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THEORY OF THE PLASTIC STABILITY OF THIN PLATES.

THEORIE DER PLASTISCHEN STABILITÄT DÜNNER PLATTEN.

THÉORIE DE LA STABILITÉ PLASTIQUE DE PLAQUES MINCES.

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According to the conceptions prevailing at present, the equations holding in the elastic domain would remain valid for centric buckling of plates in the plastic region, provided the decreased resistance to bending is taken into account by replacing in it the modulus of elasticity E by the so called reduced modulus of elasticity T [1]). The value of T is between that of E and that of the total deformation modulus $E_t = d\sigma/d\varepsilon$ at the buckling stress. At the yield stress where E_t is zero, T has the value zero too, so that the resistance to bending of the plate is also zero there.

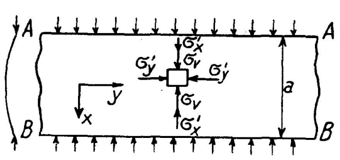

Fig. 1

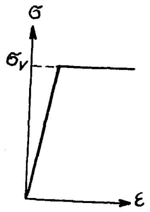

Fig. 2a

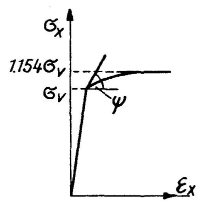

Fig. 2b

As will be explained further on, the resistance to plastic buckling of a plate is much greater than according to this conception, and while e. g. at the yield-point the resistance of the plate to further compression is indeed zero, with the resistance to bending this is not the case. I once casually mentioned that the situation is more complicated than in the case of centric compression, because a n o t h e r s t a t e o f s t r e s s i s s u p e r p o s e d b y t h e b u c k l i n g o n t h e o r i g i n a l s t a t e o f s t r e s s [2]). Now this very fact appears to cause a rise of the yield stress whenever the amount of bending is finite, as well as an increase of the resistance whenever amount of bending is infinitely small.

We consider e. g. a rectangular plate of structural steel, infinitely long in the Y direction and submitted to compression in the X direction (fig. 1). The relation between the stress σ and the strain ε for linear stress is illustrated by the diagram in fig. 2a.

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[1]) CHWALLA, Report of the 2nd Int. Congress for Bridge and Structural Engineering. Vienna, p. 321—322 (1928). BLEICH, SCHLEICHER, ROŠ and EICHINGER, Int. Association for Bridge and Structural Engineering, 1st. Congress Paris. Final Report, p. 120—149 1933). TIMOSHENKO, Theory of elastic stability, p. 386—390 (1936).

[2]) BIJLAARD, De Ingenieur, No. 26 (1934).

If now the stress σ_x increases to the yield stress σ_v, and a further shortening of the plate is opposed by preventing a further strain in the Y direction, which according to my preceding publication on local plastic deformations is possible [3]), then for a certain slenderness ratio of the plate an approach of the ends AA and BB will still be possible due to buckling. With continued buckling, an element $dx\,dy\,dz$ in the concave side of the wave (fig. 1) will suffer a considerable shortening in the X direction. Since hereby no extra strain occurs in the Y direction, the material is then in such another condition as in a strip of local plastic deformation running in the Y direction. As has been explained [3]) with a plate, locally weakened in the Y direction and subjected to tension in the X direction, considerable plastic deformations cannot occur in such a weakened strip until the stress μ, being in our case the stress σ_x, has increased up to $2\,\sigma_v/\sqrt{3} = 1{,}154\,\sigma_v$. The relation between σ_x and the strain ε_x for the concave portion of the wave is consequently according to eq. (14) of my preceding publication:

$$\varepsilon_x = \frac{(m-2)(3\,\sigma_x^2 - 2\,\sigma_v^2 + \sigma_x\sqrt{4\,\sigma_v^2 - 3\,\sigma_x^2}) - (3\,\sigma_x + \sqrt{4\,\sigma_v^2 - 3\,\sigma_x^2})\,m\,E\,\varepsilon_y}{2\,m\,E\sqrt{4\,\sigma_v^2 - 3\,\sigma_x^2}} \tag{1}$$

At the start of buckling, whereby $\sigma_x = \sigma_v$, the strain in Y direction $\varepsilon_y = -\sigma_v/mE$ and, also according to eq. (1), $\varepsilon_x = \sigma_v/E$. In this paper stresses and strains have the positive sign when they indicate compressive stresses or shortenings. During buckling ε_y remains equal to $-\sigma_v/mE$. If, with continued buckling, σ_x in the concave side of the wave e. g. has increased to $1{,}15\,\sigma_v$, that is almost to its maximum ($1{,}154\,\sigma_v$), then, as $m = 10/3$, according to eq. (1) $\varepsilon_x = 5{,}43\,\sigma_v/E$. For the concave side of the wave we have consequently to do with the relation between σ_x and ε_x defined by eq. (1) and graphically represented in fig. 2b. In the case of centric buckling, only infinitely small bending is taken into consideration, hence also infinitely small strains will have to be considered. Consequently the yield point is not raised here, but the angle ψ determining the slope of the $\sigma_x - \varepsilon_x$ diagram, which expresses the rigidity with regard to bending, nevertheless has a considerable value at the yield point. As according to eq. (1) $\sigma_x = 1{,}01\,\sigma_v$ if $\varepsilon_x = 1{,}04\,\sigma_v/E$, this shows that $\tan\psi > E/4$. So far it has been admitted that in this case $\tan\psi$ was zero.

That in the case considered here, additional stresses in the X direction will arise, may be understood as follows too. If this should not be the case, then due to the quasi-isotropic plastic deformation, an additional shortening ε'_x, arising at buckling in the concave side of the wave, would necessarily be accompanied by an additional elongation $-\varepsilon'_y = \varepsilon'_x/2$. To enable ε'_y to remain equal to zero, an additional compressive stress σ'_y will be required in the Y direction. According to the yield condition of Huber-von Mises-Hencky the following eq. is valid:

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 = \sigma_v^2 \tag{2}$$

which equation is graphically represented by the ellipse in fig. 3. Hereby the axes indicated by σ_x and σ_y are supposed to represent compressive stresses. At the moment of buckling: $\sigma_x = \sigma_v$. It may be seen readily, that an additional compressive stress σ'_y involves necessarily an additional compressive stress σ'_x, since the representative point of the state of stress should remain on the ellipse.

[3]) Bijlaard, Theory of local plastic deformations, in this volume.

In order to find in general the resistance of plates to buckling, we shall have to determine first the general relations between the additional stresses σ'_x, σ'_y and τ'_{xy} arising at buckling and the additional strains ε'_x, ε'_y and γ'_{xy} in the plastic region. The state of stress, acting in the plate at the moment of buckling, is denoted by $\varrho_1 \varrho_2$, whereby ϱ_1 and ϱ_2 act in the X and Y direction resp. The X and Y axes are to be found in the median plane of the plate. The Z direction falls perpendicularly to it. The normal stresses σ_x and σ_y, which after buckling are acting on the X and Y planes, generally will not be, strictly speaking, principal stresses, because with an arbitrary form of buckling, also twisting stresses $\tau'_{xy} = -\tau'_{yx}$ will start working. Thus after buckling we have the stresses $\sigma_x = \varrho_1 + \sigma'_x$, $\sigma_y = \varrho_2 + \sigma'_y$, $\tau_{xy} = \tau'_{xy}$ and $\tau_{yx} = \tau'_{yx}$. As to these stresses the plasticity condition runs as follows [4]).

$$\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_v^2 \tag{3}$$

By way of introduction we admit $\varrho_1 = \sigma_v$ and $\varrho_2 = 0$. The infinitely small additional stresses which result from buckling, will either cause in the elements a discharge, leading back to the domain of completely elastic deformations, or they will not take away the plasticity, the point representative of the state of stress in that case having to remain always on the limiting ellipsoid (3). We will be able, for these infinitely small additional stresses, to replace the limiting surface by the tangent plane in the point A representative of the state of stress, which existed before buckling: $\sigma_x = \varrho_1 = \sigma_v$, $\sigma_y = \varrho_2 = 0$, $\tau_{xy} = 0$ (fig. 3). From eq. (3) follows that at A:

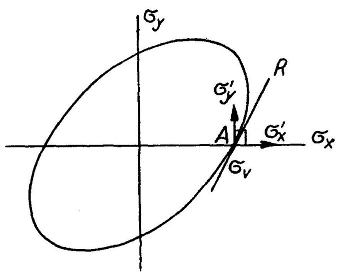

Fig. 3

$$\left.\begin{aligned} \frac{\partial \sigma_y}{\partial \sigma_x} &= \frac{2\sigma_x - \sigma_y}{\sigma_x - 2\sigma_y} = 2 \\ \text{and} \quad \frac{\partial \tau_{xy}}{\partial \sigma_x} &= -\frac{2\sigma_x - \sigma_y}{6\tau_{xy}} = -\infty \end{aligned}\right\} \tag{4}$$

so that the equation of the tangent plane, expressed in the additional stresses σ'_x and σ'_y, which have their origin in the tangent point A, becomes:

$$\sigma'_y = \sigma'_x \frac{\partial \sigma_y}{\partial \sigma_x} = 2\sigma'_x \tag{5}$$

The additional twisting stresses τ'_{xy} apparently do not affect the combination of σ'_x and σ'_y which may occur at the yield point [5]).

According to eq. (12) of our preceding publication [3]) we have:

$$\varepsilon_x = \varepsilon_{xe} + \varepsilon_{xp} = \frac{\sigma_x}{E} - \frac{\sigma_y}{mE} + \frac{2\sigma_x - \sigma_y}{2\sigma_y - \sigma_x}\left(\varepsilon_y + \frac{\sigma_x}{mE} - \frac{\sigma_y}{E}\right) \tag{6}$$

If in (6), also in connection with (5), we substitute:

$$\sigma_x = \sigma_v + \sigma'_x, \quad \sigma_y = \sigma'_y, \quad \varepsilon_x = \sigma_v/E + \varepsilon'_x \quad \text{and} \quad \varepsilon_y = -\sigma_v/mE + \varepsilon'_y \tag{7}$$

then σ'_x, ε'_x and ε'_y will be the only unknown quantities in (6), hence the extra stress σ'_x, and consequently also σ'_y, may be expressed in the extra strains ε'_x and ε'_y. We find:

[4]) This may be understood readily, since the normal and shearing stresses work independently of each other. If only shearing stresses τ_{xy} were at work, $\varrho_1 = -\varrho_2$ being τ_{xy}, then eq. (2) would change into $3\tau^2_{xy} = \sigma^2_v$. Thus when they act in combination with σ_x and σ_y, eq. (3) becomes the yield condition.

[5]) Not the extra vertical shearing stresses τ'_{xz} and τ'_{yz}, the deformations due to which are always disregarded with buckling of full sections.

$$\left.\begin{aligned}\sigma_x &= \frac{mE}{5m-4}(\varepsilon'_x + 2\varepsilon'_y)\\ \sigma_y &= \frac{2mE}{5m-4}(\varepsilon'_x + 2\varepsilon'_y)\end{aligned}\right\} \tag{8}$$

For the case illustrated in fig. 1 and fig. 2b, where ε'_y was equal to zero, we find according to (8), with $m = 10/3$: $\sigma'_x = E\varepsilon'_x/3{,}8$, so that in fig. 2b $\tan\psi = E/3{,}8$, so for a $\sigma - \varepsilon$ diagram, holding for a linear stress, according to fig. 2a, at the yield point the resistance to bending is only 3,8 times as small as in the elastic domain.

The $\sigma' - \varepsilon'$ relations will now be derived for the most general case, whilst the $\tau'_{xy} - \gamma'_{xy}$ relations will be determined too. Also when the state of stress $\varrho_1\varrho_2$ acting in the plate prior to buckling is not situated at the yield point, the deformation will take place quasi-isotropically and the shearing energy will be determinant for the appearance of the plastic deformations [6] [3]). According to our preceding publication [3]) however the function $\varrho_1^2 - \varrho_1\varrho_2 + \varrho_2^2$ in the region between the elastic limit and the yield point, which is of importance only from a practical point of view, will be determinant for thin plates when $\varrho_2 < \varrho_1/2$ only. Just as in the following, the greatest principal stress is denoted here by ϱ_1, while $\varrho_2/\varrho_1 = \beta$. The state of stress $\varrho_1\varrho_2$ is consequently equivalent to a linear stress:

$$\sigma_q = \sqrt{\varrho_1^2 - \varrho_1\varrho_2 + \varrho_2^2} \tag{9}$$

as the plastic strains are:

$$\varepsilon_{xp} = \frac{\varrho_1}{E_p} - \frac{\varrho_2}{2E_p} \quad (10\,a) \qquad \text{and} \qquad \varepsilon_{yp} = \frac{\varrho_2}{E_p} - \frac{\varrho_1}{2E_p} \quad (10\,b)$$

from which follows:

$$\varrho_1 = \frac{2}{3}E_p(\varepsilon_{xp} + 2\varepsilon_{yp}) \quad (11\,a) \qquad \text{and} \qquad \varrho_2 = \frac{2}{3}E_p(\varepsilon_{yp} + 2\varepsilon_{xp}) \quad (11\,b)$$

Substituting eq. (11 a) and (11 b) in eq. (9) we find:

$$\sigma_q = \frac{2}{\sqrt{3}}E_p\sqrt{\varepsilon_{xp}^2 + \varepsilon_{xp}\varepsilon_{yp} + \varepsilon_{yp}^2} = \frac{2}{\sqrt{3}}E_p\varepsilon_q \tag{12}$$

where:

$$\varepsilon_q = \sqrt{\varepsilon^2_{xp} + \varepsilon_{xp}\varepsilon_{yp} + \varepsilon^2_{yp}} \tag{13}$$

Relation (12) has already been found by Roš and Eichinger [6]).

We assume that for the linear stress σ_q plastic deformations $\varepsilon_p = \sigma_q/E_p = e\,\sigma_q/E$ (we put $E_p = E/e$) have arisen and that $d\sigma_q/d\varepsilon_p = \tan\varphi$ (fig. 4b).

The point B, representative of the state of stress $\sigma_x = \varrho_1$, $\sigma_y = \varrho_2$, is in analogy with (3) now situated on the ellipsoid:

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_q^2 \tag{14}$$

which intersects the σ_x axis at the point A representing the equivalent linear stress σ_q. In fig. 4a the section of (14) with the plane $\tau_{xy} = 0$ is dotted.

The point C representing the state of stress resulting after buckling, if for the element examined σ_q increases through buckling, will be situated

[6]) Roš and Eichinger, Versuche zur Klärung der Frage der Bruchgefahr. E.M.P.A., Diskussionsbericht No. 34 (1929).

on a similar ellipsoid, which intersects the σ_x axis at a point A', for which $\sigma_x = \sigma_q + d\,\sigma_q$ (fig. 4a). The two ellipsoids may in their turn be replaced by the respective tangent planes R and R' in the respective points $\sigma_x = \varrho_1$, $\sigma_y = \varrho_2$ and $\sigma_x = \varrho_1 + \sigma'_x$, $\sigma_y = \varrho_2 + \sigma'_y$, which again are parallel to the τ_{xy} axis. Since the distance BC is infinitely small with respect to the distance OB, however in fig. 4a represented on a exaggerated scale, the planes R and R' will moreover be parallel to one another. They form an angle γ with the σ_x axis, for which according to (14):

$$\tan\gamma = \frac{\partial\sigma_y}{\partial\sigma_x} = \frac{2\sigma_x - \sigma_y}{\sigma_x - 2\sigma_y} = \frac{2\varrho_1 - \varrho_2}{\varrho_1 - 2\varrho_2} = \frac{2-\beta}{1-2\beta} \tag{15}$$

Since the points A and A', like B and B', are corresponding points of the two ellipsoids, we have $BB'_x = \varrho_1 d\sigma_q/\sigma_q$ and $BB'_y = \varrho_2 d\sigma_q/\sigma_q$. The relation between σ'_x and σ'_y in accordance with fig. 4a must be:

$$\sigma'_y = BB'_y + (\sigma'_x - BB'_x)\tan\gamma = (\varrho_2 - \varrho_1\tan\gamma)\,d\sigma_q/\sigma_q + \sigma'_x\tan\gamma \tag{16}$$

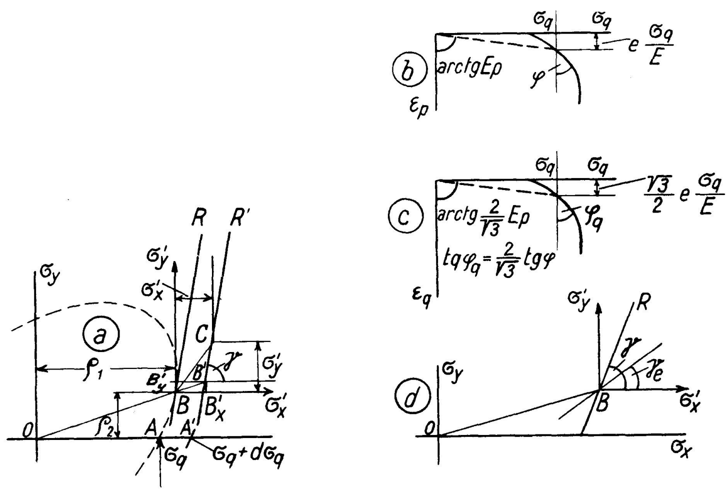

Fig. 4a — d

Hence in eq. (6) we put:

$$\left.\begin{aligned}
\sigma_x &= \varrho_1 + \sigma'_x \\
\sigma_y &= \varrho_2 + \sigma'_y = \varrho_2 + \sigma'_x\tan\gamma + (\varrho_2 - \varrho_1\tan\gamma)\,d\sigma_q/\sigma_q \\
\varepsilon_x &= \varrho_1/E - \varrho_2/mE + e\varrho_1/E - e\varrho_2/2E + \varepsilon'_x \\
\varepsilon_y &= \varrho_2/E - \varrho_1/mE + e\varrho_2/E - e\varrho_1/2E + \varepsilon'_y
\end{aligned}\right\} \tag{17}$$

As to the determination of $d\sigma_q$, it should be observed first, that the additional twisting stresses τ'_{xy} do not affect the magnitude of $d\sigma_q$, since the point C representing the state of stress remains thereby in the same tangent plane R' as with absence of the stresses τ'_{xy}, for R' is parallel to the τ_{xy} axis. This may be seen from (14) as well, since it follows from it, because before buckling $\tau_{xy} = 0$, that:

$$\partial\sigma_q/\partial\tau_{xy} = 3\tau_{xy}/\sigma_q = 0 \tag{18}$$

According to fig. 4c and eq. (12) we find then, the intermediary transformations being omitted:

$$d\sigma_q = d\varepsilon_q \tan\varphi_q = \frac{2}{\sqrt{3}}\left(\frac{\partial\varepsilon_q}{\partial\varepsilon_{xp}} d\varepsilon_{xp} + \frac{\partial\varepsilon_q}{\partial\varepsilon_{yp}} d\varepsilon_{yp}\right)\tan\varphi = (\varrho_1 d\varepsilon_{xp} + \varrho_2 d\varepsilon_{yp})\frac{\tan\varphi}{\sigma_q}$$

$$= \frac{(1-2\beta)\, mE\tan\varphi}{(1-2\beta)\, mE + (1-m\beta)\, 2\tan\varphi}\,\frac{\varrho_1}{\sigma_q}\left(\varepsilon'_x + \beta\varepsilon'_y - \frac{(m-2)(1-\beta^2)}{(1-2\beta)\, mE}\sigma'_x\right) \quad (19)$$

After a number of transformations (6) gives σ'_x as a function of ε'_x and ε'_y, and then, with (16) we find σ'_y:

$$\sigma'_x = E(A\varepsilon'_x + B\varepsilon'_y), \qquad \sigma'_y = E(C\varepsilon'_x + D\varepsilon'_y) \quad (20)$$

in which:

$$A = \varphi_1/\varphi_4, \qquad B = \varphi_2/\varphi_4 = C, \qquad D = \varphi_3/\varphi_4 \quad (21)$$

and:

$$\left.\begin{aligned}
\varphi_1 &= m^2(1-2\beta)\{(1-2\beta)^2 E + (4\eta^2+3e)\tan\varphi\} \\
\varphi_2 &= m(1-2\beta)\{(2-\beta)(1-2\beta)mE + (4\eta^2+3em\beta)\tan\varphi\} \\
\varphi_3 &= m^2 1-2\beta)\{(2-\beta)^2 E + (4\eta^2+3e\beta^2)\tan\varphi\} \\
\varphi_4 &= m(1-2\beta)\{(5m-4)(1+\beta^2)+2(5-4m)\beta+3em\eta^2\}E + [4(m^2-1)(1-2\beta)\eta^2 \\
&\qquad + 3em\{2(1-m\beta)\eta^2 + (m-2)(1-\beta^2)\}]\tan\varphi \\
\beta &= \varrho_2/\varrho_1 \quad \text{and} \quad \eta^2 = \sigma_q^2/\varrho_1^2 = \beta^2 - \beta + 1.
\end{aligned}\right\} \quad (22)$$

The strains ε'_x and ε'_y are infinitely small, hence on buckling, the finite stresses ϱ_1 and ϱ_2 produce infinitely small plastic deformations. The infinitely small twisting stresses τ'_{xy} consequently produce plastic shearing strains which are infinitely small of the second order. At the previous plastic deformation E_p was equal to E/e; hence the plastic modulus of rigidity becomes $G_p = mEp/2(m+1) = E/3e$. Thus:

$$\gamma'_{xy} = \tau'_{xy}/G + \tau'_{xy}/G_p = \{2(m+1)/m + 3e\}\tau'_{xy}/E = (2m+2+3em)\tau'_{xy}/mE$$

or

$$\tau'_{xy} = \frac{mE}{2m+2+3em}\gamma'_{xy} = EF\gamma'_{xy} \quad (23)$$

We shall now consider first the case $\tan\varphi = 0$, that is the buckling at the yield point. It follows from (21) and (22) that then:

$$B/A = D/C = C/A = D/B = \sigma'_y/\sigma'_x = (2-\beta)/(1-2\beta) = \tan\gamma \quad (24)$$

This is to be expected, since the point representing the state of stress must remain on the tangential plane to the same ellipsoid which forms the angle γ shown in equation (15) with the σ_x axis. In the elastic domain we know that:

$$\sigma'_x = \frac{m^2E}{m^2-1}\left(\varepsilon'_x + \frac{1}{m}\varepsilon'_y\right) \quad \text{and} \quad \sigma'_y = \frac{m^2E}{m^2-1}\left(\frac{1}{m}\varepsilon'_x + \varepsilon'_y\right) \quad (25)$$

as follows likewise from equation (20), in which now $\tan\varphi = \infty$ and $e = 0$; hence:

$$\sigma'_y/\sigma'_x = (\varepsilon'_x + m\varepsilon'_y)/(m\varepsilon'_x + \varepsilon'_y) = \tan\gamma_e \quad (26)$$

If we think of an arbitrary deformation of the plate at buckling, then the elements, the representative points of which for elastic deformation are situated to the left of the tangent plane R (fig. 4d) will be deformed elastically, where as those where this point would be found to the right of R would be deformed plastically, the representative point in that case remaining on R. As with bending the sections may be assumed to remain plane, ε'_x and ε'_y will vary linearly with z along the entire thickness of the plate, and so, according to (20) and (25), σ'_x and σ'_y for a certain small element $hdxdy$, will also vary linearly with z, both in the plastic and the elastic domain (fig. 5a). It should be noted, however, that equilibrium is only possible, if the plate deforms plastically along the entire height [7]) and that then in the centre plane of the plate $\sigma'_x = \sigma'_y$ is to be zero. If the strains here are ε'_{xm} and ε'_{ym} then, according to (20) and (24), $\varepsilon'_{ym}/\varepsilon'_{xm} = -A/B \doteq -(1-2\beta)/(2-\beta)$. If the strains resulting from the actual bending are given by ε''_x and ε''_y, so that e.g. $\varepsilon'_x = \varepsilon'_{xm} + \varepsilon''_x$, then it follows from (20) that:

$$\sigma'_x = E(A\,\varepsilon''_x + B\,\varepsilon''_y) \qquad \text{and} \qquad \sigma'_y = E(C\,\varepsilon''_x + D\,\varepsilon''_y) \tag{27}$$

ε'_{xm} and ε'_{ym} will have to be the same for the entire plate, in order to make $\int \tau'_{xy} dh = \int \tau'_{yx} dh$ to be zero, for then $\gamma'_{xy} = 0$ in the centre plane.

The theory of elasticity has taught us that when w represents the deflection of the plate, ε''_x, ε''_y and γ'_{xy}, as far as the top and the bottom of the plate are concerned, are given by:

$$\varepsilon''_x = -\frac{h}{2}\frac{\partial^2 w}{\partial x^2}, \qquad \varepsilon''_y = -\frac{h}{2}\frac{\partial^2 w}{\partial y^2} \qquad \text{and} \qquad \gamma'_{xy} = 2\,\frac{h}{2}\frac{\partial^2 w}{\partial x\,\partial y}.$$

Hence, according to equations (27) and (23):

$$\left.\begin{aligned} \sigma'_x &= -E\frac{h}{2}\left(A\frac{\partial^2 w}{\partial x^2} + B\frac{\partial^2 w}{\partial y^2}\right), \qquad \sigma'_y = -E\frac{h}{2}\left(C\frac{\partial^2 w}{\partial x^2} + D\frac{\partial^2 w}{\partial y^2}\right) \\ &\text{and} \qquad \tau'_{xy} = -\tau'_{yx} = EFh\frac{\partial^2 w}{\partial x\,\partial y} \end{aligned}\right\} \tag{28}$$

The bending moments M_x and M_y and the twisting moments $t_{xy} = -t_{yx}$ per unit of width, result from this by multiplication by $W = h^2/6$, as may be deduced directly from figure 5b. As $h^3/12$ represents the moment of inertia J of the plate, we have:

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[7]) The transition between the plastic and the elastic domain would have to be at a height h_p under the top of the plate (fig. 5a), where $\tan\gamma_e = \tan\gamma$ (fig. 4d), or where $\varepsilon'_x = \varepsilon'_y = 0$. The equilibrium requires $\iint \sigma'_x\,dh\,dy$ and $\iint \sigma'_y\,dh\,dx$ to be 0. As in the plastic domain σ'_y/σ'_x is constant, we note that, if $\iint \sigma'_x\,dh\,dy = 0$, $\iint \sigma'_y\,dh\,dx$ can only be 0, if also in the plastic domain σ'_y/σ'_x has the same constant ratio, and $\tan\gamma_e = \tan\gamma$, which as a rule will not be the case. In order to satisfy both conditions, the section will have to deform either altogether elastically, or altogether plastically. Entirely elastic deformation is not possible, for if the representative point of an element under the centre plane were to the left of R (fig. 4d), that of an element above the centre plane would be to the right of R, and then at that side the deformation would all the same have to take place plastically. Hence the deformation is altogether plastic. If with an uniaxial compression $\varrho_1 = \sigma_v$ there is buckling perpendicular to ϱ_1, ε'_y being 0, then, according to (5) and likewise (24) (since $\beta = 0$), in the plastic domain σ'_y would be $2\,\sigma'_x$, and in the elastic domain, according to equation (26) σ'_y would be σ'_x/m. As is shown by figure 5a, if $\int \sigma'_x\,dh$ were 0, $\int \sigma'_y\,dh$ would not be 0, but create a compressive force in the Y direction. In absence of external forces in the Y direction, this would cause an elongation in the Y direction, naturally accompanied by a shortening in the X direction, until the section completely reaches the plastic domain (fig. 5b). Hence at the bottom of the plate, according to equations (24) and (26) $\tan\gamma_e \geqq \tan\gamma$, hence as $\beta = 0$ here, ε'_{x0} will be $\geqq \frac{m-2}{2m-1}\,\varepsilon'_y$.

$$\left.\begin{aligned} M_x = - EJ\left(A \frac{\partial^2 w}{\partial x^2} + B \frac{\partial^2 w}{\partial y^2}\right), \quad & M_y = - EJ\left(C \frac{\partial^2 w}{\partial x^2} + D \frac{\partial^2 w}{\partial y^2}\right) \\ \text{and} \quad & t_{xy} = - t_{yx} = 2EJF \frac{\partial^2 w}{\partial x \, \partial y} \end{aligned}\right\} \quad (29)$$

The establishment of the condition of equilibrium leads to the equation [8]:

$$EJ\left\{A \frac{\partial^4 w}{\partial x^4} + (B + C + 4F) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D \frac{\partial^4 w}{\partial y^4}\right\} + h \varrho_1 \frac{\partial^2 w}{\partial x^2} + h \varrho_2 \frac{\partial^2 w}{\partial y^2} = 0 \quad (30)$$

We consider now the general case of $\tan \varphi > 0$.

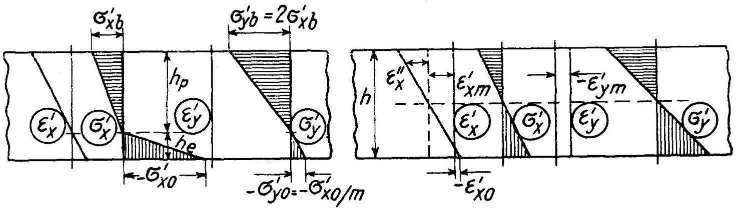

Fig. 5a Fig. 5b

When we examine the stresses on an element $h dx dy$ of a plate [9]), then on the concave side, where extra shortening takes place, the deformation generally will be plastic, and (20) will apply; whilst on the convex side, where extra extension takes place, the deformation will be elastic, and (25) will consequently apply. With a given shape of the deflection surface of the plate the stresses on the sides of the small element are fully determined by the two distances h_x and h_y — measured from the concave outside — of the surfaces where ε'_x and ε'_y respectively are equal to zero. In the case of the buckling of plates in the elastic domain three conditions are satisfied, viz. $\int \sigma'_x dh = 0$, $\int \sigma'_y dh = 0$ and $\int \tau'_{xy} dh = \int \tau'_{yx} dh = 0$, and thus far it had been assumed that this was also the case in the plastic domain. These three conditions however, in our case lead to 3 equations with only two unknown quantities h_x and h_y, and thus it will generally not be possible to satisfy them. A shortening with unchanged stresses, such as occurred in the case $\tan \varphi = 0$, considered in footnote 7, causing the whole plate to enter the plastic domain, is not possible for $\tan \varphi > 0$.

In order to get an idea of the stress distribution at buckling we consider again a rectangular plate of structural steel, infinitely long in the Y direction and submitted to compression in the X direction, as it buckles just below the yield point σ_v (fig. 6a). From the relation between the buckling stress of bars of structural steel No. 37 and their slenderness ratio $\lambda = l/r$, as established in the specifications of the German State Railways

[8]) See also Bijlaard, Proc. Royal Ac. of Sciences, Amsterdam, Vol. 41, No. 5 (1938).

[9]) For a bar, both the cross dimensions consequently being small with regard to the length, the new equations naturally show the relations applying to linear stress. With buckling in the Z direction here $\sigma'_y = 0$, as contraction may occur freely; hence it follows from equation (20) that $\varepsilon'_y = -C\varepsilon'_x/D$. When we insert this value in the first equation (20), we find, on making use of (21) and (22), and after further transformation:

$$\sigma_x' = \frac{E \tan \varphi}{E + \tan \varphi} \varepsilon_x' = \frac{(d\sigma/d\varepsilon_e)(d\sigma/d\varepsilon_p)}{d\sigma/d\varepsilon_e + d\sigma/d\varepsilon_p} \varepsilon_x' = \varepsilon_x' \frac{d\sigma}{d\varepsilon} \quad \text{or} \quad d\sigma_x = d\varepsilon_x \cdot \frac{d\sigma}{d\varepsilon}.$$

This proves that the theory of Engesser-Karman remains valid for bars.

(Vorschriften für Eisenbauwerke), on account of the relations $\sigma_B = \pi^2 T/\lambda^2$ and the relation $T = 4EE_t/(\sqrt{E} + \sqrt{E_t})^2$ holding for rectangular sections, the magnitude of the total deformation modulus $E_t = d\sigma/d\varepsilon$ may be calculated for the underlaying $\sigma - \varepsilon$ diagram, as was done already by others too. We may now write for tan φ:

$$\tan\varphi = \frac{d\sigma}{d\varepsilon_p} = \frac{d\sigma}{d\varepsilon - d\varepsilon_e} = \frac{1}{1/E_t - 1/E} = \frac{EE_t}{E - E_t} \tag{31}$$

Since $\varepsilon_p = e\sigma_q/E$ we have:

$$e = \frac{E}{\sigma_q}\varepsilon_p = \frac{E}{\sigma_q}\int \frac{d\varepsilon_p}{d\sigma}\, d\sigma = \frac{E}{\sigma_q}\int d\sigma/\tan\varphi = \frac{E}{\sigma_q}\,\Sigma \Delta\sigma/\tan\varphi \tag{32}$$

So for any stress $\tan\varphi$ and e may be computed. Just below the yield point we find $\tan\varphi = 0{,}294\,E$ and $e = 0{,}1675$. With $m = 10/3$ it follows from (20)—(23) that:

$$\left.\begin{aligned} \sigma'_x = E(0{,}421\,\varepsilon'_x + 0{,}426\,\varepsilon'_y), \qquad \sigma'_y = E(0{,}426\,\varepsilon'_x + 0{,}938\,\varepsilon'_y)\\ \tau'_{xy} = 0{,}322\,E\gamma'_{xy} \end{aligned}\right\} \tag{33}$$

whilst from (25) it follows that in the elastic domain:

$$\left.\begin{aligned} &\sigma'_x = E(1{,}099\,\varepsilon'_x + 0{,}329\,\varepsilon'_y),\\ &\sigma'_y = E(0{,}329\,\varepsilon'_x + 1{,}099\,\varepsilon'_y)\\ &\text{and finally}\\ &\qquad \tau'_{xy} = 0{,}385\,E\gamma'_{xy} \end{aligned}\right\} \tag{34}$$

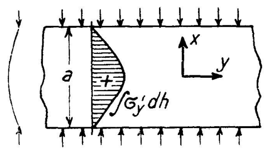

Fig. 6a

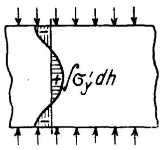

Fig. 6b

We first suppose $\varepsilon'_y = 0$ (fig. 7a). If h_x is chosen so as to make $\int \sigma'_x dh = 0$, then $\int \sigma'_y dh$ would represent a comparatively great compression, as on the upper side $\sigma'_{yb}/\sigma'_{xb} = 0{,}426/0{,}421$ whereas on the lower side $\sigma'_{y0}/\sigma'_{x0}$ is only $0{,}329/1{,}099$. The compression $\int \sigma'_y dh$ would be proportional to $\partial^2 w/\partial x^2$ (fig. 6a). It will cause an extension $-\varepsilon'_{yc}$ of the plate in the Y direction, and as the result of this there will be a shortening ε'_{xc} in the X direction. Since for $x = \pm a/2$, $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ are zero, σ'_x must in

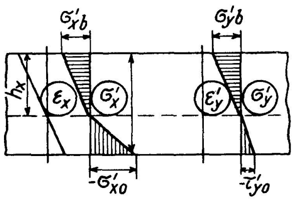

Fig. 7a

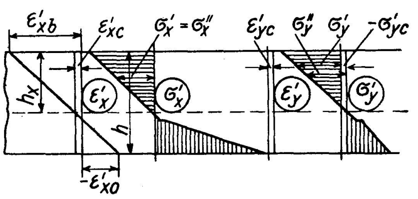

Fig. 7b

any case be zero there, as with buckling σ_x does not increase. As in the case of shortening, the plate behaves plastically over its entire height, σ'_x in (33) will have to be zero. If we insert the ratio $\varepsilon'_{xc}/\varepsilon'_{yc}$ resulting from this in the second equation (33), we find that $\sigma'_{yc} = 0{,}507\ \varepsilon'_{yc}$. The strain ε'_{yc} must be the same for the entire plate. Supposing ε'_{xc} likewise to be the same everywhere, we find $\sigma'_{yc} = 0{,}507\ \varepsilon'_{yc}$ and $\sigma'_{xc} = 0$ for the entire plate. As ε'_{yc} is negative, σ'_{yc} represents a tensile stress. The equilibrium now requires that for all values of x the distance h_x (fig. 7b) for the superposed bending is chosen in such a way, that if the superposed bending stresses are denoted by σ''_x and σ''_y,

$$\int \sigma'_x \, dh = \int \sigma''_x \, dh = 0;$$

and further that ε'_{yc} is chosen in such a way that

$$\iint \sigma'_y \, dh \, dx = a h \sigma'_{yc} + \iint \sigma''_y \, dh \, dx$$

becomes equal to 0. As could be found graphically, ε'_{yc} then must be about $-0{,}05$ $(\varepsilon'_{xb} - \varepsilon'_{x0})$ if ε'_{xb} and ε'_{x0} represent the strains at top and bottom for $x = 0$, where $\partial^2 w / \partial x^2$ is maximum. The plate remains plastic in the region where $\varepsilon'_x > (m - 2)\, \varepsilon'_y / (2m - 1)$ (see footnote 7), and thus, in connection with the strains ε'_{xc} and ε'_{yc} over the entire height, the plate, close to the boundaries $x = \pm a/2$, remains plastic over its entire height (fig. 8). More towards the centre it behaves elastically at the convex side, as shown likewise in fig. 7b. $\int \sigma'_y dh$ is shown in fig. 6b. Fig. 8 gives a section with a plane perpendicular to the Y axis (the height h has been drawn on an exaggerated scale). The concave side is supposed to be at the top. The plastic domain is indicated by cross hatching. The planes N_x and N_y, where σ'_x and σ'_y respectively are zero, also have been shown in fig. 8. Thus the plate is more rigid in the central strip than at the boundaries $x = \pm a/2$; consequently it will not buckle according to a plane $w = w_0 \cos(\pi x / a)$. As the stress distribution now is known in the various sections, it was possible graphically to determine the buckling stress σ_B as a function of a, E and h. Inversely this showed at what ratio a/h the plate will buckle just below the yield stress.

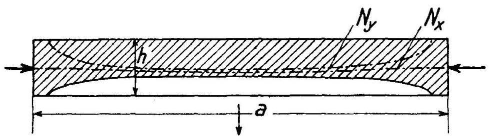

Fig. 8

It is not worth while to determine the stress distribution exactly also for other boundary conditions. If for an imaginary, entirely elastic or entirely plastic plate the buckling stresses are called σ_E and σ_P respectively, then the actual buckling stress σ_B in any case is to be found between σ_E and σ_P. With the above case it appears that approximately:

$$\sigma_B = \sigma_E / 4 + 3 \sigma_P / 4 \qquad (35)$$

This relation may be considered to hold for other cases too [10]).

[10]) With buckling at the elastic limit the deformation is wholly elastic, at higher buckling stresses it becomes partly plastic and partly elastic. At the yield point itself, so if $\tan \varphi = 0$, it is entirely plastic. It is obvious to apply for structural steel No. 37 and for buckling stresses below the yield stress, the only case that is of any importance for technical purposes, the above-mentioned relation for other boundary conditions as well, the more so when one considers the following. As has been observed by CHWALLA [11]), a discharge will only occur at buckling of rectangular bars in the plastic region when the eccentricity is less than about $h/50$. The admissable stresses are based on the assumption that such an eccentricity will always be on hand, so that as a matter of fact the reduced modulus of elasticity T is not determinant for the admissable stresses but the total deformation modulus E_t and really we should reckon with the buckling stress, calculated for the bar supposed to be plastic all over, as a basis for the determination of the admissible stress. Thus we might state that the buckling stresses from which we start are T/E_t times too large. We might therefore in a certain sense contend, that for plates as well, in order to make the calculated stresses comparable with those of bars, we should have to reckon with a buckling stress of $\sigma_B = (T/E_t)\, \sigma_P$. As may be seen readily in the region below the yield point for structural steel No. 37, $(T/E_t)\, \sigma_P$ is

For the determination of the buckling stress σ_P of plates, compressed in one direction, according to (30), since $\varrho_1 = \sigma_P$, $\varrho_2 = 0$ and $C = B$, the following equation is valid:

$$EJ\left\{A\frac{\partial^4 w}{\partial x^4} + 2(B+2F)\frac{\partial^4 w}{\partial x^2\, dy^2} + D\frac{\partial^4 w}{\partial y^4}\right\} + h\sigma_P\frac{\partial^2 w}{\partial x^2} = 0 \qquad (36)$$

For the case dealt with in the preceding (fig. 6) we assume $w = w_0 \cdot \cos(\pi x/a)$; after insertion in (36) we find $h\sigma_P = \pi^2 EJA/a^2$, from which we compute σ_B with equation (35).

We now consider the case where the boundaries $x = \pm a/2$ are simply supported, whilst the boundaries $y = \pm b/2$ are elastically built in, which is the position of the webs of compression members of steel bridges. We first consider the limit case where the boundaries $y = \pm b/2$ are also simply supported. As in the elastic domain [12]) we put

$$w = w_0 \cos(p\pi x/a)\cos(q\pi y/b),$$

after inserting which in (36) we find:

$$h\sigma_P = (\pi^2 EJ/b^2)\{Apb^2/a^2 + 2(B+2F)q^2 + Dq^4a^2/p^2b^2\} \qquad (37)$$

For minimum buckling stress, q must be 1, whilst differentiation shows that p must be $(a/b)\ (D/A)^{\frac{1}{4}}$. Substitution gives:

$$h\sigma_P = (2\pi^2 EJ/b^2)(\sqrt{AD} + B + 2F) \qquad (38)$$

When a/b is greater than 3 to 4, and this practically is always the case, (38) will also apply when $(a/b)\ (D/A)^{\frac{1}{4}}$ is not an integer. If we substitute in (38) the values for A, B, D and F used in (33), which are valid just below the yield point, then (38) yields $h\sigma_P = 3{,}40\ \pi^2 EJ/b^2$.

The buckling force $h\sigma_E$ is known, but naturally follows also from (38), when for A, B, D and F are inserted the values prevailing for the elastic region, where $\tan\varphi = \infty$ and $e = 0$. These values may be calculated with the aid of (21), (22) and (23). A, B and D may also be found directly by comparing (20) with (25) or (34) viz. $A = D = m^2/(m^2-1) = 1{,}099$. $B = m/(m^2-1) = 0{,}329$. F is equal to $m/(2m+2) = 0{,}385$.

Insertion in (38) shows $h\sigma_E = 4{,}40\ \pi^2 EJ/b^2$, so that according to (35)

$$h\sigma_B = (4{,}40/4 + 3 \cdot 3{,}40/4)\pi^2 EJ/b^2 = 3{,}65\,\pi^2 EJ/b^2 \quad \text{or} \quad \sigma_B = 3{,}00\,E(h/b)^2.$$

As slightly below σ_v the reduced modulus of elasticity $T = \lambda^2\sigma_v/\pi^2 = 3600 \cdot 2400/\pi^2$ kg/cm² $= 875\,000$ kg/cm², whilst $E = 2100000$ kg/cm², according to the existing theory [13]) we should have:

$$h\sigma'_B = (T/E)\,h\sigma_E = 1{,}83\,\pi^2 EJ/b^2 \quad \text{or} \quad \sigma'_B = 1{,}50\,E(h/b)^2.$$

Many compression members in bridge engineering have a slenderness ratio $\lambda < 60$, so that according to the German specifications the buckling

always greater than the value $\sigma_E/4 + 3\sigma_P/4$ calculated on account of other considerations, so that this latter value is indeed not too high as a basis for the determination of the admissible stresses. Eq. (35) may therefore be considered to be safe for other cases as well.

[11]) See e. g. Chwalla, Int. Ass. for Bridge and Structural Engineering. 1st. Congress Paris. Final Report, p. 58.

[12]) Timoshenko, Theory of elastic stability (1936). Hartmann, Knickung, Kippung, Beulung (1937).

[13]) Schleicher, Bauingenieur, p. 505 (1934).

stress is equal to the yield stress. It is required that the webs will not buckle before the member buckles as a whole. With webs which may be considered as simply supported and with $\sigma_v = 2400$ kg/cm², the requirement $\sigma'_B = \sigma_v$ leads to the condition $b/h = 36$[14]). According to the theory outlined here, it is sufficient when $\sigma_B = \sigma_v$ and so $b/h = 51{,}2$.

That we find σ_B to be comparatively slightly smaller than σ_E is the result of the considerable resistance to twisting in the plastic region, which expresses itself in the following quantity: $F = 0{,}322$[15]). As has been observed in my preceding publication [3]) we consider the material as a HENCKY-body, since in that case we are sure that the resistance of the material is not over-estimated[16]). When preliminary free deformations or any preliminary plastic deformations are disregarded, consequently when we consider the material as a HOHENEMSER-PRAGER or PRANDTL-REUSS body, then obviously e must be equated to zero, which for the rest does not make much difference as e is small. Thus according to (23) F would be just as great as in the elastic region i. e. $F = 0{,}385$. The results of tests of HOHENEMSER and PRAGER concerning the plastic deformation of hollow tubes of structural steel [17]), as given in their fig. 5, actually indicate that this is the case, although as far as we know, nobody has hitherto drawn the conclusions of this fact with regard to buckling phenomena[18]). The experiments of CHASE, made in connection with the design of the bridge across the Delaware River between Philadelphia and Camden[19]) also give results which are not contradictory to our theory. With his plates σ_v was 3170 KG/cM². Admitting the same values of $\tan\varphi$ and e for this case, the condition $\sigma_B = \sigma_v$ shows that σ_B will attain the yield stress at a ratio $b/h = 45$. In the tests the yield stress is attained at a ratio $b/h = 46{,}8$. Admitting e to be zero in our equations, then $\sigma_B = 3{,}30\ E\,(h/b)^2$, and thus σ_v will be reached at $b/h = 46{,}8$. The exact agreement with the experiments — the existing theory, according to condition $\sigma'_B = \sigma_v$, gives the ratio 31 — is merely accidental, as no experiments were undertaken between $b/h = 46{,}8$ and $b/h = 56{,}0$, at which latter ratio σ_v was no more reached. The plates buckled in 4 or 5 waves. According to our theory σ_B becomes minimum with $m = 4$, according to the existing theory σ'_B with $m = 3$ (but nearly 4).

For other boundary conditions at $y = b/2$ we assume in the same way as in the elastic domain, that $w = Y \cos(p\pi x/a)$, when Y is a function of y, upon which (36) transforms in the ordinary differential equation:

$$D\,d^4 Y/dy^4 - 2(B + 2F)\,\lambda^2 d^2 Y/dy^2 + (A\lambda^2 - \varphi^2)\,\lambda^2 Y = 0 \qquad (39)$$

in which

$$\lambda = p\pi/a \qquad \text{and} \qquad \varphi^2 = h\,\sigma_P/EJ$$

Assuming in (39) $Y = e^{\alpha y}$, we obtain the general solution:

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14) TIMOSHENKO, l. c., p. 406.

15) Compare equation (33).

16) This question has been discussed in detail in my publication in the proceedings of the Royal Netherlands Academy of Sciences, Vol. 41, No. 7 (1938).

17) HOHENEMSER and PRAGER, Zeitschrift für angew. Math. u. Mech., No. 1 (1932).

18) e. g. the resistance to twisting of the boundary angles of bridge-members will near the yield stress not be much less than in the elastic domain and thus also in the plastic domain give some support to the plate.

19) CHASE, Journal of the Franklin Institute, Vol. 200, p. 417. For the discussion of the edge conditions and the ratio b/h to be taken into account, cf. SCHLEICHER, First Congress of the Int. Ass. for Bridge and Structural Engineering, Final Report, p. 123. SCHLEICHER concludes that the edges could practically be considered as simply supported.

$$w = (C_1 \cosh \alpha_1 y + C_2 \sinh \alpha_1 y + C_3 \cos \alpha_2 y + C_4 \sin \alpha_2 y) \cos (p \pi x / a) \quad (40)$$

with:

$$\alpha_{1,2} = \sqrt{\pm G \lambda^2 + \lambda \sqrt{H \lambda^2 + K \varphi^2}} \quad (41)$$

in which:

$$G = \frac{B + 2F}{D}, \quad H = \frac{(B + 2F)^2 - AD}{D^2} \quad \text{and} \quad K = \frac{1}{D} \quad (42)$$

With the values of A, B, D and F introduced in (33), valid just below the yield point, we find:

$$G = 1{,}141, \quad H = 0{,}852 \text{ and } K = 1{,}066 \quad (43)$$

If both sides $y = \pm b/2$ are built in, the symmetry with regard to the X axis requires C_2 and C_4 to be zero. The boundary conditions $w = 0$ and $\partial w / \partial y = 0$ for $y = \pm b/2$ lead of course to the same condition as in the elastic domain, viz. $\alpha_1 \tanh (\alpha_1 b/2) + \alpha_2 \tan (\alpha_2 b/2) = 0$. Combination of this equation with (41) and (43) allows us to determine the value of $h\sigma_P$ for various values of p. The minimum value is:

$$h\sigma_P = 5{,}41\ \pi^2 EJ/b^2 \text{ with } a/p = 0{,}54\ b. \text{ As } h\sigma_E = 7{,}67\ \pi^2 EJ/b^2$$

we find according to (35) $h\sigma_B = 5{,}97\ \pi^2 EJ/b^2$ or $\sigma_B = 4{,}91\ E(h/b)^2$. The condition $\sigma_B = \sigma_v$ yields $b/h = 65{,}5$ (according to the existing theory b/h should be 48).

If one side is simply supported and the other is built in, then the boundary conditions lead to the equation: $\alpha_1 \coth \alpha_1 b - \alpha_2 \coth \alpha_2 b = 0$, from which in connection with (41) und (43) follows: $h\sigma_p = 4{,}38\ \pi^2 EJ/b^2$ with $a/b = 0{,}65\ b$. As $h\sigma_E = 5{,}94\, \pi^2 EJ/b^2$, (35) gives $h\sigma_B = 4{,}77\, \pi^2 EJ/b^2$ or $\sigma_B = 3{,}92\ E\,(h/b)^2$. The condition $\sigma_B = \sigma_v$ leads to: $b/h = 58{,}6$.

For sections as illustrated in Table I the plates are partly built in. The plates, which, if simply supported on the other plates, would have the lowest buckling stress, are called the "buckling" plates, with dimensions b and h; the other ones are called the "resisting" plates, with dimensions b' and h'. With the sections of Table I $b/h > b'/h'$. The calculations which have been in use till now for those sections [20]) are neither exact for the elastic region. Therefore the exact computation will be given here, which holds for the elastic region as well.

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[20]) Bleich, Theorie und Berechnung der eisernen Brücken (1924). Bleich, Int. Ass. for Bridge and Structural Engineering. 1st. Congress. Preliminary Publication p. 107 (1932). Chwalla points out already that Bleich does not consider the varying rotation of angle: Ingenieur-Archiv, p. 62 (1934). The course pursued by Chwalla with respect to the calculation of the building as given by edge angles, whereby he assumes these to be concentrated in the edges of the plate, is however not at once applicable to tubular cross sections. Moreover Chwalla neglects in this way the torsional action of the forces $-h\sigma\,(\partial^2 w/\partial x^2)\,dx\,dy$ given by the compressive stresses acting in the angles on an element $hdxdy$, in consequence of which the supporting action of the angles is over-estimated (on the other hand however the resistance to bending of the flanges of the angles is neglected). Bleich takes approximately the influence of the compressive stresses in the vertical plates (fig. 9 a) into account afterwards, but assumes that the buckling plate (fig. 9 a upper plate) may be considered to be simply supported when $(h^3 b')/(h'^3 b) = 1$ whereas this as a matter of fact will be almost the case when $h/b = h'/b'$, so when $(hb')/(h'b) = 1$. Consequently the formulas of Bleich, especially those in the second publication, procure in various cases values for b/h which are relatively considerably too high. The way in which Bleich reckons in the plastic domain, namely in the direction of compression with T and perpendicular to it with E, seems to be in better agreement with our calculations than the conception that it should be reckoned in all directions with the reduced modulus of elasticity T.

If for a section as given in fig. 9a eq. (40) holds for the upper plate, the "buckling" plate, a similar equation will be valid for the vertical plates, the "resisting" plates, viz.:

$$w' = (C'_1 \cosh \alpha'_1 y' + C'_2 \sinh \alpha'_1 y' + C'_3 \cos \alpha'_2 y' + C'_4 \sin \alpha'_2 y') \cos (p\pi x/a) \quad (44)$$

with:

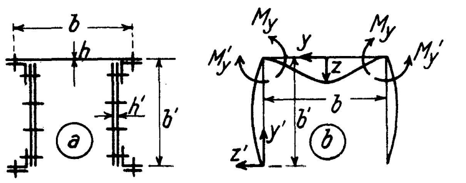

Fig. 9

$$\alpha'_{1,2} = \sqrt{\pm G\lambda^2 + \lambda\sqrt{H\lambda^2 + K\varphi'^2}} \quad (45)$$

in which $\lambda = p\pi/a$ and $\varphi'^2 = h'\sigma_P/EJ'$.

The half-wave length a/p is of course the same as that of the "buckling" plate. We assume that the upper plate, when it buckles, exerts moments $M'_y = M \cos (p\pi x/a)$ on the vertical plates. Considering first the vertical plates, the "resisting" plates, then, since M'_y is determined by (29) and since for $y' = 0$ and $y' = $ b (fig. 9b), $\partial^2 w'/\partial x^2 = 0$, the following boundary conditions will hold:

$w' = 0 \ldots$ (I) and $\partial^2 w'/\partial y'^2 = 0 \ldots$ (II) for $y' = 0$;
$w' = 0 \ldots$ (III) and $-EJ'D\,\delta^2 w'/\delta y'^2 = M \cos (p\pi x/a) \ldots$ (IV) for $y' = b'$.

The first two conditions yield, after insertion of (44): $C'_1 = C'_3 = 0$. The third condition yields:

$$C'_4 = -C'_2 \frac{\sinh \alpha'_1 b'}{\sin \alpha'_2 b'}$$

and the fourth:

$$C'_2 = -\frac{M}{EJ'D(\alpha'^2_1 + \alpha'^2_2) \sinh \alpha'_1 b'}$$

After substitution of these values in (44) we find for the angular distortion for $y' = b'$:

$$-\frac{\partial w'}{\partial y'} = M'_y \psi_a = M'_y \frac{\alpha'_1 \coth \alpha'_1 b' - \alpha'_2 \cot \alpha'_2 b'}{EJ'D(\alpha'^2_1 + \alpha'^2_2)} \quad (46)$$

We consider now the upper plate, the "buckling" plate. The bending moments M_y as indicated in fig. 9b, i. e. as negative moments, are according to (29), since $\partial^2 w/\partial x^2 = 0$ for $y = \pm b/2$: $M_y = + EJD\partial^2 w/\partial y^2$.

As at the same time for $y = \pm b/2$: $M_y = M'_y$ and $\partial w/\partial y = \pm \partial w'/\partial y'$, from equation (46) it follows that for $y = \pm b/2$:

$$\frac{\partial w}{\partial y} \pm \Theta_a \frac{\partial^2 w}{\partial y^2} = 0 \quad (47)$$

in which, since $J/J' = (h/h')^3$:

$$\Theta_a = EJD\,\psi_a = \left(\frac{h}{h'}\right)^3 \frac{\alpha'_1 \coth \alpha'_1 b' - \alpha'_2 \cot \alpha'_2 b'}{\alpha'^2_1 + \alpha'^2_2} \quad (48)$$

Further the condition $w = 0$ holds for $y = \pm b/2$.

Due to the symmetry $C_2 = C_4 = 0$ in eq. (40), applying to the upper plate. The condition (47) and $w = 0$, holding for $y = b/2$, lead then after insertion of (40) to the equations:

$$C_1\{\alpha_1 \sinh(\alpha_1 b/2) + \Theta_a \alpha_1^2 \cosh(\alpha_1 b/2)\} - C_3\{\alpha_2 \sin(\alpha_2 b/2) + \Theta_a \alpha_2^2 \cos(\alpha_2 b/2)\} = 0$$
$$C_1 \cosh(\alpha_1 b/2) + C_2 \cos(\alpha_2 b/2) = 0$$

which yield values for C_1 and C_3 different to zero only when the denominator determinant is zero. This condition gives the buckling condition:

$$\alpha_1 \tanh(\alpha_1 b/2) + \alpha_2 \tan(\alpha_2 b/2) + (\alpha_1^2 + \alpha_2^2)\Theta = 0 \tag{49}$$

in which α_1 and α_2 are given by (41) and Θ by Θ_a in (48).

For the elastic region, consequently for $\tan\varphi = \infty$ and $e = 0$, in (41) and (45): $G = 1$, $H = 0$ and $K = (m^2 - 1)/m^2$.

When h'/h is fairly great, α'_2 in (45) becomes imaginary. The solution of (36) is then:

$$w' = (C_1'' \cosh\alpha_1'' y' + C_2'' \sinh\alpha_1'' y' + C_3'' \cosh\alpha_2'' y' + C_4 \sinh\alpha_2'' y')\cos(p\pi x/a) \tag{50}$$

with:
$$\alpha''_{1,2} = \sqrt{G\lambda^2 \pm \lambda\sqrt{H\lambda^2 + K\varphi'^2}} \tag{51}$$

In the same way as before we find now:

$$\Theta_a = EJD\psi_a = \left(\frac{h}{h'}\right)^3 \frac{\alpha''_1 \coth\alpha''_1 b' - \alpha''_2 \coth\alpha''_2 b'}{\alpha''^2_1 - \alpha''^2_2} \tag{52}$$

which form can also be found directly with (48) by putting there $\alpha'_1 = \alpha''_1$ and $\alpha'_2 = i\alpha''_2$.

So if, as in fig. 9, the "resisting" plate is bent as in fig. 9b or 10a and the "buckling" plate has boundary conditions as in fig. 9b or 11a, the buckling condition is given by (49), in which $\Theta = \Theta_a$ and is given by (48) or (52). Several other combinations are possible. With section 2 in Table I the vertical "resisting" plates are bent as in fig. 10b when the two horizontal plates buckle. Due to symmetry C'_2 and C'_4 will be zero now in the general equation (44) of the "resisting" plates. From the boundary conditions for $y' = b'/2$, viz.:

$$w' = 0 \text{ and } M'_y = -EJ'D\partial^2 w'/\partial y'^2 = M\cos(p\pi x/a),$$

the values of C'_1 and C'_3 are found, and from these $-\partial w'/\partial y' = M'_y\psi_b = M_y\psi_b$ and $\Theta_b = EJD\psi_b$.

We find:

$$\Theta_b = EJD\psi_b = \left(\frac{h}{h'}\right)^3 \frac{\alpha'_1 \tanh(\alpha'_1 b'/_2) + \alpha'_2 \tan(\alpha'_2 b'/_2)}{\alpha'^2_1 + \alpha'^2_2} \tag{53}$$

and if α'_2 is imaginary, by putting $\alpha'_1 = \alpha''_1$ and $\alpha'_2 = i\alpha''_2$:

$$\Theta_b = \left(\frac{h}{h'}\right)^3 \frac{\alpha_1'' \tanh(\alpha_1'' b'/_2) - \alpha_2'' \tanh(\alpha_2'' b'/_2)}{\alpha''^2_1 - \alpha''^2_2} \tag{54}$$

In the case of a T-section, as given in fig. 12a, the "resisting" plate, viz. the horizontal plate, has still other boundary conditions. Let us consider one half of the horizontal plate, which may have a total breadth of $2b'$ and a thickness h'. It is bent as shown in fig. 10c (rotated 90 degrees). At $y' = 0$ we have the boundary conditions:

$$w' = 0 \quad \text{(I)} \qquad \text{and } M'_y = -EJ'D\partial^2 w'/\partial y'^2 = M\cos(p\pi x/a) \quad \text{(II)}$$

At $y' = b'$ the bending moment M'_y, given by equation (29), must be zero. Moreover the shearing forces Q'_y and the twisting moments t'_{yx} must be zero there. The equivalent load p'_y, transmitted from the plate to its support and including the influence of the twisting moments, is at $y' = b'$, if we choose our positive directions in the same way as TIMOSHENKO[21]) did:

[21]) TIMOSHENKO, Theory of elastic stability, p. 295—300.

$$p'_y = -Q'_y - \frac{\partial t'_{yx}}{\partial x} = -\frac{\partial M'_y}{\partial y'} + \frac{\partial t'_{xy}}{\partial x} - \frac{\partial t'_{yx}}{\partial x}$$

Introducing the values of M'_y, t'_{xy} and t'_{yx} of eq. (29), based on the same positive directions, we find for $y' = b'$, as $C = B$:

$$p'_y = EJ\left\{D\frac{\partial^3 w'}{\partial y'^3} + (B + 4F)\frac{\partial^3 w'}{\partial x^2\,\partial y'}\right\} \quad (55)$$

So the conditions $M'_y = 0$ and $p'_y = 0$ lead to the following boundary conditions for $y' = b'$:

$$B\frac{\partial^2 w'}{\partial x^2} + D\frac{\partial^2 w'}{\partial y'^2} = 0 \quad \text{(III)} \qquad \text{and} \qquad D\frac{\partial^3 w'}{\partial y'^3} + (B + 4F)\frac{\partial^3 w'}{\partial x^2\,\partial y'} = 0 \quad \text{(IV)}$$

Introducing the general equation (44) of the "resisting" plate in these four boundary conditions, we find the values of the constants C'_1, C'_2, C'_3 and C'_4 of (44).

The angular distortion $\partial w'/\partial y' = M'_y \psi_c$ for $y' = 0$ can now be computed and from this we find:

$$\Theta_c = EJD\psi_c = \left(\frac{h}{h'}\right)^3 \frac{(\alpha'^2_1 r'^2 - \alpha'^2_2 q'^2)\sinh\alpha'_1 b' \sin\alpha'_2 b' - \alpha'_1\alpha'_2(q'^2 + r'^2)\cosh\alpha'_1 b' \cos\alpha'_2 b' - 2\alpha'_1\alpha'_2 q' r'}{(\alpha'^2_1 + \alpha'^2_2)(\alpha'_1 r'^2 \cosh\alpha'_1 b' \sin\alpha'_2 b' - \alpha'_2 q'^2 \sinh\alpha'_1 b' \cos\alpha'_2 b')} \quad (56)$$

in which $q' = \alpha'^2_1 - B\lambda^2/D$ and $r' = \alpha'^2_2 + B\lambda^2/D$.

If α'_2 becomes imaginary, we have:

$$\Theta_c = EJD\psi_c = \left(\frac{h}{h'}\right)^3 \frac{(\alpha''^2_1 r''^2 + \alpha''^2_2 q''^2)\sinh\alpha''_1 b' \sin\alpha''_2 b' - \alpha''_1\alpha''_2(q''^2 + r''^2)\cosh\alpha''_1 b' \cos\alpha''_2 b' - 2\alpha''_1\alpha''_2 q'' r''}{(\alpha''^2_1 - \alpha''^2_2)(\alpha''_1 r''^2 \cosh\alpha''_1 b' \sin\alpha''_2 b' - \alpha''_2 q''^2 \sinh\alpha''_1 b' \cos\alpha''_2 b')} \quad (57)$$

in which $\alpha''_{1,2}$ is given by (51), $q'' = \alpha''^2_1 - B\lambda^2/D$ and $r'' = -\alpha''^2_2 + B\lambda^2/D$.

If a side of the "buckling" plate is connected with two or more "resisting" plates of the types given in fig. 10a, 10b or 10c, then M_y will be equal to $\Sigma M'_y$. As the bending moments, which give the same angular distortion, are inversely proportional to the angular distortions ψ by the unit of moment, and thus to the coefficients $\Theta = EJD\psi$, it can easily be perceived, that in this case the value Θ_s, which has to be introduced in the buckling condition of the "buckling" plate, is given by the following equation:

$$\frac{1}{\Theta_s} = \sum\frac{1}{\Theta} \quad (58)$$

In the case of a T-section, the upper side of the "buckling" plate (the vertical plate) is e. g. connected to two plates of the type of fig. 10c, so that $\frac{1}{\Theta_s} = \frac{2}{\Theta_c}$ and $\Theta_s = \frac{1}{2}\Theta_c$. In accordance with HARTMANN [22]) we neglect the resistance to angular distortion of the connecting angles.

Also for the "buckling" plates several boundary conditions may occur. With section 4 in Table I the angular distortion is only resisted at the upper side. The plate buckles as shown in fig. 11b (rotated 90 degrees). The boundary conditions are:

$w = 0$ (I) and $\partial^2 w/\partial y^2 = 0$ (II) for $y = 0$.

$w = 0$ (III) and $\partial w/\partial y + \Theta\,\partial^2 w/\partial y^2 = 0$ (IV) for $y = b$.

[22]) HARTMANN, l. c., p. 181.

Introducing in these the general equation (40) of the "buckling" plate, we find the buckling condition:

$$\alpha_1 \coth \alpha_1 b - \alpha_2 \cot \alpha_2 b + (\alpha_1^2 + \alpha_2^2)\,\Theta = 0 \qquad (59)$$

We had better consider now the most general case for the boundary conditions of the "buckling" plate. This is when the plate is supported at both sides $y = 0$ and $y = b$ by "beams" of different flexural rigidities B_1 and B_2 and different cross-sectional areas A_1 and A_2, whilst also the values of Θ are different, viz. Θ_1 and Θ_2 respectively (fig. 11c).

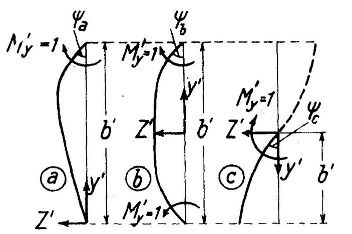

Fig. 10

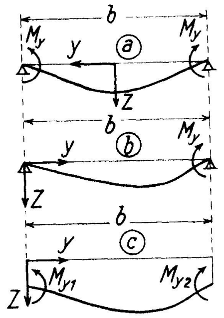

Fig. 11

At $y = 0$ the difference between the shearing forces working on an element dx of the beam will have to be in equilibrium with the load p_y transmitted by the plate and the resultant force of the compression forces $A_1\sigma_p$ working on the element dx, as is given by the equation [23]:

$$B_1 \frac{\partial^4 w}{\partial x^4} = p_y - A_1 \sigma_p \frac{\partial^2 w}{\partial x^2}$$

Proceeding in the same way as with the derivation of eq. (55), we find for $y = 0$:

$$p_y = -EJ\left\{D\frac{\partial^3 w}{\partial y^3} + (B + 4F)\frac{\partial^3 w}{\partial x^2 \partial y}\right\} \qquad (60)$$

so that our first boundary condition for $y = 0$ is:

$$B_1 \frac{\partial^4 w}{\partial x^4} + EJ\left\{D\frac{\partial^3 w}{\partial y^3} + (B + 4F)\frac{\partial^3 w}{\partial x^2 \partial y}\right\} + A_1 \sigma_p \frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{(I)}$$

If the unit of bending moment gives at $y = 0$ an angular distortion ψ_1, we know that there $\partial w/\partial y = M_{y1}\psi_1$.

The negative moment M_{y1} (fig. 11c) is given by eq. (29) by omitting the minus sign. As $C = B$ and $EJD\psi_1 = \Theta_1$ we obtain in this way our second boundary condition for $y = 0$, viz.:

$$\frac{\partial w}{\partial y} - \Theta_1\left(\frac{B}{D}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad \text{(II)}$$

For $y = b$ we obtain the third and the fourth boundary conditions:

$$B_2 \frac{\partial^4 w}{\partial x^4} - EJ\left\{D\frac{\partial^3 w}{\partial y^3} + (B + 4F)\frac{\partial^3 w}{\partial x^2 \partial y}\right\} + A_2 \sigma_p \frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{(III)}$$

[23]) Comp. TIMOSHENKO, l. c., p. 346.

$$\frac{\partial w}{\partial y} + \Theta_2 \left(\frac{B}{D}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad \text{(IV)}$$

Introducing the general solution (40) in these four equations, the following linear homogeneous equations appear:

$$\begin{aligned}
&s_1 C_1 + \alpha_1 r C_2 + s_1 C_3 - \alpha_2 q C_4 = 0\\
&-\Theta_1 q C_1 + \alpha_1 C_2 + \Theta_1 r C_3 + \alpha_2 C_4 = 0\\
&(s_2 \cosh \alpha_1 b - \alpha_1 r \sinh \alpha_1 b) C_1 + (s_2 \sinh \alpha_1 b - \alpha_1 r \cosh \alpha_1 b) C_2\\
&+ C s_2 \cos \alpha_2 b - \alpha_2 q \sin \alpha_2 b) C_3 + (s_2 \sin \alpha_2 b + \alpha_2 q \cos \alpha_2 b) C_4 = 0\\
&(\Theta_2 q \cosh \alpha_1 b + \alpha_1 \sinh \alpha_1 b) C_1 + (\Theta_2 q \sinh \alpha_1 b + \alpha_1 \cosh \alpha_1 b) C_2\\
&- (\Theta_2 r \cos \alpha_2 b + \alpha_2 \sin \alpha_2 b) C_3 - (\Theta_2 r \sin \alpha_2 b - \alpha_2 \cos \alpha_2 b) C_4 = 0
\end{aligned}$$

which give only values of the constants different from zero, if the denominator determinant $\triangle$ of the system is equal to zero.

Thus we get the buckling condition:

$$\begin{aligned}
&[\alpha_1^2 \{s_1 s_2 - \alpha_2^2 t^2 - (\Theta_1 s_2 + \Theta_2 s_1) r^2 + \Theta_1 \Theta_2 r^4\}\\
&-\alpha_2^2 \{s_1 s_2 - (\Theta_1 s_2 + \Theta_2 s_1) q^2 + \Theta_1 \Theta_2 q^4\} + \Theta_1 \Theta_2 s_1 s_2 t^2] \sinh \alpha_1 b \sin \alpha_2 b\\
&-\alpha_1 \alpha_2 [2 s_1 s_2 - (\Theta_1 + \Theta_2)(s_1 + s_2) t^2 + 2(\Theta_1 s_2 + \Theta_2 s_1) qr + 2 \Theta_1 \Theta_2 q^2 r^2] \cosh \alpha_1 b \cos \alpha_2 b\\
&+\alpha_2 t [\alpha_1^2 \{s_1 + s_2 - (\Theta_1 + \Theta_2) r^2\} - (\Theta_1 + \Theta_2) s_1 s_2 + \Theta_1 \Theta_2 (s_1 + s_2) q^2] \sinh \alpha_1 b \cos \alpha_2 b\\
&+\alpha_1 t [\alpha_2^2 \{s_1 + s_2 - (\Theta_1 + \Theta_2) q^2\} + (\Theta_1 + \Theta_2) s_1 s_2 - \Theta_1 \Theta_2 (s_1 + s_2) r^2] \cosh \alpha_1 b \sinh \alpha_2 b\\
&+2 \alpha_1 \alpha_2 [s_1 s_2 + (\Theta_1 s_2 + \Theta_2 s_1) qr + \Theta_1 \Theta_2 q^2 r^2] = 0 \qquad (61)
\end{aligned}$$

in which:

$$q = \alpha_1^2 - B\lambda^2/D,\quad r = \alpha_2^2 + B\lambda^2/D,\quad t = q + r = \alpha_1^2 + \alpha_2^2 = 2\lambda\sqrt{H\lambda^2 + K\varphi^2}$$

and
$$s_{1,2} = \frac{\lambda^2}{EJD}(B_{1,2}\lambda^2 - A_{1,2}\,\sigma_p).$$

In the elastic region, where, as we remarked already, $A = D = m^2/(m^2-1) = 1{,}099$, $B = m/(m^2-1) = 0{,}329$, $F = m/(2m+2) = 0{,}385$, $G = 1$ and $H = 0$, we consequently have:

$$\alpha_{1,2} = \sqrt{\pm \lambda^2 + \lambda\sqrt{\frac{h\,\sigma_E}{N}}},\quad q = \alpha_1^2 - \frac{\lambda^2}{m},\quad r = \alpha_2^2 + \frac{\lambda^2}{m},$$

$$t = 2\lambda\sqrt{\frac{h\,\sigma_E}{N}},\quad s_{1,2} = \frac{\lambda^2}{N}(B_{1,2}\lambda^2 - A_{1,2}\,\sigma_E),\quad N = \frac{m^2}{m^2-1}EJ$$

and in the formulae for Θ:

$$\alpha'_{1,2} = \sqrt{\pm \lambda^2 + \lambda\sqrt{\frac{h\,\sigma_E}{N}}},\quad q' = \alpha_1'^2 - \frac{\lambda^2}{m},\quad r' = \alpha_2'^2 + \frac{\lambda^2}{m}.$$

For the elastic domain (61) can be shown to be identical with the buckling condition derived by CHWALLA [20]) for the section given in fig. 12b, with the difference however that instead of our coefficients Θ in his formulae the coefficients β appear, which with our notations are equal to $\frac{N}{\lambda^2 G I_D}$, in which G is the modulus of rigidity, whilst I_D is assumed to be $\frac{1}{3}\Sigma b' h'^3$. As we remarked already [20]) the unfavourable influence of the compressive forces in the angles is neglected in this way.

With symmetrical boundary conditions of the "buckling" plate, where $B_2 = B_1$, $A_2 = A_1$ and thus $s_2 = s_1$, whilst $\Theta_2 = \Theta_1$, eq. (61) becomes much simpler. With the sections in Table I, where this is the case (1, 2, 3 and 6) the plate will buckle most easily in short half-waves a/p, so that the „beams" get the same deflections as if they were simply supported with a span of a/p, thus being very rigid. We may suppose then $B_1 = B_2 = \infty$ and $s_1 = s_2 = \infty$. As CHWALLA [20]) showed already with his formulae for the elastic domain, the buckling condition for symmetrical cases can be factorisized. Eq. (61) transforms with $\Theta_2 = \Theta_1 = \Theta$ and $s_1 = s_2 = \infty$ in:

$$\{\alpha_1 \tanh(\alpha_1 b/2) + \alpha_2 \tan(\alpha_2 b/2) + \Theta t\}\{\alpha_1 \coth(\alpha_1 b/2) - \alpha_2 \cot(\alpha_2 b/2) + \Theta t\} \cdot \cosh(\alpha_1 b/2)\cos(\alpha_2 b/2)\sinh(\alpha_1 b/2)\sin(\alpha_2 b/2) = 0$$

This does not imply that we get now 4 (or 6) buckling conditions by equating each of these factors to zero. For if e. g. $\cos(\alpha_2 b/2)$ is zero, $\tan(\alpha_2 b/2)$ in the first factor will be infinite and thus the product will not be zero. This will only be the case if the first or second factor is zero, so that we have only two buckling conditions. The first one, which gives the smallest buckling stresses, coincides of course with eq. (49).

With T-sections (fig. 12a) $s_2 = 0$ and $\Theta_2 = \infty$ in eq. (61). If Θ_1 is large, the „buckling" plate will buckle in one wave, so that with slender bars the deflection of the "beam" (the horizontal "resisting" plate), as a result of the load transmitted to it by the "buckling" plate, will diminish the buckling stress rather much. With section 5 in Table I, a single angle, the flanges will not influence each other, so that moreover $\Theta_1 = \infty$, though in the plastic domain this is not quite true.

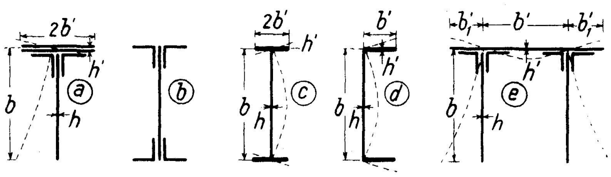

Fig. 12

The buckling condition for sections 2 and 3 in Table I is e. g. given by (49) in which $\Theta = \Theta_b$ according to (53) or (54). For section 4 the condition (59) holds, whilst $\Theta = \Theta_b$ in (53) or (54). As we neglect the angles eq. (38) determines the buckling stress of section 6. The buckling condition for a T-section (fig. 12a) is given by eq. (61) with $s_2 = 0$ and $\Theta_2 = \infty$, whilst Θ_1 is determined by (58) and (56) or (57), being $\frac{1}{2}\Theta_c$. I or U-sections (fig. 12c and d) buckle according to eq. (61) with $s_2 = s_1$ and $\Theta_2 = \Theta_1$, whilst Θ_1 is resp. $\frac{1}{2}\Theta_c$ or Θ_c. For the I or U-sections used in bridge engineering it will be allowed to put $s_2 = s_1 = \infty$, but with the sections hitherto existing buckling will not be possible below the yield stress according to my theory.

For a П-section, if buckling according to fig. 12e, eq. (61) with $s_2 = 0$, $\Theta_2 = \infty$, $s_1 = \infty$ and $\Theta_1 = \Theta_b$ according to eq. (53) or (54) will prevail. It is possible to allow also for the supporting action of the parts b'_1 of the horizontal "resisting" plate, in which case according to (58) $\frac{1}{\Theta_1} = \frac{1}{\Theta_b} + \frac{1}{\Theta_c}$, whilst Θ_c follows from (56) or (57) by replacing b' by b'_1. If the "resisting" plates are thin, the number of half waves p will only be one (or two), in which case the deflection of the "beams" of the "resisting" plate may diminish the buckling stress. We may not then put $w' = 0$ for $y' = \pm b'/2$ (fig. 10b), but Θ_b must be found by replacing this condition by a similar condition as eq. (I) following eq. (60). Allowing for the parts b' is more laborious now. If the vertical plates buckle both in the same direction, in which case the whole section is twisted, the exact solution is still more intricate.

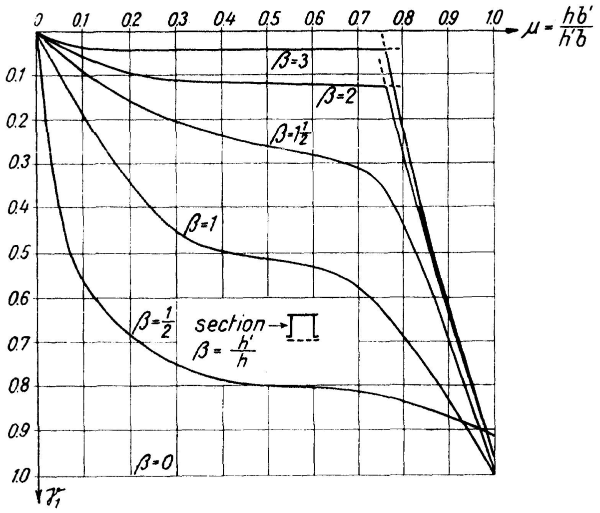

Fig. 13

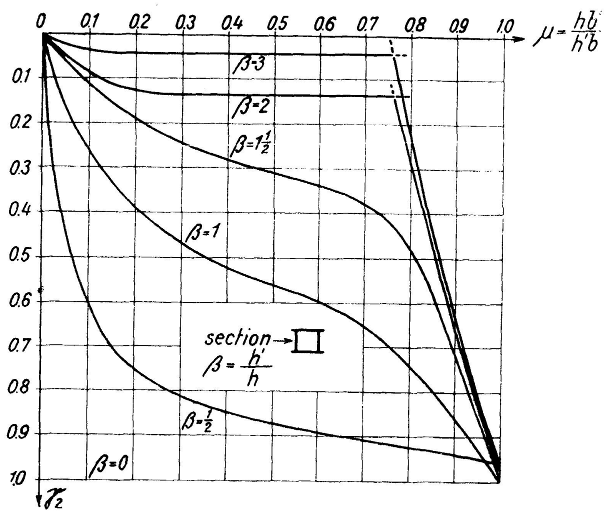

Fig. 14

In order to derive practical formulae, e. g. for section 1 in Table I, we proceed as follows. Eq. (49) with $\Theta = \Theta_a$ according to (48) oder (52) is valid here. With certain values of σ_P, h, h' and of the half-wavelength a/p, we calculate $\alpha'_{1,2}$ or $\alpha''_{1,2}$ from (45) or (51). Assuming also a certain value of b', Θ_a can then be determined. As with the assumed values we know also $\alpha_{1,2}$ from (41), in eq. (49) b is now the only unknown and can be found by trial. By variation of a/p we seek now the most unfavourable value of b/h. By doing so with other values of b' we can find with the assumed values of σ_P and $\beta = h'/h$ the ratio b/h as a function of $\mu = (hb')/(h'b)$. Now we put:

$$\frac{b}{h} = \left(\frac{b}{h}\right)_{FF} - \left\{\left(\frac{b}{h}\right)_{FF} - \left(\frac{b}{h}\right)_{SS}\right\}\gamma_1 \tag{62}$$

$\left(\frac{b}{h}\right)_{FF}$ and $\left(\frac{b}{h}\right)_{SS}$ being the values of b/h corresponding with σ_P if both sides are fixed or simply supported respectively. In this way the curves for γ_1, given in fig. 13, were determined by my assistent Ir. L. F. Cooke for different values of β and a buckling stress in the elastic domain. As can be shown, the same curves are valid for any other buckling stress in the elastic domain. It will be sufficiently accurate to reckon with the same values of γ_1 for buckling stresses in the plastic domain too. The curves for γ_2 in fig. 14, in which also Ir. P. Th. Wijnhamer cooperated, apply to the sections 2 and 3 in Table I. Approximately γ_1 and γ_2 can e. g. for values of $\frac{1}{3} < \beta < 3$ be represented by the equation:

$$\gamma = \mu^n + f \sin 2\pi\mu \tag{63}$$

in which for γ_1:

$n = 0{,}2\,\beta(1+4\,\beta-0{,}15\,\beta^3)$, $f=0{,}1\,(0{,}4+\beta)$ for $\beta<1$ and $f=0{,}07\,(3-\beta)$ for $\beta>1$ (64)

whilst for γ_2:

$n=0{,}1\,\beta(1+8\,\beta-0{,}3\,\beta^3)$, $f=0{,}12\,\beta$ for $\beta<1$ and $f=0{,}06\,(3-\beta)$ for $\beta>1$ (65)

If we demand that the ratio b/h has at the utmost such a value that the critical stress for the plates is equal to the critical stress for the entire strut, then $\left(\frac{b}{h}\right)_{SS}$ and $\left(\frac{b}{h}\right)_{FF}$ can be expressed in the slenderness ratio l/r of the strut. In the preceding we found that to reach the yield stress, $\left(\frac{b}{h}\right)_{SS}$ and $\left(\frac{b}{h}\right)_{FF}$ should be at the utmost 51,2 and 65,5 respectively (till now 36 and 48). So according to the German Specifications these ratios correspond to slenderness ratios $\leqq 60$. In the elastic domain we have e. g. $\left(\frac{b}{h}\right)_{SS} = 0{,}6\,\frac{l}{r}$. Between the proportional limit, 2073 kg/cm², corresponding with $l/r = 100$ and the yield stress, 2400 kg/cm², where l/r is a linear function of the buckling stress, the same will be almost the case with b/h, which ratio is thus also a linear function of l/r. Thus we have e. g. $\left(\frac{b}{h}\right)_{SS} = 37 + 0{,}23\,\frac{l}{r}$ if $60 < \frac{l}{r} < 100$. Insertion of these values in (62) leads to the formulae given in Table I. The coefficient γ_3 for section 4 is not yet determined, but it is sufficiently accurate to put $\gamma_3 = \gamma_2$.

If we would demand with TIMOSHENKO [14]) that the buckling stress of the plates in any case reaches the yield stress, b/h should not exceed the values given in Table I for $l/r \leqq 60$.

The formulae for the single angle we determined in a recent publication [24]), where we allowed for the fact that one flange is only elastically supported by the other, which was done by a new easy method considering the equality of inner and outer bending moment, once for one flange and then for the whole angle. If e. g. a flange is considered as simply supported, as usual, in the elastic domain b/h would have to be $< 0{,}20\ l/r$ [25]) instead of $0{,}18\ l/r$.

According to the tests of KOLLBRUNNER [26]) with angles of soft steel, with a yield stress σ_v = 3316—3334 kg/cm², the angles with $b/h = 16$ buckled practically at the yield stress. With our values $\tan\varphi = 0{,}294\,E$ and $e = 0{,}1675$ we found that according to our theory with these angles the yield stress should be reached if $b/h < 15$ and with $e = 0$ if $b/h = < 16$ [24]). Although the $\sigma-\varepsilon$ diagram of the angles is not known and the proportional limit was rather high, the behaviour of the angles does anyhow not conflict with our theory. The existing theory, supposing $\sigma_B = \frac{T}{E}\,\sigma_E$, would require here $b/h < 10$.

In some cases it is more easy to use the energy method [12]) to compute the buckling stresses. Since the strain energy of bending dV an element $h\,dx\,dy$ is [27]):

[24]) BIJLAARD, De Ingenieur in Ned. Indië, No. 3 (1939) (Dutch).
[25]) BLEICH, Theorie und Berechnung der eisernen Brücken, p. 238.
[26]) KOLLBRUNNER, Mitt. a. d. Institut für Baustatik, E. T. H. Zürich, No. 4 (1935).
[27]) TIMOSHENKO, l. c., p. 306.

$$dV = -\tfrac{1}{2}\left(M_x \partial^2 w/\partial x^2 + M_y \partial^2 w/\partial y^2 - 2\,t_{xy}\,\partial^2 w/\partial x\,\partial y\right) dx\,dy\,;$$

insertion of (29) gives us the strain energy of the entire plate:

$$V=\frac{1}{2}EJ\iint\left\{A\left(\frac{\partial^2 w}{\partial x^2}\right)^2+(B+C)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+D\left(\frac{\partial^2 w}{\partial y^2}\right)^2+4F\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right\}dx\,dy \tag{66}$$

which at the buckling stress has to be equal to the work done by the external forces. In this way we calculated the buckling stress τ_B in the plastic domain of the web of a plate girder subjected to pure shearing (fig. 15) [28]). We must remark that in this case, where the X and Y axes cannot be orientated in the directions of the principal stresses, eq. (20)—(23) have to be transformed. We can find the necessary relations by applying the well known transformation equations to (20)—(23) or directly. It follows that in this case we have to introduce in (20) and (23) and thus in (66):

$$\left.\begin{aligned} A &= D = 4m^2(1+e)/\{4(m^2-1)+4em(2m-1)+3e^2m^2\}\\ B &= C = 2m(2+em)/\{4(m^2-1)+4em(2m-1)+3e^2m^2\}\\ F &= m\tan\varphi/\{3mE+2(m+1)\tan\varphi\} \end{aligned}\right\} \tag{67}$$

It follows that $EF = d\tau/d\gamma$, the tangent of the slope angle of the $\tau-\gamma$ diagram. To compute the buckling stress τ_B in the plastic domain for a plate with infinite length, we used the method of SOUTHWELL and SKAN [29]).

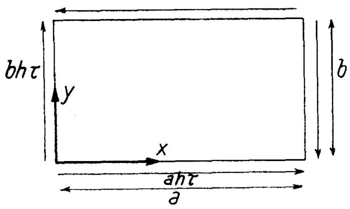

Fig. 15

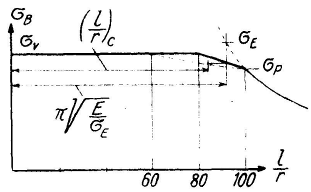

Fig. 16

We found for structural steel No. 37, with $E = 2100$ kg/cm², a proportional limit $2073/\sqrt{3}$ kg/cm² $= 1197$ kg/cm² and a yield stress $2400/\sqrt{3}$ kg/cm² $=$ 1386 kg/cm² and with $a > b$, for the plastic domain:

$$\tau_B = \left\{1197 + 189\,\frac{\left(\frac{b}{h}\right)_E - \frac{b}{h}}{\left(\frac{b}{h}\right)_E - \left(\frac{b}{h}\right)_V}\right\}\ \text{kg/cm}^2 \tag{68}$$

in which: $\left(\frac{b}{h}\right)_E = \sqrt{8456 + 6351\left(\frac{b}{a}\right)^2}$ and $\left(\frac{b}{h}\right)_V = \sqrt{5667 + 3879\left(\frac{b}{a}\right)^2}$

If we find with (68) $\tau_B > 1386$ kg/cm² we should assume $\tau_B = 1386$ kg/cm².

With a square plate the yield stress is still reached if $b/h < 98$ (according to the existing theory with $b/h < 73$).

From the several cases of plates, supported on three or four sides and subjected to compression or shearing, which we examined till now, it can

28) BIJLAARD, De Ingenieur in Ned. Indië, No. 4 (1939) (Dutch).

29) SOUTHWELL and SKAN, Proceedings of the Royal Society of London, Series A. p. 582 (1924).

be deduced that we still remain on the safe side if in the plastic domain we put:

$$\sigma_B = \frac{T_P}{E}\sigma_E \qquad \text{and} \qquad \tau_B = \frac{T_P}{E}\tau_E \tag{69}$$

In this equation σ_E or τ_E is the buckling stress which is computed on the supposition of proportionality of stress and strain. T_P is the reduced modulus of elasticity for a bar at a stress σ_B or a stress $\sigma_{qB} = \tau_B\sqrt{3}$ according to (14), however under the supposition that the relation between buckling stress and slenderness ratio is given by the full line in fig. 16, which reaches the yield stress with a slenderness ratio of 80. Thus for other cases of plates supported on three or four sides it will be possible to find an approximate value of the buckling stress. It is easy to show that for that purpose, with the real relation between buckling stress and slenderness ratio (according to the German Specifications), if the buckling stress is situated in the plastic domain, we have to put the comparable slenderness ratio [30] equal to:

$$\left(\frac{l}{r}\right)_c = 2\pi\sqrt{\frac{E}{\sigma_{qE}}} - 100 \tag{70}$$

If we find $\left(\frac{l}{r}\right)_c < 60$, then we have to put $\left(\frac{l}{r}\right)_c = \frac{3}{4}\pi\sqrt{\frac{E}{\sigma_{qE}}}$ (71)

σ_{qE} is according to (14) equivalent to the state of stress at buckling, computed in the supposition of proportionality of stress and strain.

By means of this comparable slenderness ratio we proceed in the same way as according to the existing theory to determine the real buckling stresses [30]. According to the hitherto existing theory

$$\left(\frac{l}{r}\right)_c = \pi\sqrt{\frac{E}{\sigma_{qE}}}\,. \quad ^{30)}$$

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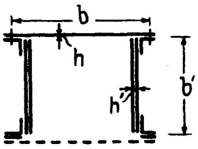
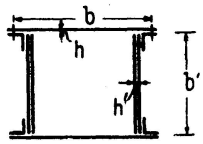
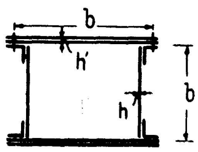
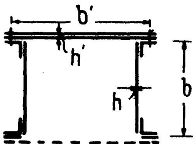
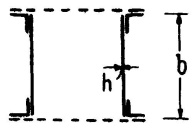
[30]) SCHLEICHER, l.c., footnote 1, p. 131. CHWALLA, Int. Ass. for Bridge and Struct. Eng., 2nd Congress Berlin, Vorbericht, p. 980 (1936).

Table I — Tafel I — Tableau I.

Section Querschnitt Coupe		Formulae for structural steel No. 37 Bemessungsformeln für Baustahl St. 37 Formules pour acier doux No. 37	
① 	$\gamma = \gamma_1$	$\frac{l}{r} \leq 60$	$\frac{b}{h} \leq 65(1 - 0{,}22\gamma)$
② 	$\gamma = \gamma_2$	$60 < \frac{l}{r} < 100$	$\frac{b}{h} \leq 44 + 0{,}36\frac{l}{r} - \left(7 + 0{,}13\frac{l}{r}\right)\gamma$
③ 	$\gamma = \gamma_2$	$\frac{l}{r} \geq 100$	$\frac{b}{h} \leq 0{,}8(1 - 0{,}25\gamma)\frac{l}{r}$
④ 	$\gamma = \gamma_2$	$\frac{l}{r} \leq 60$	$\frac{b}{h} \leq 58(1 - 0{,}12\gamma)$
		$60 < \frac{l}{r} < 100$	$\frac{b}{h} \leq 41 + 0{,}29\frac{l}{r} - \left(4 + 0{,}06\frac{l}{r}\right)\gamma$
		$\frac{l}{r} \geq 100$	$\frac{b}{h} \leq 0{,}7(1 - 0{,}14\gamma)\frac{l}{r}$
⑤ 		$\frac{l}{r} \leq 60$	$\frac{b}{h} \leq 15{,}5$
		$60 < \frac{l}{r} < 100$	$\frac{b}{h} \leq 12 + 0{,}06\frac{l}{r}$
		$\frac{l}{r} \geq 100$	$\frac{b}{h} \leq 0{,}18\frac{l}{r}$
⑥ 		$\frac{l}{r} \leq 60$	$\frac{b}{h} \leq 51$
		$60 < \frac{l}{r} < 100$	$\frac{b}{h} \leq 37 + 0{,}23\frac{l}{r}$
		$\frac{l}{r} \geq 100$	$\frac{b}{h} \leq 0{,}6\frac{l}{r}$

γ_1 and γ_2 have to be determined by interpolation by means of figs. 13 and 14 or approximately with equation (63).

On détermine les valeurs de γ_1 et γ_2 avec les fig. 13 et 14 ou approximativement avec l'équation (63).

γ_1 und γ_2 sind aus den Abb. 13 und 14 zu bestimmen oder angenähert aus Gleichung (63).

$$\frac{b'}{h'} \leq \frac{b}{h}$$

Summary.

Starting from the deformation law of quasi-isotropy and from the plasticity condition of the limited shearing energy, the relations are derived between the extra strains and extra stresses which appear in case of buckling. From these the general differential equation for buckling in the plastic domain is deduced. Due to the fact that in case of buckling the state of stress changes, the resistance to buckling is considerably greater than has been assumed hitherto and is not zero at the yield point. The theory is applied to cross sections of members used in bridge engineering, for which simple formulae are given. In this connection an exact theory for the computation of the buckling stress of such members, valid for the elastic domain as well, is developed.

Zusammenfassung.

Ausgehend vom quasi-isotropen Deformationsgesetz und von der Plastizitätsbedingung der begrenzten Gestaltänderungsarbeit werden die Beziehungen abgeleitet zwischen den bei Beulung auftretenden spezifischen Zusatzdehnungen und Zusatzspannungen. Hieraus wird die allgemeine Differentialgleichung für Beulung im plastischen Gebiet abgeleitet. Da sich beim Ausbeulen der Spannungszustand ändert, ist der Widerstand gegen Beulung viel größer als bis jetzt angenommen wurde und ist an der Fließgrenze nicht gleich Null. Die Theorie wird angewandt auf die im Brückenbau üblichen Stabprofile, für welche einfache Gebrauchsformeln gegeben werden. In diesem Zusammenhang wird auch eine strenge Theorie für die Berechnung der Beulspannungen zusammengesetzter Querschnitte entwickelt, die auch im elastischen Gebiet Gültigkeit hat.

Résumé.

Partant de la loi de déformation de la quasi-isotropie et de la condition de plasticité du travail de transfiguration limité, les relations sont derivées entre les allongements spécifiques et supplémentaires et les contraintes supplémentaires qui apparaissent au flambage. De celles-ci est tirée l'équation différentielle générale pour le flambage dans le domaine plastique. Du fait que l'état de contraintes se modifie au flambage, la résistance au flambage est beaucoup plus grande que nous l'avons admis jusqu'ici et n'est pas nulle aux paliers. Cette théorie est appliquée à des sections employées dans la construction des ponts, pour lesquelles sont données des formules simples. Dans cette étude est également développée une méthode exacte pour le calcul de la contrainte de flambage pour sections valables en même temps dans le domaine élastique.

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