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Some New Points of View in Calculating Suspension Bridges

Einige neue Gesichtspunkte zur Berechnung von Hängebrücken

Quelques points de vue nouveaux concernant le calcul des ponts suspendus

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Contents

§ 1. Introduction

1. Introduction. As it is generally known, until now only very few suspension bridges have been built for railway traffic. Intuitively one feels already that such a bridge (and specially one with a perfectly or nearly perfectly flexible main girder) will undergo great deflections, when it bears a load, which extends over a great length and moves at a high speed; for that reason it is not suitable for railway purposes.

When the author of this article was invited to demonstrate the things mentioned for a certain bridge quantitatively, it appeared, when reading the literature, that the dynamical behaviour of a suspension bridge had almost nearly not been studied until now, and that attention had only been given to the questions of free oscillations and to the so called phenomena of flutter, which appears when a horizontal current of air brushes past the girder.

For this reason he studied the question of the forced vibrations of a suspension bridge, caused by a load moving over the girder. In the article to be published the solution of this problem has been given for some simplifying suppositions mentioned a.o. at the end of this number; in § 4, where this problem has been treated, the question of free oscillations has also been studied.

In the main the derivation, given in § 2, of the formulæ which are fundamental for the further calculations, has already been published by other investigators. Nevertheless the author reproduced them in this (as succinct as possible) form, in order to get the article complete in itself.

He also paid attention (in § 3) to some points in the domain of the statical calculations, which have nearly not been studied by other investigators.

In this article we have restricted ourselves to the following simplifying assumptions:

a) The bridge has only one span, and the girder is hinged at both ends at the centre lines of the pylons. The extension to cases of a girder protruding beyond the pylons, of a bridge with more than two pylons etc. is for the rest not difficult.

b) The suspenders are perfectly inextensible.

c) The suspenders are so closed together, that they may be replaced with little objection by a continuous fastening.

d) The dead weight of the bridge is estimated to apply only on the girder of the bridge.

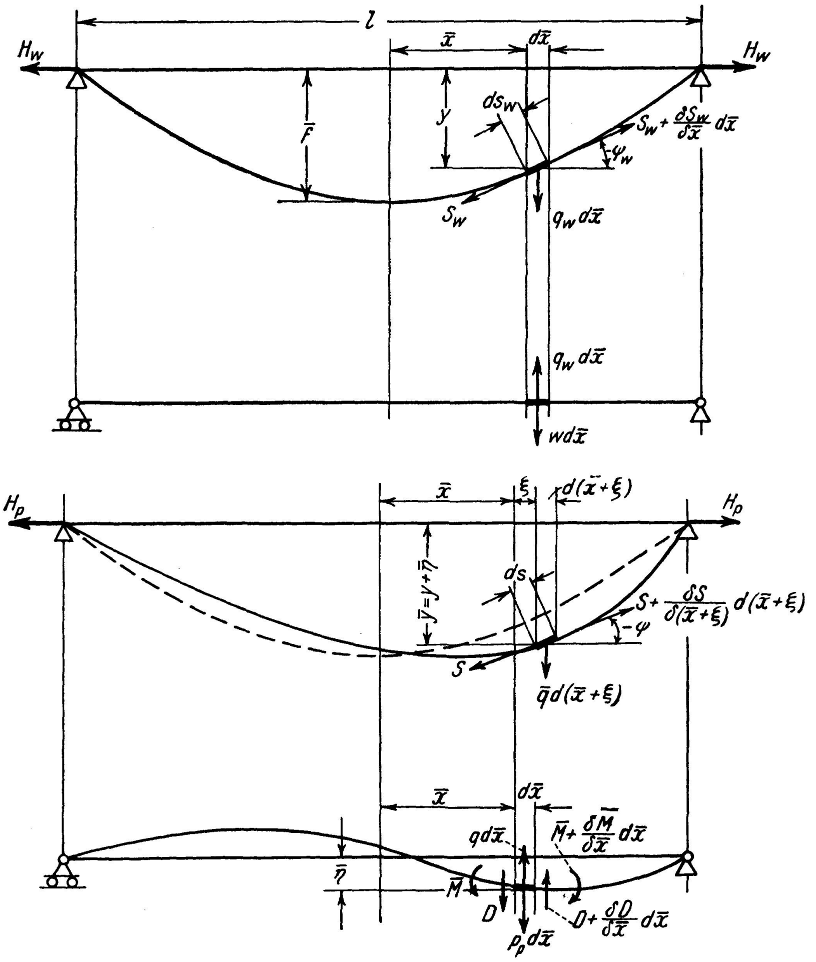
e) When no live load is placed on the bridge, this being in its state of equilibrium, the girder is perfectly straight, while the suspenders are vertical.

f) When a live load is placed on the girder, causing both a vertical and a horizontal cable displacement, the suspenders being no more perfectly vertical, the horizontal component of the force which each suspender element exerts on the cable is to be neglected.

g) In that case the vertical displacement of a cable element is equal to the vertical displacement of the girder element belonging to it.
h) The cable is of constant area throughout its whole length.
i) In this article we only deal with the case of the bridge, the live load and the displacement of the girder being symmetrical in respect to a vertical plane parallel to the longitudinal axis of the bridge. The important torsional deformations and vibrations are thus left out of consideration.
j) The derivation of the equations for the girder deflection and the cable force has been given in chapter *2* for any values of the mass per unit of length and the stiffness of the girder. But in chapter *3* and the remaining chapters we always restrict ourselves to the case of constant mass per unit of length and constant stiffness of the girder.

The author wishes to express his acknowledgements to Mr. H. J. A. Duparc, with whom he discussed the calculation of the integrals and functions mentioned in chapter *17*, and to Mr. E. J. G. Scheffer, who was so kind to assist him with a number of linguistic difficulties.

§ 2. Derivation of the fundamental equations



a) The girder bears only the dead load

b) The live load also applies on the girder

Fig. 1. Derivation of the differential equation for the stiffening girder

2. *Derivation of the equations for the girder deflection and the cable force.* In this number we shall derive the fundamental equations by considering in succession the three principal elements which form part of the bridge. The indices w herewith refer to the state indicated in the left part of fig. 1, that only the dead load w applies on the girder being in rest, while the indices p, added only if need be, indicate the state, represented in the right part of fig. 1, that the girder is deflected and bears the live load

$$p_p = \bar{p} + w, \tag{1}$$

while the cable temperature increases by an amount ϑ. As we suppose, that in the first mentioned state the suspenders are exactly vertical, the position of a cable element and that of a girder element can be indicated by the same coordinate $\bar{x}$. This quantity and the other quantities of length ξ, l, y, $\bar{y}$, s,

$$\bar{\eta}(\bar{x}) = \bar{y}(\bar{x}+\xi) - y(\bar{x}) \tag{2}$$

and $\bar{f}$ are shown in fig. 1.

At first we regard the *cable*. For a cable element we can write down the condition of length as follows:

$$ds^2 = d\bar{x}^2 + dy^2, \quad ds_p{}^2 = \{d(\bar{x}+\xi)\}^2 + \{d(y+\bar{\eta})\}^2; \tag{3}$$

when we denote the cable pull by S, the Young's modulus of the cable material by E_c, the area of the cross section of the cable by F_c, the temperature coefficient by β, we have

$$\left(1 + \frac{S - S_w}{E_c F_c}\right)(1 + \beta\vartheta)\, ds = ds_p,$$

or approximately

$$\left(1 + 2\,\frac{S - S_w}{E_c F_c} + 2\beta\vartheta\right) ds^2 = ds_p{}^2. \tag{4}$$

From (3) and (4) we can derive

$$\left(\frac{d\xi}{d\bar{x}}\right)^2 + 2\frac{d\xi}{d\bar{x}} + 2\frac{dy}{d\bar{x}}\frac{d\bar{\eta}}{d\bar{x}} + \left(\frac{d\bar{\eta}}{d\bar{x}}\right)^2 - 2\left(\frac{S - S_w}{E_c F_c} + \beta\vartheta\right)\left(\frac{ds}{d\bar{x}}\right)^2 = 0. \tag{5}$$

For the cable element also holds the condition of equilibrium, which yields for the relation between the cable pull S_w c. q. S and the suspender pull q_w c. q. $\bar{q}$ (see fig. 1):

$$\left.\begin{aligned} &\frac{d}{d\bar{x}}(S_w \cos\psi_w) + q_w = 0, \quad && \frac{d}{d\bar{x}}(S\cos\psi) + \bar{q} = 0, \\ &S_w \cos\psi_w = H_w, && S\cos\psi = H_p. \end{aligned}\right\} \tag{6}$$

By eliminating S_w, S, ψ_w and ψ from this equations there can be found, with

$$\operatorname{tg}\psi_w = \frac{dy}{d\bar{x}}, \quad \operatorname{tg}\psi = \frac{d\bar{y}}{d(\bar{x}+\xi)}: \tag{7}$$

$$q_w = -H_w \frac{d^2 y}{d\bar{x}^2}, \quad \bar{q}(\bar{x}+\xi) = -H_p \frac{d^2 \bar{y}}{d(\bar{x}+\xi)^2}. \tag{8}$$

With

$$\gamma = \frac{H_w}{E_c F_c}, \quad \bar{H} = H_p - H_w, \quad H = \frac{\bar{H}}{H_w}, \tag{9}$$

we now can substitute in (5) approximately

$$\frac{S-S_w}{E_c F_c} = \frac{H_p}{E_c F_c \cos\psi_w} - \frac{H_w}{E_c F_c \cos\psi_w} = \frac{\gamma H}{\cos\psi_w} = \gamma H \frac{ds}{dx}, \tag{10}$$

so that

$$\left(\frac{d\xi}{d\bar{x}}\right)^2 + 2\frac{d\xi}{d\bar{x}} + 2\frac{dy}{d\bar{x}}\frac{d\bar{\eta}}{d\bar{x}} + \left(\frac{d\bar{\eta}}{d\bar{x}}\right)^2 - 2\gamma H\left(\frac{ds}{d\bar{x}}\right)^3 - 2\beta\vartheta\left(\frac{ds}{d\bar{x}}\right)^2; \tag{11}$$

resolving $d\xi/d\bar{x}$ from this equation we find, suppressing all terms containing $(d\eta/d\bar{x})^3$, γ^2, $(\beta\vartheta)^2$ or higher powers of $\frac{d\eta}{d\bar{x}}$, γ or $\beta\vartheta$:

$$\frac{d\xi}{d\bar{x}} = -\frac{dy}{d\bar{x}}\frac{d\bar{\eta}}{d\bar{x}} - \frac{1}{2}\left(\frac{ds}{d\bar{x}}\right)^2\left(\frac{d\bar{\eta}}{d\bar{x}}\right)^2 + \gamma H\left(\frac{ds}{d\bar{x}}\right)^3 + \beta\vartheta\left(\frac{ds}{d\bar{x}}\right)^2. \tag{12}$$

Secondly we consider the *suspender construction*. For the case of the *non deflected girder* we have

$$q_w = w, \tag{13}$$

so that with (8)

$$\frac{d^2 y}{d\bar{x}^2} = -\frac{w}{H_w}. \tag{14}$$

This is a differential equation for the cable deflection y, which can be solved, when w is known as a function of $\bar{x}$. We shall do this in chapter *3* for the case $w = \text{const.}$

We now can derive a condition with the aid of which the cable pull can be found. Dropping the term with $(d\bar{\eta}/d\bar{x})^2$ in (12) and integrating both members of this equation to $\bar{x}$ we find, introducing the quantities

$$\bar{l}_2 = \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} \left(\frac{ds}{d\bar{x}}\right)^2 d\bar{x}, \quad \bar{l}_3 = \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} \left(\frac{ds}{d\bar{x}}\right)^3 d\bar{x}, \tag{15}$$

$$\int_{-\frac{1}{2}l}^{+\frac{1}{2}l} \frac{dy}{d\bar{x}}\frac{d\bar{\eta}}{d\bar{x}} d\bar{x} = -\xi\Big|_{-\frac{1}{2}l}^{+\frac{1}{2}l} + \gamma H \bar{l}_3 + \beta\vartheta \bar{l}_2. \tag{16}$$

By partially integrating the left member, applying the conditions

$$\bar{\eta} = 0_2 \text{ for } \bar{x} = \pm\tfrac{1}{2}l,$$

$$\xi(+\tfrac{1}{2}l) - \xi(-\tfrac{1}{2}l) = -\bar{\nu}\bar{H} \tag{17}$$

($\bar{\nu}$ being the horizontal stiffness of the pylons) and making use of (14) we find

$$\frac{1}{H_w}\int\limits_{-\frac{1}{2}l}^{+\frac{1}{2}l} w\,\bar{\eta}\,d\bar{x} = (\gamma\,\bar{l}_3+\bar{\nu}\,H_w)\,H+\beta\,\vartheta\,\bar{l}_2. \tag{18}$$

For the case of the *deflected girder* we learn from the vertical equilibrium of a suspender element

$$q(\bar{x})\,d\bar{x} = \bar{q}(\bar{x}+\xi)\,d(\bar{x}+\xi).$$

Substituting this in the second equation (8) yields, with (9),

$$q(\bar{x}) = -(H_w+\bar{H})\,\frac{d^2\bar{y}}{d(\bar{x}+\xi)^2}\,\frac{d(\bar{x}+\xi)}{d\bar{x}}. \tag{19}$$

Now we can derive from (2) and (12), neglecting all terms containing $(d\,\bar{\eta}/d\,\bar{x})^2$ and higher powers of $d\,\bar{\eta}/d\,\bar{x}$,

$$\frac{d\bar{y}(\bar{x}+\xi)}{d(\bar{x}+\xi)} = \left(\frac{dy}{d\bar{x}}+\frac{d\bar{\eta}}{d\bar{x}}\right)\frac{d\bar{x}}{d(\bar{x}+\xi)} = \left(\frac{dy}{d\bar{x}}+\frac{d\bar{\eta}}{d\bar{x}}\right)\left(1-\frac{d\xi}{d\bar{x}}\right)$$

$$= \left\{1-\gamma H\left(\frac{ds}{d\bar{x}}\right)^3-\beta\,\vartheta\left(\frac{ds}{d\bar{x}}\right)^2\right\}\frac{dy}{d\bar{x}}+\left(1-\gamma H\frac{ds}{d\bar{x}}-\beta\,\vartheta\right)\left(\frac{ds}{d\bar{x}}\right)^2\frac{d\bar{\eta}}{d\bar{x}},$$

$$\frac{d^2\bar{y}(\bar{x}+\xi)}{d(\bar{x}+\xi)^2} = \frac{d}{d\bar{x}}\left\{\frac{d\bar{y}(\bar{x}+\xi)}{d(\bar{x}+\xi)}\right\}\frac{d\bar{x}}{d(\bar{x}+\xi)}$$

$$= \left[1-\gamma H\left\{1+4\left(\frac{dy}{d\bar{x}}\right)^2\right\}\frac{ds}{d\bar{x}}-\beta\,\vartheta\left\{1+3\left(\frac{dy}{d\bar{x}}\right)^2\right\}\right]\frac{d^2y}{d\bar{x}^2}\,\frac{d\bar{x}}{d(\bar{x}+\xi)}$$

$$+\frac{d}{d\bar{x}}\left\{\left(1-\gamma H\frac{ds}{d\bar{x}}-\beta\,\vartheta\right)\left(\frac{ds}{d\bar{x}}\right)^2\frac{d\bar{\eta}}{d\bar{x}}\right\}\frac{d\bar{x}}{d(\bar{x}+\xi)};$$

combining this with (19) yields

$$q(\bar{x}) = -(H_w+\bar{H})\left[\left[1-\gamma H\left\{1+4\left(\frac{dy}{d\bar{x}}\right)^2\right\}\frac{ds}{d\bar{x}}-\beta\,\vartheta\left\{1+3\left(\frac{dy}{d\bar{x}}\right)^2\right\}\right]\frac{d^2y}{d\bar{x}^2}\right.$$
$$\left.+\frac{d}{d\bar{x}}\left\{\left(1-\gamma H\frac{ds}{d\bar{x}}-\beta\,\vartheta\right)\left(\frac{ds}{d\bar{x}}\right)^2\frac{d\bar{\eta}}{d\bar{x}}\right\}\right]. \tag{20}$$

In future we shall restrict ourselves to the case of an inextensible cable or to the case, in which $(dy/d\bar{x})^2$ is neglected with respect to 1 (so that $ds/d\bar{x}$ can be replaced by 1)[1]. In these cases we can simplify (20) respectively to

$$q(\bar{x}) = -(H_w+\bar{H})\left[\frac{d^2y}{d\bar{x}^2}+\frac{\partial}{\partial\bar{x}}\left\{\left(\frac{ds}{d\bar{x}}\right)^2\frac{\partial\bar{\eta}}{\partial\bar{x}}\right\}\right] \tag{21a}$$

and

$$q(\bar{x}) = -(H_w+\bar{H})\,(1-\gamma H-\beta\,\vartheta)\left(\frac{d^2y}{d\bar{x}^2}+\frac{\partial^2\bar{\eta}}{\partial\bar{x}^2}\right). \tag{21b}$$

In the last formulæ we replaced the ordinary differentiations of $\bar{\eta}$ to $\bar{x}$ by

[1]) The inaccuracy caused by this simplification will be computed for the case of an inextensible cable and a perfectly slack girder in chapter 7.

partial differentiations, because we shall henceforth consider $\bar{\eta}$ as a function both of $\bar{x}$ and $\bar{t}$ in general.

At last we examine the *girder* itself. From the law of Newton written down for a girder element $d\bar{x}$ (see fig. 1) we learn

$$-q(\bar{x})+p_p-\frac{\partial D}{\partial \bar{x}}=\frac{w}{g}\frac{\partial^2\bar{\eta}}{\partial \bar{t}^2}. \tag{22}$$

Denoting Young's modulus of the girder material by E and the moment of inertia of its cross section by I we have

$$D=\frac{\partial \bar{M}}{\partial \bar{x}}=\frac{\partial}{\partial \bar{x}}\left(EI\frac{\partial^2\bar{\eta}}{\partial \bar{x}^2}\right); \tag{23}$$

combining this with (22), (1) and (21a) or (21b) we obtain respectively

$$\frac{\partial^2}{\partial \bar{x}^2}\left(EI\frac{\partial^2\bar{\eta}}{\partial \bar{x}^2}\right)-(H_w+\bar{H})\frac{\partial}{\partial \bar{x}}\left\{\left(\frac{ds}{d\bar{x}}\right)^2\frac{\partial\bar{\eta}}{\partial \bar{x}}\right\}+\frac{w}{g}\frac{\partial^2\bar{\eta}}{\partial \bar{t}^2}=\bar{p}(\bar{x},\bar{t})+\bar{H}\frac{d^2y}{d\bar{x}^2} \tag{24a}$$

for the case of an inextensible cable and

$$\frac{\partial^2}{\partial \bar{x}^2}\left(EI\frac{\partial^2\bar{\eta}}{\partial \bar{x}^2}\right)-(H_w+\bar{H})(1-\gamma H-\beta\vartheta)\frac{\partial^2\bar{\eta}}{\partial \bar{x}^2}+\frac{w}{g}\frac{\partial^2\bar{\eta}}{\partial \bar{t}^2}=\bar{p}(\bar{x},\bar{t})+\bar{H}\frac{d^2y}{d\bar{x}^2}$$
$$-(H_w+\bar{H})(\gamma H+\beta\vartheta)\frac{d^2y}{d\bar{x}^2} \tag{24b}$$

for the case of an extensible cable, in which $(dy/d\bar{x})^2$ is neglected with respect to 1. In these equations $\bar{H}(\bar{t})$ always is a function of $\bar{t}$, which can be determinated by the condition (18), while the boundary conditions for $\bar{\eta}$ are

$$\bar{\eta}=0,\quad \frac{\partial^2\bar{\eta}}{\partial \bar{x}^2}=0 \qquad \text{for } \bar{x}=\pm\tfrac{1}{2}l. \tag{25}$$

In the statical case we can simplify (24b) to

$$\frac{d^2}{d\bar{x}^2}\left(EI\frac{d^2\bar{\eta}}{d\bar{x}^2}\right)-(H_w+\bar{H})(1-\gamma H-\beta\vartheta)\frac{d^2\bar{\eta}}{d\bar{x}^2}=\bar{p}(\bar{x})+\bar{H}\frac{d^2y}{d\bar{x}^2}$$
$$-(H_w+\bar{H})(\gamma H+\beta\vartheta)\frac{d^2y}{d\bar{x}^2},$$

and this equation can immediately be integrated twice to $\bar{x}$, by introducing the bending moment $\bar{M}_1(\bar{x})$ caused by the live load $\bar{p}(\bar{x})$ in the girder, when this is not coupled to the suspenders:

$$\bar{M}_1(\bar{x})=-\frac{\frac{1}{2}l-\bar{x}}{l}\int\limits_{-\frac{1}{2}l}^{\bar{x}}(\tfrac{1}{2}l+z)\,\bar{p}(z)\,dz-\frac{\frac{1}{2}l+\bar{x}}{l}\int\limits_{\bar{x}}^{\frac{1}{2}l}(\tfrac{1}{2}l-z)\,\bar{p}(z)\,dz; \tag{26}$$

then we find the differential equation for the girder deflection $\bar{\eta}$:

$$EI\frac{d^2\bar{\eta}}{d\bar{x}^2}-(H_w+\bar{H})(1-\gamma H-\beta\vartheta)\,\bar{\eta}=\bar{M}_1+\bar{H}y-(H_w+\bar{H})(\gamma H+\beta\vartheta)\,y, \tag{27}$$

and the ordinary equation for the bending moment $\overline{M}\left(=EI\dfrac{d^2\overline{\eta}}{d\overline{x}^2}\right)$ in the girder

$$\overline{M} = \overline{M}_1+\overline{H}\,y-(H_w+\overline{H})\,(\gamma\,H+\beta\,\vartheta)\,y+(H_w+H)\,(1-\gamma\,H-\beta\,\vartheta)\,\overline{\eta}. \tag{28}$$

3. *The case of constant mass per unit of length and constant stiffness of the girder. Dimensionless quantities.* For the case $w=$ const. the differential equation (*2*, 14) can be integrated twice without difficulty. With the conditions (see fig. 1)

$$y=0 \text{ for } \overline{x}=\pm\tfrac{1}{2}l,\quad y=\overline{f} \text{ for } \overline{x}=0 \tag{1}$$

we find

$$y=\overline{f}\left(1-\frac{4\overline{x}^2}{l^2}\right),\quad \frac{dy}{d\overline{x}}=-\frac{8\overline{f}\,\overline{x}}{l^2}, \tag{2}$$

$$H_w=\frac{w\,l^2}{8\overline{f}}. \tag{3}$$

As also $E\,I=$ const., the differential equations (*2*, 24a) and (*2*, 24b) and the cable conditions (*2*, 18) now can be simplified to

$$E\,I\frac{\partial^4\overline{\eta}}{\partial\overline{x}^4}-(H_w+\overline{H})\frac{\partial}{\partial\overline{x}}\left\{\left(1+64\frac{\overline{f}^2\overline{x}^2}{l^4}\right)\frac{\partial\,\overline{\eta}}{\partial\overline{x}}\right\}+\frac{w}{g}\frac{\partial^2\overline{\eta}}{\partial\overline{t}^2}=\overline{p}\,(\overline{x},\overline{t})-8\frac{\overline{f}\,\overline{H}}{l^2}, \tag{4a}$$

$$\begin{aligned}E\,I\frac{\partial^4\overline{\eta}}{\partial\overline{x}^4}-(H_w+\overline{H})\,(1-\gamma\,H-\beta\,\vartheta)\frac{\partial^2\overline{\eta}}{\partial\overline{x}^2}+\frac{w}{g}\frac{\partial^2\overline{\eta}}{\partial\overline{t}^2}&=\overline{p}\,(\overline{x},\overline{t})-8\frac{\overline{f}\,\overline{H}}{l^2}\\&\quad+\frac{8\overline{f}}{l^2}(H_w+\overline{H})\,(\gamma\,H+\beta\,\vartheta),\end{aligned} \tag{4b}$$

$$\frac{8\overline{f}}{l^2}\int_{-\frac{1}{2}l}^{+\frac{1}{2}l}\overline{\eta}\,d\overline{x}=(\gamma\,\overline{l}_3+\overline{\nu}\,H_w)\,H+\beta\,\vartheta\,\overline{l}_2. \tag{5}$$

We now introduce the dimensionless quantity

$$c_0^{\,2}=\frac{H_w\,l^2}{4\,E\,I} \tag{6a}$$

and the dimensionless quantities

$$\left.\begin{aligned}&f=\frac{4\overline{f}}{l},\quad x=\frac{2\overline{x}}{l},\quad \eta=\frac{2\,\overline{\eta}}{f\,l},\quad t=\overline{t}\sqrt{\frac{2g}{f\,l}},\quad p=\frac{\overline{p}}{w},\quad l_2=\frac{2\,\overline{l}_2}{f^2\,l}=\frac{2}{f^2}+\frac{2}{3},\\&l_3=\frac{2\,\overline{l}_3}{f^2\,l}=\frac{1}{4\,f^2}\left\{(5+2\,f^2)\sqrt{1+f^2}+3\,\frac{\operatorname{bg\,sh} f}{f}\right\},\quad \nu=\frac{2\,\overline{\nu}\,H_w}{f^2\,l},\quad M=\frac{2\,\overline{M}}{f\,H_w\,l},\quad M_1=\frac{2\,\overline{M}_1}{f\,H_w\,l}.\end{aligned}\right\} \tag{6b}$$

With their help the differential equations (4a), (4b), the cable condition (5) and the boundary conditions (*2*, 25) can be written in the form

$$\frac{1}{c_0^{\,2}}\frac{\partial^4\eta}{\partial x^4}-(1+H)\frac{\partial}{\partial x}\left\{(1+f^2x^2)\frac{\partial\,\eta}{\partial x}\right\}+\frac{\partial^2\eta}{\partial t^2}=p\,(x,t)-H\,(t), \tag{7a}$$

$$\frac{1}{c_0^{\,2}}\frac{\partial^4\eta}{\partial x^4}-(1+H)\,(1-\gamma\,H-\beta\,\vartheta)\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial t^2}=p\,(x,t)-H\,(t)+(\gamma\,H+\beta\,\vartheta)\,(1+H), \tag{7b}$$

$$\int_{-1}^{+1} \eta\, d x = (\gamma l_3 + \nu) H + l_2 \beta \vartheta, \tag{8}$$

$$\eta = 0, \quad \frac{\partial^2 \eta}{\partial x^2} = 0 \text{ for } x = \pm 1. \tag{9}$$

The differential equation (*2*, 27) and the equation (*2*, 28) can also be simplified:

$$\frac{1}{c_0^2}\frac{d^2\eta}{d x^2} - (1+H)(1-\gamma H - \beta\vartheta)\eta = M_1 + \tfrac{1}{2} H (1-x^2) - \tfrac{1}{2}(1+H)(\gamma H + \beta\vartheta)(1-x^2), \tag{10a}$$

$$M = M_1 + \tfrac{1}{2} H (1-x^2) - \tfrac{1}{2}(1+H)(\gamma H+\beta\vartheta)(1-x^2) + (1+H)(1-\gamma H - \beta\vartheta)\eta, \tag{10b}$$

in which formulæ the moment M_1 is determinated by

$$2 M_1(x) = -(1-x)\int_{-1}^{x}(1+z)\,p(z)\,dz - (1+x)\int_{x}^{1}(1-z)\,p(z)\,dz. \tag{11}$$

The form of the equations (7a), (7b) and (10a) indicates that the stiffening influence of the girder will be the less as the value of c_0 is greater. We have computed for some bridges (which are all considered here as one-span bridges) the quantity c_0 with the aid of (6a); the numerical values are shown in table I.

Table I. Values of c_0 for some suspension bridges

Bridge	w kg/cm	l cm	$\bar{f}$ cm	$f = \frac{4\bar{f}}{l}$	E kg/cm²	I cm⁴	H_w kg	c_0
Breslau 1910	108	12.000	1.200	0,4	$2{,}1 \cdot 10^6$	$0{,}26 \cdot 10^8$	$1{,}62 \cdot 10^6$	1,034
Manhattan 1910	85	44.800	4.360	0,390	$2{,}1 \cdot 10^6$	$3{,}9 \cdot 10^8$	$4{,}88 \cdot 10^6$	1,728
Delaware 1926	194	53.000	6.100	0,460	$2{,}1 \cdot 10^6$	$7{,}2 \cdot 10^8$	$11{,}18 \cdot 10^6$	2,28
Second Tacoma-bridge ± 1945	126	85.500	8.550	0,4	$2{,}1 \cdot 10^6$	5,4 à $5{,}7 \cdot 10^8$	$13{,}45 \cdot 10^6$	4,53 à 4,65
Triborough 1938	300	42.000	4.200	0,4	$2{,}1 \cdot 10^6$	$0{,}91 \cdot 10^8$	$15{,}75 \cdot 10^6$	6,03
Zweden 1941	13,6[2]	14.000	1.680	0,48	$EI =$ $0{,}196 \cdot 10^{12}$ kgcm²		$0{,}198 \cdot 10^6$[2]	7,04
Mount Hope bridge	394	36.200	3.620	0,4	$EI =$ $51{,}9 \cdot 10^{12}$ kgcm²		$17{,}81 \cdot 10^6$	10,6
Example [3]	800	43.000	4.300	0,4	$0{,}3 \cdot 10^6$	$4{,}10^8$	$43{,}10^6$	12,86
New York Washington bridge	?	106.700	?	0,4?	?	?	?	17,5[4]
First Tacoma-bridge 1940[5]	84	85.300	7.100	0,333	$EI =$ $2{,}49 \cdot 10^9$ kgcm²		$10{,}78 \cdot 10^6$	2810

[2]) Per main girder.

[3]) Conform to the figures mentioned in the article W. C. Coepijn, Enige beschouwingen over vormvastheid en de berekeningsgrondslagen van open hangbruggen, De Ingenieur 60 (1948) p. B 126.

[4]) H. H. Bleich, Die Berechnung verankerter Hängebrücken, Wien (1935) p. 27.

[5]) H. Reissner, Oscillations of suspension bridges, J. appl. mech. 10 (1943) p. A-23; J. Versloot, Het instorten van de "Tacoma" brug, Polytechnisch Tijdschrift A 5 (1950) p. 190a, B 5 (1950) p. 190b.

§ 3. Statical calculations

4. Survey of the calculations to be performed for the statical case. In this paragraph, treating some statical calculations, we are to indicate the general formulæ for the girder deflection η and the cable force H:

a) in chapter *5* and *6* for the case of an inextensible cable, in which the horizontal cable deflection is neglected;
b) in chapter *7* for the case of an inextensible cable and a perfectly slack girder, in which the horizontal cable deflection is taken into account;
c) in chapter *8* and *9* for the case of an extensible cable and a perfectly slack girder, in which the horizontal cable deflection is neglected.

Further on we shall compute in chapter *10* the amounts of work accumulated in the various parts of the bridge, a live load being placed on the girder.

5. Inextensible cable. We start from the equation (*3*, 10a) simplified to

$$\frac{1}{c_0^2}\frac{d^2\eta}{dx^2}-(1+H)\,\eta = M_1+\tfrac{1}{2}H\,(1-x^2), \tag{1}$$

with the cable condition (*3*, 8), simplified to

$$\int_{-1}^{+1}\eta\,dx = 0 \tag{2}$$

and the boundary conditions (*3*, 9). Introducing the quantities

$$c = c_0\sqrt{1+H}, \quad (3\text{a}) \qquad \mathfrak{M}\,(x) = c_0^2\{M_1(x)+\tfrac{1}{2}H\,(1-x^2)\}, \tag{3b}$$

we can immediately write down the solution of (1) which satisfies the boundary conditions (*3*, 9) with the help of the method of variation of constants:

$$c\,\eta\,\mathrm{sh}\,2c = -\mathrm{sh}\,c\,(1-x)\int_{-1}^{x}\mathfrak{M}\,(z)\,\mathrm{sh}\,c\,(1+z)\,dz - \mathrm{sh}\,c\,(1+x)\int_{x}^{+1}\mathfrak{M}\,(z)\,\mathrm{sh}\,c\,(1-z)\,dz. \tag{4}$$

In this formula we substitute the expression (3b) for $\mathfrak{M}\,(x)$ and (*3*, 11) for $M_1(x)$:

$$\begin{aligned}\frac{2\,(1+H)\,\mathrm{sh}\,2c}{c}\,\eta &= \mathrm{sh}\,c\,(1-x)\int_{-1}^{x}(1-z)\,\mathrm{sh}\,c\,(1+z)\,dz\int_{-1}^{z}(1+\zeta)\,p\,(\zeta)\,d\zeta\\ &+\mathrm{sh}\,c\,(1-x)\int_{-1}^{x}1+z)\,\mathrm{sh}\,c\,(1+z)\,dz\int_{z}^{+1}(1-\zeta)\,p\,(\zeta)\,d\zeta\\ &+\mathrm{sh}\,c\,(1+x)\int_{x}^{+1}(1-z)\,\mathrm{sh}\,c\,(1-z)\,dz\int_{-1}^{z}(1+\zeta)\,p\,(\zeta)\,d\zeta\\ &+\mathrm{sh}\,c\,(1+x)\int_{x}^{+1}(1+z)\,\mathrm{sh}\,c\,(1-z)\,dz\int_{z}^{+1}(1-\zeta)\,p\,(\zeta)\,d\zeta\\ -H\{\mathrm{sh}\,c\,(1-z)&\int_{-1}^{x}\cdot(1-z^2)\,\mathrm{sh}\,c\,(1+z)\,dz+\mathrm{sh}\,c\,(1+x)\int_{x}^{+1}(1-z^2)\,\mathrm{sh}\,c\,(1-z)\,dz.\end{aligned} \tag{5}$$

The four double integrals can be reduced to simple integrals by changing the sequence of integration, while the two last integrals may be worked out in an elementary way. This yields

$$
\begin{aligned}
2(1+H)\eta = {} & (1-x)\int_{-1}^{x}(1+z)\,p(z)\,dz+(1+x)\int_{x}^{+1}(1-z)\,p(z)\,dz \\
& -\frac{2}{c\,\mathrm{sh}\,2c}\left\{\mathrm{sh}\,c(1-x)\int_{-1}^{x}\mathrm{sh}\,c(1+z)\,p(z)\,dz+\mathrm{sh}\,c(1+x)\int_{x}^{+1}\mathrm{sh}\,c(1-z)\,p(z)\,dz\right\} \\
& -H\left\{1-x^2-\frac{2}{c^2}\left(1-\frac{\mathrm{ch}\,c\,x}{\mathrm{ch}\,c}\right)\right\}.
\end{aligned}
\tag{6}
$$

Next the cable force can be calculated with the aid of the formula (2). By applying this formula to (6) an equation is found, in which H occurs apparently only linearly; solving it to H yields

$$
H=\frac{\int_{-1}^{+1}(1-x^2)\,p(x)\,dx-\frac{2}{c^2}\int_{-1}^{+1}\left(1-\frac{\mathrm{ch}\,c\,x}{\mathrm{ch}\,c}\right)p(x)\,dx}{4\left(\frac{1}{3}-\frac{1-\frac{\mathrm{th}\,c}{c}}{c^2}\right)}.
\tag{7}
$$

But in reality the righthand term is via c (3a) a function of H, so that the determination of H from this equation requires some forethought. We shall discuss this question further when treating some special live loads $p(x)$. One also sees that for the case of more live loads applying simultaneously on the bridge, the cable force which they cause is not the sum of the cable forces caused by each live load individually, so that generally the principle of superposition holds no more for the cable force than for the deflection!

The bending moment in the girder follows from (*3*, 10b) (with $\gamma=0$, $\beta=0$), (*3*, 11) and (6):

$$
\begin{aligned}
M = {} & -\frac{\mathrm{sh}\,c(1-x)\int_{-1}^{x}\mathrm{sh}\,c(1+z)\,p(z)\,dz+\mathrm{sh}\,c(1+x)\int_{x}^{+1}\mathrm{sh}\,c(1-z)\,p(z)\,dz}{c\,\mathrm{sh}\,2c} \\
& +\frac{H}{c^2}\left(1-\frac{\mathrm{ch}\,c\,x}{\mathrm{ch}\,c}\right).
\end{aligned}
\tag{8}
$$

Formulæ like (6)—(8) were already derived by v. KÁRMÁN and BIOT [6]), AAS-JAKOBSEN [7]) and EMMEN [8]). However, AAS-JAKOBSEN restricted himself to some special cases of loading, i.e. to the concentrated force and to the load

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[6]) TH. v. KÁRMÁN and M. A. BIOT, Mathematical methods in engineering, New York and London (1940) p. 317.

[7]) A. AAS-JAKOBSEN, Berechnung der verankerten Hängebrücken für vertikale und horizontale Belastung, Mémoires de l'Ass. Int. des Ponts et Charpentes 7 (1943/44) p. 18.

[8]) J. EMMEN, Nieuwe methode voor het berekenen van hangbruggen, De Ingenieur 60 (1948) p. B 37.

uniformly distributed over a part of the girder; the formulæ for the last mentioned case were derived from the formulæ for the concentrated force, which is incorrect owing to the non linear behaviour of the considered quantities. EMMEN restricted himself to the case of a perfectly slack girder ($c=\infty$): before treating some important live load functions we shall indicate how the formulæ (6)—(8) are altered for this case and for $c=0$.

For $c=\infty$ we also have $c_0=\infty$, so that, according to (*3*, 6a), $EI=0$; in other words the girder is perfectly slack. We have in this case

$$2(1+H)\eta = (1-x)\int_{-1}^{x}(1+2)\,p(z)\,dz+(1+x)\int_{x}^{+1}(1-z)\,p(z)\,dz-H(1-x^2), \qquad (6\text{a})$$

$$H = \tfrac{3}{4}\int_{-1}^{+1}(1-x^2)\,p(x)\,dx, \;(7\text{a}) \qquad M = 0. \qquad (8\text{a})$$

For $c=0$ we also have $c_0=0$; according to (*3*, 6a) this case arises, when the live load $w=0$ or when the girder rigidity $EI=\infty$. But for $w=0$ we have, according to (*3*, 3), $H_w=0$ and consequently, according to (*2*, 9), $H=\infty$, so that the formulæ (6)—(8) lose their importance. However they remain usable for $EI=\infty$; in this case we have to apply de l'Hopital's rule and we find

$$\eta = 0, \;(6\text{b}) \qquad H = \frac{5}{32}\int_{-1}^{+1}(1-x^2)(5-x^2)\,p(x)\,dx. \qquad (7\text{b})$$

$$2M = -(1-x)\int_{-1}^{x}(1+z)\,p(z)\,dz-(1+x)\int_{x}^{+1}(1-z)\,p(z)\,dz+H(1-x^2). \qquad (8\text{b})$$

The form of the formulæ (7a) and (7b) indicates that the principle of superposition, in contradiction to the general case, does hold for the cases $c=0$ and $c=\infty$.

Now we illustrate the just found results by applying the formulæ (7), (6) and (8) to the three important cases of loading

$$\text{a)}\quad p(x)\equiv p \quad \text{for} \quad -1\leqq x\leqq 1, \qquad (9\text{a})$$

$$\text{b)}\quad p(x)\equiv \left.\begin{array}{lll} 0 & \text{for} & -1\leqq x< 0,\\ p & \text{for} & 0<x\leqq 1,\end{array}\right\} \qquad (9\text{b})$$

$$\text{c)}\quad p(x)\equiv \left.\begin{array}{lll} 0 & \text{for} & -1\leqq x< -\tfrac{1}{2},\\ p & \text{for} & -\tfrac{1}{2}<x< \tfrac{1}{2},\\ 0 & \text{for} & \tfrac{1}{2}<x\leqq 1.\end{array}\right\} \qquad (9\text{c})$$

The just mentioned formulæ yield respectively

$$H = p, \qquad (10\text{a})$$

$$H = \tfrac{1}{2}p, \qquad (10\text{b})$$

$$H = p\, f_H(c) \text{ with } f_H(c) = \frac{1}{2}\,\frac{\dfrac{11}{24} - \dfrac{1 - \dfrac{\operatorname{th} c}{c \operatorname{th} \frac{1}{2} c}}{c^2}}{\dfrac{1}{3} - \dfrac{1 - \dfrac{\operatorname{th} c}{c}}{c^2}}, \tag{10c}$$

$$\eta = 0, \tag{11a}$$

$$\left.\begin{aligned} 4\frac{1+H}{p}\eta &= x(1+x) + 2\frac{\operatorname{sh} cx - \operatorname{sh} c(1+x) + \operatorname{sh} c}{c^2 \operatorname{sh} c} \quad \text{for } -1 \leqq x \leqq 0, \\ &= x(1-x) + 2\frac{\operatorname{sh} cx + \operatorname{sh} c(1-x) - \operatorname{sh} c}{c^2 \operatorname{sh} c} \quad \text{for } 0 \leqq x \leqq 1, \end{aligned}\right\} \tag{11b}$$

$$\left.\begin{aligned} 2\frac{1+H}{p}\eta &= 1 + x - \frac{2 \operatorname{sh} \frac{1}{2} c \operatorname{sh} c(1+x)}{c^2 \operatorname{ch} c} \\ &\quad - \left\{1 - x^2 - \frac{2}{c^2}\left(1 - \frac{\operatorname{ch} cx}{\operatorname{ch} c}\right)\right\} f_H(c) \quad \text{for } -1 \leqq x \leqq -\tfrac{1}{2}, \\ &= \frac{3}{4} - x^2 - \frac{2}{c^2}\left(1 - \frac{\operatorname{ch} \frac{1}{2} c \operatorname{ch} cx}{\operatorname{ch} c}\right) \\ &\quad - \left\{1 - x^2 - \frac{2}{c^2}\left(1 - \frac{\operatorname{ch} cx}{\operatorname{ch} c}\right)\right\} f_H(c) \quad \text{for } -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}, \\ &= 1 - x - \frac{2 \operatorname{sh} \frac{1}{2} c \operatorname{sh} c(1-x)}{c^2 \operatorname{ch} c} \\ &\quad - \left\{1 - x^2 - \frac{2}{c^2}\left(1 - \frac{\operatorname{ch} cx}{\operatorname{ch} c}\right)\right\} f_H(c) \quad \text{for } \tfrac{1}{2} \leqq x \leqq 1, \end{aligned}\right\} \tag{11c}$$

and

$$M = 0, \tag{12a}$$

$$\left.\begin{aligned} 2c^2\frac{M}{p} &= \frac{\operatorname{sh} cx - \operatorname{sh} c(1+x) + \operatorname{sh} c}{\operatorname{sh} c} \quad \text{for } -1 \leqq x \leqq 0, \\ &= \frac{\operatorname{sh} cx + \operatorname{sh} c(1-x) - \operatorname{sh} c}{\operatorname{sh} c} \quad \text{for } 0 \leqq x \leqq 1, \end{aligned}\right\} \tag{12b}$$

$$\left.\begin{aligned} c^2\frac{M}{p} &= -\frac{\operatorname{sh} \frac{1}{2} c \operatorname{sh} c(1+x)}{\operatorname{ch} c} + \left(1 - \frac{\operatorname{ch} cx}{\operatorname{ch} c}\right) f_H(c) \quad \text{for } -1 \leqq x \leqq -\tfrac{1}{2}, \\ &= -\left(1 - \frac{\operatorname{ch} \frac{1}{2} c \operatorname{ch} cx}{\operatorname{ch} c}\right) + \left(1 - \frac{\operatorname{ch} cx}{\operatorname{ch} c}\right) f_H(c) \quad \text{for } -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}, \\ &= -\frac{\operatorname{sh} \frac{1}{2} c \operatorname{sh} c(1-x)}{\operatorname{ch} c} + \left(1 - \frac{\operatorname{ch} cx}{\operatorname{ch} c}\right) f_H(c) \quad \text{for } \tfrac{1}{2} \leqq x \leqq 1. \end{aligned}\right\} \tag{12c}$$

The cable force may be calculated with the aid of the formulæ (10a), (10b). For the case c there arise some difficulties, the righthand term of equation (10c) being a function of H via c (3a). But from the table II and the figure 2, in which we plotted $f_H(c)$ against c, we learn that $f_H(c)$ is almost independent

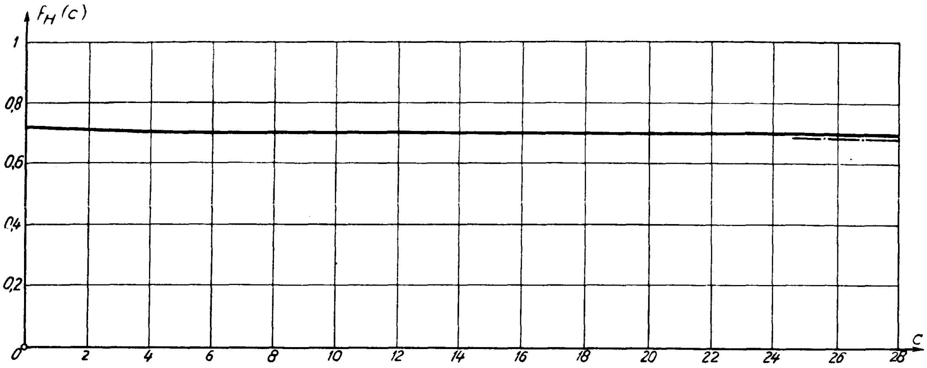

Fig. 2. Graphical representation of the function $f_H(c)$

Table II. Some values of the function $f_H(c)$

c	$f_H(c)$	c	$f_H(c)$
0	0,7051	8	0,6927
1	0,7042	9	0,6919
2	0,7026	10	0,6912
3	0,7006	15	0,6894
4	0,6981	20	0,6887
5	0,6963	50	0,6877
6	0,6947	∞	0,6875
7	0,6936		

of c, this function falling monotonously from the value $\frac{361}{512}$ (for $c=0$) to $\frac{11}{16}$ (for $c=\infty$). So we can put in a first approximation $f_H(c)=0{,}7$, $c=c_0\sqrt{1+0{,}7p}$, after which we can compute an exacter value of $f_H(c)$ with this value of c; this method may, if necessary, be repeated a number of times. For a bridge with dimensions mentioned in table I as "example", and with $\overline{p}=128$ kg/cm, we thus get from (*2*, 9), (*3*, 6b) and (10c) $p=0{,}16$, $c=13{,}55$, $f_H(c)=0{,}6899$, $H=0{,}1104$, $\overline{H}=4{,}747\cdot10^6$ kg, $H_p=47{,}747\cdot10^6$ kg for the case c; while for the same values of $\overline{p}$ and p $H=p=0{,}16$, $\overline{H}=6{,}880\cdot10^6$ kg, $H_p=49{,}880\cdot10^6$ kg for the case a; $H=\frac{1}{2}p=0{,}08$, $\overline{H}=3{,}440\cdot10^6$ kg, $H_p=46{,}440\cdot10^6$ kg, $c=13{,}36$ for the case b.

We have for the same bridge, equipped with a perfectly slack girder ($c_0=\infty$, $c=\infty$) the same values for H, $\overline{H}$ and H_p for the cases a and b, while for the case c $f_H(c)=0{,}6875$, $H=0{,}11$, $\overline{H}=4{,}730\cdot10^6$ kg, $H_p=47{,}730\cdot10^6$ kg.

For this bridge and the same values of $\overline{p}$ and p we also computed the girder deflection η for the cases b and c both for $c_0=12{,}86$ and $c_0=\infty$. The thus found values of η are given in the tables III and IV and plotted in the figures 3 and 4 against x. The dotted and the chain dotted curves drawn in these figures will be treated in the chapters *6* and *7*.

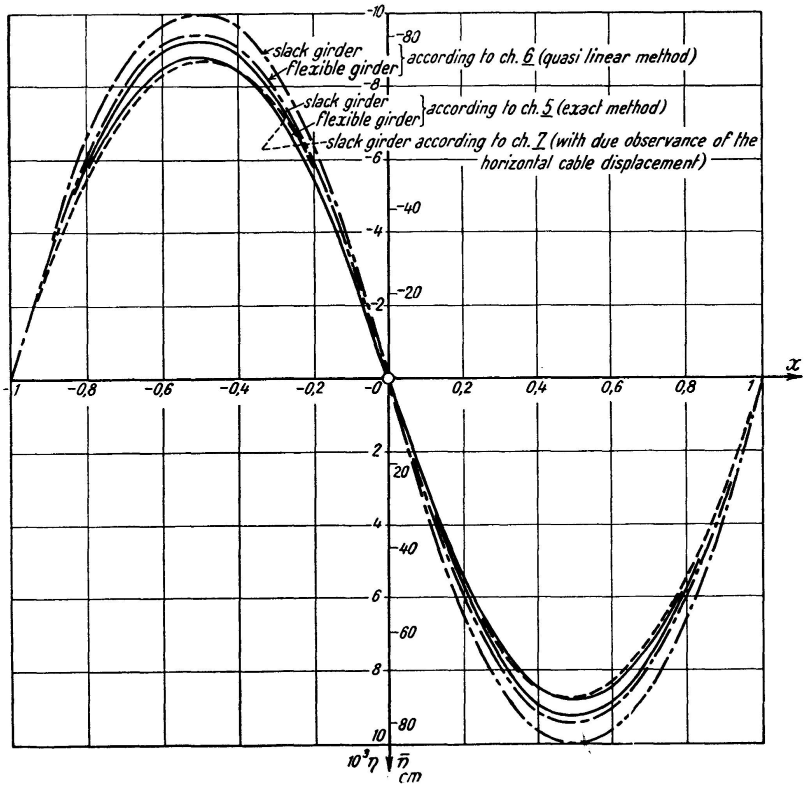

Fig. 3. The girder deflection with an inextensible cable, a constant live load being placed on one of the two halves of the girder

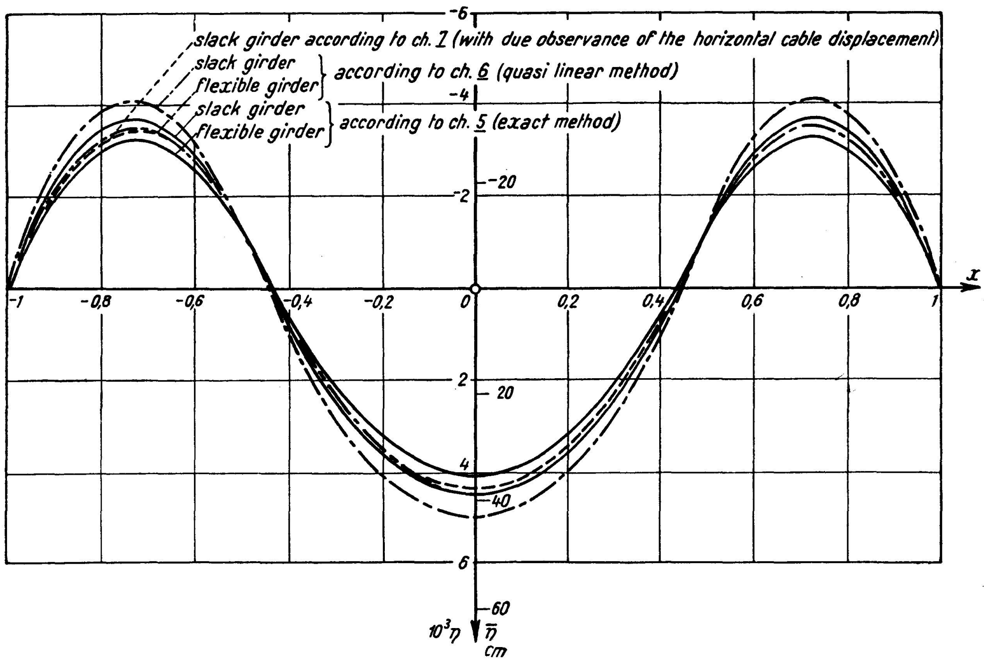

Fig. 4. The girder deflection with an inextensible cable and a constant symmetrical live load extending over half the girder

Table III. The girder deflection for some values of x with an inextensible cable, a constant live load being placed on one of the two halves of the girder

x	according to ch. *5* (exact method)				according to ch. *6* (quasi linear method)			
	flexible girder		slack girder		flexible girder		slack girder	
	$10^3\,\eta$	$\bar{\eta}$ cm	$10^3\,\eta$	$\bar{\eta}$ cm	$10^3\,\eta$	$\bar{\eta}$ cm	$10^3\,\eta$	$\bar{\eta}$ cm
-1	0	0	0	0	0	0	0	0
$-0{,}8$	$-5{,}540$	$-47{,}6$	$-5{,}926$	$-51{,}0$	$-5{,}952$	$-51{,}2$	$-6{,}4$	$-55{,}04$
$-0{,}6$	$-8{,}476$	$-72{,}9$	$-8{,}889$	$-76{,}4$	$-9{,}120$	$-78{,}4$	$-9{,}6$	$-82{,}56$
$-0{,}4$	$-8{,}476$	$-72{,}9$	$-8{,}889$	$-76{,}4$	$-9{,}120$	$-78{,}4$	$-9{,}6$	$-82{,}56$
$-0{,}2$	$-5{,}540$	$-47{,}6$	$-5{,}926$	$-51{,}0$	$-5{,}952$	$-51{,}2$	$-6{,}4$	$-55{,}04$
0	0	0	0	0	0	0	0	0
0,2	5,540	47,6	5,926	51,0	5,952	51,2	6,4	55,04
0,4	8,476	72,9	8,889	76,4	9,120	78,4	9,6	82,56
0,6	8,476	72,9	8,889	76,4	9,120	78,4	9,6	82,56
0,8	5,540	47,6	5,926	51,0	5,952	51,2	6,4	55,04
1	0	0	0	0	0	0	0	0

Table IV. The girder deflection for some values of x with an inextensible cable and a constant symmetrical live load extending over half the girder

x	according to ch. *5* (exact method)				according to ch. *6* (quasi linear method)			
	flexible girder		slack girder		flexible girder		slack girder	
	$10^3\,\eta$	$\bar{\eta}$ cm	$10^3\,\eta$	$\bar{\eta}$ cm	$10^3\,\eta$	$\bar{\eta}$ cm	$10^3\,\eta$	$\bar{\eta}$ cm
0	4,09	35,2	4,50	38,7	4,50	38,7	5	43
0,1	3,87	33,3	4,28	36,8	4,25	36,6	4,75	40,85
0,2	3,20	27,5	3,60	31,0	3,51	30,2	4	34,4
0,3	2,10	18,1	2,48	21,3	2,30	19,8	2,75	23,65
0,4	0,61	5,2	0,90	7,7	0,66	5,7	1	8,6
0,5	$-1{,}11$	$-9{,}5$	$-1{,}12$	$-9{,}6$	$-1{,}22$	$-10{,}5$	$-1{,}25$	$-10{,}75$
0,6	$-2{,}56$	$-22{,}0$	$-2{,}88$	$-24{,}8$	$-2{,}81$	$-24{,}2$	$-3{,}2$	$-27{,}52$
0,7	$-3{,}23$	$-27{,}8$	$-3{,}65$	$-31{,}4$	$-3{,}54$	$-30{,}4$	$-4{,}05$	$-34{,}83$
0,8	$-2{,}98$	$-25{,}6$	$-3{,}42$	$-29{,}4$	$-3{,}27$	$-28{,}1$	$-3{,}8$	$-32{,}68$
0,9	$-1{,}84$	$-15{,}8$	$-2{,}21$	$-19{,}0$	$-2{,}01$	$-17{,}3$	$-2{,}45$	$-21{,}07$
1	0	0	0	0	0	0	0	0

6. Approximate solution with the aid of the quasi linear method. Discussing the formula (*5*, 7) we remarked already that the principle of superposition generally does not hold for the cable force H when $c \neq 0$ and $c \neq \infty$. A look at the formulæ (*5*, 6) and (*5*, 8) shows that this principle neither holds for the deflection and the bending moment. In other words: the cable force, the deflection and the bending moment are not linearly dependent on the live load $p(x)$.

From (*5*, 6a) we learn that for $c = \infty$ the deflection neither depends linearly on $p(x)$.

We can find an approximate solution which does depend linearly on the live load by putting $H = 0$ in the formula (*5*, 3a). By this alteration the values of c are replaced by c_0 a.o. in the solutions (*5*, 6)—(*5*, 8) derived from the differential equation (*5*, 1) while moreover in the lefthand term of the formula (*5*, 6) for η $1 + H$ must be replaced by H. It is obvious that this approximation may only be executed reasonably at low values of H.

In order to check this we computed the cable forces and deflections, determined sub *b* and *c* of nr. *5*, as well according to the just mentioned so called quasi linear method[9]; in other words: by help of the formulæ (*5*, 6) and (*5*, 7) at which c is replaced by c_0, and in the lefthand term of (*5*, 6) $1 + H$ is replaced by 1. The values of the cable force are the same as the corresponding values found in chapter *5* with the exception of the case of loading *c* for a bridge equipped with a flexible girder, where we now have $f_H(c) = 0{,}6902$, $H = 0{,}1105$. The results for the deflections are shown in the tables III and IV and in the chain dotted curves in the figures 3 and 4, from which may be concluded that this approximate method can be applied to such a bridge as is mentioned under "example" in table I. Furtheron it can be verified that the bending moments increase with amounts of the same order of magnitude.

Now one of the great advantages of this method is that it allows to draw lines of influence for certain deflections and bending moments in the usual way and to determine the most unfavourable load system from these lines. This having been done the deflections and tensions may be computed afterwards, if necessary, with the aid of the more exact method treated in nr. *5*.

For the case $c_0 = \infty$ the above-mentioned method reduces the formula (*5*, 6a) for the deflection to

$$2\eta = (1-x)\int_{-1}^{x}(1+z)\,p(z)\,dz + (1+x)\int_{x}^{+1}(1-z)\,p(z)\,dz - H(1-x^2), \tag{1}$$

while the cable forces (*5*, 7a) remains equal to

$$H = \tfrac{3}{4}\int_{-1}^{+1}(1-x^2)\,p(x)\,dx. \tag{2}$$

We wrote down these formulæ explicitly here as we shall need them later on in chapters *16*—*19*.

7. *The influence of the horizontal cable displacement.* In order to examine whether it is allowed to neglect the horizontal cable displacement we shall repeat the computations of chapter *5*, but now taking in account this displacement. Now the starting point of the calculations is the differential equation (*3*, 7a). We shall restrict ourselves to the case of a perfectly slack

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[9]) H. H. Bleich, Die Berechnung verankerter Hängebrücken, Vienna (1935) p. 26.

girder, for which $c_0 = \infty$. For this value of c_0 the deflection computed observing the horizontal cable displacement will show the greatest difference with the deflection computed without this displacement: for then the balancing influence of the term $\frac{1}{c_0^2}\frac{\partial^4 \eta}{\partial x^4}$, which is apparent for other values of c_0, is eliminated.

For $c_0 = \infty$ and for the statical case (*3*, 7a) reduces to

$$(1+H)\frac{d}{dx}\left\{(1+f^2x^2)\frac{d\eta}{dx}\right\} = -p(x)+H, \tag{1}$$

with the boundary conditions

$$\eta = 0 \text{ for } x = \pm 1. \tag{2}$$

The solution of the homogeneous equation to be derived from (1) reads

$$(1+H)\,\eta = A \operatorname{bg\,tg} f x + B; \tag{3}$$

from this we calculate the general solution of (1) by the methods of variation of constants, admitting for that purpose A and B to be functions of x. This yields

$$(1+H)\frac{d\eta}{dx} = \frac{fA}{1+f^2x^2},\quad (1+H)\frac{d}{dx}\left\{(1+f^2x^2)\frac{d\eta}{dx}\right\} = f\frac{dA}{dx} = -p(x)+H, \tag{4}$$

at which we put moreover

$$\operatorname{bg\,tg} f x\frac{dA}{dx} + \frac{dB}{dx} = 0. \tag{5}$$

By substituting (4) in (1) we find that A is determinated by

$$f\frac{dA}{dx} = -p(x)+H, \tag{6a}$$

so that, according to (5), B can be found afterwards from

$$f\frac{dB}{dx} = p(x)\operatorname{bg\,tg} f x - H\operatorname{bg\,tg} f x. \tag{6b}$$

This yields for A and B

$$2fA = -\int_{-1}^{x} p(z)\,dz + \int_{x}^{+1} p(z)\,dz - \frac{1}{\operatorname{bg\,tg} f}\int_{-1}^{+1} p(x)\operatorname{bg\,tg} f x\,dx + 2Hx, \tag{7a}$$

$$2fB = \int_{-1}^{x} p(x)\operatorname{bg\,tg} f z\,dz - \int_{x}^{+1} p(z)\operatorname{bg\,tg} f z\,dz + \operatorname{bg\,tg} f\int_{-1}^{+1} p(x)\,dx$$
$$-H\left(2x\operatorname{bg\,tg} f x + \frac{1}{f}\ln\frac{1+f^2}{1+f^2x^2}\right). \tag{7b}$$

In these formulæ we have chosen the constants of integration in the solutions for A and B obtained from (6a) and (6b) in such a way that η (3) satisfies the boundary conditions (2). We then find for η

$$2f(1+H)\,\eta = -\int\limits_{-1}^{x} (\operatorname{bg}\operatorname{tg} f x - \operatorname{bg}\operatorname{tg} f z)\, p(z)\, dz + \int\limits_{x}^{+1} (\operatorname{bg}\operatorname{tg} f x - \operatorname{bg}\operatorname{tg} f z)\, p(z)\, dz$$

$$+ \operatorname{bg}\operatorname{tg} f \int\limits_{-1}^{+1} \left\{1 - \frac{\operatorname{bg}\operatorname{tg} f x \operatorname{bg}\operatorname{tg} f z}{(\operatorname{bg}\operatorname{tg} f)^2}\right\} p(z)\, dz - \frac{H}{f} \ln \frac{1+f^2}{1+f^2x^2}. \tag{8}$$

Next we can determinate the cable force H with the help of the cable condition (*5*, 2), which yields after some calculations

$$4H = \frac{\ln(1+f^2)\int\limits_{-1}^{+1} p(x)\, dx - \int\limits_{-1}^{+1} \ln(1+f^2x^2)\, p(x)\, dx}{1 - \dfrac{\operatorname{bg}\operatorname{tg} f}{f}}. \tag{9}$$

The formulæ (8) and (9) show that (just as was the case in nr. *5* for $c = \infty$) the principle of superposition does hold for the cable force, but not for the girder deflection.

We illustrate the just described method from the load cases treated in chapter *5* sub *b* and *c*. For the first mentioned case, at which the load is indicated by (5, 9b), we have

$$H = \tfrac{1}{2} p, \tag{10}$$

$$\left.\begin{aligned} 4f^2 \frac{1+H}{p}\,\eta &= \frac{\operatorname{bg}\operatorname{tg} f x}{\operatorname{bg}\operatorname{tg} f} \ln(1+f^2) + \ln(1+f^2x^2) \quad \text{for } -1 \leqq x \leqq 0, \\ &= \frac{\operatorname{bg}\operatorname{tg} f x}{\operatorname{bg}\operatorname{tg} f} \ln(1+f^2) - \ln(1+f^2x^2) \quad \text{for } 0 \leqq x \leqq 1. \end{aligned}\right\} \tag{11}$$

For the second case the formulæ (8), (9) and (*5*, 9a) yield

$$H = p\,\frac{\dfrac{1}{4}\ln\dfrac{1+f^2}{1+\frac{1}{4}f^2} + \dfrac{1}{2} - \dfrac{\operatorname{bg}\operatorname{tg}\frac{1}{2}f}{f}}{1 - \dfrac{\operatorname{bg}\operatorname{tg} f}{f}}, \tag{12}$$

$$\left.\begin{aligned} 2f^2 \frac{1+H}{p}\,\eta &= f(\operatorname{bg}\operatorname{tg} f + \operatorname{bg}\operatorname{tg} f x) - \frac{H}{p}\ln\frac{1+f^2}{1+f^2x^2} \quad \text{for } -1 \leqq x \leqq -\tfrac{1}{2}, \\ &= f(\operatorname{bg}\operatorname{tg} f - \operatorname{bg}\operatorname{tg}\tfrac{1}{2} f) + \ln\frac{1+f^2}{1+f^2x^2} - \frac{H}{p}\ln\frac{1+f^2}{1+f^2x^2} \quad \text{for } -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}, \\ &= f(\operatorname{bg}\operatorname{tg} f - \operatorname{bg}\operatorname{tg} f x) - \frac{H}{p}\ln\frac{1+f^2}{1+f^2x^2} \quad \text{for } \tfrac{1}{2} \leqq x \leqq 1. \end{aligned}\right\} \tag{13}$$

For a bridge with dimensions as mentioned in table I as "example" we found, with $\overline{p} = 128$ kg/cm, $p = 0{,}16$ for the first load case $H = 0{,}08$, and for the second case $H = 0{,}6937\ p = 0{,}1110$. The deflections belonging to them have been given in table V and they are plotted in the figures 3 and 4 (see the dotted curves).

Table V. The girder deflection for some values of x with an inextensible cable calculated with due observance of the horizontal cable displacement

constant live load placed on one of the two halves of the girder			constant symmetrical live load extending over half the girder		
x	$10^3\,\eta$	$\bar{\eta}$ cm	x	$10^3\,\eta$	$\bar{\eta}$ cm
−1	0	0	0	4,40	37,8
−0,8	−5,39	−46,4	0,1	4,18	35,9
−0,6	−8,31	−71,5	0,2	3,50	30,1
−0,4	−8,47	−72,8	0,3	2,38	20,5
−0,2	−5,74	−49,4	0,4	0,82	7,1
0	0	0	0,5	−1,16	−10,0
0,2	5,74	49,4	0,6	−2,83	−24,3
0,4	8,47	72,8	0,7	−3,51	−30,2
0,6	8,31	71,5	0,8	−3,24	−27,9
0,8	5,39	46,4	0,9	−2,06	−17,7
1	0	0	1	0	0

One may conclude from them that for the bridge treated as example the deflections are at most 15% too large, the girder rigidity and the horizontal cable deflection being neglected in the calculations, while the deflections are at most 22% too large when the quasi linear method is used in addition.

8. Extensible cable. We are now to perform calculations analogous to those made in chapter *5* for the case of an extensible cable. We here restrict ourselves to the case of a perfectly slack girder, for which $c_0 = \infty$. Then the equation (*3*, 10a) for the deflection yields immediately

$$(1+H)(1-\gamma H-\beta\vartheta)\,\eta = -M_1 - \tfrac{1}{2}H(1-x^2) + \tfrac{1}{2}(1+H)(\gamma H+\beta\vartheta)(1-x^2). \quad (1)$$

Now the quantity $\gamma = H_w/E_c F_c$ is smaller than the maximum tension admissible in the cable, divided by Young's modulus, and thus is of the same order of magnitude as $10^4\ \mathrm{kg/cm^2} : 2\cdot 10^6\ \mathrm{kg/cm^2} = 0{,}5\%$. The product γH is still smaller, while the product $\beta\vartheta$ is small compared with 1 as well. Thus we can without objection, making use of (*3*, 11), reduce (1) to

$$2(1+H)\,\eta = (1-x)\int_{-1}^{x} (1+z)\,p(z)\,dz + (1+x)\int_{x}^{+1}(1-z)\,p(z)\,dz - H(1-x^2) + (1+H)\,\beta\vartheta\,(1-x^2). \quad (2)$$

The cable force now can be calculated with the aid of the condition (*3*, 8), which yields with (*3*, 6b) the equation for H

$$\alpha H^2 + \left(1+\alpha+\frac{3}{f^2}\beta\vartheta\right)H - J + \frac{3}{f^2}\beta\vartheta = 0, \quad (3)$$

with

$$\alpha = \frac{3}{2}(\gamma l_3 + \nu), \quad J = \frac{3}{4}\int_{-1}^{+1}(1-x^2)\,p(x)\,dx. \quad (4)$$

The solution of (3) which, as it should, for $\alpha \to 0$, $\vartheta \to 0$ satisfies the relation (5, 8a), i.e. $H = J$, reads

$$\alpha H = \tfrac{1}{2}\left(1+\alpha+\frac{3}{f^2}\beta\vartheta\right)\left(-1+\sqrt{1+4\epsilon}\right) \tag{5}$$

with

$$\epsilon = \frac{\alpha\left(J-\frac{3}{f^2}\beta\vartheta\right)}{\left(1+\alpha+\frac{3}{f^2}\beta\vartheta\right)^2}. \tag{6}$$

The second factor of the righthand term of (5) being the difference of two almost equal numbers, it is advisable to develop the radical quantity in a series. This yields with (6)

$$H = \frac{J-\frac{3}{f^2}\beta\vartheta}{1+\alpha+\frac{3}{f^2}\beta\vartheta}\left(1-\epsilon+2\epsilon^2-5\epsilon^3\ldots\right). \tag{7}$$

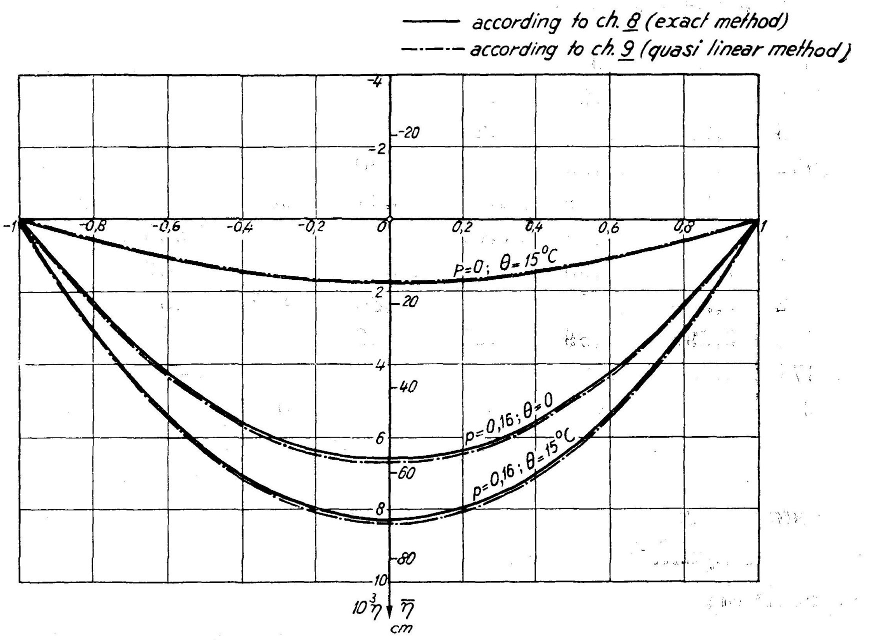

Fig. 5. The girder deflection with an extensible cable and a constant live load placed on the whole length

We illustrate the just developed method of calculations by applying it to a number of cases of loading. To this we add the case of a change of the cable temperature, no live load being placed on the girder.

a) Only a change of temperature. In this case we have $p(x) = 0$, so that (4), (6), (7) and (2) yield respectively

$$J = 0, \quad \epsilon = -\frac{\alpha\frac{3}{f^2}\beta\vartheta}{\left(1+\alpha+\frac{3}{f^2}\beta\vartheta\right)^2}, \tag{8a}$$

$$H = -\frac{\frac{3}{f^2}\beta\vartheta}{1+\alpha+\frac{3}{f^2}\beta\vartheta}(1-\epsilon+2\epsilon^2-5\epsilon^3), \quad (9a) \qquad 2\eta = \left(-\frac{H}{1+H}+\beta\vartheta\right)(1-x^2). \quad (10a)$$

With the data of table I, to which we add $E_c = 1{,}8 \cdot 10^6$ kg/cm², $F_c = 7140$ cm⁴, $\gamma = H_w/E_c F_c = 3{,}346 \cdot 10^{-3}$, $\bar{\nu} = 1{,}23 \cdot 10^{-6}$ cm/kg, $\vartheta = 15°$ C, $\beta = 12{,}5 \cdot 10^{-6}$ per °C, $\beta\vartheta = 0{,}1875 \cdot 10^{-3}$, we have $l_2 = 13{,}667$, $l_3 = 13{,}524$, $\nu = 0{,}015 \cdot 375$, $\alpha = 0{,}090 \cdot 939$, $\epsilon = -2{,}9350 \cdot 10^{-3}$, $H = -0{,}003 \cdot 221$. We assembled in table VI the values of η and $\bar{\eta}$ calculated with the aid of (10a) and plotted them in figure 5 against x.

Table VI. The girder deflection for some values of x with an extensible cable and a constant live load placed on the whole length

x	according to ch. *8* (exact method)						according to ch. *9* (quasi linear method)					
	$10^3\eta$			$\bar{\eta}$ cm			$10^3\eta$			$\bar{\eta}$ cm		
	$p=0$, $\vartheta=15°$C	$p=0{,}16$, $\vartheta=0$	$p=0{,}16$, $\vartheta=15°$C	$p=0$, $\vartheta=15°$C	$p=0{,}16$, $\vartheta=0$	$p=0{,}16$, $\vartheta=15°$C	$p=0$, $\vartheta=15°$C	$p=0{,}16$, $\vartheta=0$	$p=0{,}16$, $\vartheta=15°$C	$p=0$, $\vartheta=15°$C	$p=0{,}16$, $\vartheta=0$	$p=0{,}16$, $\vartheta=15°$C
−1	0	0	0	0	0	0	0	0	0	0	0	0
−0,8	0,614	2,371	2,980	5,28	20,39	25,63	0,614	2,401	3,015	5,28	20,65	25,93
−0,6	1,092	4,215	5,298	9,39	36,25	45,56	1,091	4,268	5,359	9,38	36,70	46,09
−0,4	1,433	5,532	6,954	12,32	47,58	59,80	1,432	5,602	7,034	12,32	48,18	60,49
−0,2	1,638	6,323	7,947	14,09	54,38	68,34	1,637	6,402	8,039	14,08	55,06	69,14
0	1,706	6,586	8,278	14,67	56,64	71,19	1,705	6,669	8,374	14,66	57,35	72,02
0,2	1,638	6,323	7,947	14,09	54,38	68,34	1,637	6,402	8,039	14,08	55,06	69,14
0,4	1,433	5,532	6,954	12,32	47,58	59,80	1,432	5,602	7,034	12,32	48,18	60,49
0,6	1,092	4,215	5,298	9,39	36,25	45,56	1,091	4,268	5,359	9,38	36,70	46,09
0,8	0,614	2,371	2,980	5,28	20,39	25,63	0,614	2,401	3,015	5,28	20,65	25,93
1	0	0	0	0	0	0	0	0	0	0	0	0

b) Constant live load placed on the whole length. In this case the load can be represented again by (*5*, 9a), with which the equations (4), (6), (7) and (2) yield respectively

$$J = p, \quad \epsilon = \frac{\alpha\left(p - \frac{3}{f^2}\beta\vartheta\right)}{\left(1+\alpha+\frac{3}{f^2}\beta\vartheta\right)^2}, \tag{8b}$$

$$H = \frac{p - \frac{3}{f^2}\beta\vartheta}{1+\alpha+\frac{3}{f^2}\beta\vartheta}(1-\epsilon+2\epsilon^2-5\epsilon^3), \quad (9b) \qquad 2\eta = \left(\frac{b-H}{1+H}+\beta\vartheta\right)(1-x^2). \quad (10b)$$

In particular for $\vartheta = 0$

$$\epsilon = \frac{\alpha p}{(1+\alpha)^2},$$

$$H = \frac{p}{1+\alpha}(1-\epsilon+2\epsilon^2-5\epsilon^3\ldots), \quad 2(1+H)\eta = (p-H)(1-x^2).$$

For this case we have, with the principal dimensions as under a and $\overline{p}=128$ kg/cm, $p=0{,}16$: $\epsilon=12{,}33\cdot10^{-3}$, $H=0{,}14492$, while the values of η and $\overline{\eta}$ belonging to it are shown in table VI and plotted in fig. 5 against x.

For the same values of p and $\vartheta=15°$ C we get: $\epsilon=11{,}88\cdot10^{-3}$, $H=0{,}14132$; the deflections η and $\overline{\eta}$ also are listed in table VI and plotted in fig. 5 against x.

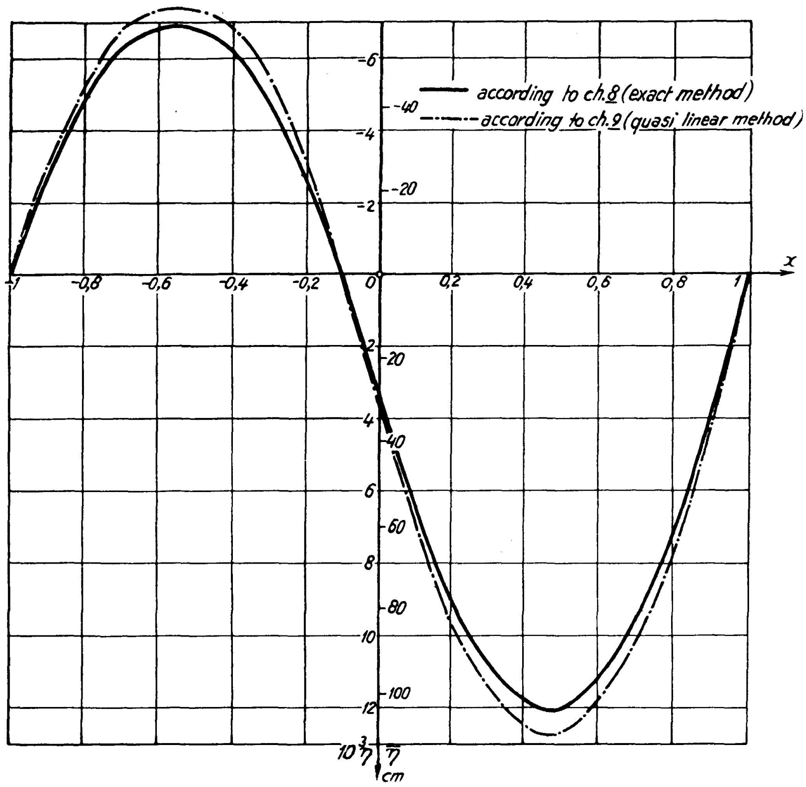

Fig. 6. The girder deflection with an extensible cable and a constant live load placed on one of the two halves of the girder

c) Constant live load on one of the two halves of the girder. For the load (5, 9b) and $\vartheta=0$ the formulæ (4), (6), (7) and (2) yield respectively

$$J=\tfrac{1}{2}p,\quad \epsilon=\frac{\alpha p}{2(1+\alpha)^2}, \tag{8c}$$

$$H=\frac{p}{2(1+\alpha)}\,(1-\epsilon+2\epsilon^2-5\epsilon^3\ldots), \tag{9c}$$

$$\left.\begin{aligned} 2\frac{1+H}{p}\,\eta &= (1+x)\left\{\frac{1}{2}-\frac{H}{p}(1-x)\right\} \quad \text{for } -1\leqq x\leqq 0,\\ &= (1-x)\left\{\frac{1}{2}+x-\frac{H}{p}(1+x)\right\} \quad \text{for } 0\leqq x\leqq 1. \end{aligned}\right\} \tag{10c}$$

With the principal dimensions as under a and $\overline{p}=128$ kg/cm, $p=0{,}16$ these formulæ yield $\epsilon=6{,}113\cdot10^{-3}$, $H/p=0{,}4556$, $H=0{,}07289$; the values of η and $\overline{\eta}$ belonging to it are shown in table VII and plotted in fig. 6 against x.

Table VII. The girder deflection for some values of x with an extensible cable and a constant live load placed on one of the two halves of the girder

x	according to ch. *8* (exact method)		according to ch. *9* (quasi linear method)	
	$10^3\,\eta$	$\bar{\eta}$ cm	$10^3\,\eta$	$\bar{\eta}$ cm
-1	0	0	0	0
$-0{,}8$	$-4{,}77$	$-41{,}0$	$-5{,}12$	$-44{,}0$
$-0{,}6$	$-6{,}83$	$-58{,}7$	$-7{,}47$	$-64{,}2$
$-0{,}4$	$-6{,}17$	$-53{,}1$	$-6{,}80$	$-58{,}5$
$-0{,}2$	$-2{,}79$	$-24{,}0$	$-3{,}20$	$-27{,}5$
0	3,31	28,5	4,14	35,6
0,2	9,14	78,6	9,60	82,6
0,4	11,73	100,9	12,40	106,6
0,6	11,07	95,2	11,74	101,0
0,8	7,16	61,6	7,60	65,4
1	0	0	0	0

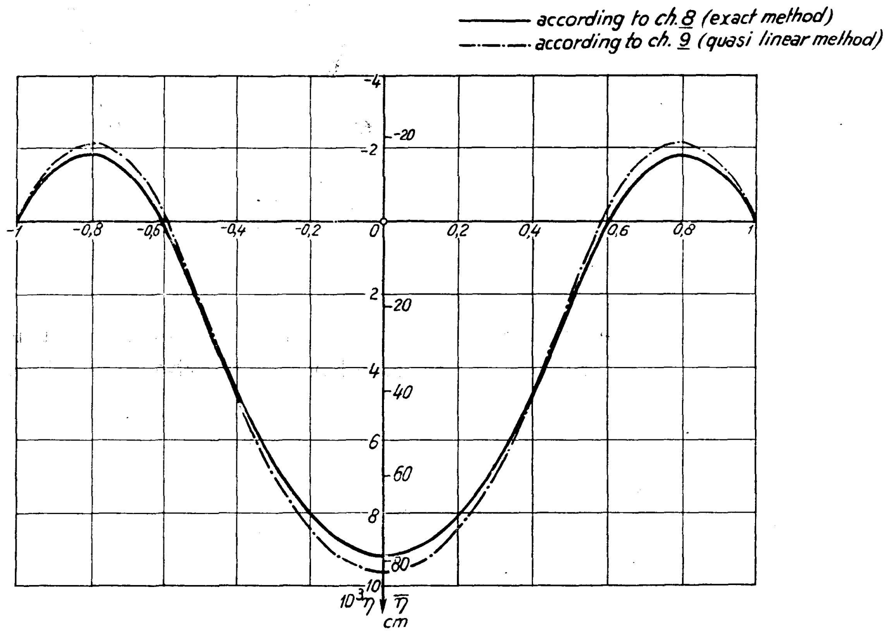

Fig. 7. The girder deflection with an extensible cable and a constant symmetrical live load extending over half the girder

d) Constant symmetrical live load extending over half the girder. For the load case (*5*, 9c) and $\vartheta = 0$ the formulæ (4), (6), (7) and (2) yield respectively

$$J = \frac{11}{16}p, \quad \epsilon = \frac{11\,\alpha\,p}{16\,(1+\alpha)^2}, \tag{8d}$$

$$H = \frac{11\,p}{16\,(1+\alpha)}\,(1 - \epsilon + 2\,\epsilon^2 - 5\,\epsilon^3 \ldots), \tag{9d}$$

$$\left.\begin{aligned} 2\,\frac{1+H}{p}\,\eta &= (1+x)\left\{1-\frac{H}{p}(1-x)\right\} && \text{for } -1 \leqq x \leqq -\tfrac{1}{2}, \\ &= \tfrac{3}{4}-x^2-\frac{H}{p}(1-x^2) && \text{for } -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}, \\ &= (1-x)\left\{1-\frac{H}{p}(1+x)\right\} && \text{for } \tfrac{1}{2} \leqq x \leqq 1. \end{aligned}\right\} \tag{10d}$$

With the principal dimensions as under a and $\bar{p}=128$ kg/cm, $p=0{,}16$ these formulæ yield $\epsilon=8{,}405\cdot 10^{-3}$, $H/p=0{,}6250$, $H=0{,}1000$; the values of η and $\bar{\eta}$ belonging to it are shown in table VIII and plotted in fig. 7 against x.

Table VIII. The girder deflection for some values of x with an extensible cable and a constant symmetrical live load extending over half the girder

x	according to ch. *8* (exact method)		according to ch. *9* (quasi linear method)	
	$10^3\,\eta$	$\bar{\eta}$ cm	$10^3\,\eta$	$\bar{\eta}$ cm
0	9,17	78,8	9,58	82,4
0,1	8,89	76,5	9,29	79,9
0,2	8,07	69,4	8,40	72,2
0,3	6,69	57,5	6,92	59,5
0,4	4,77	41,0	4,85	41,7
0,5	2,29	19,7	2,19	18,8
0,6	0,00	0,0	−0,26	− 2,3
0,7	−1,38	−11,9	−1,71	−14,7
0,8	−1,83	−15,8	−2,15	−18,5
0,9	−1,38	−11,9	−1,58	−13,6
1	0	0	0	0

9. Approximate calculation with the aid of the quasi linear method. We can execute an approximate calculation, in which the deflection depends linearly on the live load, as well with an extensible cable as with an inextensible cable. Also with an extensible cable the method is only utilisable when the live load is small compared with the dead weight of the bridge.

For that purpose we simplify (*8*, 2) to

$$2\,\eta = (1-x)\int_{-1}^{x}(1+z)\,p(z)\,dz+(1+x)\int_{x}^{+1}(1-z)\,p(z)\,dz-H\,(1-x^2)+\beta\,\vartheta\,(1-x^2). \tag{1}$$

Now the formula (*8*, 4) remains valid, but (*8*, 3) and (*8*, 5)—(*8*, 7) must be replaced by

$$H=\frac{J-\dfrac{3}{f^2}\beta\,\vartheta}{1+\alpha}\,. \tag{2}$$

We applied the just derived formulæ to the load cases which are treated in nr. *8* under a—d. No more than in chapter *6* we wrote down the formulæ for the cable force and the deflection; we found for the cable force respectively

for case a: $H = -0{,}0032$;
for case b with $\vartheta = 0$: $H = 0{,}1466$; with $\vartheta = 15°\,\mathrm{C}$: $H = 0{,}1434$;
for case c: $H/p = 0{,}4583$, $H = 0{,}0733$;
for case d: $H/p = 0{,}6302$, $H = 0{,}1008$,

while the results for the deflection are collected in tables VI—VIII and plotted in figures 5—7.

10. The work accumulated in the bridge. We shall now calculate the amounts of work accumulated in different parts of the bridge, a live load being placed on the girder. In this calculation we leave the temperature change out of consideration. In the general statical case we have three kinds of work:

a) the potential energy of gravity w acting on the girder

$$\bar{U} = -\int_{-\frac{1}{2}l}^{+\frac{1}{2}l} w\,\bar{\eta}\,d\bar{x}; \tag{1}$$

b) the deformation energy accumulated in the girder

$$\bar{A}_M = \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} \frac{\bar{M}^2}{2\,E\,I}\,d\bar{x}; \tag{2}$$

c) the deformation energy accumulated in the cable and the pylons; when a live load is placed on the girder, this energy *increases* by an amount

$$\bar{A}_H = \int_{\bar{x}=-\frac{1}{2}l}^{+\frac{1}{2}l} \frac{S^2 - S_w^{\,2}}{2\,E_c\,F_c}\,ds + \frac{1}{2}\,\bar{\nu}\,(H_p^{\,2} - H_w^{\,2}). \tag{3}$$

In the particular case of a bridge with a perfectly slack girder and an inextensible cable, the amounts of energy $\bar{A}_M$ and $\bar{A}_H$ are zero. Apparently the same holds for the energy $\bar{U}$, owing to the condition (*2*, 18), which for this case can be simplified to

$$\int_{-\frac{1}{2}l}^{+\frac{1}{2}l} w\,\bar{\eta}\,d\bar{x} = 0. \tag{4}$$

This seems to be paradoxical, as in general the live load performs a certain amount of work when it is applied to the girder. However we must consider that the condition (4) was derived from the equation (2, 12) by dropping the term with $(d\,\bar{\eta}/d\,\bar{x})^2$. In calculating the work it is not allowed to do so, but here we must start from the complete equation (2, 12).

In the more general case $\gamma \neq 0$ (but $\vartheta = 0$) this equation reads

$$\frac{d\xi}{d\bar{x}} = -\frac{dy}{d\bar{x}}\frac{d\bar{\eta}}{d\bar{x}} - \frac{1}{2}\left(\frac{ds}{d\bar{x}}\right)^2\left(\frac{d\bar{\eta}}{d\bar{x}}\right)^2 + \gamma H\left(\frac{ds}{d\bar{x}}\right)^3. \tag{5}$$

Integrating both members to $\bar{x}$ we find, with (*2*, 14), (*2*, 15) and (*2*, 17)

$$\frac{1}{H_w}\int_{-\frac{1}{2}l}^{+\frac{1}{2}l} w\,\bar{\eta}\,d\bar{x} = (\gamma\bar{l}_3 + \bar{\nu}H_w)H - \frac{1}{2}\int_{-\frac{1}{2}l}^{+\frac{1}{2}l}\left(\frac{ds}{d\bar{x}}\right)^2\left(\frac{d\bar{\eta}}{d\bar{x}}\right)^2 d\bar{x}. \tag{6}$$

Further on the expression for $\bar{A}_H$ (3) can be worked out, with (*2*, 6), (*2*, 9), (*2*, 10), (*2*, 15), $ds/d\bar{x} = 1/\cos\psi_w$ and putting approximately $S\cos\psi_w = H_p$, to

$$\bar{A}_H = H_w(1 + \tfrac{1}{2}H)(\bar{\nu}H_w + \gamma\bar{l}_3)H. \tag{7}$$

Thus the sum of the amounts $\bar{A}_H$ and $\bar{U}$ is given by

$$\bar{U} + \bar{A}_H = \frac{1}{2}H_w\int_{-\frac{1}{2}l}^{+\frac{1}{2}l}\left(\frac{ds}{d\bar{x}}\right)^2\left(\frac{d\bar{\eta}}{d\bar{x}}\right)^2 d\bar{x} + \frac{1}{2}H_w(\bar{\nu}H_w + \gamma\bar{l}_3). \tag{8}$$

For the case of constant mass per unit of length and constant stiffness of the girder we can simplify this expression, with (*3*, 2), (*3*, 3), (*3*, 6b), (*8*, 4) and

$$U = \frac{4\bar{U}}{w f l^2}, \quad A_H = \frac{4\bar{A}_H}{w f l^2}, \tag{9}$$

to

$$U + A_H = \frac{1}{2}\int_{-1}^{+1}(1 + f^2x^2)\left(\frac{d\eta}{dx}\right)^2 dx + \frac{1}{3}\alpha H^2. \tag{10}$$

With the exception of chapter *7* we always replace $ds/d\bar{x}$ in such expressions by 1:

$$U + A_H = \frac{1}{2}\int_{-1}^{+1}\left(\frac{d\eta}{dx}\right)^2 dx + \frac{1}{3}\alpha H^2. \tag{11}$$

Restricting ourselves to the case $c_0 = \infty$ we can use the expression (*8*, 2), with $\vartheta = 0$. This yields, with (*8*, 3) and (*8*, 4), after some calculations

$$\begin{aligned} 4(1+H)^2(U + A_H) &= \int_{-1}^{+1}(1-x)\,p(x)\,dx\int_{-1}^{x}(1+z)\,p(z)\,dz \\ &+ \int_{-1}^{+1}(1+x)\,p(x)\,dx\int_{x}^{+1}(1-z)\,p(z)\,dz - \frac{4}{3}H^2\{1 + \alpha(1-H^2)\}. \end{aligned} \tag{12}$$

For an inextensible cable we have $A_H = 0$, $\alpha = 0$, so that (12) can be simplified to

$$4(1+H)^2 U = \int_{-1}^{+1}(1-x)\,p(x)\,dx\int_{-1}^{x}(1+z)\,p(z)\,dz + \int_{-1}^{+1}(1+x)\,p(x)\,dx\int_{x}^{+1}(1-z)\,p(z)\,dz - \frac{4}{3}H^2. \tag{13}$$

In this case $\overline{U}$ must equal the work $\overline{A}_p$ performed by the live load, which can be calculated by assuming that the live load grows regularly from zero to the ultimate value $\overline{p}(\overline{w}) = w\,p(x)$. Putting the momentary load to

$$\overline{p}^*(\overline{x}) = \lambda \overline{p}(\overline{x}) = w \lambda p(x) = w p^*(x), \tag{14}$$

we have for the cable force belonging to it at the same moment, according to (*5*, 7a),

$$H^* = \frac{3}{4}\lambda \int_{-1}^{+1} (1-x^2)\, p(x)\, dx = H\lambda. \tag{15}$$

Now the work performed by the partial force $\overline{p}^*(\overline{x})\,d\overline{x}$, when the deflection increases with an amount $d\,\overline{\eta}$, equals

$$d\,\overline{A}_p = \overline{p}^*(\overline{x})\, d\overline{x}\, d\overline{\eta} = \overline{p}^*(\overline{x})\, d\overline{x} \frac{d\overline{\eta}}{d\lambda} d\lambda;$$

thus the total work performed by all the partial loads together, when they grow from 0 to their end values, equals

$$\overline{A}_p = \int_{\overline{x}=-\frac{1}{2}l}^{+\frac{1}{2}l} \int_{\lambda=0}^{1} \overline{p}^*(\overline{x}) \frac{d\overline{\eta}}{d\lambda}\, d\overline{x}\, d\lambda. \tag{16}$$

With (*3*, 6b) and

$$A_p = \frac{4\,\overline{A}_p}{w f l^2} \tag{17}$$

this can be reduced to

$$A_p = \int_{x=-1}^{+1} p(x)\, dx \int_{\lambda=0}^{1} \lambda \frac{d\eta}{d\lambda}\, d\lambda. \tag{18}$$

In this equation we can insert the formula (*5*, 6) for the deflection η, which now must be altered into

$$2\,\frac{1+\lambda H}{\lambda}\,\eta = (1-x)\int_{-1}^{x} (1+z)\, p(z)\, dz + (1+x)\int_{x}^{+1} (1-z)\, p(z)\, dz - H(1-x^2); \tag{19}$$

this yields, with (*5*, 8a) and

$$f_1(H) = 2\int_0^1 \frac{\lambda\, d\lambda}{(1+\lambda H)^2} = 2\,\frac{\ln(1+H) - \dfrac{H}{1+H}}{H^2}, \tag{20}$$

$$4 A_p = f_1(H)\left\{\int_{-1}^{+1}(1-x)\, p(x)\, dx \int_{-1}^{x}(1+z)\, p(z)\, dz + \int_{-1}^{+1}(1+x)\, p(x)\, dx \int_{x}^{+1}(1-z)\, p(z)\, dz - \frac{4}{3}H^2.\right. \tag{21}$$

Comparing (21) and (13) we see that U differ only slightly from A_p for small values of H. For by developing in series it may be found

$$f_1(H) = 1 - \frac{4}{3}H + \frac{3}{2}H^2 - \frac{8}{5}H^3 + \ldots,$$

$$f_2(H) = \frac{1}{(1+H)^2} = 1 - 2H + 3H^2 - 4H^3 + \ldots, \tag{22}$$

so that $\lim_{H\to 0} f_1(H) = \lim_{H\to 0} f_2(H),$

while the table IX and figure 8 show the difference between $f_1(H)$ and $f_2(H)$ for other values of H.

That the two just derived expressions (13) and (21) approach each other only for small values of H need not create surprise. For we assumed in all our calculations $\bar{\eta}$ to be small compared with y. This involves (for large values of c_0) that H is small compared with 1.

We shall now consider the deformation energy of the girder $\bar{A}_M$. With (*3*, 3), (*3*, 6b) and

$$A_M = \frac{4\bar{A}_M}{w f l^2} \tag{23}$$

we can replace (2) by

$$A_M = \frac{1}{2} c_0^2 \int_{-1}^{+1} M^2 dx = \frac{1}{2c_0^2} \int_{-1}^{+1} \left(\frac{d^2\eta}{dx^2}\right)^2 dx. \tag{24}$$

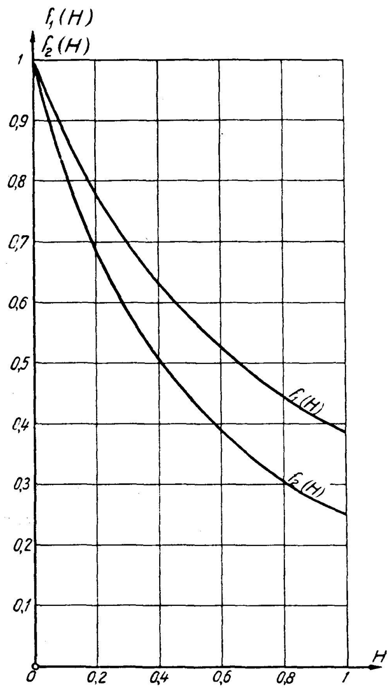

Fig. 8
Values of $f_1(H)$ and $f_2(H)$

Table IX. Values of $f_1(H)$ *and* $f_2(H)$

H	$f_1(H)$	$f_2(H)$	H	$f_1(H)$	$f_2(H)$
0	1	1	0,6	0,528	0,391
0,1	0,880	0,826	0,7	0,485	0,346
0,2	0,782	0,694	0,8	0,448	0,309
0,3	0,702	0,592	0,9	0,415	0,277
0,4	0,635	0,510	1	0,386	0,25
0,5	0,577	0,444			

We shall henceforth restrict ourselves to the case of large c_0. Then A_M is small in comparison with U and A_H, and without making a great mistake we can substitute the formula (*8*, 2) for η (with $\vartheta = 0$) in (24). This yields

$$2c_0^2(1+H)^2 A_M = \int_{-1}^{+1} \{p(x) - H\}^2 dx. \tag{25}$$

In this case also the formula (12) (which was derived for $c_0 = \infty$) can be used without great error.

Under the just mentioned conditions we compute these amounts of work for the cases of loading treated in nr. *8* sub b, c and d. For a live load working over the whole length of the bridge (case b) the load can be representated by (*5*, 9a) and (12) and (24) yield

$$3\,(1+H)^2\,(U+A_H) = p^2 - H^2\{1+\alpha\,(1-H^2)\},\; 2\,c_0^{\,2}\,(1+H)^2\,A_M = (p-H)^2. \quad (26)$$

When the live load occupies only one of the two halves of the girder (case c) we have with (*5*, 9b)

$$3\,(1+H)^2\,(U+A_H) = \frac{5}{16}\,p^2 - H^2\{1+\alpha\,(1-H^2)\},$$
$$2\,c_0^{\,2}\,(1+H)^2\,A_M = (p-H)^2 + H^2, \quad (27)$$

and for the case of a symmetrical load over the half of the girder (12) and (24) give with (*5*, 9c)

$$3\,(1+H)^2\,(U+A_H) = \tfrac{1}{2}\,p^2 - H^2\{1+\alpha\,(1-H^2)\},$$
$$2\,c_0^{\,2}\,(1+H)^2\,A_M = (p-H)^2 + H^2. \quad (28)$$

For the bridge mentioned in Table I as "example" we have computed the different amounts of work numerically for the value $\bar{p} = 128$ kg/cm, $p = 0{,}16$. The results are shown in Table X.

Table X. The amounts of work accumulated in the different parts of the bridge

load-case	extensible cable, flexible girder		inextensible cable, flexible girder			extensible cable, slack girder		inextensible cable, slack girder		
	$10^6(U+A_H)$	$10^6\,A_M$	$10^6\,U$	$10^6\,A_H$	$10^6\,A_M$	$10^6(U+A_H)$	$10^6\,A_M$	$10^6\,U$	$10^6\,A_H$	$10^6\,A_M$
b	694,5	0,5245	0	0	0	694,5	0	0	0	0
c	638,6	33,885	457,2	0	33,172	638,6	0	457,2	0	0
d	768,8	33,982	189,4	0	35,826	768,8	0	189,4	0	0
	$10^{-6}(\bar{U}+\bar{A}_H)$ kgcm	$10^{-6}\bar{A}_M$ kgcm	$10^{-6}\,\bar{U}$ kgcm	$10^{-6}\bar{A}_H$ kgcm	$10^{-6}\bar{A}_M$ kgcm	$10^{-6}(\bar{U}+\bar{A}_H)$ kgcm	$10^{-6}\bar{A}_M$ kgcm	$10^{-6}\,\bar{U}$ kgcm	$10^{-6}\bar{A}_H$ kgcm	$10^{-6}\bar{A}_M$ kgcm
b	102,7	0,078	0	0	0	102,7	0	0	0	0
c	94,5	5,01	67,6	0	4,91	94,5	0	67,6	0	0
d	113,7	5,03	28,0	0	5,30	113,7	0	28,0	0	0

§ 4. Dynamical calculations

11. Survey of the calculations to be performed for the dynamical case. As we already mentioned in chapter *1*, until now only a few treatises have been published dealing with the dynamical behaviour of suspension bridges. The most important publication is that of KLÖPPEL and LIE [10] [11], dealing both

[10]) K. KLÖPPEL and K. H. LIE, Lotrechte Schwingungen von Hängebrücken, Ingenieur-Archiv 13 (1942) p. 211.

[11]) Having completed this article, we found that an important study of the dynamical behaviour of suspension bridges was also made by G. S. VINCENT and others; a summary of this study has been published in Engineering News-Record 146 (1951) 2 p. 32. It seems not to deal with the problem of the forced vibrations caused by a moving live load.

with free and with forced vibrations. We shall discuss some of their results in the next chapters.

In our study of the dynamical behaviour we restrict ourselves to the case of a perfectly slack girder and a constant cable temperature, and we neglect the horizontal cable displacement; so we start from the differential equation (*3*, 7b), reduced to

$$-(1+H)\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial t^2} = p(x,t)-H(t), \tag{1}$$

the cable condition (*3*, 8), reduced to

$$\int_{-1}^{+1} \eta\, d x = (\gamma l_3+\nu) H, \tag{2}$$

and the boundary conditions (*3*, 9), reduced to

$$\eta = 0 \text{ for } x = \pm 1. \tag{3}$$

In the chapters *12*—*15* we are to derive the solutions of (1)—(3) for the case $p(x)=0$, corresponding with the so called free vibrations of the bridge. We shall treat the case of an inextensible cable in the chapters *12* and *13*, in chapter *14* the same case, replacing the term $(1+H)\,\partial^2\eta/\partial x^2$ in (1) by $\partial^2\eta/\partial x^2$, according to the quasi linear method, while the case of an extensible cable will be dealt with in chapter *15*, only following the quasi linear method. At last we shall treat the rather difficult problem of the forced vibrations caused by a moving live load without mass, at which we restrict ourself exclusively to the quasi linear case and to a bridge with an inextensible cable; but here we shall derive at first general formulæ for the case of a live load moving with a constant speed and conserving its form; afterwards we shall illustrate these calculations by applying them to two examples.

12. Free vibrations of a bridge with an inextensible cable. The characteristic values and the characteristic functions. As indicated above we shall at first try to answer the question whether the girder can execute free vibrations, i.e. whether it can undergo deflections, a live load $p(x,t)$ being absent; in other words whether the differential equation

$$(1+H)\frac{\partial^2 \eta}{\partial x^2}-\frac{\partial^2 \eta}{\partial t^2} = H(t), \tag{1}$$

the cable condition

$$\int_{-1}^{+1} \eta\, d x = 0 \tag{2}$$

and the boundary conditions

$$\eta = 0 \text{ for } x = \pm 1 \tag{3}$$

admit solutions for η and H different from 0. For that purpose we try, as it is generally usual with problems like this, to find a solution of the form

$$\eta = X_k T_k, \tag{4}$$

X_k being a function (the so called characteristic function) which only depends on x, and T_k, the so called normal coordinate, being only dependent on t. As we shall find in the following that there can exist more functions like that, we apply to them, in order to be able to distinguish them, already the index k, and we do the same thing with the functions H belonging to them. We substitue (4) in (1) and (2); in evaluating we shall, as henceforth anywhere in this paper, indicate differentiations with respect to x by accents and differentiations with respect to t by dots. Then we get

$$(1+H_k)\,X_k''\,T_k - X_k\,\ddot{T}_k = H_k, \tag{5}$$

$$\int_{-1}^{+1} X_k\,dx = 0. \tag{6}$$

From (5) and (6) we derive a new relation by integrating both terms of (5) with respect to x between the limits $x = -1$ and $x = +1$, and by using (6):

$$\tfrac{1}{2}(1+H_k)\,T_k\{X_k'(1) - X_k'(-1)\} = H_k. \tag{7}$$

Now the lefthand terms of (5) and (7) must be equal to each other and from this we can derive after some calculations the relation

$$\frac{X_k'' - \tfrac{1}{2}\{X_k'(1) - X_k'(-1)\}}{X_k} = \frac{\ddot{T}_k}{(1+H_k)\,T_k} = -\omega_k^2. \tag{8}$$

Here the quantity $-\omega_k^2$, to which the both terms of this equation can be equalized, must be considered as constant, as it neither depends on x nor on t. Further on it will be obvious that there can always be found solutions X_k which satisfy the boundary conditions, if this quantity is negative; for this reason it has already been written in the form of a quadratic preceded by a minus sign. The opposite quantity, ω_k^2, we call the k^{th} characteristic value of the problem. We shall not go into the question whether there can also exist solutions for X_k at negative characteristic values; it can be shown without difficulty that this is not the case.

From (8) we can derive the two following equations for X_k and T_k respectively

$$X_k'' + \omega_k^2 X_k = \tfrac{1}{2}\{X_k'(1) - X_k'(-1)\}, \quad (9\text{a}) \qquad \ddot{T}_k + \omega_k^2(1+H_k)\,T_k = 0. \quad (9\text{b})$$

The boundary conditions for X_k are, according to (3) and (4),

$$X_k = 0 \quad \text{for} \quad x = \pm 1. \tag{10}$$

In calculating the characteristic functions X_k we start from the assumption that these functions always are either even or odd, which can be demonstrated. We indicate the odd characteristic functions by an asterisk. Then we have for the even functions instead of (9a) more simple

$$X_k'' + \omega_k^2 X_k = X_k'(1). \tag{11a}$$

The even solutions of this equation are, as can easily be verified,

$$X_k = B_k \left(\cos \omega_k x - \frac{\sin \omega_k x}{\omega_k} \right). \tag{12a}$$

From the boundary condition $X_k = 0$ for $x = 1$ it is found that the quantities ω_k are roots of the so called conditional equation

$$\omega = \operatorname{tg} \omega. \tag{13a}$$

By drawing both terms of this equation as a function of ω it becomes obvious that these roots are approximately equal to

$$\omega_k \cong (k + \tfrac{1}{2})\pi \quad (k = 1, 2, \ldots \infty). \tag{14a}$$

The first 5 accurately computed roots[12] are given in the table XI. It must be observed that the root $\omega_0 = 0$ of (13a) gives rise to the function $X_0 \equiv 0$ and therefore must be left out of consideration.

Table XI. The characteristic values ω_k and $\omega_k{}^*$ for $k = 1, 2, \ldots 5$ with an inextensible cable

k	1	2	3	4	5
ω_k	4,4934	7,7253	10,9041	14,0662	17,2208
ω_k	3,1416	6,2832	9,4248	12,5864	15,7080

The characteristic functions can be written with the help of (13a) in the somewhat simplified form

$$X_k = B_k (\cos \omega_k x - \cos \omega_k) \; (k = 1, 2 \ldots \infty). \tag{15}$$

For the odd characteristic functions we have the differential equation

$$\frac{d^2 X_k{}^*}{d x^2} + \omega_k{}^{*2} X_k{}^* = 0. \tag{11b}$$

Its odd solution is

$$X_k{}^* = B_k{}^* \sin \omega_k{}^* x, \tag{12b}$$

to which the conditional equation

$$\sin \omega^* = 0 \tag{13b}$$

belongs. The roots of this equation which mean anything for our problem, are

$$\omega_k{}^* = k\pi \, (k = 1, 2 \ldots \infty). \tag{14b}$$

---

[12]) Compare F. Emde, Tables of elementary functions, Leipzig and Berlin (1940) p. 130.

The first 5 characteristic values of the odd vibrations have also been computed and reproduced in table XI.

The equations (14a) and (14b) show that indeed only for very distinct values of ω_k characteristic functions different from zero exist, which satisfy the differential equation and the boundary conditions.

At the further calculations we shall use the so called orthogonal property, which exists between two different characteristic functions,

$$\int_{-1}^{+1} X_m X_n \, dx = 0 \text{ for } \omega_m \neq \omega_n. \tag{16}$$

In this expression each of the characteristic functions X_m and X_n can be even or odd at choice. This property can be demonstrated directly by substituting the expressions for X_k (15), or as the case may be $X_k{}^*$ (12b), in (16); however the following way is more attractive. We write down equation (9a) both for the index m as for the index n, multiply these equations by X_n and X_m respectively and substract the results:

$$X_m'' X_n - X_n'' X_m + (\omega_m^2 - \omega_n{}^2) X_m X_n$$
$$= \tfrac{1}{2} [\{X_m'(1) - X_m'(-1)\} X_n - \{X_n'(1) - X_n'(-1)\} X_m].$$

Integrating both terms of this equation between the boundaries $x = -1$ and $x = 1$ yields, using (6), the result

$$(\omega_m^2 - \omega_n{}^2) \int_{-1}^{+1} X_m X_n \, dx = 0,$$

which owing to $\omega_m \neq \omega_n$ gives rise to the property (16).

By multiplying the equation (9a), written down for the index m, by X_n and integrating both terms of the thus found equation in the same way, we find with the aid of (6) and (16) that also

$$\int_{-1}^{+1} X_m'' X_n \, dx = 0 \text{ for } \omega_m \neq \omega_n. \tag{17}$$

Further on we standardize the constants of integration B_k and $B_k{}^*$ remaining in (12a) and (12b) by the so called normalizing condition

$$\int_{-1}^{+1} X_k{}^2 \, dx = 1. \tag{18}$$

From (6) and (9a) we find that in that case X_k also satisfies

$$\int_{-1}^{+1} X_k'' X_k \, dx = -\omega_k{}^2. \tag{19}$$

Applying (18) to (15) and (12b) yields

$$B_k = \frac{1}{\sin \omega_k}, \quad B_k{}^* = 1, \tag{20}$$

so that the normalized characteristic functions are

$$X_k = \frac{\cos \omega_k x - \cos \omega_k}{\sin \omega_k}, \quad X_k{}^* = \sin \omega_k{}^* x \, (k = 1, 2 \ldots \infty). \tag{21}$$

We have computed some values of X_k and $X_k{}^*$ (see table XII), which are plotted in figure 9 against x.

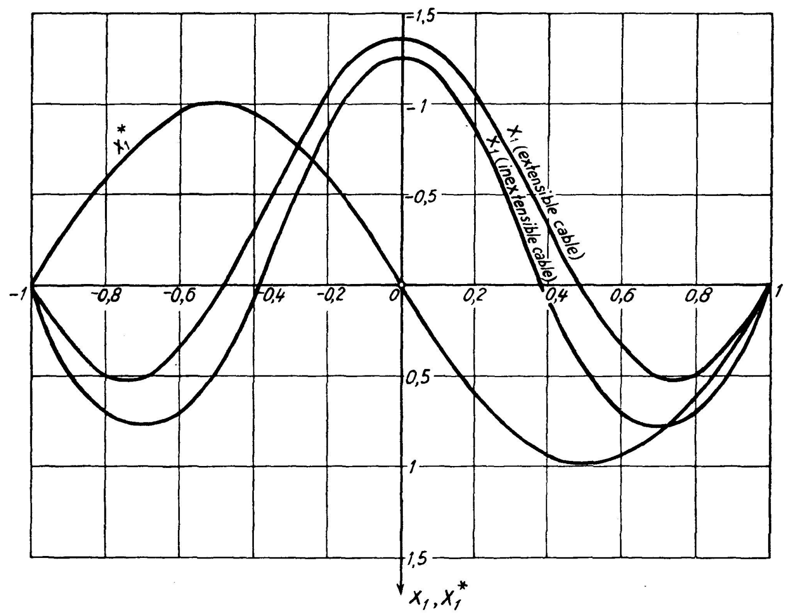

Fig. 9a. The characteristic functions X_1 and $X_1{}^*$

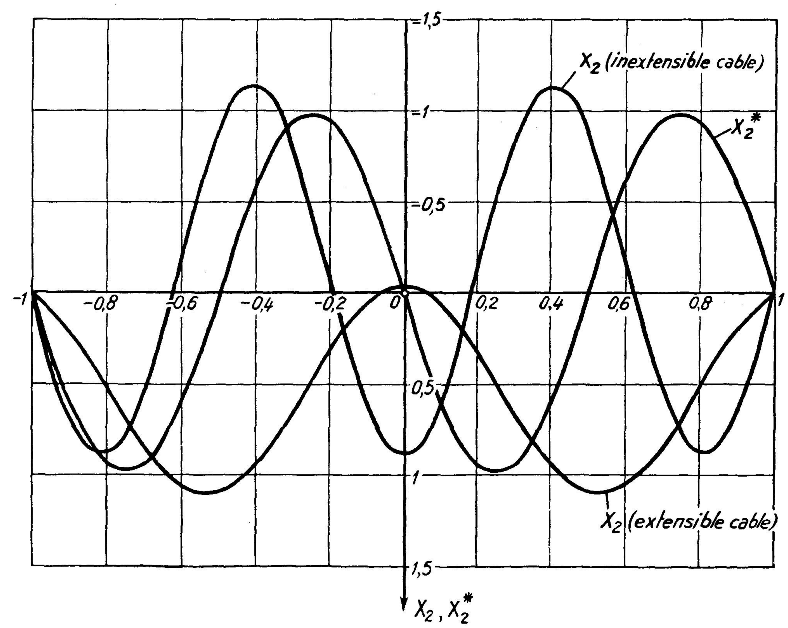

Fig. 9b. The characteristic functions X_2 and $X_2{}^*$

Table XIIa. The characteristic functions X_k and X_k^* for some values of x and $k=1$

x	-1	-0,8	-0,6	-0,4	-0,2	0	0,2	0,4	0,6	0,8	1
X_k inextensible cable	0	0,6987	0,7019	0,0076	-0,8605	-1,2472	-0,8605	0,0076	0,7019	0,6987	0
extensible cable	0	0,481	0,348	-0,309	-1,045	-1,362	-1,045	-0,309	0,348	0,481	0
X_k^* inextensible and extensible cable	0	-0,5878	-0,9511	-0,9511	-0,5878	0	0,5878	0,9511	0,9511	0,5878	0

Table XIIb. The characteristic functions X_k and X_k^* for some values of x and $k=2$

x	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
X_k inextensible cable	0,8790	0,5927	-0,1035	-0,8144	-1,1363	-0,8868	-0,2072	0,5165	0,8737	0,6612	0
extensible cable	-0,037	0,060	0,318	0,650	0,941	1,094	1,056	0,840	0,520	0,204	0
X_k^* inextensible and extensible cable	0	0,5878	0,9511	0,9511	0,5878	0	-0,5878	-0,9511	-0,9511	-0,5878	0

13. Determination of the normal coordinates. The characteristic functions and values having been found, we now can pass on to the determination of the normal coordinates. These satisfy the equation (*12*, 9b)

$$\ddot{T}_k + (1+H_k)\,\omega_k^2\, T_k = 0\,(k = 1,\, 2 \ldots \infty). \tag{1}$$

In this equation are found both the k^{th} normal coordinate and the cable force change $H_k(t)$ belonging to it. But there exists another relation between these two quantities, i.e. the relation (*12*, 7). From these two relations there can be derived formulæ with the aid of which T_k can be expressed in H_k or inversely H_k in T_k. With the even vibrations we have, according to (*12*, 21),

$$X_k'(1) = X_k'(-1) = -\omega_k, \tag{2a}$$

and these formulæ are

$$T_k = -\frac{H_k}{\omega_k(1+H_k)}, \quad (3) \quad H_k = -\frac{\omega_k T_k}{1+\omega_k T_k}. \tag{4a}$$

By substituting (4a) in (1) the non linear differential equation

$$\ddot{T}_k + \frac{\omega_k^2 T_k}{1+\omega_k T_k} = 0 \tag{5a}$$

can be derived, in which H_k is no more found.

The solution of this differential equation requires some deliberation. It can be integrated once by introducing the new variable

$$z = \frac{d T_k}{dt} \tag{6}$$

and by writing
$$\frac{d^2 T_k}{dt^2} = \frac{dz}{dt} = \frac{dz}{dT_k}\frac{dT_k}{dt} = z\frac{dz}{dT_k}:$$

$$z\,dz + \frac{\omega_k T_k d(\omega_k T_k)}{1+\omega_k T_k}; \quad \frac{1}{2}z^2 + \omega_k T_k - \ln(1+\omega_k T_k) = D_k; \tag{7}$$

D_k being a constant of integration. The ultimate result is

$$\frac{d(\omega_k T_k)}{d(\omega_k t)} = \pm\sqrt{2\{D_k + \ln(1+\omega_k T_k) - \omega_k T_k\}}. \tag{8}$$

From this it can be derived that T_k is a periodic function of t for all values D_k, but we shall not enter further into this question. For small values of D_k there can be written approximately

$$T_k \cong A_k \cos(\omega_k t + \varphi_k) \text{ with } \omega_k A_k = \sqrt{2 D_k}\,(k = 1, 2 \ldots \infty). \tag{9a}$$

With the odd vibrations we have, according to (*12*, 21) and (*12*, 14b),

$$X_k^{*\prime}(1) = 0, \; X_k^{*\prime}(-1) = 0, \tag{2b}$$

so that

$$H_k = 0 \tag{4b}$$

and (1) reduces to

$$\ddot{T}_k^* + \omega_k^{*2} T_k^* = 0 \;\; (k = 1, 2 \ldots \infty). \tag{5b}$$

The solution of the last equation is (without restrictions)

$$T_k^* = A_k^* \cos(\omega_k^* t + \varphi_k^*)\,(k = 1, 2 \ldots \infty). \tag{9b}$$

The constants of integration occurring in (8), (9a) and (9b) can be computed, when the starting conditions are known, with the aid of the properties (*12*, 16)—(*12*, 19). We shall not enter further into this question.

A motion governed by a non linear differential equation like (5a), in which the time itself does not appear, is often called a pseudo harmonic motion. Thus the even vibrations of a suspension bridge are pseudo harmonic, the odd vibratious however are harmonic.

When only one constant A_k c. q. A_k^* differs from zero, the characteristic value ω_k c. q. ω_k^* belonging to it determines the frequency with which the bridge oscillates. Restricting ourselves with respect to the even vibrations to small values of T_k, for which the formula (9a) holds sufficiently well, we can also call the characteristic values the angular frequencies of the bridge. In quantities with dimensions the angular frequencies are, according to (*3*, 6b),

$$\bar{\omega}_k = \frac{\omega_k t}{\bar{t}} = \omega_k \sqrt{\frac{g}{2f}}, \quad \bar{\omega}_k^* = \omega_k^* \sqrt{\frac{g}{2f}}. \tag{10}$$

From table XI it may be concluded that the lowest characteristic value is $\omega_1^* = \pi$, so that the lowest frequency $\bar{\nu}_1^*$ and the largest vibration time $\bar{\tau}_1^*$ of the bridge can be deduced from

$$\bar{\omega}_1^* = \pi \sqrt{\frac{g}{2f}} = 2\pi\bar{\nu}_1^*, \quad \bar{\nu}_1^* = \sqrt{\frac{g}{8f}}, \quad \bar{\tau}_1^* = \sqrt{\frac{8f}{g}}\,{}^{13)}. \tag{11}$$

For the bridge indicated in table I as "example" the last formula yields

$$\bar{\tau}_1^* = 5{,}91 \text{ sec.}$$

14. Application of the quasi linear method. In chapters *6* and *9* we applied already the so called quasi linear method, by which the non linear connection between the girder deflection and the live load is simplified to a linear relation. A reduction like that we can also execute with dynamical calculations; applying it on the equation (*12*, 1) yields

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial t^2} = H(t). \tag{1}$$

Again we can try to substitue the solution (*12*, 4), which gives rise to the simpler relation

$$X_k'' T_k - X_k \ddot{T}_k = H_k, \tag{2}$$

instead of (*12*, 5), while the relation (*12*, 6) remains valid. The relations (*12*, 7) and (*12*, 8) must here be replaced by

$$\tfrac{1}{2} T_k \{X_k'(1) - X_k'(-1)\} = H_k \tag{3}$$

and
$$\frac{X_k'' - \frac{1}{2}\{X_k'(1) - X_k'(-1)\}}{X_k} = \frac{\ddot{T}_k}{T_k} = -\omega_k^2. \tag{4}$$

Now it can be easily shown that the differential equations (*12*, 9a), (*12*, 11a) and (*12*, 11b), the boundary conditions (*12*, 10), the formulæ (*12*, 12a), (*12*, 12b), (*12*, 21) and the properties (*12*, 16)—(*12*, 19) for the characteristic

[13]) This result has already been found by H. Granholm, Beräkning av hängbroar 2, Göteborg (1945) p. 68.

functions as well as the equations (*12*, 13a), (*12*, 13b) and the formulæ (*12*, 14a), (*12*, 14b) for the characteristic values remain valid, while all the normal coordinates T_k and T_k^* now satisfy the equation

$$\ddot{T}_k + \omega_k^2 T_k = 0; \tag{5}$$

thus the latter are equal to

$$T_k = A_k \cos(\omega_k t + \varphi_k), \ T_k^* = A_k^* \cos(\omega_k^* t + \varphi_k^*) \ (k = 1, 2 \ldots \infty). \tag{6}$$

The cable force is at this case, owing to (3), determined by

$$H_k = -\omega_k T_k, \ H_k^* = 0. \tag{7}$$

15. Free vibrations of a bridge with an extensible cable. We shall now calculate the free vibrations of a suspension bridge, which occur when the extensibility of the cable is taken into account. Here we start from the differential equation (*11*, 1), reduced to

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial t^2} = H(t), \tag{1}$$

and the conditions (*11*, 2) and (*11*, 3).

Again we try to write down as solution from (1)

$$\eta = X_k T_k, \tag{2}$$

X_k being only dependent from x, T_k only from t. Substituting (2) in (1) yields the equation

$$X_k'' T_k - X_k \ddot{T}_k = H_k \tag{3}$$

at which the corresponding cable force $H(t)$ now again is indicated with the index k. Integrating both terms of the last equation to x between the boundaries -1 and $+1$ we find

$$\tfrac{1}{2}\{X_k'(1) - X_k'(-1)\} T_k = \left\{1 + \tfrac{1}{2}(\gamma l_3 + \nu)\frac{\ddot{T}_k}{T_k}\right\} H_k. \tag{4}$$

By eliminating H_k from (3) and (4) there comes

$$\frac{X_k'' - \dfrac{1}{2}\dfrac{X_k'(1) - X_k'(-1)}{1 + \dfrac{1}{3}\alpha\dfrac{\ddot{T}_k}{T_k}}}{X_k} = \frac{\ddot{T}_k}{T_k}, \tag{5}$$

in which α is determined by the formula (*8*, 4). From (5) we can solve $\ddot{T}_k/T_k$ as a function of X_k'' and X_k; so, X_k being only dependent from x and T_k only from t, the ratio $\ddot{T}_k/T_k$ must again be constant. Hence, putting $\ddot{T}_k/T_k = -\omega_k^2$, we can write down the relations

$$X_k'' + \omega_k^2 X_k = \frac{1}{2}\,\frac{X_k'(1) - X_k'(-1)}{1 - \frac{1}{3}\alpha\,\omega_k^2}, \quad \text{(6a)} \qquad \ddot{T}_k + \omega_k^2 T_k = 0. \qquad \text{(6b)}$$

For the even functions (6a) can be further reduced to

$$X_k'' + \omega_k^2 X_k = \frac{X_k'(1)}{1 - \frac{1}{3}\alpha\,\omega_k^2}; \tag{7}$$

the even solution of this equation is

$$X_k = B_k\left\{\cos\omega_k x - \frac{\sin\omega_k}{\omega_k\,(1 - \frac{1}{3}\,\alpha\,\omega_k^2)}\right\}, \tag{8}$$

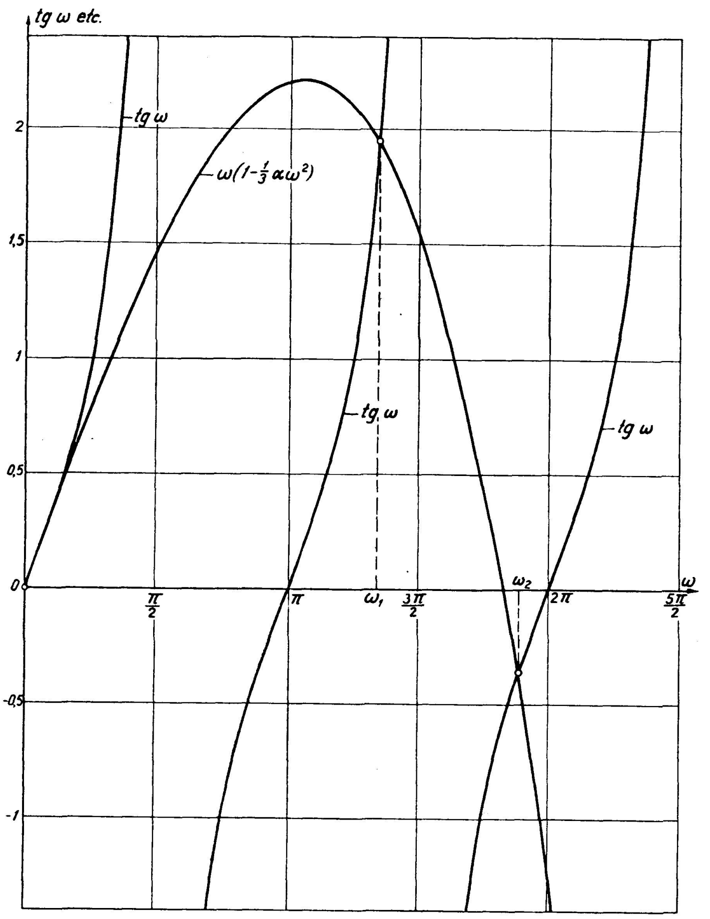

Fig. 10
Determination of the roots of the frequency equation

which, together with the boundary conditions (*11*, 3) gives rise to the conditional equation

$$\operatorname{tg}\omega = \omega\,(1 - \tfrac{1}{3}\,\alpha\,\omega^2), \tag{9}$$

of which ω_k must be the roots. This equation may best be solved in a graphical way, as has been drawn in figure 10 for the bridge treated in this paper as ,,example'', for which, according to nr. *8*, $\alpha = 0{,}090.939$. Here we find $\omega_1 = 4{,}24$; $\omega_2 = 5{,}92$.

The root $\omega_0 = 0$ gives rise to a characteristic function $X_0 = 0$, and so it makes no sense. The other characteristic functions (8) can, the quantities ω_k being roots of (9), be written down in the more simple form

$$X_k = B_k\,(\cos\omega_k x - \cos\omega_k)\ (k = 1, 2, \ldots \infty), \tag{10}$$

which expression can be normalized with the aid of the condition (*12*, 18) to

$$X_k = \frac{\cos\omega_k x - \cos\omega_k}{\sin\omega_k\sqrt{1 + \dfrac{\alpha}{(1 - \frac{1}{3}\alpha\,\omega_k^2)^2}}}\ (k = 1, 2, \ldots \infty). \tag{11}$$

For the bridge treated as example we have computed some values of X_k; these are given in table XII and plotted in fig. 9 against x.

With the odd vibrations (6a) also in this case reduces to (*12*, 11b). Thus the characteristic values belonging to it are fixed by (*12*, 12b), while the normalized odd characteristic functions are determined by the second formula (*12*, 21).

At last the equation (6b) corresponds with (*14*, 5). So the normal coordinates are, like as in the corresponding case of an inextensible cable, determined by (*14*, 6), while the cable forces belonging to it are equal to

$$H_k = -\frac{\omega_k T_k}{1-\frac{1}{3}\alpha\,\omega_k^2}, \quad H_k{}^* = 0. \tag{12}$$

A particular case is $\alpha = \frac{3}{\pi^2}$, in which we have $\omega_1 = \pi$, so that the characteristic value belonging to the even vibrations is equal to the characteristic value belonging to the odd vibrations; here $X_1 = \frac{\cos \pi x + 1}{\sqrt{3}}$. For this value of α and for greater values of this quantity the first even vibration has only two modes, situated at the ends of the bridge. For $\alpha > 3/\pi^2$ $\omega_1{}^*$ is no more the lowest characteristic value, but now the set of characteristic values, ranged in growing order, is: ω_1, $\omega_1{}^*$, ω_2, $\omega_2{}^*$ etc.

For the bridge treated as example we have computed the first two even characteristic functions X_k; these are given in table XII and plotted in fig. 9 against x. We learn that for this bridge the set of characteristic values, ranged in growing order, is: $\omega_1{}^*$, ω_1, ω_2, $\omega_2{}^*$ etc.

Klöppel and Lie [14]) have treated the problem of free oscillations of a bridge with an extensible cable and a girder of any stiffness, but they restricted themself to the linearised case. They solved it by developping the deflection in a Fourier serie. As however the even characteristic functions of the problem are no cosinus functions at all, this way of solution must have an approximative character. Moreover it can not be applied at the case of an inextensible cable, as the cosinus functions do not satisfy the condition that for this case the integral of the function taken over the length of the girder is equal to zero.

16. Forced vibrations caused by a moving live load. We now shall enter into the question how the girder deflection and the cable force can be calculated when the bridge undergoes a forced vibration, i. e. when the live load $p(x,t)$ differs from zero. We restrict ourselves exclusively to the case of the quasi linear method, which is naturally admissible when the load $p(x,t)$ is small compared to 1.

Although we could use for this case the method of the characteristic functions as well as in the nrs. *12*—*15*, it gives much advantage to make use

[14]) K. Klöppel and K. H. Lie, loc. cit. p. 220.

of the so called operational or symbolic calculus, based on the one sided Laplace integral. For the principles of this calculus we refer to the handbooks[15]).

The starting point now is the differential equation (*11*, 1), reduced to

$$-\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial t^2} = p(x,t) - H(t), \tag{1}$$

with the cable condition

$$\int_{-1}^{+1} \eta\, dx = 0 \tag{2}$$

and the boundary conditions

$$\eta = 0 \qquad \text{for } x = \pm 1, \tag{3}$$

to which we now add the starting conditions

$$\eta = 0, \quad \frac{\partial \eta}{\partial t} = 0 \quad \text{for} \quad t = 0. \tag{4}$$

Following the rules of the symbolic calculus we introduce the new variable s and we join to the functions $\eta(x,t)$, $p(x,t)$ and $H(t)$ the so called image functions

$$\left.\begin{aligned} \eta^*(x,s) = \int_0^\infty e^{-st}\eta(x,t)\,dt,\ p^*(x,s) = \int_0^\infty e^{-st} p(x,t)\,dt,\\ H^*(s) = \int_0^\infty e^{-st} H(t)\,dt. \end{aligned}\right\} \tag{5}$$

By multiplying both terms of the partial differential equation (1) with e^{-st}, integrating with respect to t between 0 and ∞ and making use of (5) and the starting conditions (4) we find the ordinary differential equation

$$-\frac{d^2 \eta^*}{d x^2} + s^2 \eta^* = p^*(x,s) - H^*(s), \tag{6}$$

to which belongs the cable condition

$$\int_{-1}^{+1} \eta^*\, dx = 0 \tag{7}$$

and the boundary conditions

$$\eta^* = 0 \text{ for } x = \pm 1, \tag{8}$$

as can be shown from (2), (3) and (5).

With the help of the method of variation of constants we find as the general solution of (6) and (8)

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15) For instance: G. Doetsch, Theorie und Anwendung der Laplacetransformation, Berlin (1937); H. S. Carslaw and J. C. Jaeger, Operational methods in applied mathematics, Oxford (1941); R. V. Churchill, Modern operational mathematics in engineering, New York and London (1944).

$$2s\eta^*\operatorname{ch}s = \int_{-1}^{x} p^*(\xi,s)\operatorname{sh}s(1+\xi-x)\,d\xi + \int_{x}^{+1} p^*(\xi,s)\operatorname{sh}s(1+x-\xi)\,d\xi$$
$$-\frac{\operatorname{sh}sx}{\operatorname{sh}s}\int_{-1}^{+1} p^*(\xi,s)\operatorname{sh}s\xi\,d\xi - 2\frac{\operatorname{ch}s-\operatorname{ch}sx}{s}H^*(s), \tag{9}$$

from which the cable force $H^*(s)$ (5) can be calculated with the help of the cable condition (7):

$$2\left(1-\frac{\operatorname{th}s}{s}\right)H^*(s) = \int_{-1}^{+1} p^*(x,s)\,dx - \frac{1}{\operatorname{ch}s}\int_{-1}^{+1} p^*(x,s)\operatorname{ch}sx\,dx. \tag{10}$$

So far the live load could be any function of x and t. But now we shall use the stipulation, mentioned already in chapter *11*, that this load moves, conserving its form, with a constant speed $\bar{v}$, so that we have at any time

$$\bar{p}(\bar{x},\bar{t}) \equiv \bar{p}_1(\bar{x}-\bar{v}\bar{t}), \tag{11}$$

or with (*3*, 6b) and

$$v = \frac{\bar{v}}{v_1}, \quad v_1 = \sqrt{\frac{gl}{2f}}: \tag{12}$$

$$p(x,t) \equiv p_1(x-vt). \tag{13}$$

Moreover we shall assume

$$p_1(x) \equiv 0 \quad \text{for} \quad x > -1. \tag{14}$$

Now we have

$$p^*(x,s) = \int_0^\infty e^{-st} p_1(x-vt)\,dt;$$

introducing the new variable

$$z = x - vt \tag{15}$$

yields, with (14),

$$p^*(x,s) = \frac{1}{v}\int_{-\infty}^{-1} e^{-\frac{s}{v}(x-z)} p_1(z)\,dz. \tag{16}$$

At first we substitute (16) in the formula (10) for the cable force; then we get

$$2v\left(1-\frac{\operatorname{th}s}{s}\right)H^*(s) = \int_{-1}^{+1} dx \int_{-\infty}^{-1} e^{-\frac{s}{v}(x-z)} p_1(z)\,dz - \frac{1}{\operatorname{ch}s}\int_{-1}^{+1}\operatorname{ch}sx\,dx\int_{-\infty}^{-1} e^{-\frac{s}{v}(x-z)} p_1(z)\,dz.$$

After changing the order of integration in both double integrals, we can reduce them to single ones:

$$2(1-v^2)(s-\operatorname{sh}s)H^*(s) = v(v+\operatorname{th}s)\int_{-\infty}^{-1} e^{-\frac{s}{v}(1-z)} p_1(z)\,dz - v(v-\operatorname{th}s)\int_{-\infty}^{-1} e^{\frac{s}{v}(1+z)} p_1(z)\,dz. \tag{17}$$

Now we apply the reversing formula of the operational calculus:

$$H(t) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st} H^*(s)\,ds; \tag{18}$$

this yields, after changing the orders of integration another time,

$$4\pi i(1-v^2)H(t) = v\int_{-\infty}^{-1} p_1(z)\,dz \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{v+\operatorname{th}s}{s-\operatorname{th}s}\, e^{st-\frac{s}{v}(1-z)}\,ds$$

$$-v\int_{-\infty}^{-1} p_1(z)\,dz \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{v-\operatorname{th}s}{s-\operatorname{th}s}\, e^{st+\frac{s}{v}(1+z)}\,ds. \tag{19}$$

Now it will be shown in nr. *17* that

$$\left.\begin{aligned} J_2 = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{e^{us}}{s-\operatorname{th}s}\,ds &= 0 && \text{for } u<0,\\ &= \frac{6}{5}+\frac{3}{2}u^2-2f_2(u) && \text{for } u>0, \end{aligned}\right\} \tag{20a}$$

$$\left.\begin{aligned} J_1 = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{\operatorname{th}s\, e^{us}}{s-\operatorname{th}s}\,ds &= 0 && \text{for } u<0,\\ &= 3u+2f_1(u) && \text{for } u>0, \end{aligned}\right\} \tag{20b}$$

with
$$f_1(u) = \sum_{k=1}^{\infty} \frac{\sin u\,\omega_k}{\omega_k}, \quad f_2(u) = \sum_{k=1}^{\infty} \frac{\cos u\,\omega_k}{\omega_k^{\,2}}, \tag{21}$$

ω_k being, according to nr. *12*, the k^{th} root of the equation (*12*, 13a). We shall also demonstrate in nr. *17* that

$$\left.\begin{aligned} f_1(u) &= \frac{1}{2}e^u-\frac{3}{2}u && \text{for } 0<u<2,\\ &= -(u-1)e^{u-2}+\frac{1}{2}e^u-\frac{3}{2}u && \text{for } 2<u<4,\\ &= (u^2-5u+5)e^{u-4}-(u-1)e^{u-2}+\frac{1}{2}e^u-\frac{3}{2}u && \text{for } 4<u<6,\\ &= -\left(\frac{2}{3}u^3-8u^2+29u-29\right)e^{u-6}+(u^2-5u+5)e^{u-4}-(u-1)e^{u-2}+\frac{1}{2}e^u-\frac{3}{2}u\\ & && \text{for } 6<u<8, \end{aligned}\right\} \tag{22a}$$

$$\left.\begin{aligned} f_2(u) &= -\frac{1}{2}e^u+\frac{3}{4}u^2+\frac{3}{5} && \text{for } 0\leqq u\leqq 2,\\ &= (u-2)e^{u-2}-\frac{1}{2}e^u+\frac{3}{4}u^2+\frac{3}{5} && \text{for } 2\leqq u\leqq 4,\\ &= -(u-3)(u-4)e^{u-4}+(u-2)e^{u-2}-\frac{1}{2}e^u+\frac{3}{4}u^2+\frac{3}{5} && \text{for } 4\leqq u\leqq 6,\\ &= \left(\frac{2}{3}u^2-6u+13\right)(u-6)e^{u-6}-(u-3)(u-4)e^{u-4}+(u-2)e^{u-2}\\ &\qquad -\frac{1}{2}e^u+\frac{3}{4}u^2+\frac{3}{5} && \text{for } 6\leqq u\leqq 8. \end{aligned}\right\} \tag{22b}$$

With the aid of (20a) and (20b) the formula (19) can, replacing z by $z-vt$, be reduced to

$$(1-v^2)\,H(t) = \int\limits_{-1}^{vt-1} p_1(z-vt)\left\{\frac{3}{4}(1-z^2)-\frac{3}{5}v^2+v f_1\left(\frac{z+1}{v}\right)+v^2 f_2\left(\frac{z+1}{v}\right)\right\}dz$$
$$+\int\limits_{vt-1}^{+1} p_1(z-vt)\left\{\frac{3}{4}(1-z^2)-\frac{3}{5}v^2-v f_1\left(\frac{z-1}{v}\right)+v^2 f_2\left(\frac{z-1}{v}\right)\right\}dz. \quad (23)$$

By using (22a) and (22b) we can split $H(t)$ into parts:

$$H(t)=\sum_{j=1}^{\infty} H_j(t) \quad (24)$$

with

$$\left.\begin{aligned}
H_1(t) &= \frac{v}{2(1+v)}\int\limits_{-1}^{vt-1} p_1(z-vt)\,e^{\frac{z+1}{v}}\,dz,\\
H_2(t) &= -\frac{1}{1+v}\int\limits_{2v-1}^{vt-1} p_1(z-vt)\left(z-2v+\frac{1}{1-v}\right)e^{\frac{z+1}{v}-2}dz,\\
H_3(t) &= \frac{v}{2(1-v)}\int\limits_{+1}^{vt-1} p_1(z-vt)\,e^{\frac{z-1}{v}}\,dz,\\
H_4(t) &= \frac{v}{1+v}\int\limits_{4v-1}^{vt-1} p_1(z-vt)\left\{\left(\frac{z+1}{v}\right)^2-5\frac{z+1}{v}+12+\frac{2z-5}{1-v}\right\}e^{\frac{z+1}{v}-4}dz,\\
H_5(t) &= -\frac{1}{1-v}\int\limits_{2v+1}^{vt-1} p_1(z-vt)\left(z-2v-\frac{1}{1+v}\right)e^{\frac{z-1}{v}-2}dz,
\end{aligned}\right\} \quad (25)$$

etc.

For the speed $v=1$ the functions $H_2(t)$, $H_3(t)$ etc. become indefinite. But then we can take together H_2+H_3, H_4+H_5 etc. and apply the rule of de l'Hôpital; thus we find

$$\left.\begin{aligned}
H_1(t) &= \tfrac{1}{4}\int_{-1}^{t-1} p_1(z-t)\,e^{z+1}dz,\\
H_2(t)+H_3(t) &= -\tfrac{1}{4}\int_{+1}^{t-1} p_1(z-t)\,(2z+3)\,e^{z-1}dz-p_1(-t+1),\\
H_4(t)+H_5(t) &= \tfrac{1}{2}\int_{+3}^{t-1} p_1(z-t)\,(z^2-3)\,e^{z-3}dz+p_1(-t+3),
\end{aligned}\right\} \quad (26)$$

etc.

Next we shall compute the deflection. For this purpose we substitute the expression (16) for $p^*(x,s)$ in (9); after changing the order of integration and evaluating the integrals as far as possible in the now well known way, we get

$$(1-v^2)\,s^2\,\eta^* = -v\int_{-\infty}^{-1} p_1(z)\,e^{-s\frac{x-z}{v}}\,dz + \frac{v}{\operatorname{sh}2s}\int_{-\infty}^{-1} p_1(z)\operatorname{sh}s\,(1-x)\,e^{s\frac{1+z}{v}}\,dz$$

$$+\frac{v}{\operatorname{sh}2s}\int_{-\infty}^{-1} p_1(z)\operatorname{sh}s\,(1+x)\,e^{-s\frac{1-z}{v}}\,dz - (1-v^2)\left(1-\frac{\operatorname{ch}sz}{\operatorname{ch}s}\right)H^*(s). \tag{27}$$

We now apply the reversing formula and also the composition product rule of the operational calculus, which reads, $\varphi(t)$ being any new function of t and $\varphi^*(s)$ its image,

$$\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \varphi^*(s)\,H^*(s)\,e^{st}\,ds = \int_0^t \varphi(t-\tau)\,H(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\int_0^t H(\tau)\,d\tau\int_{\epsilon-i\infty}^{\epsilon+i\infty} \varphi^*(s)\,e^{s(t-\tau)}\,d\tau. \tag{28}$$

With their help and by renewed changing of the order of integration the formula (27) can be reduced to

$$2\pi i\,(1-v^2)\,\eta = -v\int_{-\infty}^{-1} p_1(z)\,dz\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{1}{s^2}\,e^{s\left(t-\frac{x-z}{v}\right)}\,ds$$

$$+v\int_{-\infty}^{-1} p_1(z)\,dz\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{\operatorname{sh}s\,(1-x)\,e^{s\left(t+\frac{1+z}{v}\right)}}{s^2\operatorname{sh}2s}\,ds + v\int_{-\infty}^{-1} p_1(z)\,dz\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{\operatorname{sh}s\,(1+x)\,e^{s\left(t-\frac{1-z}{v}\right)}}{s^2\operatorname{sh}2s}\,ds$$

$$-(1-v^2)\int_0^t H(\tau)\,d\tau\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{s^2}\left(1-\frac{\operatorname{ch}sx}{\operatorname{ch}s}\right)e^{s(t-\tau)}\,ds. \tag{29}$$

The four complex integrals can be evaluated in a way similar to that shown in nr. *17* for the complex integrals (20a) and (20b). We omit the calculations and write down only the result:

$$\eta = \frac{2}{1-v^2}\left[\frac{1}{4}(1-x)\int_{v(1-x)-1}^{x} p_1(z-vt)\,(1+z)\,dz + \frac{1}{4}(1+x)\int_{x}^{v(1+x)+1} p_1(z-vt)\,(1-z)\,dz\right.$$

$$-v\int_{-v(1-x)-1}^{vt-1} p_1(z-vt)\,f_2^*\left(1-x+\frac{1+z}{v}\right)dz + v\int_{v(1-x)-1}^{vt-1} p_1(z-vt)\,f_2^*\left(1-x-\frac{1+z}{v}\right)dz$$

$$-v\int_{-v(1+x)+1}^{vt-1} p_1(z-vt)\,f_2^*\left(1+x-\frac{1-z}{v}\right)dz + v\int_{v(1+x)+1}^{vt-1} p_1(z-vt)\,f_2^*\left(1+x+\frac{1-z}{v}\right)dz$$

$$+v\int_{-v(1-x)-1}^{v(1-x)-1} p_1(z-vt)\left\{\frac{1}{16}\left(1-x+\frac{1+z}{v}\right)^2-\frac{1}{12}\right\}dz$$

$$+v \int\limits_{-v(1+x)+1}^{v(1+x)+1} p_1(z-vt)\left\{\frac{1}{16}\left(1+x-\frac{1-z}{v}\right)^2-\frac{1}{12}\right\}dz\Big]$$

$$-\int\limits_0^t H(\tau)\{4f_2^*(x+t-\tau+1)-f_2^*(2x+2t-2\tau)-\tfrac{1}{2}(x-t+\tau)\}d\tau$$

$$+\int\limits_0^{t-a} H(\tau)\{4f_2^*(x-t+\tau+1)-f_2^*(2x-2t+2\tau)-\tfrac{1}{2}(x-t+\tau)\}d\tau \tag{30}$$

Here a and $f_2^*(u)$ are defined by

$$a=t \text{ for } t\leqq x, \quad a=x \text{ for } t\geqq x, \tag{31}$$

$$f_2^*(u)=\sum_{k=1}^{\infty}(-1)^k\frac{\cos\frac{1}{2}k\pi u}{k^2\pi^2}. \tag{32}$$

In nr. *17* we shall demonstrate that

$$\left.\begin{aligned} f_2^*(u) &= \frac{1}{16}u^2-\frac{1}{12} && \text{for } -2\leqq u\leqq 2,\\ &= \frac{1}{16}(u-4)^2-\frac{1}{12} && \text{for } 2\leqq u\leqq 6,\\ &\text{etc.} \end{aligned}\right\} \tag{33}$$

With the help of (33) we can further reduce the deflection η (30). Splitting this function into parts we have

$$\eta=\sum_{m=1}^{3}\sum_{n=1}^{\infty}\eta_{mn} \tag{34}$$

with

$$\left.\begin{aligned}
(1-v^2)\,\eta_{11} &= \int\limits_x^{vt-1} p_1(z-vt)(x-z)\,dz,\\
(1-v^2)\,\eta_{12} &= \int\limits_{v(1+x)-1}^{vt-1} p_1(z-vt)\{1+z-v(1+x)\}\,dz,\\
(1-v^2)\,\eta_{13} &= -\int\limits_{v(1-x)+1}^{vt-1} p_1(z-vt)\{1-z+v(1-x)\}\,dz,\\
(1-v^2)\,\eta_{14} &= -\int\limits_{v(3-x)-1}^{vt-1} p_1(z-vt)\{1+z-v(3-x)\}\,dz,\\
(1-v^2)\,\eta_{15} &= \int\limits_{v(3+x)+1}^{vt-1} p_1(z-vt)\{1-z+v(3+x)\}\,dz,\\
(1-v^2)\,\eta_{16} &= \int\limits_{v(5+x)-1]}^{vt-1} p_1(z-vt)\{1+z-v(5+x)\}\,dz,\\
(1-v^2)\,\eta_{17} &= -\int\limits_{v(5-x)+1}^{vt-1} p_1(z-vt)\{1-z+v(5-x)\}\,dz,\\
(1-v^2)\,\eta_{18} &= -\int\limits_{v(7-x)-1}^{vt-1} p_1(z-vt)\{1+z-v(7-x)\}\,dz,\\
(1-v^2)\,\eta_{19} &= \int\limits_{v(7+x)+1}^{vt-1} p_1(z-vt)\{1-z+v(7+x)\}\,dz,\\
&\text{etc.,}
\end{aligned}\right\} \tag{35a}$$

$$\left.\begin{aligned}
\eta_{21} &= \int_0^t H(\tau)(\tau-t)\,d\tau && \text{for } 0 \leqq t \leqq 1-x,\\
&= \int_{t+x-1}^t H(\tau)(\tau-t)\,d\tau && \text{for } t \geqq 1-x,\\
\eta_{22} &= 0 && \text{for } 0 \leqq t \leqq 1-x,\\
&= -(1-x)\int_0^{t+x-1} H(\tau)\,d\tau && \text{for } 1-x \leqq t \leqq 3-x,\\
&= -(1-x)\int_{t+x-3}^{t+x-1} H(\tau)\,d\tau && \text{for } t \geqq 3-x,\\
\eta_{23} &= 0 && \text{for } 0 \leqq t \leqq 3-x,\\
&= \int_0^{t+x-3} H(\tau)(\tau-t+2)\,d\tau && [\text{for } 3-x \leqq t \leqq 5-x,\\
&= \int_{t+x-5}^{t+x-3} H(\tau)(\tau-t+2)\,d\tau && \text{for } t \geqq 5-x,\\
\eta_{24} &= 0 && \text{for } 0 \leqq t \leqq 5-x,\\
&= -(3-x)\int_0^{t+x-5} H(\tau)\,d\tau && \text{for } 5-x \leqq t \leqq 7-x,
\end{aligned}\right\}\quad (35\,\text{b})$$

etc.,

$$\left.\begin{aligned}
\eta_{31} &= 0 && \text{for } 0 \leqq t \leqq 1+x,\\
&= -\int_0^{t-x-1} H(\tau)(\tau-t+1+x)\,d\tau && \text{for } 1+x \leqq t \leqq 3+x,\\
&= -\int_{t-x-3}^{t-x-1} H(\tau)(\tau-t+1+x)\,d\tau && \text{for } t \geqq 3+x,\\
\eta_{32} &= 0 && \text{for } 0 \leqq t \leqq 3+x,\\
&= 2\int_0^{t-x-3} H(\tau)\,d\tau && \text{for } 3+x \leqq t \leqq 5+x,\\
&= 2\int_{t-x-5}^{t-x-3} H(\tau)\,d\tau && \text{for } t \geqq 5+x,\\
\eta_{33} &= 0 && \text{for } 0 \leqq t \leqq 5+x,\\
&= -\int_0^{t-x-5} H(\tau)(\tau-t+3+x)\,d\tau && \text{for } 5+x \leqq t \leqq 7+x,
\end{aligned}\right\}\quad (35\,\text{c})$$

etc.

For the speed $v=1$ the functions η_{1n} become indefinite; with the aid of de l'Hôpitals rule we can write

$$\left.\begin{aligned}
\eta_{11}+\eta_{12} &= \tfrac{1}{2}(1+x)\int_x^{t-1} p_1(z-t)\,dz,\\
\eta_{13}+\eta_{14} &= -\int_{2-x}^{t-1} p_1(z-t)\,dz,\\
\eta_{15}+\eta_{16} &= \int_{4+x}^{t-1} p_1(z-t)\,dz,\\
\eta_{17}+\eta_{18} &= -\int_{6-x}^{t-1} p_1(z-t)\,dz,
\end{aligned}\right\}\quad (36)$$

etc.

In practical calculations we shall split the functions η_{mn} $(m = 2, 3)$ further on:

$$\eta_{mn} = \sum_{j=1}^{n} \eta_{mnj} \, (m=2,3;\ n=1,2,\ldots\infty); \tag{37}$$

here η_{mnj} are calculated with the help of (35b) and (35c), in which formulæ $H(\tau)$ must be replaced by $H_j(\tau)$ $(j=1,2,\ldots\infty)$ respectively.

We now have found formulæ which give us the cable force (24)—(26) and the girder deflection (34)—(37) for any live load. In nrs. *18* and *19* we shall illustrate these formulæ by applying them to the two special cases of a moving concentrated force and a moving homogeneous load extending over half the length of the bridge.

17. Some integrals and formulæ which act a part in the preceding chapter. At first we shall demonstrate the relations (*16*, 20a) and (*16*, 20b). For abbreviation we write

$$F_1(s,u) = \frac{\operatorname{th} s\, e^{su}}{s - \operatorname{th} s} = \frac{\operatorname{sh} s\, e^{su}}{s \operatorname{ch} s - \operatorname{sh} s}, \quad F_2(s,u) = \frac{e^{su}}{s - \operatorname{th} s}. \tag{1}$$

We now have to consider the behaviour of F_1 and F_2 for all complex values of s. It can be shown easily that these functions only have poles s_k on the imaginary axis (see fig. 11), s_k being the k^{th} root of the equation

$$s = \operatorname{th} s, \tag{2}$$

so that, according to (*12*, 13a),

$$\left.\begin{array}{l} s_0 = 0, \ s_k = i\,\omega_k \, (k = 1, 2, \ldots \infty), \\ s_k = -i\,\omega_{-k} \, (k = -1, -2, \ldots -\infty). \end{array}\right\} \tag{3}$$

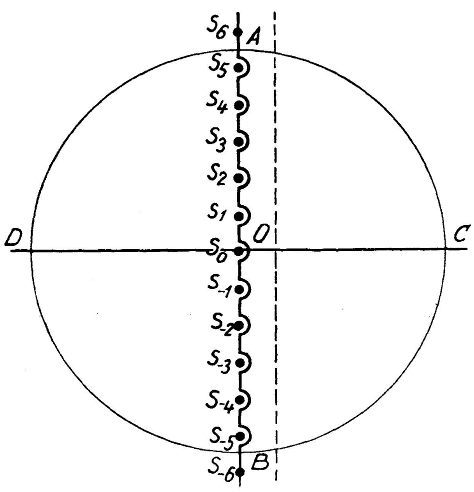

Fig. 11. The contours for the integration of the functions F_1 and F_2

The poles s_k $(k \neq 0)$ are simple, but the pole s_0 is a triple pole both for F_1 and F_2.

Both functions must be integrated along the path dotted in fig. 11. We can replace this path by the curve drawn in heavy lines in this figure. Further on we draw a circle $ACBDA$ with its mid point at the origin 0 and with such a radius R that the circle is sufficiently far removed from any pole of $F(s)$. By taking R larger and larger, $2\pi J_1$ and $2\pi J_2$ approach to $\int\limits_{BOA} F_1 ds$ and $\int\limits_{BOA} F_2 ds$ respectively.

For $u < 0$ we complete the path of integration to the contour $0ACB0$. It then can be proved that

$$\lim_{R\to\infty} \int\limits_{ACB} F_1 ds = 0, \quad \lim_{R\to\infty} \int\limits_{ACB} F_2 ds = 0,$$

and as there are no poles lying within the contour, we also have

$$\int_{0ACB0} F_1 \, ds = 0, \quad \int_{0ACB0} F_2 \, ds = 0,$$

so that in this case

$$J_1 = 0, \quad J_2 = 0.$$

For $u > 0$ we complete the path of integration to the contour $0ADB0$. Here we can prove

$$\lim_{R\to\infty} \int_{ADB} F_1 \, ds = 0, \quad \lim_{R\to\infty} \int_{ADB} F_2 \, ds = 0.$$

Now we have to apply the remainder theorem of the function theory; with its aid and considering the fact that $s_k \, (k \neq 0)$ is a simple pole but s_0 a triple pole both of the functions F_1 and F_2, we easily find the formulæ (*16*, 20a) and (*16*, 20b).

Next we come to the functions $f_1(u)$ and $f_2(u)$. At first we shall prove

$$f_2(0) = \sum_{k=1}^{\infty} \frac{1}{\omega_k^2} = \frac{1}{10}. \tag{4}$$

For this purpose we consider the integral

$$J = \frac{1}{2\pi i} \int_C F(s) \, ds, \quad F(s) = \frac{f'(s)}{s^2 f(s)}, \quad f(s) = \operatorname{tg} s - s, \tag{5}$$

C being a circle in the complex s plane with 0 as centre and with such a radius R, that C is sufficiently far removed from any pole of $F(s)$. The function $F(s)$ has a triple pole at 0, simple poles at the points $s = s_n = (n + \frac{1}{2})\pi \, (n = -\infty, \ldots -1, 0, +1, \ldots +\infty)$ and simple poles at the points $s = \omega_k \, (k \neq 0)$. Now the remainder theorem yields for $R \to \infty$

$$J = \frac{4}{5} - \frac{1}{\pi^2} \sum_{n=-\infty}^{+\infty} \frac{1}{(n + \frac{1}{2})^2} + \sum_{\substack{k=-\infty \\ \neq 0}}^{+\infty} \frac{1}{\omega_k^2},$$

but it also can be proved that for $R \to \infty$ $\int_C F(s) \, ds \to 0$, so that

$$f_2(0) = \frac{1}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^2} - \frac{2}{5} = \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} - \frac{2}{5}$$

$$= \frac{4}{\pi^2} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \right\} - \frac{2}{5} = \frac{1}{10},$$

as we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{6}$$

which property can be proved in an analogous manner by evaluating

$$\frac{1}{2\pi i} \int_C \frac{\operatorname{cotg} s \, ds}{s^2}.$$

Now we consider the function X_k (*12*, 21). With the aid of the orthogonal property (*12*, 16), the normalizing condition (*12*, 18) and the property that ω_k is a root of (*12*, 13a), we can prove

$$\tfrac{1}{4}(3x^2-1) = \sum_{n=1}^{\infty} \frac{X_k}{\omega_k} \qquad \text{for } -1<x<+1. \tag{7}$$

Introducing the new variable

$$u = x+1 \tag{8}$$

yields, with the formulæ (*16*, 21),

$$f_1(u)+f_2(u) = f_2(0)+\tfrac{1}{4}(3u^2-6u+2) \quad \text{for } 0 < u < 2. \tag{9a}$$

Further, it can be proved with the aid of the same formulæ

$$f_1(u)+f_2(u) = -f_1(u-2)+f_2(u-2) \quad \text{for all values of } u. \tag{9b}$$

As we also have

$$f_1(u) = -\frac{d f_2(u)}{d u}, \tag{10}$$

the function $f_2(u)$ can be solved from the differential equation

$$-\frac{d f_2}{d u}+f_2 = \frac{3}{4}u^2-\frac{3}{2}u+\frac{3}{5} \quad \text{for } 0<u<2, \tag{11a}$$

after which it can be solved in other intervals of u, using also (10), from the equation

$$-\frac{d f_2}{d u}+f_2 = -f_1(u-2)+f_2(u-2) \quad \text{for all values of } u. \tag{11b}$$

Thus the formulæ (*16*, 22a) and (16, 22b) can easily be found.

At last we consider the formulæ (*16*, 32). These can be proved by an integration of the Fourier expansion

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^k \frac{\sin\frac{1}{2}k\pi u}{k\pi} &= -\frac{1}{4}u && \text{for } -2<u<2,\\ &= -\frac{1}{4}(u-4) && \text{for } 2<u<6,\\ &\text{etc.,}\end{aligned}$$

making use of (6).

18. Continuation; the moving concentrated load. As a first example of a moving live load we shall treat the concentrated load. This can at the moment $\bar{t}$ be represented by the limit for $\Delta\bar{x}\to 0$ of

$$\left.\begin{aligned}\bar{p}(\bar{x},\bar{t}) &= 0 && \text{for } \bar{x}>\bar{v}\bar{t}-1 \text{ or } \bar{x}<\bar{v}\bar{t}-1-\Delta\bar{x},\\ &= \frac{\bar{P}}{\Delta\bar{x}} && \text{for } \bar{v}\bar{t}-1-\Delta\bar{x}<\bar{x}<\bar{v}\bar{t}-1,\end{aligned}\right\} \tag{1}$$

or with
$$P=\frac{2\,\overline{P}}{w\,l},\quad \Delta x=\frac{2}{l}\Delta\,\overline{x},$$
(*3*, 6b) and (*16*, 12), by the limit for $\Delta x\to 0$ of
$$\left.\begin{aligned}p(x,t)&=0 &&\text{for } x>v\,t-1 \text{ or } x<v\,t-1-\Delta x,\\ &=\frac{P}{\Delta x} &&\text{for } v\,t-1-\Delta x<x<v\,t-1.\end{aligned}\right\}\tag{2}$$
Then, according to (*16*, 11), $p_1(x)$ is the limit for $\Delta x\to 0$ of
$$\left.\begin{aligned}p_1(x)&=0 &&\text{for } x>-1 \text{ or } x<-1-\Delta x,\\ &=\frac{P}{\Delta x} &&\text{for } -1-\Delta x<x<1,\end{aligned}\right\}\tag{3}$$
so that the condition (*16*, 14) is satisfied.

At first we calculate the cable force. For $v=0$ this quantity is determined by the formula (*6*, 2) which yields with (2),
$$\frac{H}{P}=\frac{3}{4}(1-x_1^2)\ \text{for}\ -1\leqq x_1\leqq 1,\quad \frac{H}{P}=0\quad\text{for}\ x_1\geqq 1.\tag{4}$$
Here we replaced the quantities v and t, which become respectively zero and infinite for this case, by
$$x_1=v\,t-1.\tag{5}$$
For $v\neq 0$ we calculate with the help of (*16*, 25) the partial cable forces $H_j(t)$ $(j=1,2,\ldots)$ with (*18*, 3):
$$\left.\begin{aligned}\frac{H_1}{P}&=\frac{v}{2(1+v)}e^{t} &&\text{for } t>0,\\ \frac{H_2}{P}&=0 &&\text{for } 0<t<2,\\ &=-\frac{v}{1+v}\left(t-1+\frac{v}{1-v}\right)e^{t-2} &&\text{for } t>2,\\ \frac{H_3}{P}&=0 &&\text{for } 0<t<\frac{2}{v},\\ &=\frac{v}{2(1-v)}e^{t-\frac{2}{v}} &&\text{for } t>\frac{2}{v},\\ \frac{H_4}{P}&=0 &&\text{for } 0<t<4,\\ &=\frac{v}{1+v}\left(t^2-5\,t+12+\frac{2\,v\,t-7}{1-v}\right)e^{t-4} &&\text{for } t>4,\\ \frac{H_5}{P}&=0 &&\text{for } 0<t<2+\frac{2}{v},\\ &=-\frac{1}{1-v}\left\{v(t-2)-1-\frac{1}{1+v}\right\}e^{t-\frac{2}{v}-2} &&\text{for } t>2+\frac{2}{v},\end{aligned}\right\}\tag{6}$$
etc.

For $v=1$ we can write, according to (*16*, 26):

$$\left.\begin{aligned}
\frac{H_1}{P} &= \frac{1}{4}\,e^t && \text{for } t>0,\\
\frac{H_2+H_3}{P} &= 0 && \text{for } 0<t<2,\\
&= -\infty && \text{for } t=2,\\
&= -\tfrac{1}{4}(2t+1)\,e^{t-2} && \text{for } t>2,\\
\frac{H_4+H_5}{P} &= 0 && \text{for } 0<t<4,\\
&= +\infty && \text{for } t=4,\\
&= \tfrac{1}{2}(t^2-2t-2)\,e^{t-4} && \text{for } t<4,
\end{aligned}\right\} \qquad (7)$$

etc.

With the help of these formulæ and the relation (*16*, 24) we calculated $H(t)$ for $0 \leqq vt \leqq 3$ and for $v=0$; 0,5; 1 and 2. These values are represented in table XIII and figure 12; they will be discussed afterwards at the end of this chapter.

Table XIII. The cable force H/P for some values of vt with a moving concentrated load

vt \ v	0	0,5	1	2
0	0	{0; 0,1667}	{0; 0,25}	{0; 0,3333}
0,2	0,27	0,2486	0,3054	0,3684
0,4	0,48	0,3709	0,3730	0,4071
0,6	0,63	0,5534	0,4555	0,4500
0,8	0,72	0,8255	0,5564	0,4973
1	0,75	{1,2315; 0,5648}	0,6796	0,5496
1,2	0,72	0,6438	0,8300	0,6074
1,4	0,63	0,6637	1,0138	0,6713
1,6	0,48	0,5472	1,2382	0,7418
1,8	0,27	0,1561	1,5124	0,8199
2	0	{-0,7522; 0,4145}	{1,8473; $-\infty$; 0,5973}	{0,9061; -0,0939}
2,2	0	0,2220	0,6073	-0,1038
2,4	0	-0,0311	0,5927	-0,1147
2,6	0	-0,2029	0,5416	-0,1268
2,8	0	-0,0326	0,4391	-0,1405
3	0	{1,1449; 0,4782}	0,2644	-0,1548

Next we calculate the girder deflection. For the statical case the formula (*6*, 1) yields

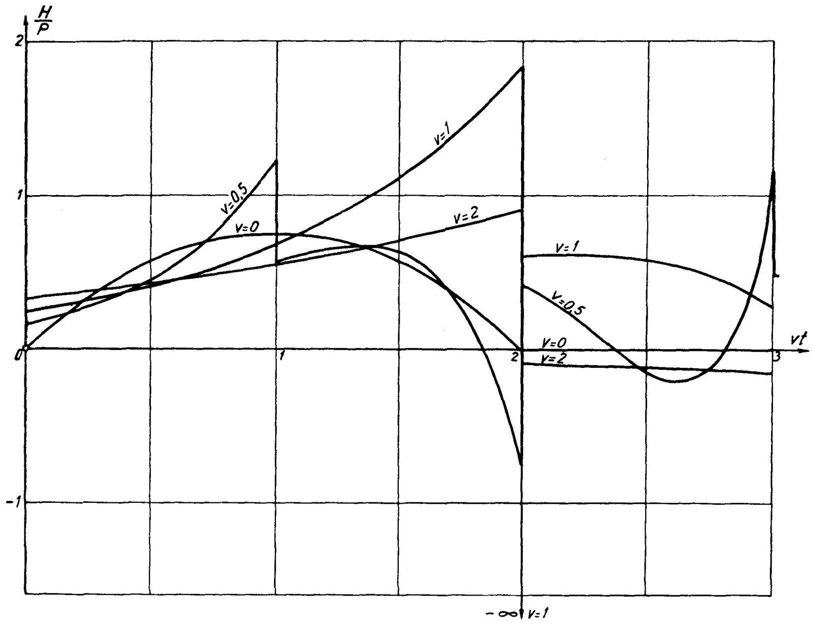

Fig. 12
The cable force as a function of vt with a moving concentrated load

$$\left.\begin{aligned}\frac{2\eta}{P} &= (1+x)(1-x_1)\{1-\tfrac{3}{4}(1+x_1)(1-x)\} \quad \text{for } -1 \leqq x \leqq x_1,\\ &= (1-x)(1+x_1)\{1-\tfrac{3}{4}(1-x_1)(1+x)\} \quad \text{for } x_1 \leqq x \leqq +1.\end{aligned}\right\} \tag{8}$$

For the dynamical case we must calculate the functions η_{mnj} (*17*, 34) – (*17*, 36). Restricting ourselves to the cases $0{,}5 \leqq v \leqq 2$ and $0 \leqq vt \leqq 2$ (i. e. to the time in which the force is on the girder), we can find that it is only necessary to determine

$$\left.\begin{aligned}\eta_1 &= \eta_{11}+\eta_{12}+\eta_{14},\\ \eta_2 &= \eta_{211}+\eta_{221}+\eta_{231}+\eta_{311}+\eta_{321},\\ \eta_3 &= \eta_{212}+\eta_{222}+\eta_{312},\end{aligned}\right\} \tag{9}$$

being

$$\eta = \eta_1+\eta_2+\eta_3. \tag{10}$$

We here omit the elementary intermediate calculations; having executed them, we get the formulæ

$$\left.\begin{array}{lll}\dfrac{1+v}{P}\,\eta_1 = 1+x & \text{for } -1 \leqq x \leqq vt-1, & \\ \quad = -\dfrac{v}{1-v}(1+x-t) & \text{for } vt-1 \leqq x \leqq t-1, & 0 \leqq t \leqq 2,\\ \quad = 0 & \text{for } t-1 \leqq x \leqq +1, & \\ \quad = 1+x & \text{for } -1 \leqq x \leqq vt-1, & \\ \quad = -\dfrac{v}{1-v}(1+x-t) & \text{for } vt-1 \leqq x \leqq 3-t, & 2 \leqq t \leqq \dfrac{4}{1+v},\\ \quad = \dfrac{2v}{1-v}(1-x) & \text{for } 3-t \leqq x \leqq +1, & \\ \quad = 1+x & \text{for } -1 \leqq x \leqq 3-t, & \\ \quad = \dfrac{1}{1-v}(1+x-vt)+\dfrac{2v}{1-v}(1-x) & \text{for } 3-t \leqq x \leqq vt-1, & \dfrac{4}{1+v} \leqq t \leqq \dfrac{2}{v}\\ \quad = \dfrac{2v}{1-v}(1-x) & \text{for } vt-1 \leqq x \leqq +1, & \end{array}\right\} \text{and } \tfrac{1}{2} \leqq v \leqq 1, \tag{11a}$$

$$
2\frac{1+v}{v}\frac{\eta_2}{P} = \left.\begin{array}{ll}
e^{t-x-1}-e^t+1+x & \text{for } -1\leqq x\leqq t-1, \\
= -e^t+t+1 & \text{for } t-1\leqq x\leqq 1-t, \\
= e^{t+x-1}-e^t+1-x & \text{for } 1-t\leqq x\leqq +1,
\end{array}\right\} 0\leqq t\leqq 1,
$$

$$
\left.\begin{array}{ll}
=e^{t-x-1}-e^t+1+x & \text{for } -1\leqq x\leqq 1-t, \\
=e^{t-x-1}+e^{t+x-1}-e^t-t+1 & \text{for } 1-t\leqq x\leqq t-1, \\
=e^{t+x-1}-e^t+1-x & \text{for } t-1\leqq x\leqq +1,
\end{array}\right\} 1\leqq t\leqq 2,
$$

$$
=e^{t-x-1}+e^{t+x-1}-e^t-t+1 \quad \text{for } -1\leqq x\leqq +1,\ 2\leqq t\leqq \frac{2}{v} \qquad \text{and } \tfrac{1}{2}\leqq v\leqq 1, \qquad (11\text{b})
$$

$$
2\frac{1+v}{v}\frac{\eta_3}{P} = 0 \qquad \text{for } 0\leqq t\leqq 2 \text{ and } \tfrac{1}{2}\leqq v\leqq 1, \qquad (11\text{c})
$$

$$
\left.\begin{array}{l}
=2\left(t-3+\dfrac{v}{1-v}\right)e^{t-2}-\left(2t-2x-7+\dfrac{2v}{1-v}\right)e^{t-x-3} \\
\quad +t-1-\dfrac{1+v}{1-v}(1+x) \quad \text{for } -1\leqq x\leqq t-3, \\
=2\left(t-3+\dfrac{v}{1-v}\right)e^{t-2}-\dfrac{2v}{1-v}(t-1)+2 \\
\qquad \text{for } t-3\leqq x\leqq 3-t, \\
=2\left(t-3+\dfrac{v}{1-v}\right)e^{t-2}-\left(2t+2x-7+\dfrac{2v}{1-v}\right)e^{t+x-3} \\
\quad +t-1-\dfrac{1+v}{1-v}(1-x) \quad \text{for } 3-t\leqq x\leqq +1
\end{array}\right|
\begin{array}{l}
\text{and} \\ 2\leqq t\leqq 3, \\ \tfrac{1}{2}\leqq v\leqq \tfrac{2}{3} \\ \\ \text{or} \\ 2\leqq t\leqq \tfrac{2}{v}, \\ \tfrac{2}{3}\leqq v\leqq 1,
\end{array} \qquad (11\text{d})
$$

$$
\left.\begin{array}{l}
=2\left(t-3+\dfrac{v}{1-v}\right)e^{t-2}-\left(2t-2x-7+\dfrac{2v}{1-v}\right)e^{t-x-3} \\
\quad +t-1-\dfrac{1+v}{1-v}(1+x) \quad \text{for } -1\leqq x\leqq 3-t, \\
=2\left(t-3+\dfrac{v}{1-v}\right)e^{t-2}-\left(2t-2x-7+\dfrac{2v}{1-v}\right)e^{t-x-3} \\
\quad -\left(2t+2x-7+\dfrac{2v}{1-v}\right)e^{t+x-3}+\dfrac{2}{1-v}(t-3) \\
\qquad \text{for } 3-t\leqq x\leqq t-3, \\
=2\left(t-3+\dfrac{v}{1-v}\right)e^{t-2}-\left(2t+2x-7+\dfrac{2v}{1-v}\right)e^{t+x-3} \\
\quad +t-1-\dfrac{1+v}{1-v}(1-x) \quad \text{for } t-3\leqq x\leqq +1
\end{array}\right\}
\begin{array}{l}
\text{and} \\ 3\leqq t\leqq \tfrac{2}{v}, \\ \tfrac{1}{2}\leqq v\leqq \tfrac{2}{3},
\end{array} \qquad (11\text{e})
$$

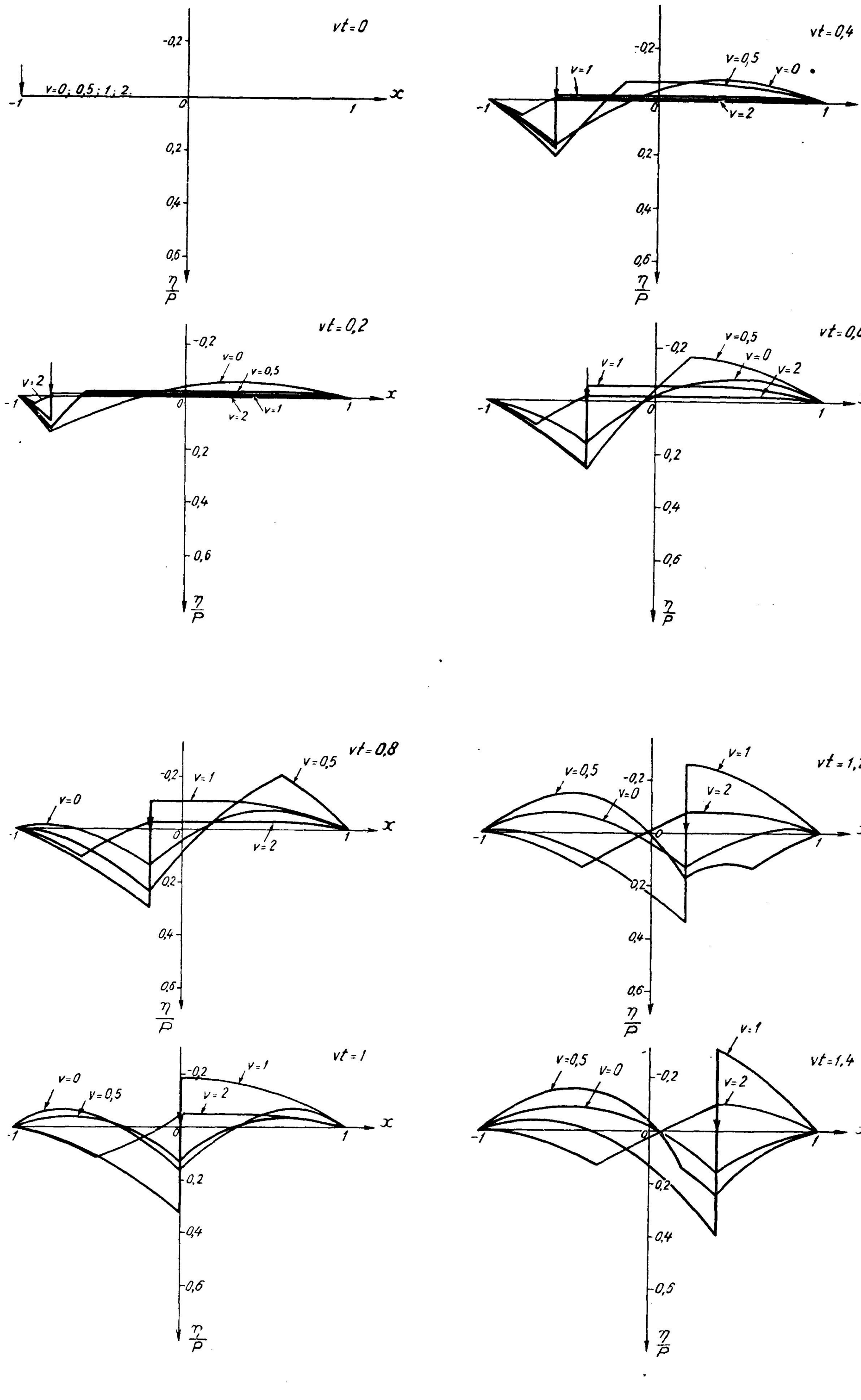

Fig. 13a. The girder deflection as a function of x and vt with a moving concentrated load

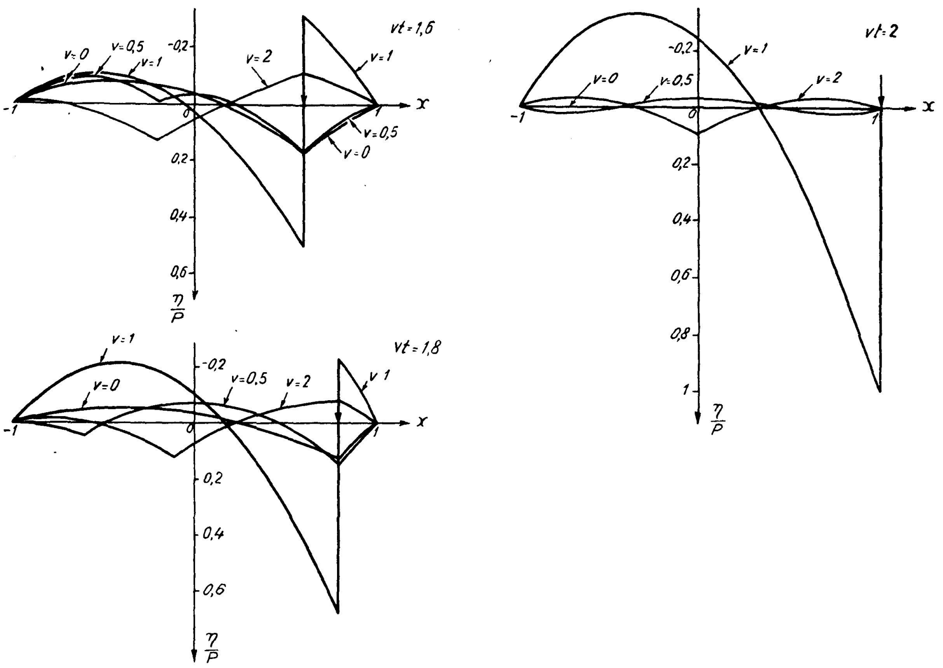

Fig. 13b. The girder deflection as a function of x and vt with a moving concentrated load

$$\left.\begin{array}{lll}
\left.\begin{array}{ll}
\dfrac{2\eta_1}{P} = 1+x & \text{for } -1 \leqq x < t-1, \\
\quad = 0 & \text{for } t-1 < x \leqq +1,
\end{array}\right\} & 0 \leqq t \leqq 2, & \\
\left.\begin{array}{ll}
4\,\dfrac{\eta_2+\eta_3}{P} = e^{t-x-1} - e^{t} + 1 + x & \text{for } -1 \leqq x \leqq t-1, \\
\quad = -e^{t} + t + 1 & \text{for } t-1 \leqq x \leqq 1-t, \\
\quad = e^{t+x-1} - e^{t} + 1 - x & \text{for } 1-t \leqq x \leqq +1,
\end{array}\right\} & 0 \leqq t \leqq 1, & \\
\left.\begin{array}{ll}
\quad = e^{t-x-1} - e^{t} + 1 + x & \text{for } -1 \leqq x \leqq 1-t, \\
\quad = e^{t+x-1} + e^{t-x-1} - e^{t} + 1 - t & \text{for } 1-t \leqq x \leqq t-1, \\
\quad = e^{t+x-1} - e^{t} + 1 - x & \text{for } t-1 \leqq x \leqq +1,
\end{array}\right\} & 1 \leqq t \leqq 2 &
\end{array}\right\} \text{and } v = 1, \qquad (12)$$

$$\left.\begin{array}{lll}
\left.\begin{array}{ll}
\dfrac{1+v}{P}\,\eta_1 = 1+x & \text{for } -1 \leqq x \leqq t-1, \\
\quad = \dfrac{1}{1-v}(1+x-vt) & \text{for } t-1 \leqq x \leqq vt-1, \\
\quad = 0 & \text{for } vt-1 \leqq x \leqq +1,
\end{array}\right\} & 0 \leqq t \leqq \frac{2}{v}, & \\
\left.\begin{array}{ll}
2\,\dfrac{1+v}{v}\,\dfrac{\eta_2+\eta_3}{P} = e^{t-x-1} - e^{t} + 1 + x & \text{for } -1 \leqq x \leqq t-1, \\
\quad = -e^{t} + t + 1 & \text{for } t-1 \leqq x \leqq 1-t, \\
\quad = e^{t+x-1} - e^{t} + 1 - x & \text{for } 1-t \leqq x \leqq +1
\end{array}\right\} & 0 \leqq t \leqq 1, & \\
\left.\begin{array}{ll}
\quad = e^{t-x-1} - e^{t} + 1 + x & \text{for } -1 \leqq x \leqq 1-t, \\
\quad = e^{t+x-1} + e^{t-x-1} - e^{t} + 1 - t & \text{for } 1-t \leqq x \leqq t-1, \\
\quad = e^{t+x-1} - e^{t} + 1 - x & \text{for } t-1 \leqq x \leqq +1,
\end{array}\right\} & 1 \leqq t \leqq \frac{2}{v} &
\end{array}\right\} \text{and } 1 \leqq v \leqq 2. \qquad (13)$$

With their help we calculated η for $v=0$; 0,5; 1 and 2. The values are represented in tables XIV and figure 13.

The figures 12 and 13 show us the following phenomena, somewhat analogous to those occurring with a strained string:

Table XIVa. The deflection η/P for some values of vt and x with a moving concentrated load, for $v=0$

$vt \backslash x$	-1	-0,8	-0,6	-0,4	-0,2	0	0,2	0,4	0,6	0,8	1
0	0	0	0	0	0	0	0	0	0	0	0
0,2	0	0,1314	0,0736	0,0266	-0,0096	-0,035	-0,0496	-0,0534	-0,0464	-0,0286	0
0,4	0	0,0736	0,1664	0,0784	0,0096	-0,04	-0,0704	-0,0816	-0,0736	-0,0464	0
0,6	0	0,0266	0,0784	0,1554	0,0576	-0,0150	-0,0624	-0,0846	-0,0816	-0,0534	0
0,8	0	-0,0096	0,0096	0,0576	0,1344	0,0400	-0,0256	-0,0624	-0,0704	-0,0496	0
1	0	-0,035	-0,04	-0,015	0,04	0,125	0,04	-0,015	-0,04	-0,035	0
1,2	0	-0,0496	-0,0704	-0,0624	-0,0256	0,0400	0,1344	0,0576	0,0096	-0,0096	0
1,4	0	-0,0534	-0,0816	-0,0846	-0,0624	-0,0150	0,0576	0,1554	0,0784	0,0266	0
1,6	0	-0,0464	-0,0736	-0,0816	-0,0704	-0,04	0,0096	0,0784	0,1664	0,0736	0
1,8	0	-0,0286	-0,0464	-0,0534	-0,0496	-0,035	-0,0096	0,0266	0,0736	0,1314	0
2	0	0	0	0	0	0	0	0	0	0	0

Table XIVb. The deflection η/P for some values of vt and x with a moving concentrated load, for $v=0{,}5$

$vt \backslash x$	-1	-0,8	-0,6	-0,4	-0,2	0	0,2	0,4	0,6	0,8	1
0	0	0	0	0	0	0	0	0	0	0	0
0,2	0	0,1216	-0,0153	-0,0153	-0,0153	-0,0153	-0,0153	-0,0153	-0,0153	-0,0117	0
0,4	0	0,0994	0,2111	0,0659	-0,0709	-0,0709	-0,0709	-0,0674	-0,0556	-0,0339	0
0,6	0	0,0663	0,1509	0,2503	0,0953	-0,0463	-0,1714	-0,1497	-0,1158	-0,0670	0
0,8	0	0,0170	0,0612	0,1311	0,2273	0,0818	-0,0393	-0,1356	-0,2055	-0,1163	0
1	0	-0,0531	-0,0574	-0,0187	0,0593	0,1745	0,0593	-0,0187	-0,0574	-0,0531	0
1,2	0	-0,0822	-0,1359	-0,1438	-0,0929	0,0133	0,1738	0,1229	0,1307	0,0512	0
1,4	0	-0,0851	-0,1377	-0,1524	-0,1238	-0,0300	0,1429	0,2476	0,1289	0,0483	0
1,6	0	-0,0632	-0,0922	-0,0767	-0,0063	-0,0255	-0,0063	0,0566	0,1744	0,0702	0
1,8	0	0,0092	0,0474	-0,0172	-0,0587	-0,0731	-0,0587	-0,0172	0,0474	0,1426	0
2	0	0,0244	0,0153	-0,0054	-0,0232	-0,0301	-0,0232	-0,0054	0,0153	0,0244	0

Table XIV c. The deflection η/P for some values of vt and x with a moving concentrated load, for $v = 1$

vt \ x	−1	−0,8	−0,6	−0,4	−0,2	0	0,2	0,4	0,6	0,8	1
0	0	0	0	0	0	0	0	0	0	0	0
0,2	0	−0,0054 0,0946	−0,0054	−0,0054	−0,0054	−0,0054	−0,0054	−0,0054	−0,0054	−0,0054	0
0,4	0	0,0824	−0,0230 0,1770	−0,0230	−0,0230	−0,0230	−0,0230	−0,0230	−0,0230	−0,0176	0
0,6	0	0,0674	0,1498	−0,0555 0,2445	−0,0555	−0,0555	−0,0555	−0,0555	−0,0502	−0,0326	0
0,8	0	0,0492	0,1166	0,1990	−0,1064 0,2936	−0,1064	−0,1064	−0,1010	−0,0834	−0,0508	0
2	0	0,0268	0,0760	0,1434	0,2258	−0,1796 0,3204	−0,1742	−0,1566	−0,1240	−0,0732	0
1,2	0	−0,0004	0,0264	0,0755	0,1429	0,2307	−0,2571 0,3429	−0,2245	−0,1736	−0,1004	0
1,4	0	−0,0338	−0,0342	−0,0074	0,0471	0,1321	0,2471	−0,3074 0,3926	−0,2342	−0,1338	0
1,6	0	−0,0744	−0,1082	−0,1033	−0,0589	0,0228	0,1411	0,2967	−0,3082 0,4918	−0,1744	0
1,8	0	−0,1242	−0,1932	−0,2094	−0,1773	−0,0996	0,0227	0,1906	0,4068	−0,2242 0,6758	0
2	0	−0,1795	−0,2861	−0,3280	−0,3109	−0,2381	−0,1109	0,0720	0,3139	0,6205	0 1

Table XIV d. The deflection η/P for some values of vt and x with a moving concentrated load, for $v = 2$

vt \ x	−1	−0,8	−0,6	−0,4	−0,2	0	0,2	0,4	0,6	0,8	1
0	0	0	0	0	0	0	0	0	0	0	0
0,2	0	−0,0017	−0,0017	−0,0017	−0,0017	−0,0017	−0,0017	−0,0017	−0,0017	−0,0017	0
0,4	0	0,0596	−0,0071	−0,0071	−0,0071	−0,0071	−0,0071	−0,0071	−0,0071	−0,0071	0
0,6	0	0,0528	0,0501	−0,0166	−0,0166	−0,0166	−0,0166	−0,0166	−0,0166	−0,0149	0
0,8	0	0,0432	0,1027	0,0361	−0,0306	−0,0306	−0,0306	−0,0306	−0,0306	−0,0235	0
1	0	0,0338	0,0855	0,0837	0,0271	−0,0496	−0,0496	−0,0496	−0,0478	−0,0329	0
1,2	0	0,0233	0,0664	0,1260	0,0593	−0,0073	−0,0740	−0,0740	−0,0669	−0,0434	0
1,4	0	0,0117	0,0453	0,0971	0,0954	0,0287	−0,0379	−0,1029	−0,0880	−0,0550	0
1,6	0	−0,0011	0,0221	0,0653	0,1249	0,0582	−0,0085	−0,0680	−0,1112	−0,0676	0
1,8	0	−0,0152	−0,0037	0,0301	0,0831	0,0802	0,0152	−0,0366	−0,0703	−0,0819	0
2	0	−0,0309	−0,0321	−0,0088	0,0344	0,0939	−0,0344	−0,0088	−0,0321	−0,0309	0

1. There is a nick in the girder at the place of the load, moving with the speed v. For $v < 1$ the moving force is in a trough and for $v > 1$ it is on a peak.

2. For $v \neq 1$ there exists another nick in the girder which moves with the speed 1. When this nick reaches one of the ends of the girder, the cable force shows a jump.

3. For $v = 1$ the two nicks mentioned under 1 and 2 and the intermediate part of the girder change into a jump in the girder. When this jump reaches the right end of the girder (for $t = 2$), the cable force becomes $-\infty$ and gets the character of an impact, so that the velocities of the girder elements undergo a sudden change. Physically this is impossible, as the total cable force H_p cannot become negative in reality, let alone that the cable force cannot be infinite at all.

However, we must keep in mind that the reduction of the non linear differential equation (*11*, 1) to the linear equation (*16*, 1) only has significance for $p(x,t)$ and $H(t)$ being small with respect to 1. When in the solution of (16, 1) $H(t) = \infty$ this solution is not valid any more, so that we now must try to solve the equation (*11*, 1) with the conditions (*16*, 2) and (*16*, 3), $p(x,t)$ being a concentrated load P moving with the speed 1. Owing to the non linear character of the problem η/P is no more independent of P, so that P becomes a parameter. It would be very interesting to solve this problem.

Further on the figures 12 and 13 show that it is necessary to execute the calculations for many more values of v if one desires to know H and η plotted against v.

19. Continuation; the moving load, uniformly distributed. We shall now consider the case in which the moving load is constant and extends over half the length of the girder. Here the load $p(x,t)$ can be represented by

$$\left.\begin{aligned} p(x,t) &= 0 \quad \text{for } x > vt - 1 \text{ or } x < vt - 2, \\ &= p \quad \text{for } vt - 2 < x < vt - 1, \end{aligned}\right\} \tag{1}$$

so that we have, according to (*16*, 11),

$$\left.\begin{aligned} p_1(x) &= 0 \quad \text{for } x > -1 \text{ or } x < -2, \\ &= p \quad \text{for } -2 < x < -1, \end{aligned}\right\} \tag{2}$$

and the condition (*16*, 14) is satisfied.

For $v = 0$ we define the place of the front of the moving load by the coordinate x_1, satisfying (*18*, 5). Then the formula (*6*, 2) yields for the cable force

$$\left.\begin{aligned} \frac{H}{p} &= \tfrac{1}{4}(1+x_1)^2(2-x_1) \text{ for } -1 \leqq x_1 \leqq 0, \quad \frac{H}{p} = \tfrac{1}{4}(2+3x_1-3x_1^2) \text{ for } 0 \leqq x_1 \leqq 1, \\ \frac{H}{p} &= \tfrac{1}{4}(2-x_1)^2(1+x_1) \text{ for } 1 \leqq x_1 \leqq 2, \quad \frac{H}{p} = 0 \text{ for } x_1 \geqq 2. \end{aligned}\right\} \tag{3}$$

For $v \neq 0$ we use the formulæ (*16*, 25) and (*16*, 26). Restricting ourselves to the interval in which the moving load is on the girder, i.e. the interval $0 \leqq vt \leqq 3$, and to the values $0{,}5 \leqq v \leqq 2$, it is apparent that we only have to compute

$$\left.\begin{aligned}
\frac{H_1}{p} &= \frac{v^2}{2(1+v)}(e^t-1) && \text{for } 0 \leqq t \leqq \frac{1}{v},\\
&= \frac{v^2}{2(1+v)}\left(e^t - e^{t-\frac{1}{v}}\right) && \text{for } t \geqq \frac{1}{v},\\
\frac{H_2}{p} &= 0 && \text{for } 0 \leqq t \leqq 2,\\
&= -\frac{v^2}{1+v}(t-2)\,e^{t-2} - \frac{v^3}{1-v^2}(e^{t-2}-1) && \text{for } 2 \leqq t \leqq 2+\frac{1}{v},\\
&= -\frac{v^2}{1+v}\left\{(t-2)\,e^{t-2} - \left(t-2-\frac{1}{v}\right)e^{t-2-\frac{1}{v}}\right\}\\
&\quad - \frac{v^3}{1-v^2}\left(e^{t-2} - e^{t-2-\frac{1}{v}}\right) && \text{for } t \geqq 2+\frac{1}{v},\\
\frac{H_3}{p} &= 0 && \text{for } 0 \leqq t \leqq \frac{2}{v},\\
&= \frac{v^2}{2(1-v)}\left(e^{t-\frac{2}{v}} - 1\right) && \text{for } \frac{2}{v} \leqq t \leqq \frac{3}{v},\\
\frac{H_4}{p} &= 0 && \text{for } 0 \leqq t \leqq 4,\\
&= \frac{v^2}{1+v}(t^2-9t+21)\,e^{t-4} + \frac{v^2}{1-v^2}\{(2t-9)\,e^{t-4}+v\} && \text{for } 4 \leqq t \leqq 4+\frac{1}{v},
\end{aligned}\right\} \quad (4)$$

while for $v = 1$

$$\left.\begin{aligned}
\frac{H_1}{p} &= \tfrac{1}{4}(e^t-1) && \text{for } 0 \leqq t \leqq 1,\\
&= \tfrac{1}{4}(e^t - e^{t-1}) && \text{for } t \geqq 1,\\
\frac{H_2+H_3}{p} &= 0 && \text{for } 0 \leqq t < 2,\\
&= -\tfrac{1}{4}\{(2t-1)\,e^{t-2}+1\} && \text{for } 2 < t < 3,\\
&= -\tfrac{1}{4}\{(2t-1)\,e^{t-2}-(2t-3)\,e^{t-3}\} && \text{for } t > 3.
\end{aligned}\right\} \quad (5)$$

With the help of these formulæ we calculated $H(t)$ for $0 \leqq vt \leqq 2$ and for $v = 0$; 0,5; 1 and 2. The values are represented in table XV and figure 14.

Next we compute the girder deflection, but owing to the great amount of computing labour we shall restrict ourselves to the cases $v = 0$ and $v = 1$. For $v = 0$ the formula (*6*, 1) yields with (1) and (*18*,5)

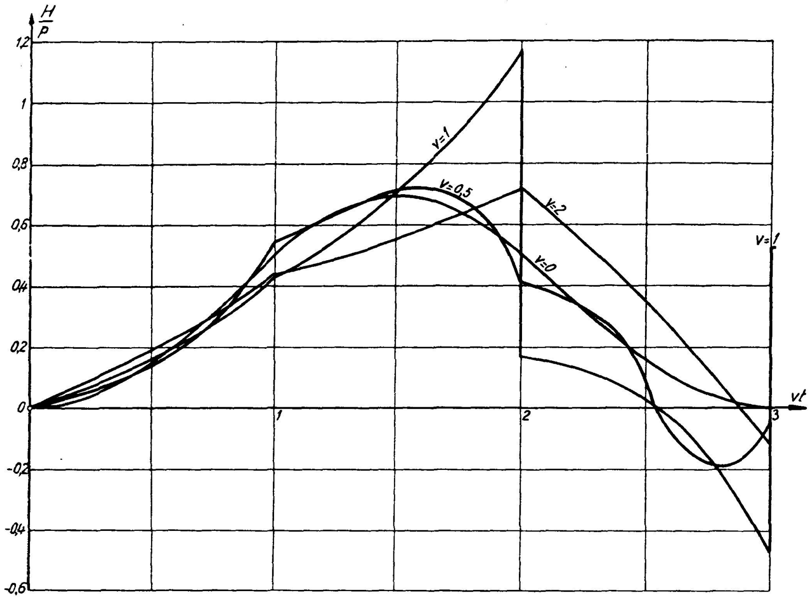

Fig. 14. The cable force as a function of vt with a moving load, uniformly distributed

Table XV. The cable force H/p for some values of vt with a moving load, uniformly distributed

vt \ v	0	0,5	1	2
0	0	0	0	0
0,2	0,028	0,0410	0,0554	0,0701
0,4	0,104	0,1021	0,1230	0,1476
0,6	0,216	0,1933	0,2055	0,2332
0,8	0,352	0,3294	0,3064	0,3279
1	0,5	0,5324	0,4296	0,4325
1,2	0,62	0,6128	0,5247	0,4780
1,4	0,68	0,6839	0,6408	0,5282
1,6	0,68	0,7170	0,7827	0,5838
1,8	0,62	0,6574	0,9560	0,6452
2	0,5	0,4062	{ 1,1677	0,7130
			0,1677 }	
2,2	0,352	0,3497	0,1380	0,5777
2,4	0,216	0,2379	0,0748	0,4281
2,6	0,104	–0,0783	–0,0355	0,2628
2,8	0,028	–0,1883	–0,2106	0,0801
3	0	–0,0540	{ –0,4737	–0,1218
			0,5263 }	

$$\left.\begin{aligned}
\frac{4\eta}{p} &= \{2(1-x)-(1-x_1)^2\}(1+x)-\frac{2H}{p}(1-x^2) \\
&\qquad\qquad\qquad\qquad \text{for } -1\leqq x\leqq x_1, \\
&= (1+x_1)^2(1-x)-\frac{2H}{p}(1-x^2) \quad \text{for } x_1\leqq x\leqq 1
\end{aligned}\right\}\ \text{and } -1\leqq x_1\leqq 0,$$

$$\left.\begin{aligned}
&= (3-2x_1)(1+x)-\frac{2H}{p}(1-x^2) \quad \text{for } -1\leqq x\leqq x_1-1, \\
&= 2(1-x^2)-x_1^2(1-x)-(1-x_1)^2(1+x)-\frac{2H}{p}(1-x^2) \\
&\qquad\qquad\qquad\qquad \text{for } x_1-1\leqq x\leqq x_1, \\
&= (1+2x_1)(1-x)-\frac{2H}{p}(1-x^2) \quad \text{for } x_1\leqq x\leqq 1
\end{aligned}\right\}\ \text{and } 0\leqq x_1\leqq 1,$$

$$\left.\begin{aligned}
&= (2-x_1)^2(1+x)-\frac{2H}{p}(1-x^2) \quad \text{for } -1\leqq x\leqq x_1-1, \\
&= \{2(1+x)-x_1^2\}(1-x)-\frac{2H}{p}(1-x^2) \quad \text{for } x_1-1\leqq x\leqq 1
\end{aligned}\right\}\ \text{and } 1\leqq x_1\leqq 2,$$

$$= 0 \qquad \text{for } x_1\geqq 2. \tag{6}$$

For $v=1$ we write down the formulæ for all partial deflections η_{11}, $\eta_{12}\dots$, η_{211} etc. being considered, as it presents no advantage at all to combine them. We find with (*16*, 36) for η_{1n}

$$\left.\begin{aligned}
\frac{2}{p}(\eta_{11}+\eta_{12}) &= 1+x && \text{for } x\leqq t-2, \\
&= (1+x)(t-1-x) && \text{for } t-2\leqq x\leqq t-1, \\
&= 0 && \text{for } x\geqq t-1, \\
\frac{2}{p}(\eta_{13}+\eta_{14}) &= 0 && \text{for } x\leqq 3-t, \\
&= 2(3-x-t) && \text{for } 3-t\leqq x\leqq 4-t.
\end{aligned}\right\} \tag{7}$$

Further on we find with (*16*, 35b), (*16*, 35c) and (5) for η_{mnj} ($m=2$ and 3)

$$\left.\begin{aligned}
\frac{4}{p}\eta_{211} &= -e^t+\tfrac{1}{2}t^2+t+1 && \text{for } x\leqq 1-t, \\
&= -e^t+(2-x)e^{t+x-1}+\tfrac{1}{2}(1-x)^2 && \text{for } x\geqq 1-t
\end{aligned}\right\}\ \text{and } 0\leqq t\leqq 1,$$

$$\left.\begin{aligned}
&= -e^t+e^{t-1}+t+\tfrac{1}{2} \qquad \text{for } x\leqq 1-t, \\
&= -e^t+e^{t-1}+(2-x)e^{t+x-1}-\tfrac{1}{2}t^2+\tfrac{1}{2}(1-x)^2-\tfrac{1}{2} \\
&\qquad\qquad \text{for } 1-t\leqq x\leqq 2-t, \\
&= -e^t+e^{t-1}+(2-x)(e^{t+x-1}-e^{t+x-2}) \\
&\qquad\qquad \text{for } x\geqq 2-t
\end{aligned}\right\}\ \text{and } t\geqq 1,$$

$$\frac{4}{p}(\eta_{212}+\eta_{213}) = 0 \qquad \text{for } 0\leqq t\leqq 2,$$

$$\left.\begin{aligned}
&= (2t-5)e^{t-2}+\tfrac{1}{2}t^2-3t+5 \qquad \text{for } x\leqq 3-t, \\
&= (2t-5)e^{t-2}+\{2(3-x)^2-(2t+3)(2-x)\}e^{t+x-3} \\
&\quad +\tfrac{1}{2}(1-x)^2 \qquad \text{for } x\geqq 3-t
\end{aligned}\right\}\ \text{and } 2\leqq t\leqq 3, \tag{8a}$$

$$\left.\begin{aligned}
\frac{4}{p}\,\eta_{221} &= 0 && \text{for } 0 \leqq t \leqq 1-x,\\
&= -(1-x)\,(e^{t+x-1}-t-x) && \text{for } 1-x \leqq t \leqq 2-x,\\
&= -(1-x)\,(e^{t+x-1}-e^{t+x-2}-1) && \text{for } 2-x \leqq t \leqq 3-x,\\
&= -(1-x)\,(e^{t+x-1}-e^{t+x-2}-e^{t+x-3}+t+x-3) && \text{for } 3-x \leqq t \leqq 4-x,\\
\frac{4}{p}\,(\eta_{222}+\eta_{223}) &= 0 && \text{for } 0 \leqq t \leqq 3-x,\\
&= (1-x)\{(2t+2x-5)\,e^{t+x-3}+t+x+4\} && \text{for } 3-x \leqq t \leqq 4-x,
\end{aligned}\right\} \qquad (8\text{b})$$

$$\left.\begin{aligned}
\frac{4}{p}\,\eta_{231} &= 0 && \text{for } 0 \leqq t \leqq 3-x,\\
&= -(2-x)\,e^{t+x-3}-\tfrac{1}{2}(1-x)^2+\tfrac{1}{2}(t^2-2t+2) && \text{for } 3-x \leqq t \leqq 4-x,
\end{aligned}\right\} \qquad (8\text{c})$$

$$\left.\begin{aligned}
\frac{4}{p}\,\eta_{311} &= 0 && \text{for } 0 \leqq t \leqq 1+x,\\
&= e^{t-x-1}-\tfrac{1}{2}(t-x)^2-\tfrac{1}{2} && \text{for } 1+x \leqq t \leqq 2+x,\\
&= e^{t-x-1}-e^{t-x-2}-t+x+\tfrac{1}{2} && \text{for } 2+x \leqq t \leqq 3+x,\\
&= e^{t-x-1}-e^{t-x-2}-3\,e^{t-x-3}+\frac{1}{2}(t-x-1)^2-\frac{3}{2} && \text{for } 3+x \leqq t \leqq 4+x,\\
\frac{4}{p}\,(\eta_{312}+\eta_{313}) &= 0 && \text{for } 0 \leqq t \leqq 3+x,\\
&= -(2t-2x-7)\,e^{t-x-3}-\tfrac{1}{2}(t-x-4)^2-\tfrac{1}{2} && \text{for } 3+x \leqq t \leqq 4+x,
\end{aligned}\right\} \qquad (9\text{a})$$

$$\left.\begin{aligned}
\frac{4}{p}\,\eta_{321} &= 0 && \text{for } 0 \leqq t \leqq 3+x,\\
&= 2\,e^{t-x-3}-2t+2x+4 && \text{for } 3+x \leqq t \leqq 4+x.
\end{aligned}\right\} \qquad (9\text{b})$$

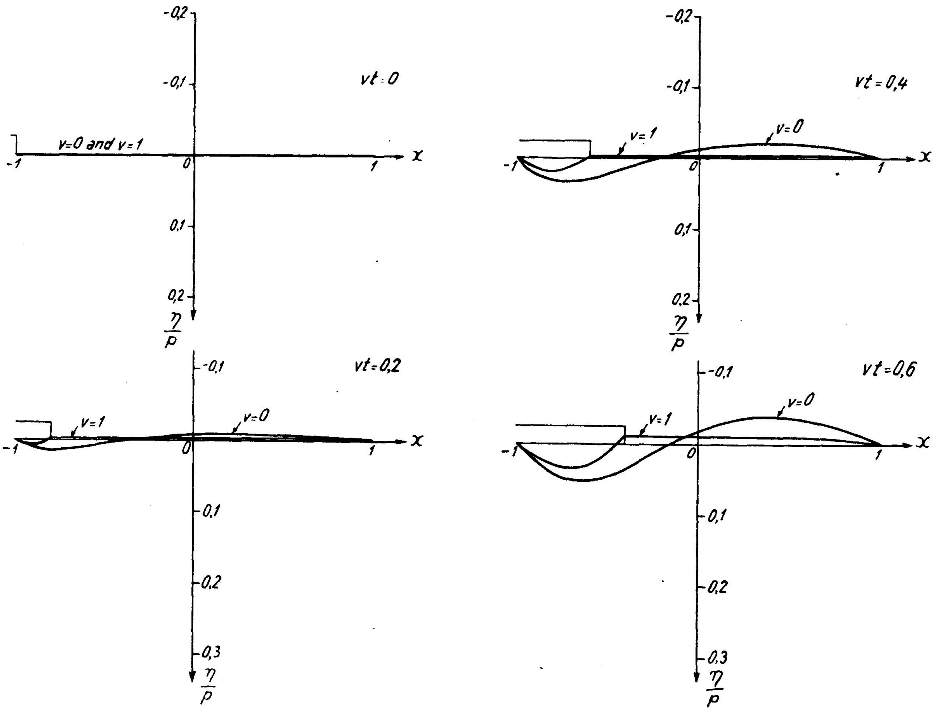

Fig. 15a. The girder deflection as a function of x and vt with a moving uniformly distributed load

With the help of these formulæ we calculated η for $v=0$ and $v=1$. The values are represented in tables XVIa, XVIb and figures 15a—15c.

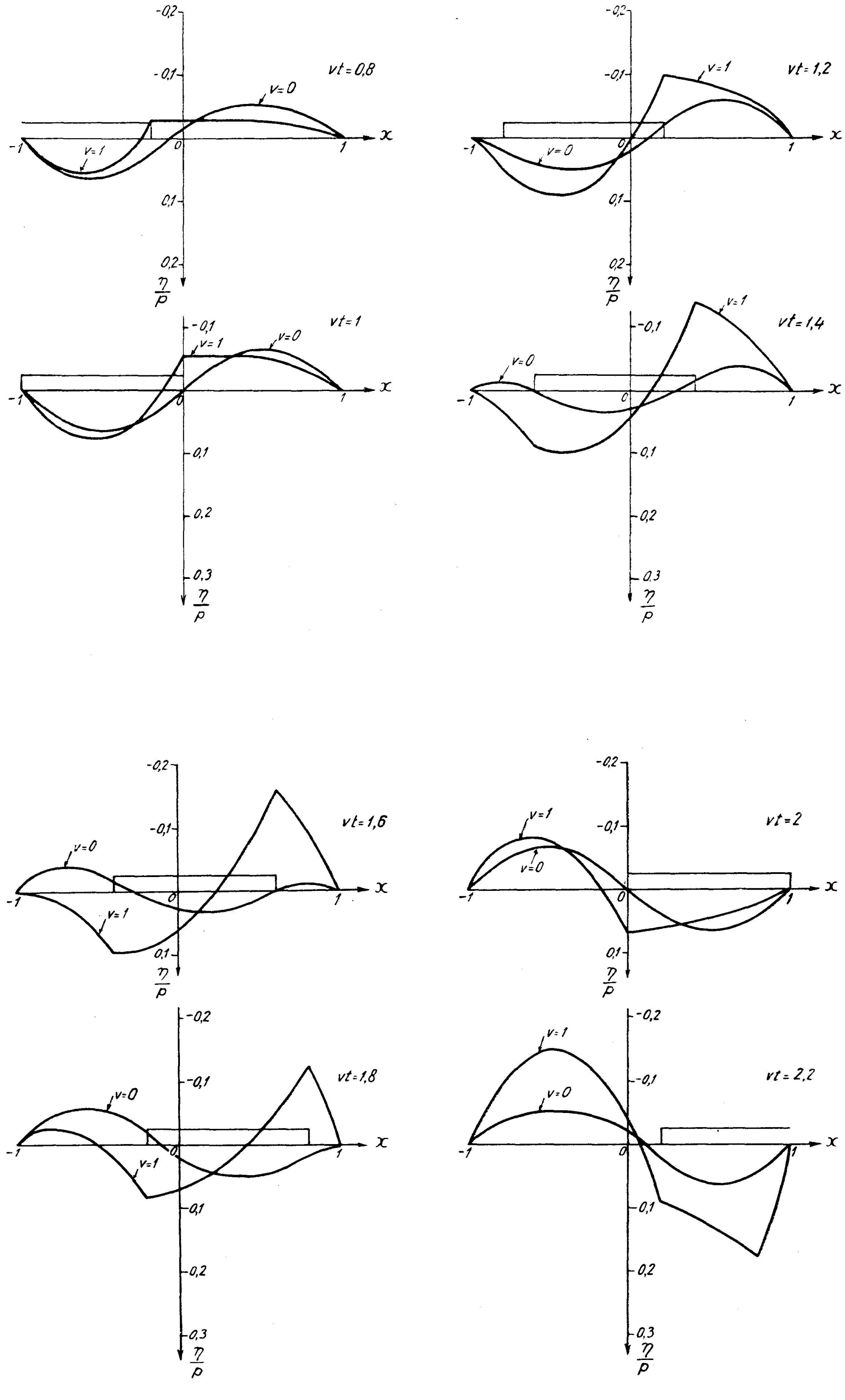

Fig. 15b. The girder deflection as a function of x and vt with a moving load, uniformly distributed

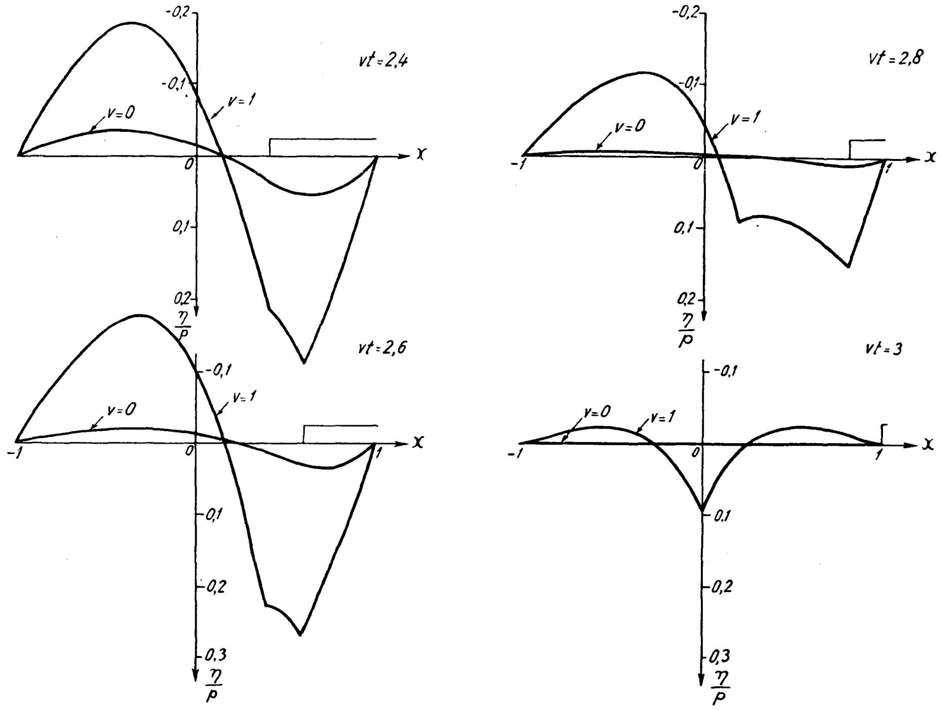

Fig. 15c. The girder deflection as a function of x and vt with a moving uniformly distributed load

Table XVIa. The deflection $10^3\,\eta/p$ for some values of $v\,t$ and x with a moving uniformly distributed load, for $v=0$

vt \ x	−1	−0,8	−0,6	−0,4	−0,2	0	0,2	0,4	0,6	0,8	1
0	0	0	0	0	0	0	0	0	0	0	0
0,2	0	12,96	7,04	2,24	−1,44	− 4	− 5,44	− 5,76	− 4,96	− 3,04	0
0,4	0	33,28	30,72	12,32	−1,92	−12	−17,92	−19,68	−17,28	−10,72	0
0,6	0	43,12	54,80	35,28	4,32	−18	−31,68	−36,72	−33,12	−20,88	0
0,8	0	44,64	63,36	56,16	23,04	−16	−40,96	−51,84	−48,64	−31,36	0
1	0	40	60	60	40	0	−40	−60	−60	−40	0
1,2	0	18,4	41,6	49,6	42,4	20	−17,6	−50,4	−58,4	−41,6	0
1,4	0	−12,4	2,4	24,4	33,6	30	13,6	−15,6	−37,6	−32,4	0
1,6	0	−32,4	−37,6	−15,6	13,6	30	33,6	24,4	2,4	−12,4	0
1,8	0	−41,6	−58,4	−50,4	−17,6	20	42,4	49,6	41,6	18,4	0
2	0	−40	−60	−60	−40	0	40	60	60	40	0
2,2	0	−31,36	−48,64	−51,84	−40,96	−16	23,04	56,16	63,36	44,64	0
2,4	0	−20,88	−33,12	−36,72	−31,68	−18	4,32	35,28	54,88	43,12	0
2,6	0	−10,72	−17,28	−19,68	−17,92	−12	− 1,92	12,32	30,72	33,28	0
2,8	0	− 3,04	− 4,96	− 5,76	− 5,44	− 4	− 1,44	2,24	7,04	12,96	0
3	0	0	0	0	0	0	0	0	0	0	0

Table XVIb. The deflection $10^3 \eta/p$ for some values of vt and x with a moving uniformly distributed load, for $v = 1$

vt \ x	−1	−0,8	−0,6	−0,4	−0,2	0	0,2	0,4	0,6	0,8	1
0	0	0	0	0	0	0	0	0	0	0	0
0,2	0	−0,351	−0,351	− 0,351	− 0,351	− 0,351	− 0,351	− 0,351	− 0,351	− 0,351	0
0,4	0	17,394	−2,956	− 2,956	− 2,956	− 2,956	− 2,956	− 2,956	− 2,956	− 2,606	0
0,6	0	32,426	29,821	−10,530	−10,530	−10,530	−10,530	−10,530	−10,179	− 7,574	0
0,8	0	44,144	56,571	33,966	−26,385	−26,385	−26,385	−26,034	−23,429	−15,856	0
1	0	51,815	75,959	68,386	25,780	−54,570	−54,220	−51,614	−44,041	−28,185	0
1,2	0	54,892	86,707	90,851	63,278	1,023	−96,722	−89,149	−73,293	−45,108	0
1,4	0	33,835	88,727	100,542	85,037	40,069	−34,963	−139,458	−111,273	−66,165	0
1,6	0	8,116	41,950	97,193	91,613	63,331	11,613	−62,807	−158,050	−91,884	0
1,8	0	−23,298	−14,832	21,608	84,074	71,744	44,074	1,608	−54,832	−123,298	0
2	0	−61,316	−82,009	−66,320	−16,630	66,448	63,370	53,680	37,991	18,684	0
2,2	0	−83,785	−137,878	−145,321	−109,020	−30,078	90,980	114,679	142,122	176,215	0
2,4	0	−81,972	−152,506	−185,988	−164,178	−89,898	35,822	214,012	287,494	158,028	0
2,6	0	−71,556	−132,916	−174,198	−169,700	−101,114	30,300	225,802	267,084	148,444	0
2,8	0	−49,308	−91,611	−114,991	−109,497	−47,866	90,503	85,008	108,389	150,692	0
3	0	−11,338	−22,665	−18,192	15,560	90,837	15,560	−18,192	−22,665	−11,338	0

20. Final remarks and conclusions. We shall now discuss the results of the calculations performed in chapters *16*—*19*. Although we determined the cable force and the girder deflection only for a few values of v, the figures 12—15c and the form of the formulæ (*16*, 25), (*16*, 35a)—(*16*, 35c), (*18*, 6) (*18*, 11a)—(*18*, 13) and (*19*, 4) show that, when the speed takes the value $v = 1$, the cable force and the girder deflection can have values much higher than in the corresponding statical case, while for $v \gg 1$ the two quantities mentioned tend to zero.

According to (*16*, 12) the speed $v = 1$ corresponds to a "critical speed"

$$v_1 = \sqrt{\frac{g\,l}{2f}}. \tag{1}$$

For the bridges mentioned in table I for which $c_0 > 10$ we calculated v_1 and we found for the

Mount Hope bridge:	$v_1 = 240$ km/h;
Example:	$v_1 = 261$ km/h;
New York Washington bridge:	$v_1 = 411$ km/h;
First Tacoma bridge:	$v_1 = 403$ km/h.

At ordinary girder bridges carrying moving live loads, there also exist critical speeds, i. e. speeds at which the deflections show important deviations from their statical value. However here the deviations are much smaller, while moreover the order of magnitude of the lowest critical speed is 1000 km/h, thus being much higher than the critical speed v_1. The figures 12—15c show that at a suspension bridge the deviations are already important at a speed $v = \frac{1}{2}$, corresponding to a speed $\bar{v} = \frac{1}{2} v_1$, which is of the same order of magnitude as the speeds usual in railway traffic.

These results are quite different from the results obtained by Klöppel and Lie[16]). These authors calculated the forced vibrations for the linearised case of a bridge with an extensible cable and a girder of any stiffness. Just as at the free vibrations they found a solution by developing the girder deflection in a Fourier series, thus applying an approximate method. For this reason their results cannot inevitably be considered as right.

Of course our treatment of the problem has not been complete at all. It would be worth while to perform the following supplementary work:

a) Determination of the forced vibrations for other values of v.
b) Determination of the free vibrations after the live load has left the girder. For another source of danger at a suspension bridge being used for railway traffic can be that at the moment at which a train has reached the bridge, it shows heavy oscillations caused by a previous train, as it is not unlikely that a suspension bridge has a rather small amount damping.
c) Study of the forced vibrations for the non linear case.
d) Study of the forced vibrations of a bridge with an extensible cable.
e) Study of the forced vibrations of a bridge with a girder of any stiffness.
f) Study of the forced vibrations when taking into account the damping of the bridge.
g) Study of the forced vibrations when taking into account the mass of the moving live load.
h) Study of the torsional vibrations.

Nevertheless it seems very unlikely that the result of these additional calculations will show that a suspension bridge with a perfectly, or a nearly perfectly, slack girder in general can undoubtedly be used for railway traffic, also because reducing the non linear differential equation for the deflection into a linear equation tends to increase the deflection; the diminution of the cable elasticity however decreases the deflection.

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[16]) K. Klöppel and K. H. Lie, loc. cit. p. 230—266.

Summary

This article shows that a single-span suspension bridge with perfectly or nearly flexible girder is not suitable for railway traffic.

The static investigations comprise calculation of the cable pull and the deflection of the stiffening girder for any given live loads and for various cases of girder stiffness and cable extension. By reducing the problem to a linear one, approximate solutions are obtained. A simple method is given for taking horizontal displacements of the cable into consideration. Finally, the work accumulated in various parts of the loaded bridge is discussed.

The dynamical problems of the natural and forced vibrations are examined for such a bridge under different assumptions. The author shows that the natural vibrations can be calculated by the method of the characteristic functions. As regards the forced vibrations, with the aid of the symbolic calculus, formulæ are derived for calculating the cable pull and the girder deflections in connection with live loads of any form, moving at a constant speed and illustrated by applying them on two special cases.

The results demonstrate that, contrary to those obtained by KLÖPPEL and LIE, the cable pull and the deflections of the stiffening girder caused by moving loads become much greater than those caused by dead loads. The critical speed is given at which these values attain their maxima. This critical speed is comparatively low. It is probable that similar results would be obtained for other cases than those investigated.

Dimensionless quantities have been used as far as possible in all calculations, whereby the work involved could be reduced considerably.

Zusammenfassung

Im vorliegenden Aufsatz wird gezeigt, daß eine einfeldrige Hängebrücke mit vollkommen oder weitgehend biegsamem Versteifungsträger für den Eisenbahnverkehr nicht geeignet ist.

Die statischen Untersuchungen umfassen die Berechnung der Kabelzugkraft und der Durchbiegungen des Versteifungsträgers für beliebige Nutzlasten und verschiedene Fälle von Trägersteifigkeit und Dehnbarkeit des Kabels. Durch Linearisierung des Problems ergeben sich Näherungslösungen. Ein einfacher Weg zur Berücksichtigung der waagrechten Kabelverschiebungen wird angegeben. Schließlich werden für die belastete Brücke Energiebetrachtungen durchgeführt.

Die dynamischen Probleme der Eigenschwingungen und erzwungenen Schwingungen einer solchen Brücke unter verschiedenen Voraussetzungen werden studiert. Der Verfasser zeigt, daß die Eigenschwingungen mit der Methode der Eigenfunktionen berechnet werden können. Hinsichtlich der erzwungenen Schwingungen werden mit Hilfe einer symbolischen Rechnungsweise Formeln für die Berechnung der Kabelzugkraft und der Trägerdurchbiegungen für beliebig verteilte Nutzlasten abgeleitet, die sich mit konstanter Geschwindigkeit bewegen; diese Formeln werden durch Anwendung auf zwei besondere Fälle veranschaulicht.

Die Resultate zeigen, daß, im Gegensatz zu den Ergebnissen von Klöppel und Lie, die Kabelzugkraft und die Durchbiegungen des Versteifungsträgers infolge sich bewegender Lasten bedeutend größer werden als infolge ruhender Belastungen. Es wird die kritische Geschwindigkeit angegeben, bei der diese Werte ihr Maximum erreichen. Diese kritische Geschwindigkeit ist verhältnismäßig klein. Für andere als die untersuchten Fälle dürften sich ähnliche Resultate ergeben.

Allen Berechnungen wurden dimensionslose Größen zugrunde gelegt, wodurch der Arbeitsaufwand beträchtlich beschränkt werden konnte.

Résumé

L'auteur montre qu'un pont suspendu à une seule travée avec poutre raidisseuse intégralement ou, tout au moins, largement flexible, ne convient pas pour le trafic ferroviaire.

Les investigations d'ordre statique portent sur le calcul de l'effort de traction dans les câbles et des fléchissements de la poutre raidisseuse pour des charges utiles arbitraires et différents cas de rigidité de la poutre et d'extensibilité du câble. Le traitement linéaire du problème permet d'obtenir des solutions approchées. L'auteur indique un moyen simple pour tenir compte des déformations horizontales des câbles. Enfin, il fait intervenir des considérations d'énergie dans le cas de la mise en charge du pont.

Il étudie les problèmes dynamiques des oscillations propres et des oscillations forcées d'un tel pont dans différentes hypothèses. Il montre que les oscillations propres peuvent être calculées par la méthode des fonctions propres. En ce qui concerne les oscillations forcées et à l'aide d'un mode de calcul symbolique, il établit des formules pour le calcul de l'effort de traction dans les câbles et des fléchissements des poutres, dans le cas de charges utiles d'une forme quelconque, se déplaçant avec une vitesse constante; il éclaircit ces formules en les appliquant à deux cas spéciaux.

Les résultats montrent que, contrairement aux conclusions de Klöppel et Lie, les efforts de traction dans les câbles et les fléchissements des poutres raidisseuses provoqués par les charges mobiles sont beaucoup plus grands que ceux qui résultent des charges fixes. L'auteur indique la vitesse critique pour laquelle ces valeurs atteignent leur maximum. Cette vitesse critique est relativement faible. Des résultats semblables doivent être obtenus dans des cas différents de ceux qui sont ici étudiés.

Tous les calculs sont basés sur des grandeurs non dimensionnelles, de sorte que le travail effectif à prévoir est considérablement limité.