no reactions. Consequently, corresponding internal forces produced in the upper and lower chords at any vertical section $s-s$ are in equilibrium, fig. 13c. In other words, the chord moments in each panel are proportional to the corresponding height. Thus every panel provides one unknown value and the number of closed panels m gives the degree of redundancy of the system.

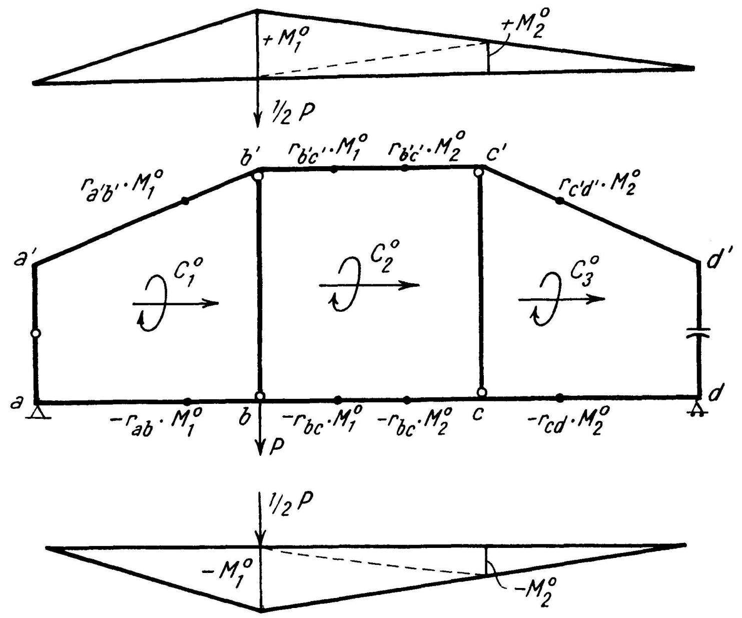


$$\because \quad \frac{I_U}{s_U} = \frac{I_L}{s_L}$$

$$\therefore \quad M_L = -M_U$$

And $M_b = -M_{b'}$

Fig. 13a. Vierendeel Girder of Equal Chord Stiffness



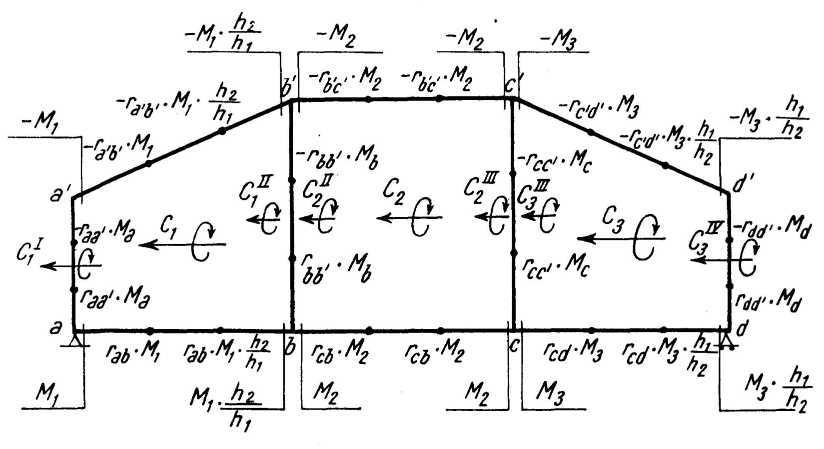
M^0-Diagram of Upper Chord

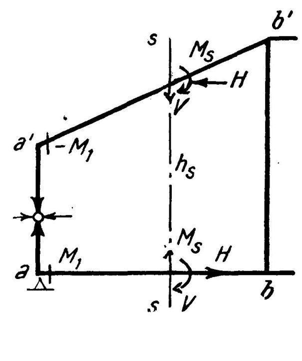
Fig. 13b

Main System and Elastic Couples C^0

M^0-Diagram of Lower Chord

Equilibrium at Section $s-s$





$2\,M_s = H \cdot h_s$

Fig. 13c. Elastic Couples C_r

The equilibrium of the elastic couples in every end panel involves two unknown moments, while that of an intermediate panel involves 3 unknown moments. The solution can be carried out in a progressive manner from panel to panel. Starting, for example, at panel I, the equilibrium of the elastic couples supplies a relation between the two moments M_1 and M_2. This relation can be utilized in eliminating M_1 from the elastic couples of panel II. In this way, the equilibrium of the elastic weights in panel II supplies a relation between M_2 and M_3.

Finally, for panel III, M_2 is first eliminated. All the elastic couples in this panel are then functions of M_3, which can be determined. The relations between the different moments hitherto obtained enable the calculation of the remaining values. The introduction of the term "elastic couple" reduces the equilibrium of the elastic weights to a simple algebraic summation of the elastic couples in each panel.

In order to render the solution independent of the different cases of external loading, the effect of the elastic couples C^0 is first neglected. The relations obtained in this way between M_1, M_2, and M_3 will be valid for all cases of loading. For a certain load, however, panel I is again considered, and the effect of the next panel is neglected. M_1 is determined as function of $C_1{}^0$, and introduced in the couple C_1^{II} of the vertical bb'. The new value of C_1^{II} is then added to the elastic couple $C_2{}^0$ giving C^0_{L2}. Equilibrium of the elastic couples in panel II is then considered, leaving out the effect of panel III. All elastic couples are now functions of M_2 which can be determined from C^0_{L2}. This value is used in determining the couple C_2^{III} of the vertical cc'. The new value of this couple is added to $C_3{}^0$ to get C^0_{L3}. M_3 is found from equilibrium of the elastic couples in panel III. Going back to panel II, M_2 is determined. Finally, M_1 is determined from panel I. The progressive method just explained is in a way similar to the elimination method of Gauss which is generally used in solving the matrix of the elastic equations.

Method of Successive Approximations

In order to avoid the solution of the complicated elastic equations of a highly indeterminate structure, iteration methods are used. The general rule is to simplify the original system by making a suitable assumption of hinges, certain end conditions, approximate joint displacements, etc. The corresponding forces and moments are then determined. They are of course not the same as those of the original structure. Nevertheless, they are used in correcting the first assumption, and the whole calculation is repeated for the new conditions. This second step is followed by a third, and so on until the required degree of accuracy is obtained.

The success of iteration depends on its convergency. Results of successive approximations should gradually approach the correct values. The number of steps needed for the solution differs according to the nature of the problem, and depends on the choice of a suitable assumption. The first obtained values should be real approximations, which are not very far from the accurate results. In this way, successive corrections tend to vanish. Otherwise, the computation will take a long time, involving many steps and corrections, and may not lead to the required results.

In the case of a truss with rigid connections, the joint displacements are first assumed to be the same as those of a hinged system. The end moments are then calculated, either directly from the joint rotations by Mohr's method, or successively by relaxation methods. In the latter case, the members are assumed to be fixed at their ends and then relaxed one by one until finally all the system is eased. Either the moments themselves, or rotations of the joints are successively corrected. The end moments obtained by the first approximation can be used in correcting the displacements of the joints which have been assumed at first. The calculations are then repeated for the new values. and a second approximation of the end moments obtained. The whole process may be continued until no further corrections are needed. This condition, however, is not always fulfilled, specially for a complicated system of triangulations.

Unfortunately, the Vierendeel girder cannot be treated in the same way. It is impossible to assume all members to be hinged at their ends. However, a girder with equal chord stiffness can be solved successfully by the "Panel Method".

The Panel Method

The idea involved in this method is to split up the Vierendeel girder into single closed panels, fig. 14a, and to consider the equilibrium of each panel separately. Assuming every panel to be hinged at both sides to the rest of the structure, the effect of external loads gives the so-called primary moments, fig. 14b. The connecting moments at the introduced hinges just outside the four corners produce secondary moments in the closed panel, fig. 14c. The sums of the primary and secondary moments give the required moments of the Vierendeel girder.

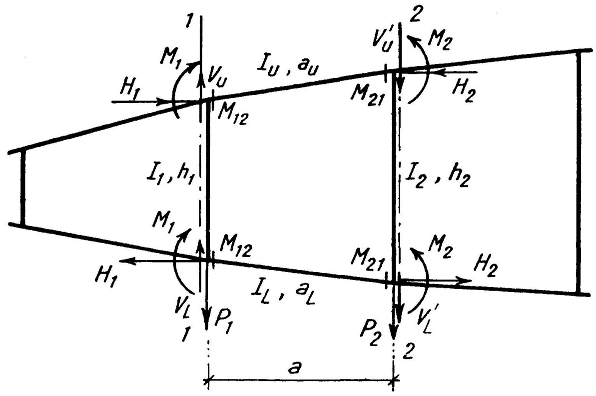
In the special case of a Vierendeel girder with equal chord stiffness, the corresponding moments in the upper and lower chords will be equal. It is, therefore, possible to derive a simple expression for the primary and secondary moments which can be applied to every closed panel in the system. Referring to figs. 14b and 14c:

$$M'_{12} = \frac{\alpha M - V a}{2D} [3 + s + \alpha(2 + s)] \qquad M''_{12} = \frac{r}{D} \cdot M_1 - \frac{s(1+\alpha)}{D} \cdot M_2$$

$$M'_{21} = \frac{\alpha M - V a}{2D} \cdot (3 + r + \alpha) \qquad M''_{21} = -\frac{r(1+\alpha)}{D} \cdot M_1 + \frac{s(1+\alpha)^2}{D} \cdot M_2$$

$$\text{Hence} \quad M_{12} = M'_{12} + M''_{12} \quad \text{and} \quad M_{21} = M'_{21} + M''_{21}$$

Fig. 14a. Equilibrium of Panel 1—2



$$k_1 = \frac{I_1}{h_1}, \qquad k_2 = \frac{I_2}{h_2}, \qquad k = \frac{I_U}{a_U} = \frac{I_L}{a_L}$$

$$r = \frac{k}{k_1}, \qquad s = \frac{k}{k_2}, \qquad \alpha = \frac{h_2 - h_1}{h_1}$$

$$D = 6 + r + s + \alpha\,(2\,\alpha + \alpha\, s + 2\, s + 6)$$

$$V_1 = V_U + V_L, \qquad V = V_1 - P_1, \qquad H_1 = \frac{M - 2\,M_1}{h_1}$$

Fig. 14b. Primary Moments

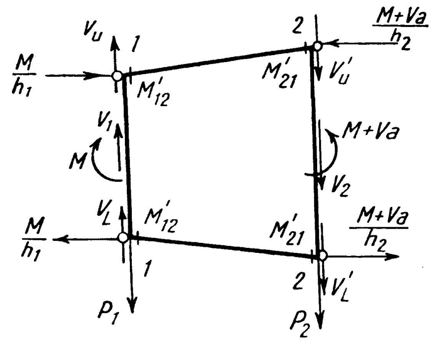
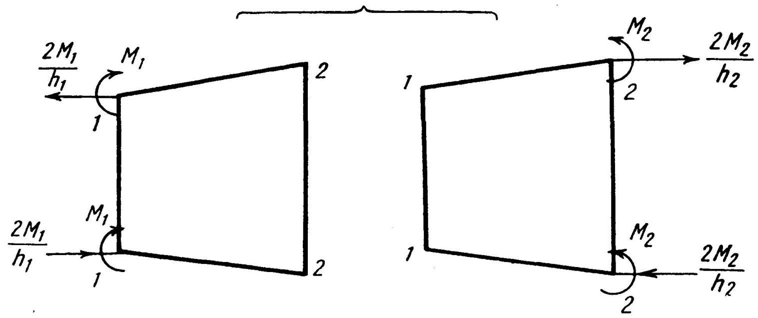


Fig. 14c. Secondary Moments



At first, the primary moments are determined for the whole girder. They are considered to be first approximations of the actual end moments. Consequently, the secondary moments in each panel can be calculated. The sums of corresponding values give a second approximation of the end moments, which can be used in correcting the secondary moments obtained before. In this way a third approximation of the end moments is obtained, and so on. The process is continued until it converges.

Modification of the "Panel Method"

The panel method just explained assumes every closed panel to be hinged at its four corners to the rest of the structure. Unfortunately, the bending moments of the Vierendeel girder are maximum at these points, and big values of connecting moments are expected. Moreover, the so-called secondary moments will not be small compared with the primary moments, being sometimes almost of the same order. This explains why several steps are needed to bring the successive approximations to an end.

In order to simplify calculations, and to reduce the number of corrections, it is necessary to adapt the assumption to the real behaviour of the structure. This can be done by introducing the hinges at the mid-points of the different members where the bending moments are small. In this way, every element is considered to be built up of a closed panel with overhanging arms, which extend to the middle of the adjacent panels, fig. 15a. The secondary moments produced by the connecting moments, fig. 15c, are very small compared with the primary moments due to the external loading, fig. 15b. They are real corrections of small magnitude. The process converges after very few successive approximations.

In determining the values of the primary moments, the forces acting at the ends of the overhanging arms, fig. 15b, are shifted parallel to themselves to sections $1-1$ and $2-2$ respectively. Four couples are introduced at the 4 corners of the closed frame. Owing to the fact that the Vierendeel girder has in this case equal chord stiffness, the additional couples at the corresponding joints of the upper and lower chords will be the same. They are represented by M_1' and M_2' respectively. Thus:

$$M = M_m + V_1 \cdot \frac{a}{2} = \frac{M_m}{h_m} \cdot h_1 + 2\,M_1'$$

and
$$M' = M_n - V_2 \cdot \frac{a}{2} = \frac{M_n}{h_n} \cdot h_2 + 2\,M_2'$$

or
$$M_1' = \frac{M_m\,(h_m - h_1)}{2\,h_m} + V_1 \cdot \frac{a}{4} = \frac{h_1}{2}\left(\frac{M}{h_1} - \frac{M_m}{h_m}\right)$$

and
$$M_2' = \frac{M_n\,(h_n - h_2)}{2\,h_n} - V_2 \cdot \frac{a}{4} = \frac{h_2}{2}\left(\frac{M'}{h_2} - \frac{M_n}{h_n}\right)$$

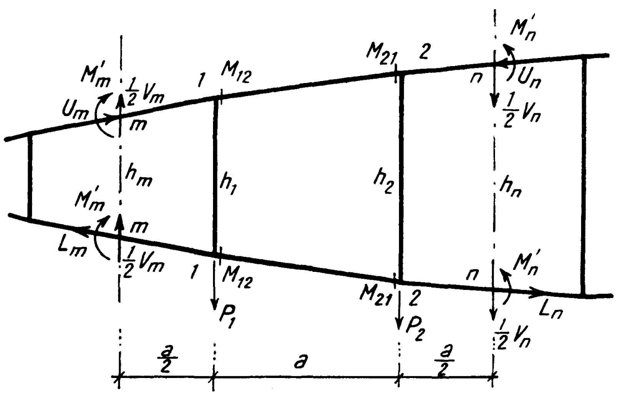


Fig. 15a. Equilibrium of Element $m-n$

Fig. 15b. Primary Moments

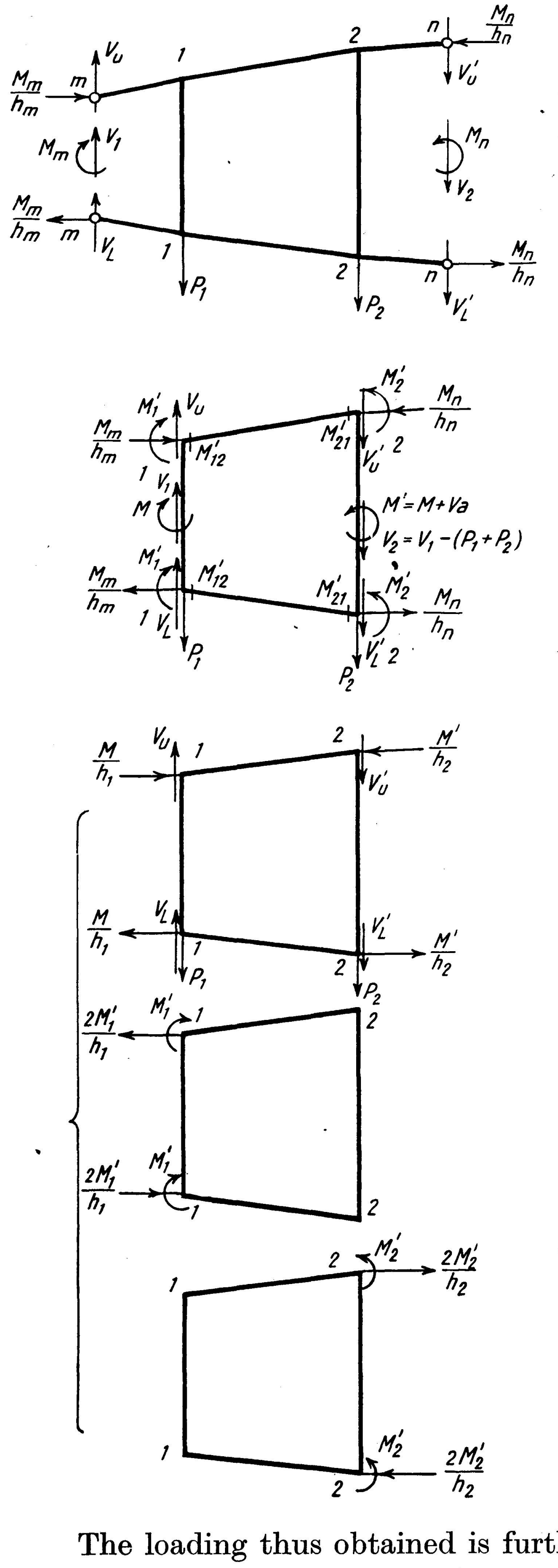
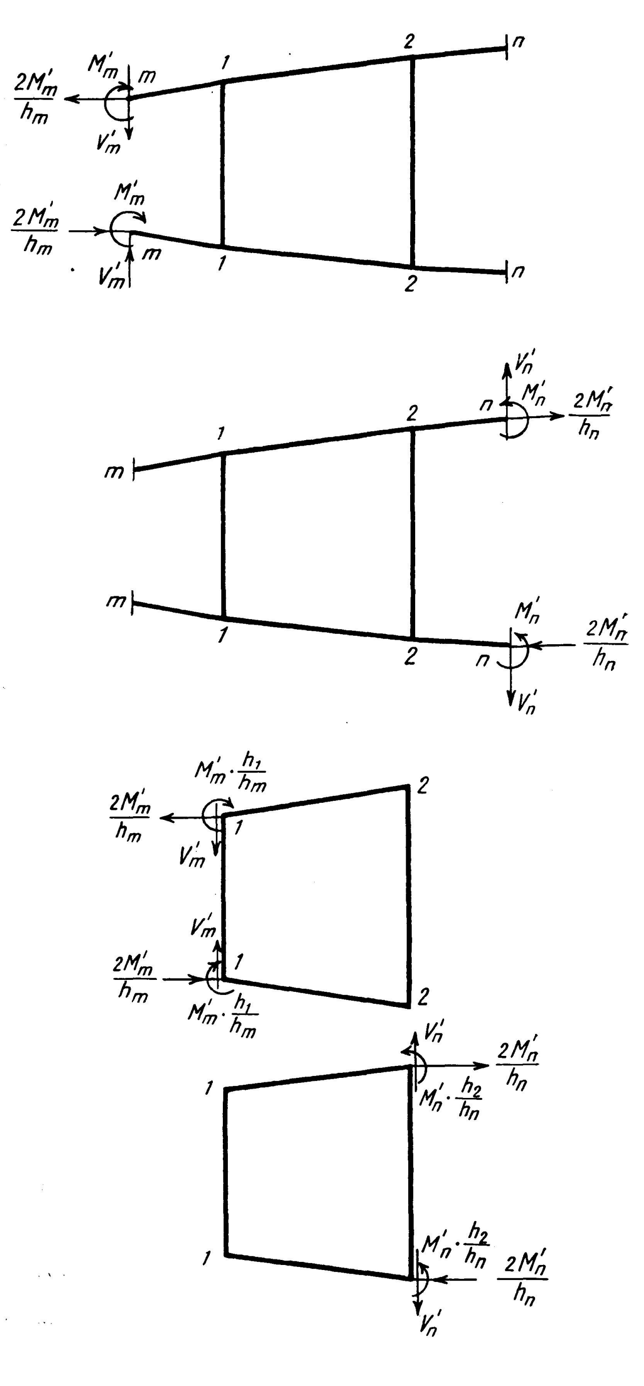


Fig. 15c. Secondary Moments



The loading thus obtained is further split up into cases of partial loading similar to those of fig. 14. In this way, it is possible to use the relations found before in determining the new primary moments M'_{12} and M'_{21}. Thus:

$$M'_{12} = \frac{\alpha M - Va}{2D}[3+s+\alpha(2+s)] + \frac{r}{D}\cdot M_1' - \frac{s(1+\alpha)}{D}\cdot M_2',$$

and $$M'_{21} = \frac{\alpha M - Va}{2D}[3+r+\alpha] - \frac{r(1+\alpha)}{D}\cdot M_1' + \frac{s(1+\alpha)^2}{D}\cdot M_2',$$

or $$M'_{12} = \frac{\alpha(M_m + \frac{1}{2}\cdot V_1\cdot a) - V\cdot a}{2D}[3+s+\alpha(2+s] + \frac{r}{D}\cdot\left[\frac{M_m}{2h_m}(h_m-h_1) + V_1\cdot\frac{a}{4}\right] - \frac{s(1+\alpha)}{D}\cdot\left[\frac{M_n(h_n-h_2)}{2h_n} - V_2\cdot\frac{a}{4}\right]$$

and $$M'_{21} = \frac{\alpha(M_m + \frac{1}{2}V_1\cdot a) - V\cdot a}{2D}[3+r+\alpha] - \frac{r(1+\alpha)}{D}\cdot\left[\frac{M_m}{2h_m}(h_m-h_1) + V_1\cdot\frac{a}{4}\right] + \frac{s(1+\alpha)^2}{D}\cdot\left[\frac{M_n}{2h_n}(h_n-h_2) - V_2\cdot\frac{a}{4}\right]$$

The other cases of secondary moments are treated in a similar manner. They supply the new secondary moments M''_{12} and M''_{21}, thus:

$$M''_{12} = \frac{r}{D}\cdot\frac{h_1}{h_m}\cdot M'_m - \frac{s(1+\alpha)}{D}\cdot\frac{h_2}{h_n}\cdot M'_n$$

$$M''_{21} = -\frac{r(1+\alpha)}{D}\cdot\frac{h_1}{h_m}\cdot M'_m + \frac{s(1+\alpha)^2}{D}\cdot\frac{h_2}{h_n}\cdot M'_n$$

Finally: $M_{12} = M'_{12} + M''_{12}$, and $M_{21} = M'_{21} + M''_{21}$.

The steps followed in the successive solution are the same as for the panel method. The primary moments, which serve as first approximation of the actual moments, are first determined. The secondary moments are then calculated, and a second approximation of the actual moments is obtained. There is generally no need for a further correction.

Needless to say that the panel method explaind before as well as its modification are suitable only for indirect loading and equal chord stiffness of the Vierendeel girder. If, however, the loads are applied directly between the panel points, the deformations of the loaded chord will not be followed by the other chord. Further, for different chord stiffness, the moments of the upper and lower chords will not be equal, rendering the calculations more difficult.

However, in the case of a Vierendeel girder with a constant ratio between the upper and lower chord stiffness up to say 1.3, the panel method may be used as a fair approximation. The two unequal chords are assumed to have an equal average stiffness. The approximate moments obtained in this way lie between the two unequal real values of the upper and lower chords. Consequently, the normal forces found by the approximate method will be almost equal to the real values. Remembering, now, that the stresses are due partly to the bending moments, and partly to the axial forces, the error involved in the total stresses found approximately will be less. This proves that the proposed method of average chord stiffness may be used as a practical approximation for differences of 30 and even 50% between the stiffness of the two chords.

Conclusion

The statics of the Vierendeel girder should no more be considered as an obstacle in the way of adopting this type in competitive structural work. The exact calculations can be often replaced by successive approximations. Even if such a procedure seems to be complicated, the designer can resort to fairly good approximate methods by making appropriate assumptions, or by experimental and semi-experimental methods.

Specially in the case of steel, the construction of the joints in the Vierendeel girder is simplified by welding. There is also no need of having plate girder sections, the members of the Vierendeel girder may be of the open web or truss form. Finally, the statics of the Vierendeel girder may be applied in the design of battened compression members, framed buildings, and in similar structures.

Bibliography

1. H. Schwyzer, "Bow String Girders", Civil Engineering Society, Faculty of Engineering, Fouad I University, Giza, Year Book, 1935.
2. L. C. Maugh, "Stresses and Deformations in two-hinged Vierendeel Truss Arches", Proceedings of the fifth International Congress of Applied Mechanics, 1938.
3. M. Megahid, "The Free Main System", Ph. D. thesis, Faculty of Engineering, Fouad I University, Giza, 1948.
4. F. Stüssi, „Zur Berechnung des Vierendeelträgers", Internationale Vereinigung für Brückenbau und Hochbau, Abhandlungen, 1950.
5. A. E. Shaâban, "Stress Analysis of a Vierendeel Girder", M. sc. thesis, Faculty of Engineering, Fouad I University, Giza, 1951.

Summary

The Vierendeel girder is a highly indeterminate system whose exact calculation necessitates the solution of a big number of elastic equations. However, if the verticals are assumed to be hinged at their ends the number of redundants is heavily reduced. Further, if a constant ratio of stiffness is maintained between the two chords, the problem can be referred to an once indeterminate system. Moreover, if the top or lower chord is non-stiff, the Vierendeel girder becomes a bow string girder or a tied arch with one redundant only. A statically determinate system, which has more or less a similar statical behaviour as the indeterminate Vierendeel girder, may be formed by introducing hinges at the mid-points of the different members.

The equilibrium of the elastic weights in any closed panel with rigid joints can be easily proved by using a "free main system" i. e. a main system which is changed for every case of virtual loading. This fact can be utilized in solving the Vierendeel girder. The conditions of equilibrium of the elastic weights in every panel replace the ordinary elastic equations.

Moreover, the elastic weights due to the redundant values in a closed panel may be combined into three resultants acting in the so-called "elastic poles", and depending only on the dimensions of the panel.

In the special case of a Vierendeel girder with equal chord stiffness, the elastic weights of the upper and lower chords, as well as those of the verticals form elastic couples. The equilibrium of the elastic weights in each panel is reduced to an algebraic summation of the corresponding elastic couples. This special case of equal chord stiffness may be also solved by successive approximations using the so-called "panel method".

Finally, a Vierendeel girder whose two chords have constant ratio of stiffness may be approximately solved by assuming an equal average stiffness. This approximation gives good practical results up to 30 and even 50% difference in the stiffness of the two chords.

Zusammenfassung

Der Vierendeel-Träger ist ein vielfach statisch unbestimmtes System, dessen Berechnung die Auflösung einer großen Zahl von Elastizitätsgleichungen erfordert. Wenn die Pfosten als an den Enden gelenkig angenommen werden, kann die Zahl der Überzähligen stark vermindert werden. Überdies kann das Problem auf ein einfach statisch unbestimmtes System zurückgeführt werden, wenn ein konstantes Verhältnis zwischen den Steifigkeiten der beiden Gurtungen beibehalten wird. Ferner wird der Vierendeel-Träger zu einem versteiften Stabbogen oder zu einem Bogen mit Zugband mit nur einer Überzähligen, wenn die obere oder die untere Gurtung keine Steifigkeit hat. Ein statisch bestimmtes System, das mehr oder weniger ein dem statisch unbestimmten Vierendeel-Träger ähnliches statisches Verhalten zeigt, kann durch Einführen von Gelenken in den Mittelpunkten der einzelnen Stäbe erhalten werden.

Das Gleichgewicht der elastischen Gewichte in jedem geschlossenen Feld mit steifen Knotenpunkten kann leicht überprüft werden, indem man ein „freies Grundsystem" einführt, d.h. ein Grundsystem, das für jeden virtuellen Belastungsfall geändert wird. Diese Tatsache kann für die Berechnung des Vierendeel-Trägers ausgenützt werden. Die Gleichgewichtsbedingungen der elastischen Gewichte in jedem Feld ersetzen die gewöhnlichen Elastizitätsgleichungen.

Die elastischen Gewichte infolge der überzähligen Größen können in jedem Feld zu drei Resultierenden zusammengesetzt werden, die in den sogenannten „elastischen Polen" angreifen und nur von den Abmessungen des Feldes abhängen.

Im Spezialfall eines Vierendeel-Trägers mit gleichen Steifigkeiten der Gurtungen bilden die elastischen Gewichte sowohl der Gurtungen als auch der Pfosten „elastische Kräftepaare". Die Herstellung des Gleichgewichts der

elastischen Gewichte in jedem Feld wird zurückgeführt auf eine algebraische Summation der entsprechenden elastischen Kräftepaare. Dieser Spezialfall der gleichen Gurtsteifigkeiten kann auch durch eine sukzessive Approximation gelöst werden bei Verwendung der sogenannten ,,Feld-Methode".

Schließlich kann ein Vierendeel-Träger, dessen beide Gurtungen konstantes Steifigkeitsverhältnis besitzen, näherungsweise unter der Annahme einer konstanten mittleren Steifigkeit berechnet werden. Diese Näherung ergibt praktisch gute Resultate bis zu Steifigkeitsdifferenzen der Gurtungen von 30 oder sogar 50%.

Résumé

La poutre Vierendeel est un système qui présente un degré multiple d'hyperstatisme, dont le calcul exige la résolution d'un grand nombre d'équations d'élasticité. Si l'on admet que les éléments verticaux sont articulés à leurs extrémités, le nombre des barres surabondantes peut être considérablement réduit. De plus, il est possible de ramener le problème à une indétermination statique simple en conservant un rapport constant entre les rigidités des deux membrures. La poutre Vierendeel peut en outre prendre la forme d'un arc simple renforcé ou d'un arc avec tirant avec un seul élément surabondant, lorsque l'une des deux membrures, supérieure ou inférieure, ne présente aucune rigidité. Un système isostatique qui offre un comportement statique plus ou moins analogue à celui de la poutre Vierendeel hyperstatique peut être réalisé par introduction d'articulations aux milieux des différentes barres.

Il est possible de contrôler aisément les conditions effectives d'équilibre des poids élastiques dans tout champ fermé avec nœuds rigides, en y introduisant un ,,système de base libre", c'est-à-dire un système de base se modifiant pour chaque cas virtuel de charge. Ce fait peut être utilisé pour le calcul de la poutre Vierendeel. Les conditions d'équilibre des poids élastiques dans chaque champ de charge peuvent être composées en trois résultantes, appliquées aux points dits ,,pôles élastiques" et qui ne dépendent que des dimensions du champ.

Dans le cas particulier d'une poutre Vierendeel dont les deux membrures présentent une même rigidité, les poids élastiques des membrures aussi bien que ceux des montants forment des ,,couples élastiques". La réalisation de l'équilibre des poids élastiques dans chaque champ est ramené à une sommation algébrique des couples élastiques correspondants. Ce cas particulier de l'égalité des rigidités peut également être résolu par approximations successives, en appliquant la méthode dite ,,du champ".

Enfin, une poutre Vierendeel dont les deux membrures présentent un rapport constant de rigidité peut être calculée d'une manière approchée dans l'hypothèse d'une rigidité constante moyenne. Cette approximation donne de bons résultats pratiques jusqu'à des écarts de rigidité atteignant 30, voire 50% entre les deux membrures.