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Autor: Hrennikoff, A.

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Framework Method and its Technique for Solving Plane Stress Problems

Die „Fachwerkmethode" und ihre Anwendung zur Lösung von ebenen Spannungsproblemen

La méthode du treillis et son application à la résolution des problèmes de contrainte plane

A. HRENNIKOFF, Sc. D., Professor of Civil Engineering,
University of British Columbia, Vancouver, B. C., Canada

Introduction

The framework method discussed in this paper is intended to solve the so-called plane stress problems in which an elastic thin plate is acted upon by forces lying in its plane. A number of problems of engineering come into this category. Stresses in the gusset plates of steel framed structures is one of them. Stresses produced by interaction of floors and walls of reinforced concrete buildings as a result of the temperature and shrinkage changes belong largely to the same class, as well as some problems in machine design and in the design of the aeroplane structures. Most of these problems are insoluble by the formal mathematical analysis, except in a very crude way, — a circumstance which stimulated the development of photoelasticity, a study whose primary purpose lies exactly in solving this type of problem. Both on the theoretical and the experimental sides, photoelasticity has evolved into an exceedingly complicated science, whose very complexity testifies to the importance of problems for which it has been devised.

The framework method covers roughly the same ground as photoelasticity. Its basic principles will be covered here only briefly, since they have been presented in detail elsewhere,[1] and most of the paper will be devoted to a description of the working procedure of the method and its application to a specific example of the gusset plate of a truss.

[1] "Solution of Problems of Elasticity by the Framework Method", by A. HRENNIKOFF. Journal of Applied Mechanics, December, 1941.

Framework Method [2])

The basic idea of the framework method consists in replacing the continuous material of the elastic plate under investigation by a framework of articulated elastic bars arranged into identical units of some definite pattern. The framework is given the external outline and the boundary conditions of the plate prototype, and it is subjected to the same loads as the latter. The bar stresses produced by these loads are analysed by an appropriate procedure, after which they are spread over the tributary areas for the purpose of obtaining the stresses in the plate prototype. When the units are infinitesimal in size the framework of this kind is rigorously equivalent to the prototype with regard to the stresses and deformations. On the other hand, when the units are finite, the framework method is not exact, but it still gives a close approximation of the problem. Bars forming the individual units receive all their loads at the ends, and for this reason their stresses are only axial. The bars are endowed with the same modulus of elasticity as the prototype and with some appropriate cross-sectional areas. The size of the unit is arbitrary, as long as fits the shape of the plate prototype. The smaller the unit the more laborious the solution, but the results are closer to the truth.

The pattern of bars in the cells of the framework is not arbitrary, although it is not unique. In order to reproduce faithfully the behaviour of the plate, the framework must have the same deformability as the plate. In other words, the corresponding deformations of the framework and the plate must be identical in conditions of any arbitrary uniform stress. This statement is the criterion of equivalence of the framework and the plate.

Determination of the cell pattern consists in assumption of a tentative form and in testing its suitability by the above stated criterion, deriving at the same time the expressions for the necessary pattern characteristics, such as the cross-sectional areas of the bars and the angles between them. For the sake of simplicity, only forms possessing two axes of symmetry have been investigated, and in the following discussion these axes are used consistently as the coordinate axes.

From the viewpoint of geometry, framework, unlike the plate, possesses some preferred directions. Its deformability however must be the same in all directions.

Keeping in mind this important peculiarity of the framework, the criterion of its equivalence to the plate may be conveniently stated in terms of the

[2]) For early applications of the framework method to the plane stress problem in a different and more restricted manner see: K. Wiegardt, Verhandlungen des Vereins zur Beförderung des Gewerbefleißes, Bd. 85, 1906; W. Riedel, Zeitschrift für angewandte Mathematik und Mechanik, 1927. See also discussion by C. Weber, Z.f.a.M.M., 1928.

following three conditions, although other equivalent formulations are also possible.

1. The framework is loaded uniformly with the normal loads p per unit length on the X plane and μp on the Y plane. The resultant deformations must be the same as in the plate prototype. The normal unit strains in the framework ϵ_x and ϵ_y are expressed in terms of the framework characteristics and they are equated to the corresponding strains in the plate, loaded with the same loads. The resultant equations are used for derivation of expressions for the characteristics in terms of the quantities of μ, a and t, signifying respectively the Poisson's ratio, the size of the cell and the thickness of the plate. These equations are as follows:

$$\epsilon_x = \frac{p(1-\mu^2)}{Et} \tag{a}$$

$$\epsilon_y = 0 \tag{b}$$

2. Reversing the planes X and Y, on which the normal loads p and μp are applied, two similar equations are set up:

$$\epsilon_x = 0 \tag{c}$$

$$\epsilon_y = \frac{p(1-\mu^2)}{Et} \tag{d}$$

Equation (c) is not an independent one, but it follows from (b) by Betti's reciprocal theorem. The second condition thus gives only one new equation (d). It is clear that if the framework pattern is identical in X and Y directions the Condition 2 is superfluous.

3. A uniform tangential load p per unit length is applied, both on the X and Y planes, and the resultant unit shear deformation of the framework is

$$\gamma_{xy} = \frac{2(1+\mu)p}{Et}\,. \tag{e}$$

It will be realized that a proper combination of the three conditions just considered will reproduce any conceivable state of uniform stress, and consequently, a framework obeying the four equations (a), (b), (d) and (e), will satisfy the criterion of equivalence.

From the number of independent equations involved in the three conditions it follows that a framework pattern with two axes of symmetry X and Y must in general possess four independent characteristics, in order to comply with the criterion, provided the geometry of the framework is different in X and Y directions. Should the properties of the pattern be the same in the directions of both axes, the number of necessary characteristics reduces to three, the equation (d) being superfluous.

In patterns having more characteristics than necessary the excess ones may be assigned at will. On the other hand, when the pattern is deficient in characteristics by one, the equations of deformability may be satisfied only for one particular value of Poisson's ratio, which in this case plays the part of the missing characteristic.

Several patterns of the framework have been found valid, and the most convenient of them is the square pattern represented in Fig. 1, and consisting of squares of the size a by a, containing interior squares of the size $\frac{1}{2}a$ by $\frac{1}{2}a$. Three kinds of bars enter the construction of each cell:

Bars lying along the sides of the main squares, except those coinciding with the periphery of the plate, each have the cross-sectional area

$$A = \frac{a\,t}{1+\mu}, \tag{1a}$$

where t and μ are the thickness and the Poisson's ratio of the plate prototype.

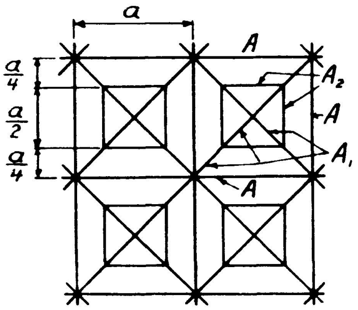

Fig. 1

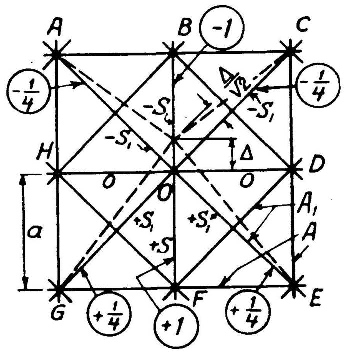

Fig. 2

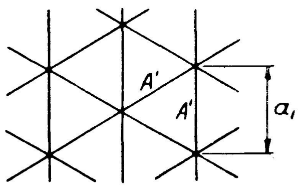

Fig. 3

Diagonal bars have on their whole lengths the cross-section area

$$A_1 = \frac{a\,t}{\sqrt{2}\,(1+\mu)} \tag{1b}$$

Secondary bars of the length $\frac{1}{2}a$ inside the squares have the area

$$A_2 = \frac{(3\mu - 1)}{2\,(1+\mu)\,(1-2\mu)}\,a\,t \tag{1c}$$

Marginal bars, i.e. the bars lying along the periphery of the framework, have their areas only half as large as the areas A in Eq. (1a).

The areas A, A_1 and A_2 are the functions of the Poisson's ratio μ. For $\mu = {}^1/_3$ the area A_2 becomes zero, and the pattern reduces to the simple square type shown in Fig. 2, with the areas:

$$A = \frac{3}{4}\,a\,t \text{ [3]} \tag{2a}$$

[3]) Simple square framework was also used by D. McHenry in the paper "A Lattice Analogy for the Solution of Stress Problems", Journal of the Institution of Civil Engineers, London, December, 1943.

$$A_1 = \frac{3}{4\sqrt{2}}\, a\, t \tag{2b}$$

(For the derivation of these formulae see Appendix 2).

Pattern in the form of equilateral triangles (Fig. 3) is also valid for one particular value of the Poissons's ratio, namely again for $\mu = {}^1/_3$. All interior bars in this pattern must have the cross-section area

$$A' = \frac{\sqrt{3}}{2}\, a_1 t\,, \tag{3}$$

where a_1 is the length of the side of the triangle. The area of the marginal bars must be again only half as large as that of the interior ones. The triangular type of the pattern will not be considered here any further.

Methods of Stress Analysis in Square Frameworks

Frameworks of any pattern discussed above are structures highly indeterminate. Thus in the square framework of Fig. 2, consisting of several rows of units there is one static unknown in each unit of the first row and in the first unit of every subsequent row, whereas two unknowns are present in each of the other units. This means that a free framework made of 2 by 3 units has 8 unknowns. Equal number of static unknowns is present also in the framework of the pattern of Fig. 1. If some of the joints are restrained the number of the unknowns may be greater. The great number of static unknowns shows the formidable character of stress analysis of the framework and indicates that a solution based on equations derived by the method of virtual work is impracticable.

Analysis of the framework can also be based on the determination of the joint displacements. Each joint possesses two degrees of freedom and consequently has two components of displacement brought about by the elastic distortion of the structure, caused by the applied loads. These displacements may be taken as the unknowns, and equations may be set up for their determination, expressing the conditions of equilibrium of each joint. This method is equally impracticable in view of the multiplicity of the unknowns. The 2 by 3 free framework mentioned above would require 21 simultaneous equations by this method.

The most practicable method of analysis of the square framework is an adaptation of the method of joint displacements, just referred to. If the elastic displacements of the joints are found, and the joints are brought into their true displaced positions, the bar stresses and the external forces acting at each joint are mutually balanced. Instead of finding these displacements from equations, it is possible to guess them roughly on the basis of the applied loads, displace the joints one by one by the amounts guessed, compute the bar stresses caused by these displacements and then determine the remaining unbalanced joint forces. This operation is repeated many times until a close balance is established at all joints. This procedure resembles somewhat the method of moment distribution of Professor Hardy Cross and forms the essence of the method of successive joint movements to be presented below.

A circumstance highly favourable to the use of this method in the framework stress analysis is the identity of the pattern in all parts of the framework. For this reason if a joint gets a unit horizontal displacement while the adjacent joints remain fixed, stresses produced in the members radiating from this joint are the same as the stresses in corresponding members caused by unit horizontal displacement of any other joint. Such stresses, or rather values proportional to them, have been termed the distribution factors.

Distribution Factors in Simple Square Framework

Let the joint 0 (Fig. 2) move upward a distance Δ, while all the adjacent joints are held against any movement. The changes in length induced by this distortion are as follows: the two vertical members deform by the amount Δ, the four inclined members radiating from the central joint – by the amount $\frac{\Delta}{\sqrt{2}}$, while the other inclined members as well as the horizontal members, retain their original lengths, and consequently remain unstressed.

Using now the expressions (2) for the cross-sectional areas of the members, the stress induced in the vertical members is found to be:

$$S_1 = \frac{A E \Delta}{a} = \frac{3}{4} E t \Delta ,$$

while in the stressed diagonals the stress is

$$S_2 = \frac{A E \frac{\Delta}{\sqrt{2}}}{a\sqrt{2}} = \frac{3\sqrt{2}}{16} E t \Delta ,$$

and its vertical or horizontal component is $\frac{3}{16} Et\Delta$.

It may be easily seen that the ratio of the horizontal or vertical component of S_2 to S_1 is $^1/_4$ to 1. These figures, $^1/_4$ and 1, are the distribution factors of the simple square framework. They are merely the stress components corresponding to a certain movement of one of the joints in the direction parallel to one of the axes, while the adjacent joints remain fixed. The magnitude of the movement itself is not stated as immaterial. Distribution factors possess signs: on the side towards which the movement is made they are negative for compression, and on the opposite side — positive for tension. Their values are shown in circles in Fig. 2. It is emphasized here that the distribution factors of the diagonals represent not the stresses themselves but their vertical or horizontal components, since it is the balance of the components of the joint forces that is pursued in the course of the distribution. The distribution factors of the marginal members are only half as large as the ones belonging to the interior members, – this is indicated in Fig. 4.

The use of the distribution factors is illustrated in Fig. 5, representing a portion of the framework whose central joint is acted upon by a vertical force of 100 units and by a horizontal force of 50 units, indicated by arrows. It is required to move this joint towards the balance. The joint is first moved upward arbitrarily 32 units, which causes the stresses in the verticals and the diagonals 32 and 8 units respectively, as recorded on the corresponding members, these figures standing in the ratio of the distribution factors 1 to $^1/_4$. As a result of this movement unbalanced vertical joint forces appear at the adjacent joints. They are equal to 8 units at the joints A, C, E and G, and to 32 units at the joints B and F, while an unbalance of $100 - [4\,(8) + 2\,(32)] = 4$ still remains at the central joint. It is easy to see that the total amount of the vertical unbalance still remains 100, although most of it has shifted from the central joint to the neighbouring joints. A similar procedure disposes of the most of the horizontal unbalanced force. The additional stresses caused by this new movement and the resultant unbalanced forces are recorded on the diagram.

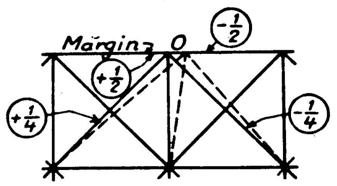

Fig. 4

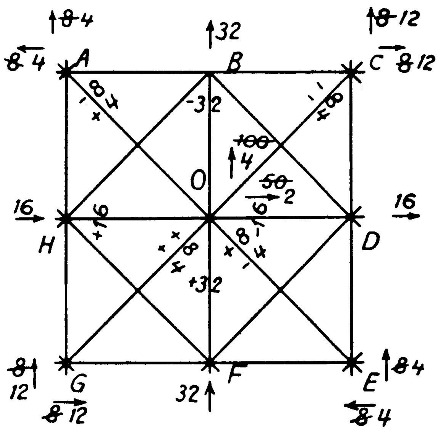

Fig. 5

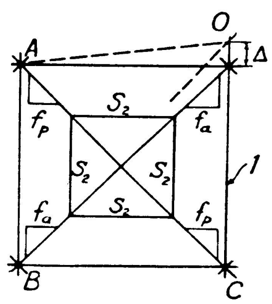

Fig. 6

Distribution Factors in Square Pattern with Auxiliary Members

With regard to the square units with auxiliary members (Fig. 1) it is necessary to meet at the outset two possible objections: first, that the cross-section area of the auxiliary members A_2, Eq. (1c), comes out negative when $\mu < {}^1/_3$, and second, that the central part of the unit becomes unstable in the conditions of the compression stress. In the light of these objections the construction of a mechanical model of this pattern is plainly impossible, but in spite of being a mathematical fiction the framework of this type is perfectly suitable for stress computations.

Application of equations of statics to the four interior joints (Fig. 1) shows that under all possible conditions of loading the four auxiliary members of each unit are stressed equally, and so are the two outer parts of each diagonal.

Let one of the main joints of the framework unit, such as the joint 0 in Fig. 6, move vertically through a distance Δ, and let the other main joints remain fixed, while the interior joints are allowed to move as much as is necessary for the equilibrium. The ensuing stresses in different members of the cell can be easily found in terms of Δ. If now the displacement Δ is so chosen that the stress in the vertical member OC is equal to unity, then this stress unity and the corresponding vertical or horizontal components of stresses in the outer parts of the diagonals f_a and f_p are defined as the distribution factors. The subscripts a and p indicate the words active and passive, the active diagonal being the one whose end moves, and the passive — the one whose both ends remain stationary. A rather elementary application of statically indeterminate analysis leads to the following expressions for the distribution factors:

$$\left.\begin{aligned} f_a &= \frac{m+1}{8\,(m-1)}, \\ f_p &= \frac{3-m}{8\,(m-1)}. \end{aligned}\right\} \tag{4}$$

where m $= \dfrac{1}{\mu}$.

Numerical values of these factors, corresponding to different values of m, are stated in table, Fig. 7. It is peculiar that some of these factors, particularly the ones corresponding to $m = 2, 3, 5, 7, 11$ and infinity, come out as simple round fractions, which fact contributes much to the ease of carrying out the distribution procedure.

$m=\frac{1}{\mu}$	2	2.5	3	$3\frac{1}{3}$	4	5	6	7	8	9	10	11	12	15	19	20	∞
f_a	3/8 0.375	7/24 0.2917	1/4 0.25	13/56 0.2321	5/24 0.2083	3/16 0.1875	7/40 0.175	1/6 0.1667	9/56 0.1607	5/32 0.1562	11/72 0.1527	3/20 0.15	13/88 0.1477	1/7 0.1428	5/36 0.1389	21/152 0.1381	1/8 0.125
f_p	1/8 0.125	1/24 0.0417	0	−1/56 −0.0178	−1/24 −0.0417	−1/16 −0.0625	−3/40 −0.075	−1/12 −0.0833	−5/56 −0.0893	−3/32 −0.0937	−7/72 −0.0972	−1/10 −0.1	−9/88 −0.1022	−3/28 −0.1072	−1/9 −0.1111	−17/152 −0.1119	−1/8 −0.125

Fig. 7. Distribution Factors of Square Framework

When $m = 3$ the auxiliary members disappear, and the framework becomes transformed into the simple square type with $f_a = {}^1/_4$ and $f_p = 0$.

The distribution factors f_a and f_p are quite sufficient for stress analysis of a framework with auxiliary members. Stresses in the interior members of the framework are not needed and do not enter the distribution.

Apart from the existence of stresses in passive diagonals, the distribution procedure in the framework with auxiliary members is in no way different from the similar procedure in the simple square framework.

Distribution Procedure

The single joint movement parallel to one of the axes, explained earlier, forms the basis of the distribution procedure, but, although it has its proper place, its exclusive use would be very cumbersome. In order to shorten the distribution, block movements are resorted to. Such movements are quite legitimate as long as the stresses brought about by them can be easily visualized. The purpose of these movements is to work closer toward the state of equilibrium at all joints.

Several typical movements are stated below. Although these movements are applicable to any kind of the framework, the values of stresses given presuppose simple square pattern. It is reminded that the values recorded on the diagonals represent the vertical or horizontal components of stresses in these members.

1. Movement in a row, Fig. 8. All the joints, with the exception of F, G and H, remain fixed. The joint F moves a units to the right, which causes stress a in the member EF and stresses $\pm\frac{1}{4}a$ in the four diagonals radiating from F. At the same time the joint G moves b units away from F, i.e. $(a+b)$ units altogether, and the joint H c units away from G, i.e. $(a+b+c)$ units altogether. It may be seen that while the horizontal members affected have stresses b and c, the diagonals meeting at G carry stresses $\pm\frac{1}{4}(a+b)$, and the diagonals CH and MH carry stresses $+\frac{1}{4}(a+b+c)$. The numerical values of the stresses thus produced can be easily computed mentally, and there is no difficulty in visualizing their signs.

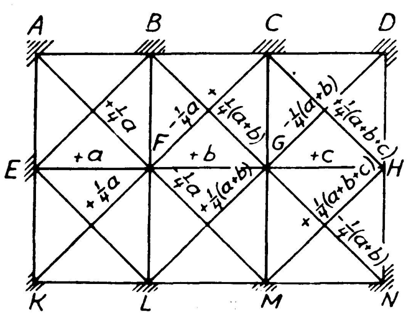

Fig. 8

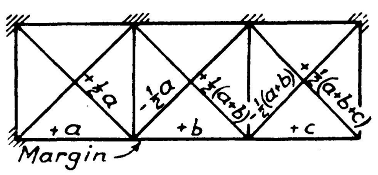

Fig. 9

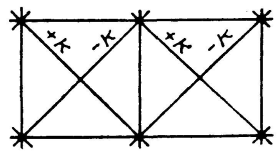

Fig. 10

2. Movement in a row in *a* marginal panel, Fig. 9, is analogous to the movement just discussed. The only distinction lies in twice smaller stresses in the marginal members in view of their twice smaller cross-sectional areas.

3. Shear distortion, Fig. 10. The lower row of joints, together with the whole lower part of the framework, is moved horizontally $4k$ units, causing stresses in all diagonals of the panel $\pm k$.
4. Direct stress, Fig. 11. All lower part of the framework is moved bodily down a units. All verticals in the panel, except the marginal ones, are stressed $+a$, and all diagonals $+\frac{1}{4}a$.
5. Interior block displacement, Fig. 12. The block $BDHF$ is moved to the right a units. Members AB and EF are stressed a, while all the diagonals affected carry stresses $\pm\frac{1}{4}a$.

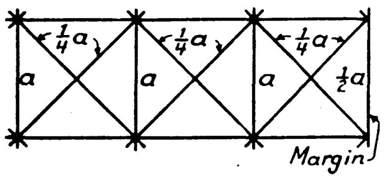

Fig. 11

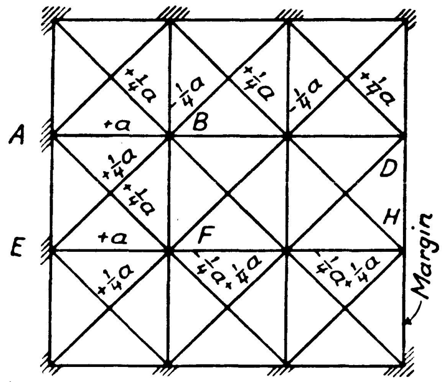

Fig. 12

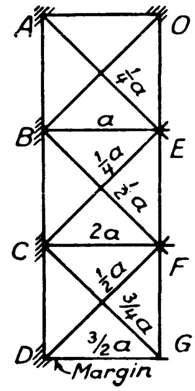

Fig. 13

6. Rotation about an exterior joint, Fig. 13. The right hand part of the framework is rotated about the joint 0 in such a manner that the joint E moves horizontally a units to the right. The joints F and G then move $2a$ and $3a$ units respectively, which explains the stresses produced.

A modification of this movement is illustrated in Fig. 14., representing two equal rotations in opposite directions about the joints 0_1 and 0_2. A similar rotation about an interior joint can also be easily visualized.

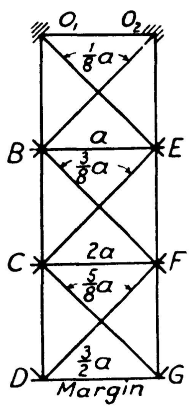

Fig. 14

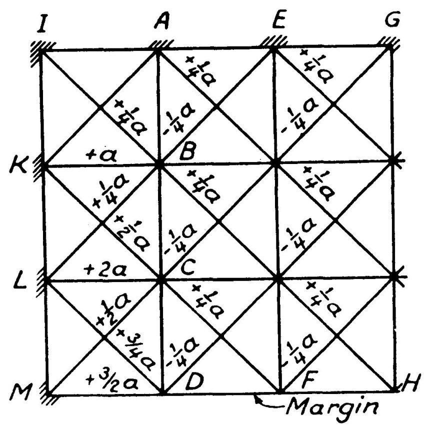

Fig. 15

7. Rotation and shear combined, Fig. 15. Here the lines AD, EF and GH are rotated separately through equal angles in the same direction about the points A, E and F respectively.

All these movements and some of their combinations, as well as the single joint displacements, are used in the distribution. Just which movement should be used at each stage and how great should be the distortion is decided by judgment. The general principle is of course to work towards the equilibrium and to move those joints first that are most unbalanced. Improper movements do not invalidate the work, but merely retard the progress.

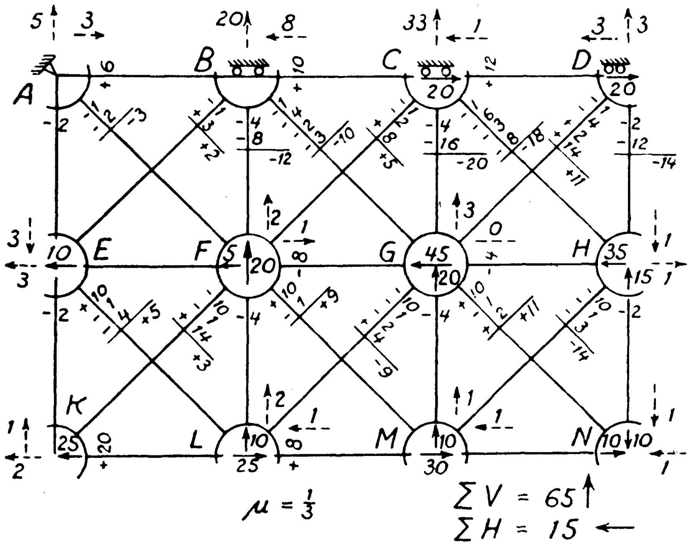

Fig. 16

The technique and the form used in carrying out the distribution are explained in an imaginary example of Fig. 16, representing a two by three simple square framework acted upon at most of the joints by the forces shown in circles. The four top joints are assumed to be prevented from moving vertically, while one of these joints, A, is fixed in both directions. This problem may represent one half of a symmetrically loaded plate, whose axis of symmetry lies along the row of the restrained joints.

As has been pointed out, movements of the joints are accompanied by shifting of the unbalanced forces from joint to joint, while the algebraic sum of the unbalanced components of all the joint forces remains constant. For this reason, the purpose of the distribution in the present problem is to shift all vertical unbalanced forces to the upper edge, where they can be resisted by the restraining reactions, and to move those horizontal forces that do not get mutually balanced, to the joint A, the only joint restrained against a horizontal motion.

Each movement is accompanied by recording the stresses produced by it on all the members concerned, using plus sign for tension and minus for compression. Having done this, all the joints on both ends of all the members affected are gone over, and the unbalanced forces acting on these joints are modified by the values of the newly added bar stresses. The resultant state of equilibrium of each joint is expressed by means of vertical and horizontal arrows with appropriate numbers, signifying the remaining unbalanced forces, while the original figures are simply rubbed out.

For the purpose of demonstrating the procedure, the framework of Fig. 16 can be brought to a state close to equilibrium by the following, somewhat artificial, movements:

1. Shear movement of the bottom row of joints, causing stresses ± 10 in all diagonals of the bottom bay.
2. Vertical compression movement producing stresses —4, —2 and —1 in the interior verticals, marginal verticals and the diagonals respectively.
3. Rotation of the block $EHNK$ about the joint E. The resultant stresses in the verticals BF, CG and DH are respectively —8, —16 and —12, while the stresses in the corresponding diagonals are one quarter as great.
4. Horizontal movement on the middle line, in which the joint G moves 8 units to the left, and the joint H approaches G by 4 units.
5. Similar movement on the bottom line, with the joint L moving away from M 8 units, and the joint K — away from L 20 units.
6. Similar movement on the top line, with the members AB, BC and CD being stressed in tension 6, 10 and 12 units respectively.

The joint forces remaining after these movements are shown by dotted arrows. While most of these arrows represent unbalanced forces, the ones at the point A and the vertical arrows at the other top joints are balanced by the reactions. In order to distinguish these forces from the unbalanced forces they may be written with a coloured pencil, but even though they are balanced, they must not be left out, since they are needed for checking. It is emphasized here that it is not the reactions but the forces equal and opposite to the reactions that must be stated at the joints where restraints are present.

In carrying out the distribution it is not practicable to attempt to achieve a close balance at any of the joints at once, and the movements attempted should preferably be of the same order, or of the order one decimal place lower, than the unbalanced forces. At the same time the numerical values of the joint movements should be multiples of four, so as not to bring in prematurely any fractional forces. In each problem roughly equal intervals of time are needed to reduce the order of unbalanced forces by one place, i.e., from thousands to hundreds or from hundreds to tens, e.t.c.

In spite of great simplicity of the numerical procedure, consisting in addition, subtraction and division by four and two of round numbers, after hundreds and thousands of these actions, made mostly mentally, errors are bound to crop in, and it is unwise to proceed too far with the distribution without making some effective checks at regular intervals.

One possible type of error is recording a wrong stress or failing to record any stress at all in a member affected. In order to remedy this error it may be necessary to go over the work after every 4—8 hours, dotting with a coloured pencil every stress which has been found correct. Experience shows that even though the original sequence of movements may be forgotten by the time of checking, the kind of every movement can always be reconstructed and the checking can always be accomplished. After experience has been gained, this kind of error is not likely to occur, if the work is done carefully.

The second kind of error is incorrect addition of the unbalanced joint forces. Apparently no reasonable amount of care insures freedom from this error. Fortunately however, an easy check is available, consisting in adding up all the horizontal and separately all the vertical unbalanced joint forces and comparing these sums with the original sums. In the example of Fig. 16 the sums of the joint forces, including the coloured forces at the top edge, have been found both before and after the distribution to be 65 up and 15 to the left, which agreement constitutes the check. If a discrepancy between the two sums is discovered, all the joints are gone over, and all the joint forces are recomputed from the recorded bar stresses and compared with the previously stated values. As a preliminary step to this operation, the resultant bar stresses are found by adding up several figures resulting from individual movements. It has been found most satisfactory to use this check after reduction of the order of the unbalanced forces by approximately two places.

When all the unbalanced forces become reduced to sufficiently low values the distribution may be considered complete, and it is desirable then to make the final check of the framework stresses. The first step in this procedure consists in distorting the structure in such manner that all horizontal and vertical members are stressed to the values found in the solution. This can be accomplished in a number of ways, each of which is permissible, if it is consistent with the restraints of the structure. Stresses present in the diagonals after this first distorting operation do not as a rule represent their true values. Second step consists in producing shear distortions of the structure of such kind that stresses in some of the diagonals assume their true values. Stresses in the remaining diagonals must then also become equal to their true values, which agreement will check the original solution. The shear distortion, referred to herein, must not of course violate the conditions of restraint; thus in Fig. 16 the top edge joints must not be moved across this edge.

Principle of Symmetry

When the number of units in the framework is too great, and the ordinary distribution becomes too laborious, it is advantageous to make use of the principle of symmetry. This principle can be successfully applied in relation to one or two axes when the following two conditions are satisfied:

1. The framework itself is symmetrical about one or two axes.
2 a. The framework is acted upon only by known or statically determinate forces.
 b. Or else, it has in addition to the known forces some known joint displacements along the same coordinate axes at the joints symmetrically located.

The latter condition means that if a joint A has a known displacement δ_{1x} along the X axis, the joint B located symmetrically across the X (or the Y) axis must necessarily have a known displacement δ_{2x}, which may in a special case be zero, along the same X axis. Absence of a known displacement at the joint B, or presence of a known displacement at B along the other axis, would invalidate the application of the principle of symmetry in relation to the axis in question, i.e., the X (or the Y) axis.

If the above stated conditions are satisfied in relation to two axes of symmetry, the framework problem may be broken up into four symmetrical and antisymmetrical cases, each of which has similar stresses and displacements in its four quadrants. The problem of stress distribution in the given framework may therefore be replaced by four separate distributions in the frameworks one quarter as large as the original one. Superposition of the four solutions will give the answer to the original problem.

For the reason of suggestiveness it is convenient to designate the axes of symmetry of the framework by the capital letters S with subscripts x or y, and the axes of antisymmetry – by the letters A with the same subscripts. This has been done everywhere in the following discussion and in the figures related to it.

The method of forming the four component cases out of the given loading in the presence of two axes of symmetry of the framework is illustrated in Fig. 17 (a) to (e). Each component case in this figure contains four forces, one quarter as large as the given single force, applied parallel to it at the four symmetrical locations, and pointed symmetrically or antisymmetrically,

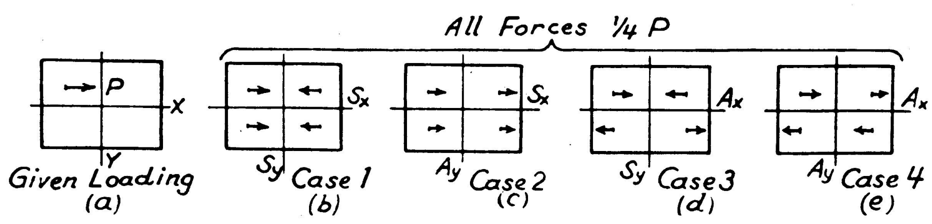

Fig. 17 (a) to (e)

depending on whether the axis in question is an S axis or an A axis. If the loads happen to be applied at the joints lying on the axes of symmetry of the framework, the break up of the given loading into the constituent cases comes out somewhat different from the general case, described in the example

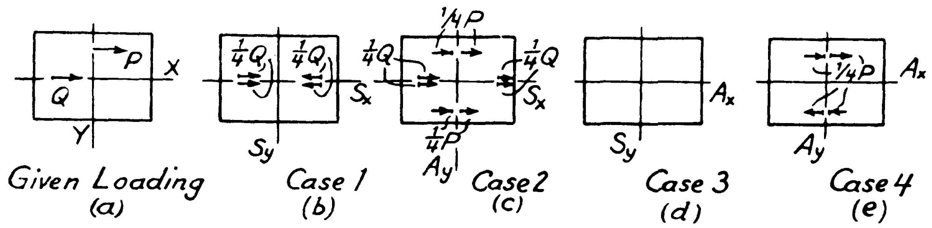

Fig. 18 (a) to (e)

of Fig. 17. This is illustrated in Fig. 18 (a) to (e), from which it may be seen that one of the component cases, namely $A_x S_y$, is missing. In the three remaining cases forces $\frac{1}{4} P$ and $\frac{1}{4} Q$ are assumed to be applied to their respective quadrants. A more restricted special case, in which a load is applied at the centre of the framework, is shown in Fig. 19. With this loading three of the four component cases are missing.

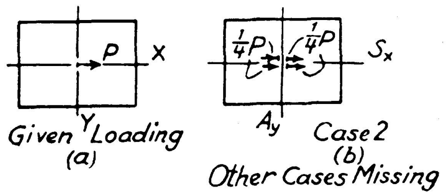

Fig. 19 (a) to (b)

Forces acting at an angle to the framework axes must be resolved into their X and Y components, prior to formation of the constituent cases.

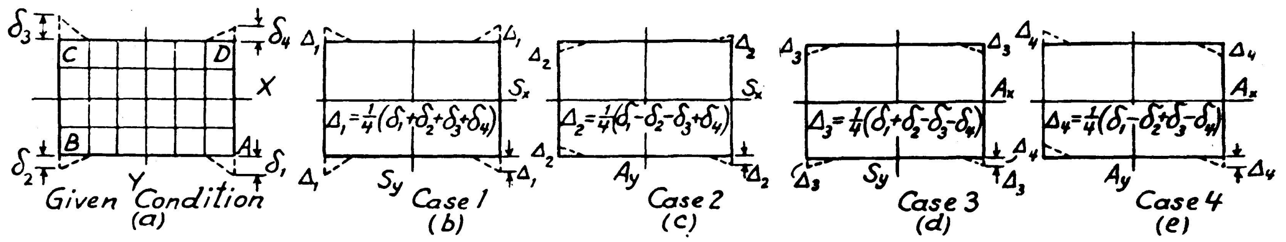

Fig. 20 (a) to (e)

Breaking up of the known restraints at the symmetrical points, when forming the constituent cases, is illustrated in Fig. 20. The method followed in this operation is similar to the one described above with reference to the forces. The following features of the component cases, into which the given problem may be broken up, will be noted.

1. Each case represents a self contained problem with the forces and the reactions being in a state of equilibrium.
2. Superposition of the component cases results in the parent problem.

3. In each component case stresses and displacements in each of the four quadrants of the framework with two axes of symmetry resemble closely the stresses and displacements in the three other quadrants. If two quadrants lie across an S axis the conditions at the corresponding points in these quadrants are as follows: displacements are symmetrical, normal stresses and strains along the corresponding axis are identical, and shearing stresses and strains along the same axes are equal and opposite in sign. In the quadrants located opposite each other across an A axis, displacements are antisymmetrical, normal stresses and strains are equal and opposite in sign, while shearing stresses and strains are identical. This mechanical similarity of the four quadrants explains why it is sufficient to consider only one quarter of the original framework in each component case.

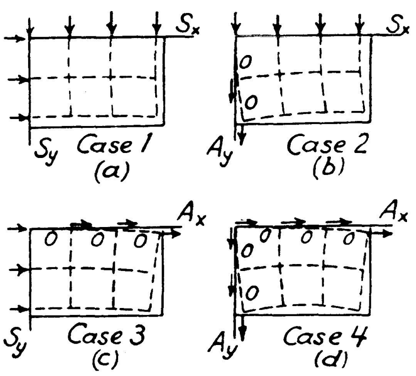

Fig. 21 (a) to (d)

When analyzing the component cases, the following conditions present at the axes must be allowed for, see Fig. 21 (a) to (d). Joints located on the S axes possess no displacements normal to these axes, the restraints being effected by the reactions perpendicular to these axes. Joints located on the A axes can move only perpendicularly to these axes, the restraints being produced by the reactions parallel to the A axes. Bars lying along the A axes are unstressed. The unknown reactions at the axes are found in the course of the distribution. Bars lying along the axes are considered as ordinary marginal members.

Further subdivision of each quadrant into four smaller quadrants is generally impossible, since the second condition necessary for the applicability of the principle of symmetry does not hold, i.e., the joints located on the axes are restrained, while the peripheral joints are not.

Correlation between the Plate and the Framework

Two problems come under this heading: first, – how to apply the forces acting on the plate prototype to its framework analogue; and second, – how to convert the bar stresses into the plate stresses. Some of the questions arising in connection with these problems are not susceptible to rigorous mathematical treatment, and this part of the framework theory, perhaps more than any other, requires some additional thought.

Since the framework members do not possess any flexural rigidity, loads must be applied to the framework at the joints only, and evidently in such a way as to preserve fully their static effect. Since joints are available only at a few definite locations, this may involve shifting of forces away from their true points of application, leading to some errors in stresses in the neighbourhood of these points. These errors however, according to St. Venant's Principle, are purely local.

Transformation of loading distributed over the edges of the plate into the marginal joint concentrations is the converse of the problem of reduction of the bar stresses to the plate stresses, and it will be understood after that problem has been solved.

When converting the bar stresses into plate stresses the latter must first be calculated on the planes of the framework axes, and only after that stresses on other planes may be determined if necessary by the ordinary formulae.

Plate Stresses when Interior Loads are Absent

Determination of the plate stresses from the bar stresses will first be discussed for the conditions when no applied loads are present at the interior framework joints. Normal and shearing stresses will be considered separately.

Normal Stresses

In computing normal stresses on the plane AE (Fig. 22), the normal joint concentrations N_A, N_B etc., are calculated first from the bar stresses, as in the following example:

$$N_B = S_1 + S_2 + S_3 .$$

These concentrations come out the same no matter whether a section on the right or on the left of AE is taken.

The plate stress at an intermediate joint, such as C, is found by the formula:

$$\sigma_c = \frac{N_c}{a t}, \tag{5}$$

where a is the length of the side of the square unit and t is the thickness of the plate. Calculation of stress by the equation (5) is equivalent to replacement of the concentration N_C by the triangular area $bd\,3$, having an equivalent static effect. The stress diagram on the plane AE thus comes out in the form of a polygon, which may be considered as an approximation of the true stress curve.

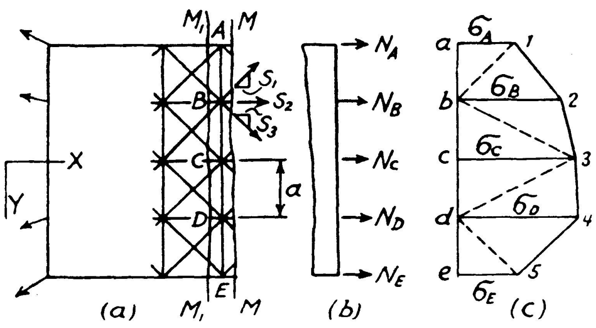

Fig. 22 (a) to (c)

The edge stress, such as σ_A, cannot in general be found by Eq. (5) without violating statics, since the centre of the triangle $ab\,1$, into which the concentration N_A is converted, does not coincide with the point A. Three cases should be distinguished in this connection:

1. The two edge concentrations N_A and N_E (Fig. 22) are equal and of the same sign. In this case it is consistent with statics to find σ_A by Eq. (6), which agrees with Eq. (5).

$$\sigma_A = \frac{2\,N_A}{a\,t}. \tag{6}$$

2. The edge concentrations N_A and N_E are numerically equal and opposite in sign, as in Fig. 23. In order to preserve statics in this case it is necessary to take

$$\sigma_A = \frac{2}{2 - {}^1/_3}\;\frac{2\,N_A}{a\,t}, \quad \text{or in general}$$

$$\sigma_A = \frac{n}{n - {}^1/_3}\;\frac{2\,N_A}{a\,t}, \tag{7}$$

where $2\,n$ is the number of the units in the section $A\,E$.

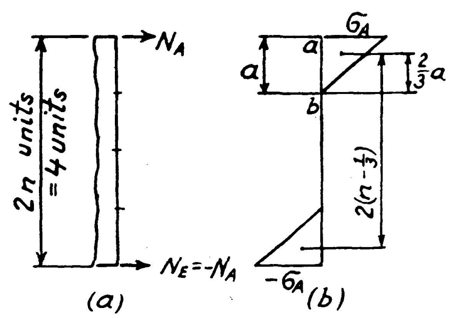

Fig. 23 (a) and (b)

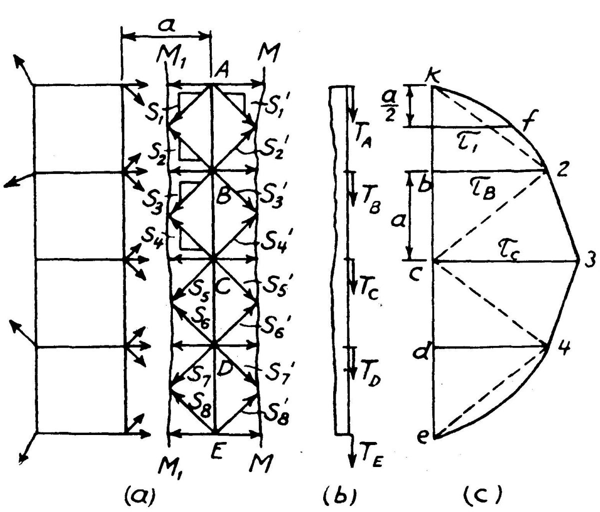

Fig. 24 (a) to (c)

Equation (7) preserves the equality of the moments caused by the stresses σ_A and the concentrations N_A. The equality of the normal effects is of course preserved automatically, since $N_A + N_E = 0$.

3. General case. The end concentrations N_A and N_E are broken up into symmetrical and antisymmetrical parts:

$$N_{Sym.} = \frac{N_A + N_E}{2}, \text{ both at } A \text{ and } E,$$

$$N_{Ant.} = \frac{N_A - N_E}{2}, \text{ acting at } A, \text{ and } N_{Ant.} = \frac{N_E - N_A}{2}, \text{ acting at } E.$$

These two parts are treated as in the first two cases.

Shearing Stresses

When finding shearing stresses in the plate on the plane AE (Fig. 24), it is necessary first to compute the tangential joint concentrations. These however, unlike the normal joint concentrations, are different on the two sides of the joint, and therefore the mean of the two should be used for calculating the shearing stress. Then, following the method of reduction of normal stresses:

$$\tau_B = \frac{T_B}{a t}, \tag{8}$$

where $T_B = \frac{1}{2}[(S_2 - S_3) + (S'_3 - S'_2)]$.

The signs of the stresses S in this expression must be carefully noted. Note also that the average tangential concentration at B, found by the preceding expression, is the same on the vertical and on the horizontal planes, which is in agreement with the principle of equality of shearing stresses on the two perpendicular planes.

The edge concentration T_A should be taken as

$$T_A = \tfrac{1}{2}(S'_1 - S_1).$$

It would be incorrect to transform this tangential concentration into the triangular stress area, as has been done with the normal edge concentration, since the shearing stress at A, in the absence of the edge loads, must be zero. The most convenient disposal of the edge concentration T_A has been found in transforming it into a parabolic stress area $kf2$, superimposed on the triangular stress area $k2c$, caused by the concentration T_B. Therefore, the shearing stress at the middle of the edge square AB becomes:

$$\tau_1 = \frac{\tau_B}{2} + \frac{3\,T_A}{2\,a t} = \frac{T_B + 3\,T_A}{2\,a t} \tag{9}$$

This method of reduction of the shearing concentration at the egde is not fully satisfactory, because it often results in too great a value of the stress τ_1 and in an unreasonable reversal of slope in the stress diagram at the point 2. This irregularity however is confined to the region near the edge, where shearing stresses are not the greatest.

Loads at the Interior Joints

Presence of an external load at an interior framework joint results in inequality of the normal joint concentrations on the two sides of the joint. Thus in Fig. 25 the normal concentration at the joint 0 on the plane $M_1 M_1$, equal to the sum $S_1 + S_2 + S_3$, differs from the concentration on the plane $M_2 M_2$, equal to $S_4 + S_5 + S_6$, by the value of the load P.

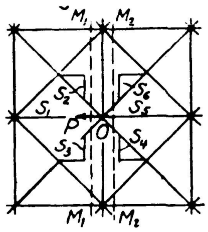

Fig. 25

If this external load represents the effect of a loading distributed over an area of the plate around the joint, it is correct to average up the two normal concentrations and to compute the normal stress σ at the joint by the ordinary method explained above using the mean value of the concentration. A similar procedure must also be used under the same circumstances when computing the shearing stresses on the horizontal planes at the point 0 (Fig. 25).

If one or several concentrated or sharply localized plate loads are responsible for the external load applied at an interior framework joint then the situation is more complicated. It is demonstrated in the Appendix that in this case the stress condition in the plate at the location of the joint in question may be represented by the sum of the two states of stress: the general state of stress found by using the average of the joint concentrations on the two sides of the joint, as has been done previously, and of the purely local effect of the concentrated load acting alone on the plate which is extended infinitely in all directions.

In a purely theoretical case of a concentrated point load, this local effect is expressed by the well known formulae of the theory of elasticity[4]), and at the immediate point of application of the load the local maximum stresses, both normal and tangential, are infinite. Away from this point these stresses are quickly dampened in accordance with the formulae.

[4]) Timoshenko, Theory of Elasticity, page 111. McGraw-Hill.

It may be reminded here that in dealing with the static loads, i.e. when fatigue is not a factor, a very high stress acting over a small area is not considered significant for the strength of the member, since it is usually relieved by yielding. Furthermore, a truly concentrated point load is a physical impossibility. The nearest approach to it in the usual structural practice is the rivet load, which is by no means a concentrated point load; however the local effect even of such load must not be left out.

The local normal stress produced by rivets on the plane perpendicular to their line of action may be taken as

$$\sigma_l = \frac{P_r}{td}, \text{ compression,} \tag{10}$$

where P_r is the rivet load and not the joint load, since the latter may represent the effect of several rivets.

In this discussion of the local effects of rivets it has been tacitly assumed that the rivets fill the holes completely, and at the same time the general state of stress in their vicinity is compression on all planes, since tension would bring in the effect of stress concentration by the hole, — an extraneous effect, for investigation of which the framework method is unsuitable.

The local normal stress on the plane coinciding with the rivet load is also present, although its value is problematical. By the way of approximation this may be taken as

$$\sigma_l' = \mu \frac{P_r}{td}, \text{ compression.} \tag{11}$$

Local shearing effect in the neighbourhood of rivets also must not be overlooked, although only its average value may be computed. Let Fig. 26 represent a plate of thickness t extended infinitely in all directions and acted upon by a single rivet force P_r. An elongated strip $ABCD$ is assumed to be sepa-

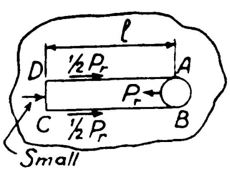

Fig. 26

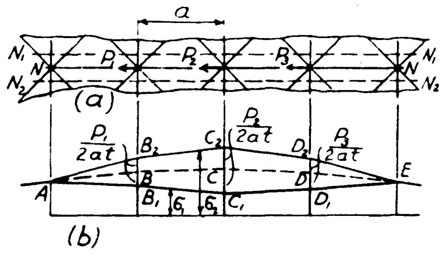

Fig. 27 (a) and (b)

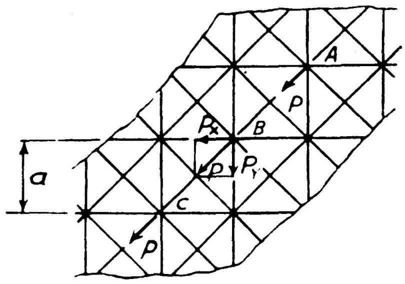

Fig. 28

rated from the plate, and its equilibrium is considered. If the length of the strip l is sufficiently great, the stress at the end DC is small, and the average shearing stresses on the planes AD and CB become $\pm \frac{P_r}{2lt}$, as follows from statics and symmetry. With a single rivet load acting on the plate there seems to be no special reason for selecting any particular value of the length l, and consequently there is no justification for assigning any definite value to the average local shearing stress. If however the infinite plate in question is subjected to a line loading, represented in the framework analogue by several joint forces P_1, P_2, P_3 etc. (Fig. 27a), then it seems reasonable to select the length of the framework unit a as the shear length tributary to each joint force P. The average local shearing stresses on the planes $N_1 N_1$ and $N_2 N_2$ may then be taken as $-\frac{P}{2at}$ and $+\frac{P}{2at}$ respectively, the loads P being taken with their proper subscripts. These local stresses should be superimposed on the general shearing stresses, found by the method explained earlier from the average joint concentrations on the planes above and below the line of the joints. If the general shearing stress is represented in Fig. 27b by the diagram $ABCDE$ then the combined shearing stress on the planes $N_1 N_1$ and $N_2 N_2$ may be represented approximately by the lines $AB_1C_1D_1E$ and $AB_2C_2D_2E$ respectively.

Proper treatment of the local effect of concentrated loads acting at an angle with the framework axes may be exemplified by the case of a line loading illustrated in Fig. 28. In this example the general state of stress in the vicinity of the joints, found from the average joint concentrations, should be increased by the following local stresses: the bearing stress $\sigma_l = \frac{P_r}{td}$, acting on the planes normal to the joint loads; the compression stress $\mu \sigma_l$, acting on the plane AC coinciding with the loads; and the shearing stress $\tau_l = \pm \frac{P}{2(AB)t} = \pm \frac{P_x}{2at}$, acting on both sides of the plane AC. This additional shearing stress is positive on one side of this plane and negative on the other. Since the basic or general stresses in the plate are determined on the planes of the framework axes, it is necessary to reduce the local stresses also to their σ_x, σ_y and τ_{xy} equivalents, which may be found by the usual formulae.

Gusset Plate Problem

In addition to the problem of bending of a wide beam, which was reported in the previous paper[5]), and which showed a good agreement with the results found by the theory of elasticity, the framework method has been also applied to the analysis of stresses in the gusset plate of a riveted truss. Although this problem contains a number of features for investigation of which the framework method is largely unsuitable, such as the presence of rivet holes, stress division among several rivets connecting each member, the nature of structural action between the rivet and the plate, including friction under heads,

[5]) See footnote on page 215.

and the influence of plastic deformations at the points of stress concentration, still on the whole this is essentially a plane stress problem, susceptible in simplest cases to the analysis by framework.

The problem is stated by means of Fig. 29. Five structural members are joined by means of this gusset plate, and their stresses are assumed to be distributed uniformly over the lengths of their attachments. The loaded joints in the framework do not necessarily represent the individual rivets, but may signify the whole groups of rivets, which would make the state of stress in the plate somewhat generalized. The Poisson's ratio is assumed to be $^1/_3$, which is not far from 0,3, the value generally accepted for steel. The plate is divided into four by six units, and for the purpose of solution it is broken up into four symmetrical and antisymmetrical cases.

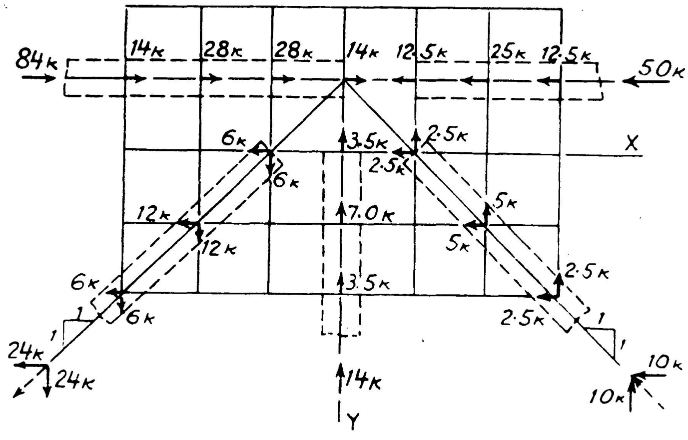

Fig. 29

The results of stress analysis in the first quadrants of these four cases are presented in Fig. 30. The acting joint forces are stated inside the circles. The arrows outside of the axes represent the reactions of the axis joints. The remaining unbalanced joint forces have been reduced to values less than one pound. The stress distribution in each separate case has taken, together with the necessary checks, some 8 to 10 hours of work. Using the laws of symmetry, the bar stresses present in the first quadrants (Fig. 30) have been extended over the whole framework, and the four constituent cases have been combined into the solution of the parent problem, after which the bar stresses have been converted into the plate stresses.

Some of the typical plate stresses σ_x, σ_y and τ_{xy} are shown in Figs. 31 to 34. Stress discontinuities occur on some of these diagrams at the points where the planes of stress are intersected by the lines of loading. The local bearing effects of rivets appear on some of these diagrams as arrows $\sigma_l = \frac{P_r}{t\,d}$ and $\mu\,\sigma_l$. These local stresses have not been merged with the general stresses on account of difference in symbols.

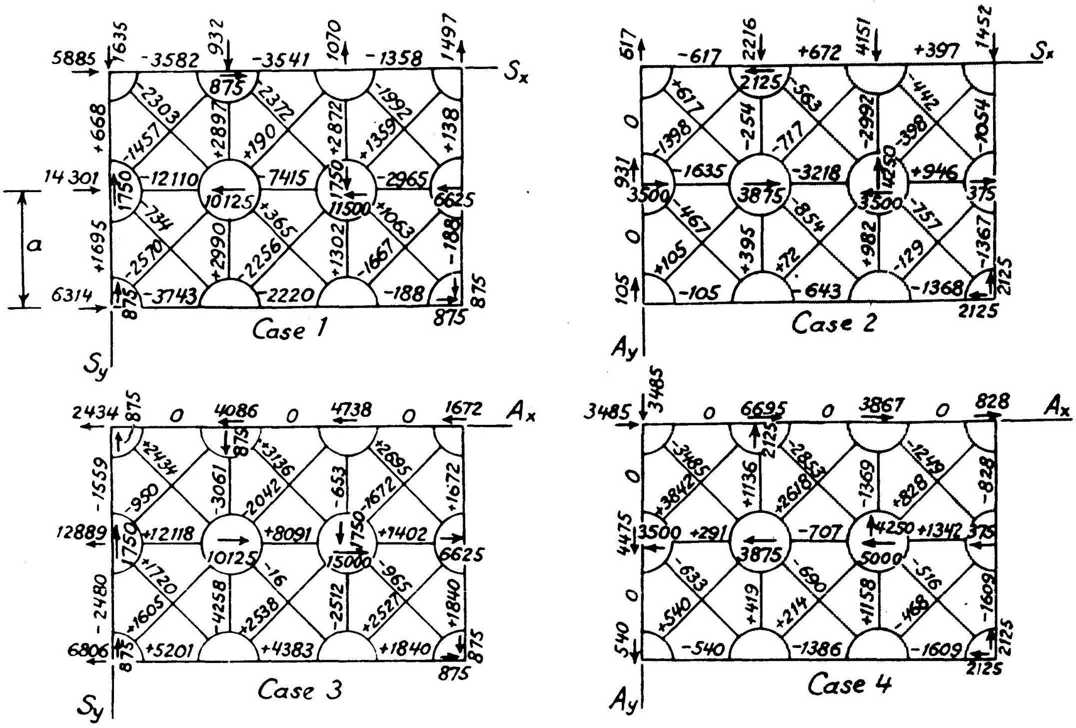

Fig. 30

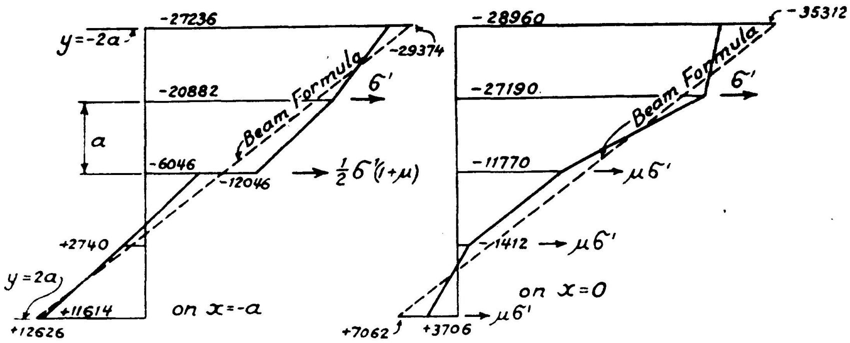

Fig. 31. σ_x Stresses

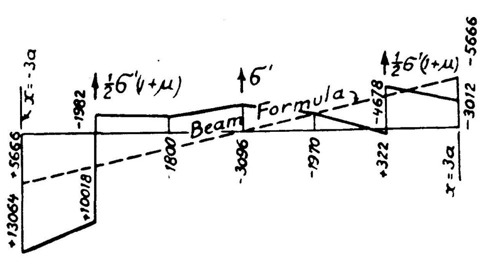

Fig. 32. σ_y Stresses on $y=a$

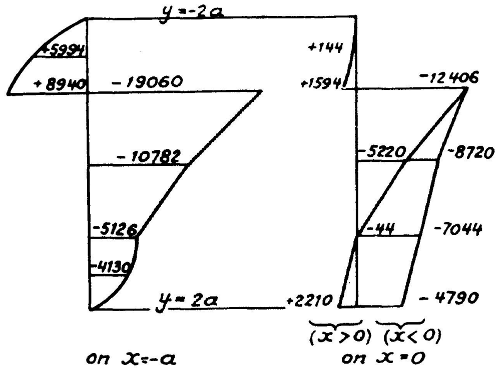

Fig. 33. τ_{xy} Stresses

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[6]) General Note re Fig. 33ff.: All figures represent pounds.

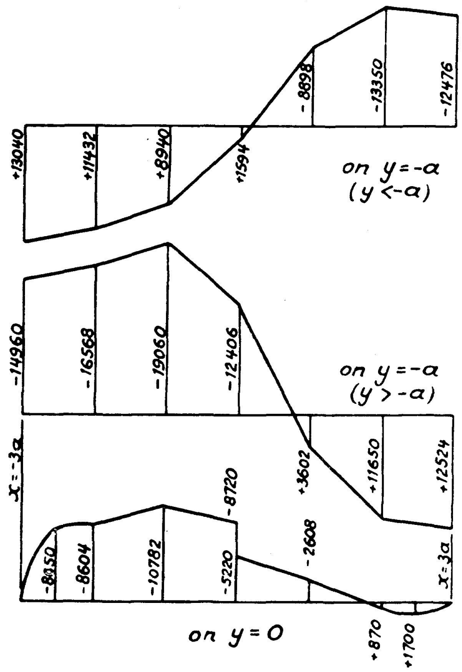

Fig. 34. τ_{xy} Stresses

Fig. 31—34: All figures to be divided by at. Arrows with σ' signify rivet bearing stress to be added to σ_x and σ_y

It is interesting to point out that the normal stresses σ_x have turned out to be not completely dissimilar to the stresses found by the ordinary beam formula $\sigma = \frac{N}{A} + \frac{My}{I}$. The variation of the shearing stresses over the cross-section does not however resemble even remotely the parabolic shape characteristic of the beam with a rectangular cross-section.

The same gusset plate was also analyzed by means of an 8 by 12 framework. The results were found to be in close agreement with the ones reported herein, indicating that structural behaviour of the plate prototype is simulated fairly well by its framework analogue, even when the latter contains only comparatively few units.

Conclusion

Framework Method in Relation to Other Methods

1. The framework method is suitable for solution of the plane stress problems. As long as the outline of the plate suits the shape of the framework unit, the framework method is nearly independent of the boundary conditions and for this reason can be used for problems unsolved by the theory of elasticity. The gusset plate problem belongs to this category, and its solution in this paper is the only one known to the author.

2. In some plane stress problems, including the gusset plate problem, the stress pattern depends on the value of the Poisson's ratio. Under these circumstances the photoelastic method will not give the correct solution if the value of μ in the model differs from the value in the prototype, as is usually the case. This limitation does not apply to the framework method, which can be used with all values of μ, although the distribution procedure comes out the simplest when $\mu = {}^1/_3$.
3. The photoelastic method requires a complicated technique for conversion of the fringe pattern into the stresses. The framework method on the other hand uses a simple arithmetical procedure, although the solution itself is quite lengthy.
4. The framework method is rigorously correct only when the size of the cell is infinitesimal. Of necessity the cells must be finite in size, and even fairly large. This condition leads to some error. It is fortunate however that, judging by the examples solved, this error is moderate even when the units are made quite large, in order to reduce the labour of computation.
5. Employment of half size and other fractional cells in the vicinity of heavy loads and high stresses, or for the purpose of following more closely the curved boundary of the plate, is theoretically possible, but in most cases it increases greatly the labour of computation, and by that makes the method unworkable. The difficulty of fitting the regular shape of the cell to a curved or irregular boundary of the plate is the greatest disadvantage of the method.
6. Most of the labour associated with the framework method lies in determination of the framework stresses, and the proficiency in this procedure depends greatly on the experience of the computor. A beginner may spend much time without any progress, while an experienced person would quickly sense the path taken by stresses from loads to reactions, and by making shrewd guesses would soon reduce the order of the unbalanced loads.
7. The framework method is susceptible to extension into the fields lying outside of the conventional plane stress, such as the action of a reinforced concrete plate, the gusset plate problem allowing for nonequality of the rivet stresses and even the three dimensional stress conditions.
8. Experimental study employing framework models also suggests itself as a by-product of the framework method.

This outline of the general characteristics of the framework method reveals its place in stress analysis among the other methods and the possibilities for the enlargement of its scope in future. When the author thinks of the limitations of the purely mathematical and the photoelastic methods, with full realization of the enormous amount of theoretical and experimental work associated with their progress and involved in their application, he feels amply justified in presenting the framework method before the engineering profession for the study, constructive criticism and further advancement.

Appendix 1

General and Local Stresses in Plates in the Presence of Concentrated Loads

Let Fig. 35 represent a plate acted upon by several (in this case three) concentrated forces which are in equilibrium among themselves. Imagine now this plate under the same forces to be extended infinitely in all directions, and

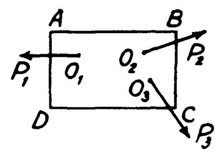

Fig. 35

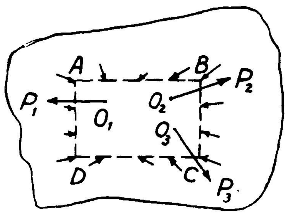

Fig. 36

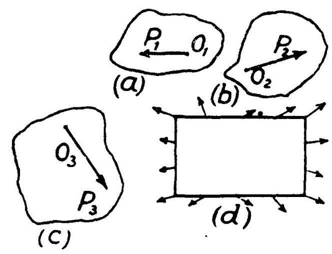

Fig. 37 (a) to (d)

let the stresses existing in this infinite plate along the obliterated now former boundaries be as shown in Fig. 36, these stresses representing the action of the outer part of the plate on the inner part $ABCD$. The state of stress in the original finite plate (Fig. 35) may evidently be considered as a combination of the following four states, shown separately in Fig. 37 (a) to (d).

a) An infinite plate acted upon by a single force P_1, applied at the same point O_1, as before.
b) The same infinite plate with a single force P_2 at the point O_2.
c) The same with the force P_3 at the point O_3.
d) The original finite plate under the action of the boundary stresses equal and opposite to the ones shown in Fig. 36.

It is worthy of notice that the stresses of the case (d) are mutually balanced, because the three forces P_1, P_2 and P_3 are in equilibrium. Superposition of the last stress condition on the first three evidently results in removal of all stresses from the boundary $ABCD$, thus transforming this part of the infinite plate of Fig. 36 into the finite plate with free edges, as in Fig. 35.

Suppose now that the plate of Fig. 35 is analyzed by the framework method, the points O_1, O_2 and O_3 being made coincident with some of the framework joints. The stresses in the vicinity of the joint O_1 are now under investigation, the force P_1 for simplicity of reasoning being assumed horizontal, i.e. parallel to one of the framework axes.

The problem considered may be investigated in two different ways: as a complete single problem, or as a combination of the stress condition (a), i.e. a single concentrated load P_1 applied to an infinite plate at the point O_1, with the combination of conditions (b), (c) and (d). These two methods must lead to the same result. In the former method a difficulty of converting the bar stresses into the plate stresses at the point O_1 is encountered, as a result of inequality of the normal concentrations on the two sides of the joint. In the second approach the part of the problem corresponding to the conditions (b), (c) and (d) combined causes no difficulty of this kind with regard to the point O_1, so that the plate stresses at this point may be computed in the usual manner. The framework stresses corresponding to the condition (a) on the other hand are again characterized by inequality of the normal concentrations on the two sides of the joint O_1, but this case, in view of its symmetry, is more simple than the combined problem. The following features of this case become apparent from symmetry (Fig. 38a). The normal joint concentrations on the

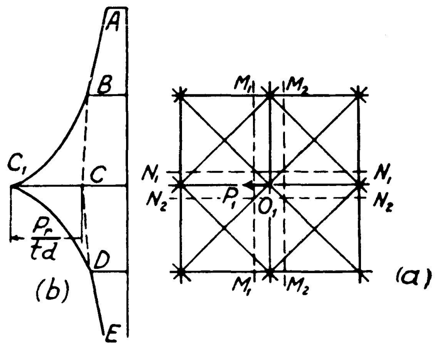

Fig. 38 (a) and (b)

planes M_1M_1 and M_2M_2 are $-\frac{1}{2}P_1$ and $+\frac{1}{2}P_1$ respectively; the normal concentrations on the horizontal planes and the shearing concentrations on the vertical planes are zeroes on both sides of the joint, and the shearing concentrations on the horizontal planes above and below the joint are equal in magnitude and opposite in sign. All this means that the averages of either normal or shearing concentrations on the two sides of the joint, taken either on the horizontal or on the vertical planes are all zeroes. Therefore, addition of the load condition (a) to the combination of the conditions (b), (c) and (d), for the purpose of obtaining the parent problem, does not change the averages of the joint concentrations on the two sides of the joint peculiar to the combination (b) + (c) + (d). Expressed differently, this result means that in the analysis of the complete parent problem, computation of plate stresses in the usual manner on the basis of the average joint concentrations on the two sides of the joint does not give the complete picture of stress, but only its part caused by the conditions (b), (c) and (d) combined, which may be termed the general state of stress. To this must be added the stresses peculiar to the condition (a), the local stresses. In case of a truly concentrated point load these local stresses are expressed by the formulae of elasticity already referred to, while in case of rivet loads the local normal stresses may be taken as $\frac{P_r}{td}$ in the direction of the load and $\frac{\mu P_r}{td}$ in the perpendicular direction, P_r being the rivet load. There are also some local shearing stresses present, as has already been explained.

The combination of the general and the local stresses may be further illustrated by reference to Fig. 38b. If $ABCDE$ is the diagram of the general normal stresses on the vertical plane through the joint O_1, then the local normal stress $\frac{P_r}{td}$ should be added at the point O_1, while no such addition is needed at the adjacent joints not acted upon by the concentrated loads. The resultant stress may then be represented by the diagram ABC_1DE. The shape of the portions of the curve BC_1 and DC_1 is uncertain.

Appendix 2

Determination of Characteristics of the Framework of Simple Square Pattern

First Condition. The framework is loaded with horizontal forces pa per joint and vertical forces μpa per joint (Fig. 39). The symbols for the areas and the stresses in different members are stated in Fig. 39 opposite the corresponding members.

Since the framework must have no vertical deformation, $S_2 = O$.

From the vertical equilibrium of an outside joint $S_1 = \frac{\mu pa}{\sqrt{2}}$, and from the horizontal equilibrium of an outside joint $S + \sqrt{2} S_1 = pa$ or $S = (1 - \mu) pa$.

The horizontal unit strain in the framework therefore is

$$\epsilon_x = \frac{S}{AE} = \frac{(1-\mu)\, pa}{AE} \qquad \text{(f)}$$

The unit strain along the diagonal is

$$\epsilon_l = \frac{S_1}{A_1 E} = \frac{\mu pa}{\sqrt{2}\, A_1 E} \qquad \text{(g)}$$

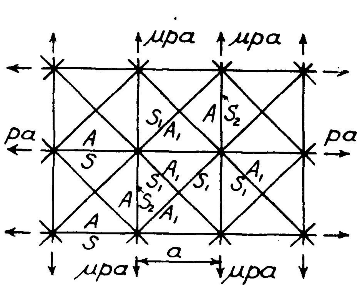

Fig. 39

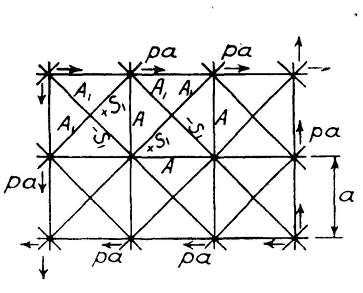

Fig. 40

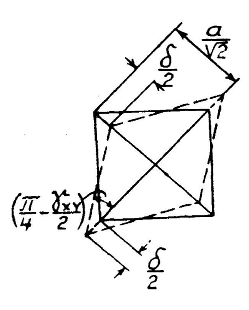

Fig. 41

The strain ϵ_y, although being zero, is also a function of ϵ_x and ϵ_l, and may be expressed through them. For a unit remaining rectangular after the deformation $b^2 = l^2 - a^2$, where a, b and l are respectively the lengths of the horizontal, vertical and diagonal members. Differentiating this expression: $bdb = ldl - ada$. Substituting $da = \epsilon_x a$, $db = \epsilon_y b = \epsilon_y a$ and $dl = l\epsilon_l = \sqrt{2}a\epsilon_l$, the following expression results after the cancellation

$$\epsilon_y = 2\,\epsilon_l - \epsilon_x \tag{h}$$

This equation, on substitution of the values of ϵ_x and ϵ_l leads to the expression

$$\epsilon_y = \frac{pa}{E}\left(\frac{\mu\sqrt{2}}{A_1} - \frac{1-\mu}{A}\right) \tag{k}$$

Equating ϵ_x and ϵ_y to the corresponding strains in the similarly loaded plate the following two equations are produced for determination of characteristics A and A_1.

$$\frac{(1-\mu)\,pa}{AE} = \frac{(1-\mu^2)\,p}{t\,E} \tag{l}$$

and
$$\frac{pa}{E}\left(\frac{\mu\sqrt{2}}{A_1} - \frac{1-\mu}{A}\right) = 0\,. \tag{m}$$

From these equations
$$A = \frac{a\,t}{1+\mu} \tag{n}$$

and
$$A_1 = \frac{\mu}{1-\mu^2}\sqrt{2}\,at \tag{p}$$

Second Condition. In case when tangential loads pa are applied at the outside joints, as shown in Fig. 40, all horizontal and vertical bars are unstressed, which follows from symmetry, while all the diagonals are stressed with equal tensions and compressions $S_1 = \frac{pa}{\sqrt{2}}$, and, consequently, are changed in length by the amount

$$\delta = \frac{S_1\,l}{A_1 E} = \frac{pa^2}{A_1 E}, \tag{t}$$

which transforms each square into a rhombus (Fig. 41).

The shear strain in the framework, γ_{xy}, is found from

$$\tan\left(\frac{\pi}{4} - \frac{\gamma_{xy}}{2}\right) = \frac{\frac{a}{\sqrt{2}} - \frac{\delta}{2}}{\frac{a}{\sqrt{2}} + \frac{\delta}{2}} = \frac{1 - \frac{\delta}{a\sqrt{2}}}{1 + \frac{\delta}{a\sqrt{2}}} \tag{u}$$

Since $$\tan\left(\frac{\pi}{4}-\frac{\gamma_{xy}}{2}\right)=\frac{\tan\frac{\pi}{4}-\tan\frac{1}{2}\gamma_{xy}}{1+\tan\frac{\pi}{4}\tan\frac{1}{2}\gamma_{xy}}=\frac{1-\frac{1}{2}\gamma_{xy}}{1+\frac{1}{2}\gamma_{xy}},$$

the equations (u) and (t) give

$$\gamma_{xy}=\frac{\delta\sqrt{2}}{a}=\frac{pa\sqrt{2}}{A_1 E} \tag{v}$$

Equating this expression for γ_{xy} to the shear strain of the plate gives the third framework equation

$$\frac{pa\sqrt{2}}{A_1 E}=\frac{2(1+\mu)p}{tE},$$

from which $$A_1=\frac{at}{\sqrt{2}(1+\mu)} \tag{w}$$

A_1 can satisfy the expressions (w) and (p) only if

$$\frac{at}{\sqrt{2}(1+\mu)}=\frac{\mu\sqrt{2}}{1-\mu^2}at, \text{ from which } \mu=\tfrac{1}{3}.$$

Therefore, from (n) and (p)

$$A=\tfrac{3}{4}at \tag{2a}$$

and $$A_1=\frac{3}{4\sqrt{2}}at \tag{2b}$$

Summary

The framework method is intended for the solution of problems of plane stress, insoluble for formal mathematics. The plate stressed by forces lying in its plane, is replaced by a framework of elastic bars hinged together, forming a large number of identical cells of some appropriate pattern.

The framework is brought into a state of near equilibrium by successive movements of the unbalanced joints, after which the bar stresses are found and spread over the tributary areas to give an approximation of stresses in the plate prototype. The method is demonstrated on the example of a gusset plate of a truss.

Zusammenfassung

Die „Fachwerkmethode“ wurde entwickelt zur Lösung von ebenen Spannungsproblemen, deren genaue mathematische Behandlung unmöglich ist. Die durch Kräfte in ihrer Ebene beanspruchte Platte wird ersetzt durch ein System von miteinander verbundenen, elastischen Stäben, sodaß eine große Anzahl identischer Zellen eines günstigen Musters entsteht.

Das Rahmensystem wird näherungsweise ins Gleichgewicht gebracht durch sukzessives Verschieben der unausgeglichenen Knoten. Danach werden die Schnittkräfte in den einzelnen Gliedern gefunden und auf die angrenzenden Flächen verteilt. Dies ergibt eine Kräfteverteilung, die die wirklich vorhandene gut annähert.

Die beschriebene Methode wird auf ein Knotenblech eines Fachwerks angewandt.

Résumé

La méthode du treillis a été mise au point en vue de la résolution de problèmes portant sur des contraintes s'exerçant dans le plan et dont il n'est pas possible de faire une étude mathématique précise. La dalle sollicitée par des efforts se manifestant dans son plan est remplacée par un système de barres élastiques liées entre elles, de telle sorte qu'il intervient à sa place un grand nombre d'éléments identiques présentant une disposition favorable.

Le système à cadres ainsi constitué est amené en équilibre d'une manière approchée par déplacements successifs des noeuds non équilibrés. Ceci permet de trouver les sollicitations dans les différentes barres, sollicitations que l'on répartit ensuite sur les surfaces que limitent ces barres. On obtient ainsi une distribution des efforts qui correspond avec une bonne approximation à la distribution réelle.

La méthode ici exposée est appliquée au calcul du gousset d'un treillis.