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THEORY OF LOCAL PLASTIC DEFORMATIONS.

THEORIE DER ÖRTLICHEN PLASTISCHEN FORMÄNDERUNGEN.

THÉORIE DES DÉFORMATIONS PLASTIQUES LOCALES.

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1. Resistance of a weakened section of a thin plate.

With respect to the calculation of the resistance of a steel plate between rivets in an oblique row [1]), we have considered the more fundamental theoretical case of a thin plate of sufficient width, showing a weakened strip AA, which embraces an angle β with the normal plane [2]) and may be obtained by a local reduction of the original thickness (fig. 1). Under a certain load minor plastic deformations will occur in the weakened strip. In consequence of this the state of stress in the strip will deviate from that in the elastic domain. The state of stress which will establish itself before the yielding of the section, that is to say before considerable deformations arise, depends, so far as the ratio and the direction of the principal stresses ϱ_1 and ϱ_2 are concerned, on the mechanism of the plastic deformations, so far as their magnitude depends on the plasticity condition.

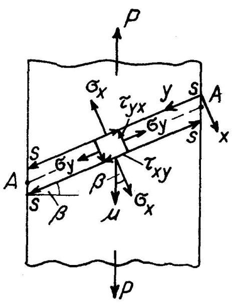

Fig. 1

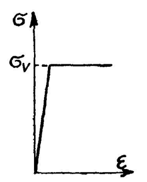

Fig. 2

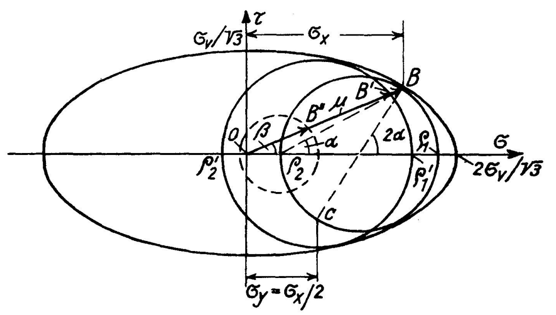

Fig. 3

If e. g. the shearing energy was assumed to be determinant for the appearance of plastic deformations, so that the hypothesis of HUBER-VON MISES-HENCKY is valid, and in this connection, since there is no preference for any plane, that the material behaves quasi-isotropically under the plastic deformation, then it appeared that the state of stress established itself prior to the yielding of the section in such a manner that a maximum resistance is obtained.

To prove this we adopt the $\sigma - \varepsilon$ diagram for structural steel subjected to pure tension according to fig. 2 and compute the envelope of the stress circles $\varrho_1 \varrho_2$ of all the states of stress $\varrho_1 \varrho_2 o$ located at the yield point according to the yield condition of the limited shearing energy. This envelope,

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1) BIJLAARD, De Ingenieur, No. 8 (1931) (Dutch).
2) BIJLAARD, De Ingenieur, No. 37 (1931) (Dutch).

which to the author's knowledge, was introduced for the first time by himself in 1931 [2]), is not identical with the envelope according to MOHR, which comprises the largest stress circles and which is with the yield condition of HUBER-VON MISES-HENCKY an impossibility. If the yield stress in the case of pure tension is denoted by σ_v, the combinations of principal stresses situated at the yield point for a plane state of stress, according to the mentioned yield condition, are given by the equation:

$$\varrho_1^2 - \varrho_1 \varrho_2 + \varrho_2^2 = \sigma_v^2 \tag{1}$$

As to the stress circles $\varrho_1\varrho_2$, the following equation holds:

$$\left(\sigma - \frac{\varrho_1 + \varrho_2}{2}\right)^2 + \tau^2 = \left(\frac{\varrho_1 - \varrho_2}{2}\right)^2 \tag{2}$$

Partial differentiation of (2) with respect to ϱ_1 gives:

$$-\left(\sigma - \frac{\varrho_1 + \varrho_2}{2}\right)\left(1 + \frac{d\varrho_2}{d\varrho_1}\right) = \frac{\varrho_1 - \varrho_2}{2}\left(1 - \frac{d\varrho_2}{d\varrho_1}\right)$$

as from (1) it follows that:

$$\frac{d\varrho_2}{d\varrho_1} = \frac{\varrho_2 - 2\varrho_1}{2\varrho_2 - \varrho_1}$$

Substitution of this in the foregoing equation yields:

$$2(\varrho_1 + \varrho_2) = 3\sigma \tag{3}$$

Elimination of ϱ_1 and ϱ_2 from (1), (2) and (3) yields as equation of the envelope:

$$3\sigma^2 + 12\tau^2 - 4\sigma_v^2 = 0 \tag{4}$$

This is obviously an ellipse with semi-axes $2\sigma_v/\sqrt{3}$ and $\sigma_v/\sqrt{3}$ (fig. 3).

Now in the elastic region the state of stress in the strip, apart from the disturbances at the boundaries A, is represented e. g. by the dotted circle (fig. 3), as the stress in the weakened plane AA is indicated by the point B'' [3]). Thus it follows from the equilibrium that $\angle B''O\sigma = \tau_{xy}/\sigma_x = \tan\beta$ and that $\mu = \sqrt{\sigma_x^2 + \tau^2_{xy}}$ represents the oblique stress on the plane AA. The stress on the plane AA, and only this one stress on that one plane, is consequently determined by the equilibrium, thus statically determined. For the stresses on other planes depend on the forces S, which, in view of the junction of the strip at the unweakened plate portions, are exerted by these on it, in order to keep the elongation of the strip in the direction AA equal to that of the unweakened plate portions.

If now with increasing tensile force P the stress circle touches the envelope, plastic deformations will occur, which are very small and of the same order of magnitude as the elastic deformations. *The state of stress will subsequently change in such a way that prior to the yielding of the section, that is before considerable deformations occur, the oblique stress μ increases up to the envelope and the state of stress is indicated by the circle $\varrho_1\varrho_2$ that touches the envelope at B.* For differentiation

[3]) By a plane is understood in the following, a plane perpendicular to the stressless plane, unless it concerns planes of maximum shearing stress, which, when ϱ_1 and ϱ_2 have the same sign, are not perpendicular to the stressless plane.

of (4) yields for the slope angle of the envelope $d\tau/d\sigma = -\sigma/4\tau$. Thus for the slope angle 2α of the diameter BC of circle $\varrho_1\varrho_2$ we find:

$$\tan 2\alpha = -\frac{d\sigma}{d\tau} = 4\frac{\tau_{xy}}{\sigma_x} = 4\tan\beta \tag{5}$$

Since point C represents the stress on a plane perpendicular to the weakened plane, the normal stress σ_y acting in the longitudinal direction of the weakened strip AA (fig. 1), is consequently equal to:

$$\sigma_y = \sigma_x - \frac{2\tau_{xy}}{\tan 2\alpha} = \sigma_x - \frac{\sigma_x}{2} = \frac{\sigma_x}{2} \tag{6}$$

which relation may also be deducted from eq. (3).

Indeed σ_y should be $\sigma_x/2$ for the following reason:

As compared with the greater plastic deformations, as will be proved below, the elastic deformations may be disregarded. In view of the junction at the not yet plastically deformed unweakened plate portions, the plastic elongation of the strip in the direction AA must remain zero. For quasi-isotropic deformation, as is known, the relation between the plastic deformations and the state of stress prevailing, is the same as between the deformations and stresses for an isotropic elastic material, whereby however instead of the modulus of elasticity E a variable modulus of deformation E_p appears, as the coefficient of lateral contraction [9]) m is equal to 2, because plastical deformations do not involve an alteration of the volume [4]). In order that the plastic strain ε_{yp} of the strip in the direction AA be equal to zero the following condition has to be fulfilled:

$$\varepsilon_{yp} = \frac{\sigma_y}{E_p} - \frac{\sigma_x}{2E_p} = 0 \qquad \text{or} \qquad \sigma_y = \frac{\sigma_x}{2}. \tag{7}$$

According to equation (6) this will be the case when the stress circle in the representative point B of the weakened plane touches the envelope (fig. 3). The state of stress establishes itself in such a manner that a maximum resistance to deformation is obtained, since for any other state of stress for which the stress circle should remain within the envelope, the oblique stress μ would be smaller than OB. As may be understood readily, the same holds when the plate is acted upon by any arbitrary plane state of stress, for it is quite indifferent through what cause the oblique stress is produced; but then $\angle BO\sigma$ will not be equal to β.

2. Resistance of a symmetrically situated weakened section. Direction of yielding.

In case the weakened strip has not the same direction over its whole length, but e. g. is situated symmetrically with respect to the middle of the plate (fig. 4), then a more favourable establishment of the stresses will be possible [1]) [2]). For it is not in conflict with the equilibrium, when the oblique stresses μ deviate over an angle γ from the direction of the tensile force P, so that they make an angle $\beta-\gamma$ with the normal on the weakened plane. Then the oblique stress μ will, according to fig. 5, in which the envelope (4) is drawn, attain the magnitude OB, prior to the yielding of the section, corresponding with the effective oblique stress $OD = \mu_e = \mu\cos\gamma$ in the direction of the tensile force. μ_e will now reach a greatest possible

[4]) Roš and Eichinger, Diskussionsbericht E. M. P. A. Zürich, No. 34 (1929).

value when the normal BD at B touches the envelope. Whereas for a weakened strip according to fig. 1, the envelope (4) is the limit curve for the oblique stress μ, so the pedal curve belonging to (4), dotted line in fig. 5, must be the limit curve for the effective oblique stress μ_e with a weakened strip according to fig. 4.

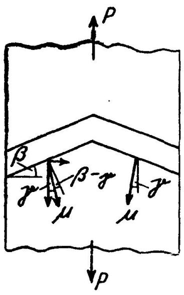

Fig. 4

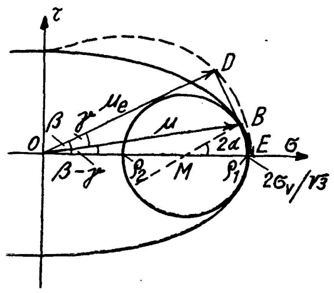

Fig. 5

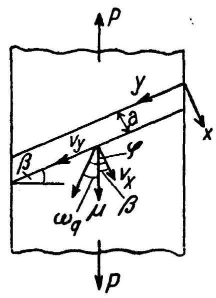

Fig. 6

That this is indeed the case, results from the fact that during the yielding of a weakened strip according to fig. 1, the lower part of the plate moves with respect to the upper part in a direction which we have termed the direction of yielding, and which embraces an angle ω_q with the axis of the plate (fig. 6). The displacement v_x of the lower portion of the plate with respect to the upper portion is namely, also in connection with (6):

$$v_x = a\,\varepsilon_x = a\left(\frac{\sigma_x}{E_p} - \frac{\sigma_y}{2E_p}\right) = \frac{3}{4}\,a\,\frac{\sigma_x}{E_p}$$

The displacement v_y in the Y-direction depends on the plastic modulus of rigidity G_p, which, with $m = 2$, is equal to $G_p = \dfrac{m}{2(m+1)}\,E_p = \frac{1}{3}E_p$ so that, as according to fig. 1, $\tau_{xy} = \sigma_x \tan\beta$:

$$v_y = a\gamma = a\,\frac{\tau_{xy}}{G_p} = 3\,a\,\frac{\sigma_x}{E_p}\tan\beta$$

Hence it follows that (fig. 6):

$$\tan\varphi = v_y / v_x = 4\tan\beta \tag{8}$$

and thus:

$$\tan\omega_q = \tan(\varphi - \beta) = \frac{\tan\varphi - \tan\beta}{1 + \tan\varphi\tan\beta} = \frac{3\tan\beta}{1 + 4\tan^2\beta} \tag{9}$$

For a symmetrical position of the weakened strip however the direction of yielding will ultimately necessarily coincide with the axis of the plate, so that the oblique stress μ will have to deviate with respect to the axis of the plate over such an angle γ that this will indeed be the case. Therefore the angle φ in fig. 6 between the direction of yielding and the normal to the strip will have to be equal to β. According to (5) und (8) $\tan\varphi = \tan 2\alpha$ when α denotes the angle between the plane upon which ϱ_1 acts and the weakened plane, so that 2α has to be equal to β. This is indeed the case when the angle γ increases in such a manner, that μ_e has increased up to the pedal curve of (4), for according to fig. 5, then both MB and OD will be perpendicular to BD, so that $2\alpha = \beta$ [5]).

[5]) This may also be derived in a different way, see litt. footnote 2.

3. Determination of the laws of deformation and of the condition of yielding.

So the stresses will always establish themselves in such a way that a maximum resistance to deformation is developed. The tangent points of the stress circles on the envelope represent always planes in which the lines parallel to the stressless plane show a plastic strain of zero:

$$\varepsilon_{yp} = \sigma_y / E_p - \sigma_x / 2 E_p = \sigma_x / 2 E_p - \sigma_x / 2 E_p = 0 .$$

These planes we have termed the dangerous planes.

The same appeared to be the case when the maximum shearing stress was assumed to be determinant for yielding, that is to say, when the hypothesis due to Coulomb was assumed and in connection with this, that the plastic deformation took place through sliding along the planes of maximum shearing stress [1]).

It should be considered as a universal law of nature, which however has only been proved for the elastic region, viz. the law of minimum strain energy, that the stresses in a body will generally establish themselves in such a manner that a maximum of resistance to the deformation due to external forces arises.

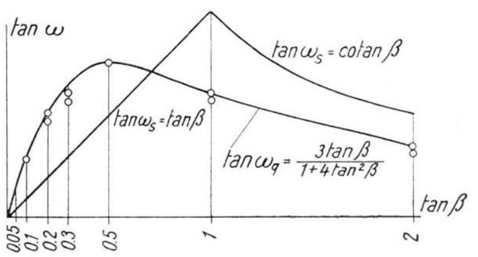

Fig. 7

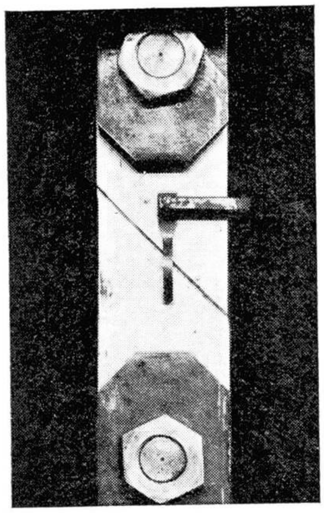

Fig. 8

From the experimental determination of the oblique stress μ it could not be definitively concluded which yield condition was the right one; that according to HUBER-VON MISES-HENCKY procures values which are at most 15,4 % higher than those according to COULOMB. On account of the preceding, if the actual law of deformation should be determined, the conclusion might be drawn directly with regard to the yield condition too. For this purpose a favourable opportunity presented itself, since $\tan \omega_q$ according to (9), through which the direction of yielding for quasi-isotropic deformation is determined, for certain values of β appeared to be over 100 % higher than $\tan \omega_s$ for sliding along the planes of maximum shearing stress. As may be proved readily [1]), for $\beta < 45^0$, in case of sliding along the planes of max. shearing stress through the line AA: $\tan \omega_s = \tan \beta$ and for $\beta > 45^0$, whereby sliding takes place along the weakened plane itself, $\tan \omega_s = \operatorname{cotan} \beta$. In fig. 7 $\tan \omega_q$ and $\tan \omega_s$ have been plotted as functions of $\tan \beta$. With the aid of locally weakened test plates (fig. 8) $\tan\omega$ has been measured by fixing to the upper part of the test bar above the weakened strip a scratch-pen, which on pulling traces a line on the lower part. The values which have been found are indicated in fig. 7 by circles. They are quite in agreement with what may be expected for quasi-isotropic deformation, so that the correctness of this law of deformation may be concluded. At the same time we concluded from this regarding the correctness of the condition of yielding of HUBER-VON MISES-HENCKY, since only in accordance with this condition of yielding a max. resistance is developed[6]).

This may more generally be understood as follows. Since in the weakened strip the same state of stress prevails everywhere and also the same plastic deformations occur, the modulus of deformation E_p for these deformations will be everywhere the same. Since the elastic deformations may be disregarded with respect to the plastic deformations, so that only in the weakened strip work is done, the stresses will establish themselves in the same way as for elastic deformations, consequently in such a manner that the strain energy is a minimum.

However for the plastic deformations, for which $m = 2$, the energy due to the change of volume is zero, so that the total strain energy is exclusively shearing energy and so the stresses will establish themselves in such a manner, that the shearing energy becomes a minimum. Only if the function of the stresses, which becomes a minimum for plastic deformation, and it has been proved that this is the shearing energy, is assumed to be determinant with regard to yielding, the stresses will establish themselves in such a manner that under a certain load as small as possible deformations occur and consequently so that the max. resistance to deformation is developed. At the same time it follows from this, that in that case the oblique stress μ will reach a value as large as possible, consequently will increase up to the envelope, whereby the stress circle in the representative point of the weakened plane will have to touch the envelope. This tangent point of a stress circle on the envelope may always represent such a weakened plane, so that the plastic strain of the lines situated in such a plane and parallel to the stressless plane is always zero.

4. Influence of the elastic deformations.

That in the foregoing considerations the elastic deformations may be disregarded with respect to the plastic deformations will be explained e. g. for the case that $\beta = 0$ [6]) [10]). We adopt a $\sigma - \varepsilon$ diagram according to fig. 2

[6]) BIJLAARD, De Ingenieur, No. 23 (1933) (Dutch). — Also VON MISES established a hypothesis, from which from quasi-isotropic deformation the plasticity condition of HUBER-VON MISES-HENCKY can be concluded: Zeitschrift für angew. Math. u. Mech., No. 3 (1928). The hypotheses of quasi-isotropy and of HUBER-VON MISES-HENCKY are rather well confirmed also by the well-known experiments of ROŠ and EICHINGER on tubes under pressure: Diskussionsbericht E. M. P. A. Zürich, No. 34 and LODE: Forsch.arb. V. D. I. Heft 303, though for the former hypotheses in my opinion, my experiments allow a more definite conclusion, viz. 1 st, due to the considerable discrepancy between the results to be expected in my experiments according to the different hypotheses, leaving no doubt about the correctness of the hypotheses of quasi-isotropy (fig. 7), 2 nd, in consequence of the fact that for tubes the inner and outer strains are not equal, and 3 rd, because with tubes the strains in directions perpendicular to each other have to be measured at different points, so that, due to irregularities, spreading occurs.

and assume further that at any given moment the total deformation and the then prevailing state of stress determine each other reciprocally, that is to say, that the state of stress is independent of the course of deformation along which this total deformation is reached and that the components of the stress deviator are always proportional to those of the deformation deviator. According to HOHENEMSER and PRAGER however mild steel does not behave in this way, like a so-called HENCKY body, as after a free deformation i. e. a deformation during which the ratios of the deviator components do not change, the preceding deformations have no influence on the state of stress[7]).

They found deviations however towards the assumption of PRANDTL and REUSS, according to which the preceding deformations have no influence at all, so that the increase of the plastic deformation is at any moment determined by the then existing state of stress only. For a changing state of stress a PRANDTL-REUSS body will show a higher resistance than a HOHENEMSER-PRAGER body and such a body again a higher resistance than a HENCKY body, so that we are perfectly safe and we may draw a more universally valid conclusion when we conceive the material as a HENCKY body[8]). With a $\sigma-\varepsilon$ diagram according to fig. 2 the preceding free deformations are zero, so that a HOHENEMSER-PRAGER body will behave in the same way as a HENCKY body.

Thus in the weakened strip (fig. 9) the plastic strains are:

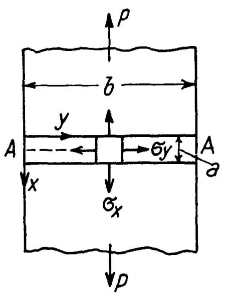

Fig. 9

$$\varepsilon_{xp} = \frac{\sigma_x}{E_p} - \frac{\sigma_y}{2E_p} \tag{10a}$$

and

$$\varepsilon_{yp} = \frac{\sigma_y}{E_p} - \frac{\sigma_x}{2E_p} = \frac{2\sigma_y - \sigma_x}{2\sigma_x - \sigma_y}\,\varepsilon_{xp} \tag{10b}$$

The total strain in the Y-direction, including the elastic strain, is[9]):

$$\varepsilon_y = \varepsilon_{ye} + \varepsilon_{yp} = \frac{\sigma_y}{E} - \frac{\sigma_x}{mE} + \frac{2\sigma_y - \sigma_x}{2\sigma_x - \sigma_y}\,\varepsilon_{xp} \tag{11}$$

So the total strain in the X-direction is:

$$\varepsilon_x = \varepsilon_{xe} + \varepsilon_{xp} = \frac{\sigma_x}{E} - \frac{\sigma_y}{mE} + \frac{2\sigma_x - \sigma_y}{2\sigma_y - \sigma_x}\left(\varepsilon_y + \frac{\sigma_x}{mE} - \frac{\sigma_y}{E}\right) \tag{12}$$

As σ_x and σ_y must continue to satisfy eq. (1) ($\varrho_1 = \sigma_x$, $\varrho_2 = \sigma_y$):

$$\sigma_y = \frac{1}{2}\,\sigma_x \pm \sqrt{\sigma_v^2 - \frac{3}{4}\,\sigma_x^2} \tag{13}$$

In our case σ_y will always be less than $\frac{1}{2}\sigma_x$, so the minus sign in (13) will have to be taken into account. It follows from (12) and (13) that:

$$\varepsilon_x = \frac{(m-2)(3\sigma_x^2 - 2\sigma_v^2 + \sigma_x\sqrt{4\sigma_v^2 - 3\sigma_x^2}) - (3\sigma_x + \sqrt{4\sigma_v^2 - 3\sigma_x^2})\,mE\varepsilon_y}{2mE\sqrt{4\sigma_v^2 - 3\sigma_x^2}} \tag{14}$$

The value ε_y is determined in this case by the shortening in transverse direction of the unweakened plate portions. If the thickness h of the plate in

7) HOHENEMSER and PRAGER, Zeitschr. für angew. Math. und Mech., No. 1 (1932).
8) BIJLAARD, Proceeding Royal Netherlands Academy of Sciences, Vol. 41, No. 7 (1938).
9) $m = 1/r$. r is Poisson's ratio.

the weakened strip has been reduced e. g. to one third, then a stress of $\sigma_x/3$ prevails in the unweakened plate in the X direction. In the Y direction compressive forces $S = \frac{1}{2} \cdot \frac{1}{3} h a \sigma_y$ act in each half of the plate. The strain arising through the forces S depends on the width b of the plate, which should always be assumed to be large with respect to the thickness, and is e. g. as large as with axially loaded plates with a dimension $3a$ in the X direction. So the total strain in the Y direction is in connection with (13) [10].

$$\varepsilon_y = -\frac{\sigma_x}{3mE} - \frac{h a \sigma_y}{18 h a E} = -\frac{(m + 12)\sigma_x - m\sqrt{4\sigma_v^2 - 3\sigma_x^2}}{36 m E} \tag{15}$$

According to (12) and apart from disturbances at the boundaries A, plastic deformations will arise when $\varepsilon_y + \sigma_x/mE - \sigma_y/E$ becomes positive, consequently in connection with (13) and (15), with $m = 10/3$, when $\sigma_x = 1.09\ \sigma_v$. Further according to (14) and (15), when $\sigma_x = 1{,}14\ \sigma_v$ and $1{,}15\ \sigma_v$ respectively, ε_x is $2{,}27\ \sigma_v/E$ and $3{,}87\ \sigma_v/E$ respectively, so with $\sigma_v = 2400$ KG/cM^2: $2{,}6\ ^0/_{00}$ and $4{,}4\ ^0/_{00}$ respectively. When increasing up to the envelope, in this case ($\beta = 0$), according to fig. 3, $\mu = \sigma_x = 2\sigma_v/\sqrt{3} = 1{,}154\ \sigma_v$, so that, when disregarding the elastic deformations, plastic deformations would only arise when $\sigma_x = 1{,}154\ \sigma_v$. As a matter of fact this limit is practically already reached for a strain ε_x of about $5\ ^0/_{00}$, so that the horizontal portion of the $\sigma_x - \varepsilon_x$ diagram, which for a linear stress extends to about $\varepsilon = 15\ ^0/_{00}$, lies as high as when the elastic deformations are disregarded.

Since for linear stress, in reality plastic deformations arise already at the proportional limit, that is below the yield point, it may be stated that due to the elastic deformations only the proportional limit of the weakened strip is reduced slightly, but that the yield point is determined by the oblique stress μ which increases up to the envelope. As may be proved, under the conditions assumed here, minor plastic deformations will arise in general in a weakened strip under an angle β when the oblique stress amounts to:

$$\mu' = \mu \sqrt{\frac{27075(1 + 3\sin^2\beta)}{(4900\tan^4\beta + 100040\tan^2\beta + 30556)\cos^2\beta}}$$

in which μ represents the oblique stress at the yield point of the strip, so the oblique stress which increases up to the envelope.

5. Explanation for the direction of flow lines and necking in thin plates.

Since according to fig. 3: $\sigma = \mu\cos\beta$ and $\tau = \mu\sin\beta$ holds for the envelope, it follows from eq. (4):

$$\mu = \frac{2}{\sqrt{3}} \frac{\sigma_v}{\sqrt{1 + 3\sin^2\beta}} = \frac{2}{\sqrt{3}} \frac{\sigma_v}{\sqrt{4 - 3\cos^2\beta}} \tag{16}$$

If the section of the weakened plane AA be denoted by f_s and its projection on the normal plane by $f_s\cos\beta = f$, then for the resistance of the weakened strip we may write:

[10]) ε_y has not been substituted directly in (14), because eq. (14) has then more universal validity and may be used in connection with our theory on the buckling of plates in the plastic domain, which is published in these publications, just as well. This derivation is also of importance for understanding the phenomena occuring with the plastic deformation in case of non-homogeneous stress distribution.

$$P = f_s \mu = \frac{2}{\sqrt{3}} \frac{1}{\cos \beta \sqrt{4 - 3 \cos^2 \beta}} f \sigma_v \tag{17}$$

This resistance P has been plotted in fig. 10 as a function of β. $f = f_s \cos \beta$ has for any angle β the same value when the weakened strips have the same thickness. P becomes a minimum when $\cos \beta = \sqrt{2/3}$ or $\tan \beta = 1/\sqrt{2}$ and $\beta = 35^0$, as also follows readly from the differentiation of P with respect to $\cos \beta$. P gets then the value $f \sigma_v$. For increasing values of β the weakened strip is obviously stronger than the unweakened plate, which shows a resistance of $p f \sigma_v$ at the utmost, when the thickness of the plate h in the strip is reduced to h/p. In that case the strength of the plate is not reduced by the weakening[11]).

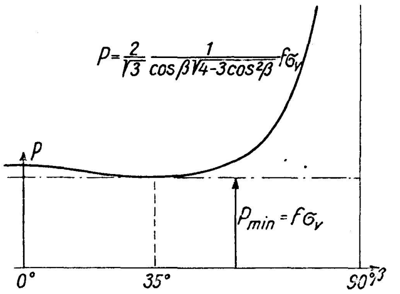

Fig. 10

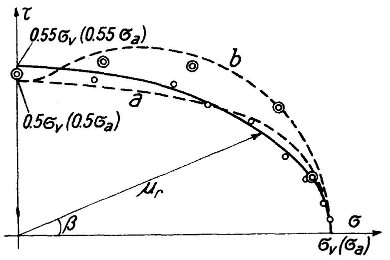

Fig. 11

[11]) In "De Ingenieur", No. 8 (1931), the author has calculated the resistance of a plate between rivet holes in an oblique row, whose normal makes an angle β with the direction of the force, starting from the yield condition of the limited shearing stress. In fig. 11 the dotted line a represents the limit curve which has been determined for the oblique stress μ_r for rivet holes placed in one line; the dotted line b is the pedal curve for a and holds for symmetrically situated rivet holes. The results of the experiments carried out by the authour for single and symmetrically situated sections have been indicated by single and double circles. According to the theory of the limited shearing energy it might be expected that for $\beta = 0$, μ_r would be in excess of σ_v, though smaller than $2\sigma_v/\sqrt{3}$, since the contraction in transverse direction cannot be prevented entirely. Nevertheless it has not been found that μ_r was in excess of σ_v, which should be attributed to the inhomogeneity of the stress distribution in the section. For $\beta = 0$ we may indeed not permit more than the allowable stress σ_a. This comes down to neglecting stresses σ_y acting in the direction of the line connecting the centres of the rivet holes. It is then self-evident to neglect these stresses σ_y (see fig. 1) also for other values of β, so that $\sigma_x = \mu_r \cos \beta$, $\sigma_y = 0$, $\tau_{xy} = -\tau_{yx} = \mu_r \sin \beta$. This stress distribution underlies the graph for such sections published in Waddell Bridge Engineering, Vol. 1, p. 295, whereby however an incorrect yield condition has been started from. According to the yield condition of the constant shearing energy as limit curve for μ_r the ellipse $\sigma_x^2 + 3\tau_{xy}^2 = \sigma_v^2$ holds, which is sometimes used nowadays in computations. As appears from my above-mentioned publication, the stress will not, after plastic deformation, be uniformly distributed over the plane through the line connecting the centres of the rivet holes, which may readily be understood upon examination of the course of the stress-trajectories. Thus we arrive at the limit curve $\sigma^2 + 3{,}3\,\tau^2 = \sigma_v^2$. (The same holds for the theory of the limited shearing stress, but on the strength of our experiments we neglected this and got the curves a und b.) The oblique stress, assumed to be uniformly distributed over the section through the connecting line of the rivet holes, will consequently, since $\sigma = \mu_r \cos \beta$ and $\tau = \mu_r \sin \beta$, not be allowed to be in excess of

$$\mu_r = \frac{\sigma_a}{\sqrt{1 + 2{,}3 \sin^2 \beta}} \tag{18}$$

when σ_a denotes the allowable stress for linear stress. This limit curve $\sigma^2 + 3{,}3\,\tau^2 = \sigma_v^2$ belonging to μ_r has been illustrated in fig. 11 by the full line. For symmetrically

If now an unweakened plate of structural steel is subjected to tension, then the flow lines arising at the yield point and the necking at the point of rupture indicate the places where the material is locally plastically deformed. In these portions of the $\sigma-\varepsilon$ diagram where $d\sigma/d\varepsilon \leqq 0$, local plastic deformations will be possible. For the plate will always show, due to minor irregularities, in some plane less resistance than in all other planes. If such a plane at the yield point gives way a little, then the resistance of this section will not increase, so that further deformations are only confined to this weakest strip. This continues until, due to strain hardening, the resistance of the strip increases again a little and another plane comes to its turn. Various flow lines arise in this way. At the point of rupture where $d\sigma/d\varepsilon = 0$ just as well, an increase in resistance does not any longer occur on further deformation; therefore at that place the local plastic deformation, the so called necking, will be confined to one single section, where finally rupture occurs as well. Since in the yield lines the plastic strain in the longitudinal direction of the flow lines, due to the junction with the not yet plastically deformed material, for a sufficient width of the plates, must be zero, the stresses will try to establish themselves as in a weakened section. As may be seen readily from fig. 10, at the yield point, so for a tensile force of $f\sigma_v$, only a section inclined at 35^0 will be able to deform plastically. In general, according to the preceding, for any arbitrary plane state of stress $\varrho_1\varrho_2$, when the yield point is reached, so eq. (1) is satisfied, so that the relative stress circle $\varrho_1\varrho_2$ touches the envelope, the oblique stress μ' on an arbitrary plane would have to increase first to the envelope to make greater local plastic deformation possible.

However there are two planes for which the oblique stress μ' has already increased to the envelope. These are the planes which show the least resistance to local plastic deformation, and in these planes the flow lines will appear. The necking will later on appear as well here. For these are also the only planes for which the lines coinciding with the plane of the plate show a plastic strain equal to zero, and this should be just the case for the flow lines. From the fact that for these planes according to (6) $\sigma_y = \sigma_x/2$, it follows according to fig. 12 that $\varrho_m = (\varrho_1 + \varrho_2)/2 = 3\sigma_x/4$, so that:

$$\cos 2\alpha = \frac{\sigma_x/4}{(\varrho_1 - \varrho_2)/2} = \frac{\varrho_1 + \varrho_2}{3(\varrho_1 - \varrho_2)},$$

as α denotes the angle between these planes and the plane on which ϱ_1 acts. So:

$$\tan \alpha = \sqrt{\frac{\varrho_1 - 2\varrho_2}{2\varrho_1 - \varrho_2}} \tag{19}$$

So these are also the planes in which a weakening is most dangerous, and we termed these planes the dangerous planes.

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situated sections the pedal curve of this limit curve holds, but for practical use it is more convenient to calculate for any section with (18). Strictly speaking (18) holds alone when failure due to fatigue has not to be feared. If however the plausible assumption is made that the influence of the secondary stresses, that is the difference between the stresses computed according to the plasticity theory and elasticity theory, for values $\beta > 0$ is not greater than for $\beta = 0$, then it may be used as well in that case.

For linear stress $\varrho_2 = 0$, so eq. (19) yields $\tan\alpha = 1/\sqrt{2}$ and $\alpha = 35^0$, as appeared already from fig. 10. For plates for which the width is sufficient with respect to the thickness, e. g. more than 10 to 15 times, the yield lines, the necking and finally rupture as well, occur in these planes (fig. 13) [12]. For plates of aluminium and brass (fig. 13 on the right) likewise necking and fracture occur in these planes.

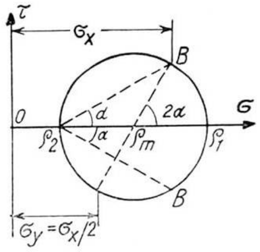

Fig. 12

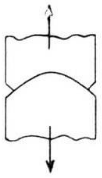

Fig. 14

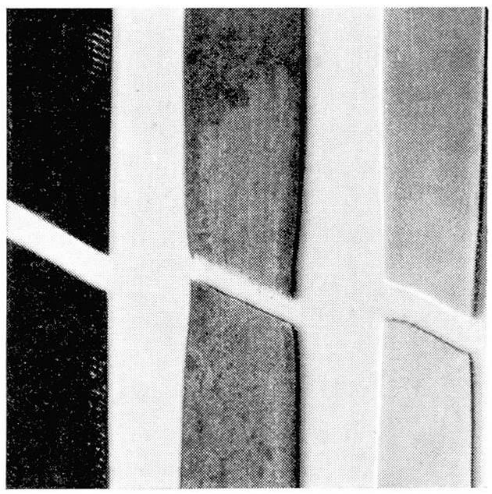

Fig. 13

Also without using the yield condition of the limited shearing energy this may be understood. For relatively thin plates the yielding in the direction of the thickness is not impeded. Only when neither the yielding in longitudinal direction of the yield lines is impeded by the adjacent material, the resistance to yielding in the yield lines will not be in excess of that for unimpeded yielding. This will only be the case when the state of stress $\varrho_1 \varrho_2$ existing in the plate, in the direction of the flow lines, shows already a plastic strain $\varepsilon_{yp} = 0$, so when $\varepsilon_{yp} = \sigma_y/E_p - \sigma_x/2E_p = 0$ and $\sigma_y = \sigma_x/2$, as applies to the planes indicated by the points B in fig. 12.

Hence when considered macroscopically the material does not yield in the direction of the flow lines, so that the yielding may be visualized as the pulling from one another of merely elastically deformed lines, situated in the dangerous planes and parallel to the width of the plate.

If along the sides of the plate sufficiently large grooves are made, situated opposite each other, then a symmetrical necking and accordingly a symmetrical plane of fracture may arise (fig. 14). In publications in which we applied the preceding theory to *the earth's crust* [13]), in which the local plastic deformations (bands of negative gravity anomalies, geosynclines,

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[12]) KÖRBER and SIEBEL, Mitt. Kaiser Wilhelminstitut für Eisenforschung, p. 189 (1928), give an explanation of the angle of rupture, founded on the supposition of two hypothetic sliding planes, which supposition is rather arbitrary — as may be proved, sliding along these sliding planes in general does not take place in the direction of the shearing stress — and does not indicate the real cause of the establishment of a certain angle of rupture.

[13]) BIJLAARD Int. Congress of the geodetic and geophysical Union, Edinburgh (1936), 3rd. Engineering Congress, Tokio (1936). De Ingenieur in Ned. Indië, No. 11 and 12 (1935); No. 4, 7 und 11 (1936).

isles series, chains of mountains) will arise as well in the dangerous planes according to eq. (19), we have explained the origin of such planes of fracture. It should only be observed here that the necking in the middle of the plate coincides with a plane perpendicular to the axis of the plate, which is due to the fact that when the yielding, as in that case, has to take place precisely in the direction of the axis of the plate, such a plane has a smaller resistance than a symmetrically situated oblique section. For in fig. 5 this resistance is equal to $2\sigma_v/\sqrt{3}$ and thus always smaller than the stress $\mu_e/\cos\beta = OE$, which has, if working in the normal plane, the same effect as the effective oblique stress μ_e working in an oblique plane.

If ϱ_1 is the greatest principal stress and $\varrho_2 > \varrho_1/2$, then the circles $\varrho_1 \varrho_2$ will not touch the envelope. Equation (19) gives for this case imaginary values. There is not a single plane in which $\varepsilon_{yp} = 0$. As may be seen readily the plane with the greatest principal stress ϱ_1 is the plane in which μ' has to increase the least to reach the envelope. Flow lines can arise only when ϱ_1 increases first up to $2\sigma_v/\sqrt{3}$, that is to a higher value than is required for unimpeded yielding. So the angle α is here 0^0.

6. Explanation of the flow lines in thick bars.

For bars, for which the thickness is not too small with respect to the width, e. g. bars with a circular or square cross section (fig. 15), the flow lines for tension or compression arise in planes which make an angle of about 45^0 with the axis of the bar, that is approximately in the planes of maximum shearing stress. Therefore it is often assumed that for the so-called yielding in layers [14]), that is the local yielding in the flow lines, the yield condition of COULOMB would be valid. In that case however, the yield lines would for a wide plate make also angles with the normal plane which range in value from 0^0 to 45^0; in reality this angle is however 35^0. It might also be expected that at yielding in the planes of max. shearing stress, in the beginning indeed a sliding along these planes took place. Then at yielding in planes perpendicular to the front surface $ABCD$ of the square bar illustrated in fig. 15, there would be no tendency in a plastically deformed strip EFG ($\beta = \alpha = 45^0$) to become shorter in the direction FG, that is in the Z direction.

In reality a shortening does occur; a cross section according to the plane V_y is illustrated in fig. 15 a [15]). This mode of deformation points to quasi isotropic deformation and thus to the validity of the yield condition of the limited shearing energy.

This may therefore be explained in the same way as for wide and thin plates. Though according to the yield condition of HUBER-VON MISES-HENCKY there are no planes, which as far as plastic deformation is concerned, stand in a special position, it may be proved that *for certain planes the resistance to local plastic deformations is a minimum, and these are again the planes in which the yield lines appear.*

Suppose that, in a bar under tension, local plastic deformations appear in a plane EFG, denoted by V_x, and perpendicular to the front surface V_z (fig. 15). If the thickness of the deformation strip is still small with respect

[14]) LODE, Forschungsarbeiten V. D. I., Heft 303 (1928). — KOLLBRUNNER, Publications Int. Association for Bridge and Structural Engineering, Third Volume (1935).

[15]) NADAI, Der bildsame Zustand der Werkstoffe, Fig. 58 a.

to the lengths EF and FG, then both the plastic strains ε_{yp} and ε_{zp} will have to be equal to zero. Thus

$$\left.\begin{aligned} \varepsilon_{yp} &= \frac{\sigma_y}{E_p} - \frac{\sigma_x}{2E_p} - \frac{\sigma_z}{2E_p} = 0 \\ \varepsilon_{zp} &= \frac{\sigma_z}{E_p} - \frac{\sigma_x}{2E_p} - \frac{\sigma_y}{2E_p} = 0 \end{aligned}\right\} \tag{20}$$

from which follows:

$$\sigma_x = \sigma_y = \sigma_z \tag{21}$$

This state of stress is illustrated in fig. 16. Before considerable plastic deformations appear, the oblique stress

$$\mu = \sqrt{\sigma_x^2 + \tau_{xy}^2}$$

in V_x will have to increase so that the yield condition of HUBER-VON MISES-HENCKY:

$$(\varrho_1 - \varrho_2)^2 + (\varrho_2 - \varrho_3)^2 + (\varrho_3 - \varrho_1)^2 = 2\sigma_v^2 \quad . \tag{22}$$

is satisfied. According to fig. 16:

$$\varrho_1 = \sigma_2 + \tau_{xy}, \qquad \varrho_2 = \sigma_z - \tau_{xy}, \qquad \varrho_3 = \sigma_z$$

from which, after substitution in eq. (22), follows

$$\tau = \tau_{xy} = \sigma_v/\sqrt{3} \tag{23}$$

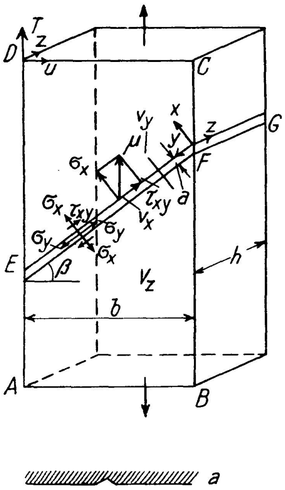

Fig. 15

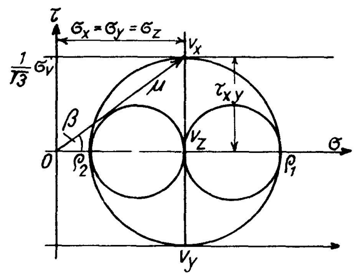

Fig. 16

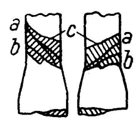

Fig. 17

That this must be, may be seen readily, since the deviator of the state of stress is the same as with pure shearing stress, for which $\tau_v = \sigma_v/\sqrt{3}$ just as well.

Before local plastic deformations appear in any plane V_x, the shearing stress τ in that plane will therefore have to increase up to $\sigma_v/\sqrt{3}$ and the oblique stress μ acting on it will have to increase until it reaches one of the lines $\tau = \pm\sigma_v/\sqrt{3}$ (fig. 16). These lines form consequently the limit curve for μ.

Since it is indifferent whether the oblique stress μ is due to pure tension in the bar or due to any other arbitrary homogeneous state of stress, the preceding holds for any arbitrary state of stress which might act on the bar.

Since according to eq. (23) flow lines appear as soon as the shearing stress in that plane becomes equal to $\sigma_v/\sqrt{3}$, they will therefore appear in the planes with max. shearing stress, consequently in the same planes in which according to the hypothesis of COULOMB the sliding planes would arise.

That e. g. for a thick bar subjected to tension the flow lines will lie in planes inclined at 45^0 to the axis, may be understood as well as follows. In fig. 15 the contraction in the Z direction is impeded anyhow. Thus $\varepsilon_{zp} = 0$. If now the plastic deformation in the strip were not impeded further on, the directions T and U (fig. 15) for the deformation strip would remain directions of principal stress, for which the plastic strains are denoted by ε_{tp} and ε_{up}. Since $\varepsilon_{tp} + \varepsilon_{up} + \varepsilon_{zp} = 0$, it follows that $\varepsilon_{up} = -\varepsilon_{tp}$. The plastic strain ε_{yp} is now:

$$\varepsilon_{yp} = \varepsilon_{up} \cos^2 \beta + \varepsilon_{tp} \cos^2 (90^0 - \beta) = \varepsilon_{tp} (\sin^2 \beta - \cos^2 \beta)$$

The plastic deformation will not any longer be impeded when $\varepsilon_{yp} = 0$ and consequently $\tan \beta = \pm 1$, that is when $\beta = \pm 45^0$.

Obviously the yield lines will in general appear in the planes of max. shearing stress when their width a (fig. 15) is (still) small with respect to the thickness h of the bar, whilst when a is of the same magnitude as h, so that the contraction in the direction of the thickness is almost not impeded (any longer), for a plane state of stress they will appear in the planes according to eq. (19).

Since yielding is impeded only in the direction FG (fig 15), the yield lines will establish themselves in such a way for a bar with rectangular cross section, that this impediment is as small as possible, so such that the horizontal lines FG appear along the narrow sides and so the lines inclined at 45^0 along the wide sides, which is in agreement with the observations. Fig. 17, taken from the book of NADAI[16]), illustrates the course of the yield lines. First of all the dark thin lines a appeared. These lines have consequently according to the preceding to be inclined at 45^0, which they are indeed. Afterwards the lines b appeared, which extend up to the broad strip c, shaded in fig. 17. This will have to be inclined at about 35^0 with the normal plane of the bar, which is indeed the case.

7. Yield conditions for local yielding. Upper and lower yield point.

Instead of the yield condition (1), holding for unimpeded yielding, when the stress circles for unimpeded yielding do not touch the envelope, namely when ϱ_1 and ϱ_2 have the same sign and $2\varrho_1 > \varrho_2 > \varrho_1/2$, for local yielding comes for thin plates the condition: $\varrho_2 < 2\sigma_v/\sqrt{3}$ or $\varrho_1 < 2\sigma_v/\sqrt{3}$. As in fig. 18 the ellipse represents the limit curve for unimpeded yielding, the figure *SABASABAS,* in fig. 19 indicated separately, represents the limit curve for local yielding in the yield lines.

For thick bars, instead of the condition (1) for local yielding, the condition (23) holds, so that therefore the difference of the ultimate principal stresses has to reach the value $2\sigma_v/\sqrt{3}$. The geometric interpretation of this

[16]) NADAI, Der bildsame Zustand der Werkstoffe, Fig. 55.

yield condition in the $\varrho_1\varrho_2\varrho_3$ space is a six-sided prism, that is circumscribed around the circular cylinder, which represents the yield condition (22). For plane states of stress the limit figure is the hexagon *CBCCBC* (fig. 18) circumscribed around the ellipse (1). The limit figure of COULOMB is dotted in fig. 18. For the so-called yielding in layers this latter hypothesis is only apparently valid. In the $\sigma - \tau$ diagram the envelope of the critical stress circles for local yielding for plane states of stress consists of two straight lines $\tau = \pm \sigma_v/\sqrt{3}$ and two semicircles with radius $\sigma_v/\sqrt{3}$ and centre points at $\sigma = \pm \sigma_v \sqrt{3}$ respectively (fig. 20). The yield lines in the stressless plane of the bar, being the lines of transition of it with the planes of max. shearing stress, occur here as well in the planes, indicated by the tangent points of the circles $\varrho_1\varrho_2$ on the envelope.

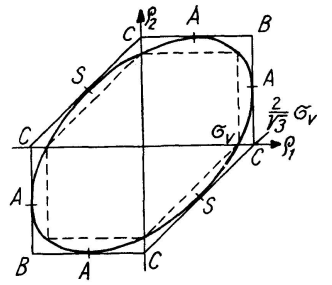
Fig. 18

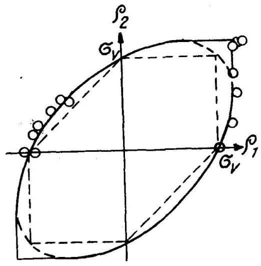
Fig. 19

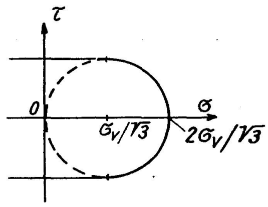
Fig. 20

The stress required for the appearance of the first yield lines is the upper yield point; the stress to which the yield stress decreases on its further spreading, whereby the contraction is practically not impeded any longer, is the lower yield point. The upper yield point and probably also the elastic limit is consequently determined by the limit figures for local yielding, the lower yield point consequently by those for unimpeded yielding.

The limit figures for local yielding are only of a qualitative value, since e. g. the hexagon *CBCCBC* in fig. 18 will hold completely only for very thick bars, in which very thin yield lines appear at first. Moreover, as to the difference between upper and lower yield point, other factors play a part as well, such as the speed of testing and such like[17]).

According to fig. 18 e. g., for thick bars subjected to tension, a difference between upper and lower yield stress will appear, whereas for thin plates this is not the case. We found this proved by experiments carried out by ourselves. For bars 24×24 mM² in size we found a difference between upper and lower yield stress; with bars 24×3 mM² and $24 \times 1{,}5$ mM², cut from the same material, we found no difference. The tests of BACH[18]), indicated in fig. 21 (each value is an average of three) show clearly that the difference between upper and lower yield stress is smaller if the ratio between the thickness and the width of the bars is smaller. KÖRBER and POMP found e. g. for the difference between upper and lower yield point for bars of 18×18 mM², $25 \times 12{,}5$ mM² and 36×9 mM² in size resp. max. 400 KG/cM², 250 KG/cM² and 160 KG/cM²[19]).

[17]) KÖRBER, Congress for testing materials. Amsterdam (1927).
[18]) BACH, Elastizität und Festigkeit, p. 164, Fig. 22.
[19]) KÖRBER and POMP, Mitt. Kaiser Wilhelminstitut für Eisenforschung (1934).

When, after the yield point is reached, the first flow lines do not continue to spread out, additional flow lines can appear after a small increase in stress, after which the stress decreases again, which may elucidate the fluctuations in the $\sigma - \varepsilon$ diagram at the yield point [20]).

According to fig. 18 and 19 the limit figure *SABASABAS* will hold for thin plates for the upper yield point and for the proportional limit. Experiments carried out with soft steel tubes, which refer to the proportional limit, have been taken by BECKER [21]).

They have been indicated in fig. 19 by small circles. They are very satisfactorily approximated by the new limit figure.

8. Apparent raising of the yield point with non-homogeneous stress distribution.

For a non-homogeneous stress distribution, since plastic deformation appears at first in a small region, the yielding will be impeded for rather thin plates as well. Moreover in general the yield lines will not be able to arise in the planes showing the least resistance to local plastic deformation, so that in the other direction as well plastic deformation is impeded (as e. g. for the bar of fig. 15 the yielding in the direction EF would be impeded also if β were not 45^0). Also the parts of the plate in which the stresses have not yet reached the yield point will prevent a relative motion of the portions situated on both sides of the flow lines in the direction of the yielding; in consequence of this the forces establish themselves in another way, as for instance was the case with the symmetrically situated sections in fig. 4,

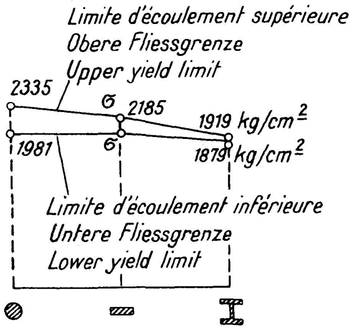

Fig. 21

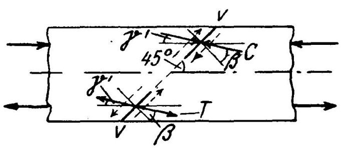

Fig. 22

through which the resistance to local yielding is increased. It may be understood now why the flow lines, i. e. larger plastic deformations, appear only after the maximum stress, calculated according to the theory of elasticity, is considerably beyond the yield point, whereby of course the fact that beyond the proportional limit the actual max. stress is already lower than the stress calculated in this way, plays also a part.

Before the appearance of flow lines the material yields indeed, but very slightly only. So in this respect we agree with RINAGL [22]). In consequence of those minor plastic deformations however, another state of stress is created

[20]) FRÉMONT (Le Génie Civil, No. 8, Tome 82 — 1923) gives another explanation of the fluctuations in the $\sigma - \varepsilon$ diagram at the yield point, which however, as we have proved (De Ingenieur in Ned. Indië, No. 8, 1936), is in conflict with experience.

[21]) BECKER, University of Illinois. Bulletin No. 85 (1916).

[22]) RINAGL, Bauingenieur No. 41/42 (1936).

by the forces, exerted by the elastically deformed portion, so that for a progressing yielding higher stresses are required. Also when the max. stress does not appear locally, such as for a rectangular girder subjected to pure bending, the resistance to local yielding will neverthless be increased considerably. The flow lines appear here in planes V inclined at 45^0 with the centerline of the bar. (fig. 22, see also fig. 15). If a free relative motion of the bounding portions would be possible, these flow lines, according to the preceding, would require a linear stress $\sigma = 2\sigma_v/\sqrt{3}$ in the extreme fibres of the girder. A relative motion of the bounding parts in the direction of yielding will however be impeded here as well [23]), even when the flow line ran from the upper to the lower side, because the upper and lower halves of the girder would have to move in opposite directions (the dotted arrows in fig. 22).

Since such a motion is not possible, the forces C and T will rotate through an angle γ', in such another way as has been proved for the symmetrically weakened section in fig. 4, through which, as may be derived from fig. 16, because $\beta < 45^0$, the oblique stress μ required for local yielding is increased, as well as the normal stress on the plane V. The moment required for local yielding will consequently be considerably in excess of $W\ (2\sigma_v/\sqrt{3})$, as W denotes the resisting moment $bh^2/6$, which is also in agreement with the experiments.

In this way all phenomena connected with local yielding may be explained by means of the yield condition of the limited shearing energy.

Summary.

Starting from quasi-isotropic plastic deformation and the yield condition of the limited shearing energy, the state of stress establishes itself in a weakened oblique section of a wide plate in such a manner that the oblique stress μ increases up to the envelope of the critical stress circles $\varrho_1\varrho_2$. In symmetrically situated sections the effective oblique stress μ_e increases up to the pedal curve of the envelope, because yielding in the so-called direction of yielding is impeded. From the experimental determination of the yield direction, the correctness of the deformation law of quasi-isotropy is concluded, and from this the correctness of the yield condition of the limited shearing energy, due to HUBER-VON MISES-HENCKY. The points of tangency of the circles $\varrho_1\varrho_2$ on the envelope, whose lines of transition with the wide surface of the plate show a plastic strain equal to zero, the so-called dangerous planes, are the planes with the least resistance to local plastic deformation. In these planes both the flow lines and necking appear. In thick bars the flow lines appear just as well in the planes, which according to the yield condition of the limited shearing energy, show the least resistance to local plastic deformation. The difference between upper and lower yield point is explained. The apparent raising of the yield point in case of non-homogeneous states of stress is explained just as well with the yield condition of the limited shearing energy. The theory forms an introduction to our theory of the plastic buckling of plates which is based on this plasticity condition too.

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[23]) Also FRITSCHE (Stahlbau, No. 7/8, 1938) points out, that in a bent girder unimpeded yielding at the outset is not possible. His conclusions are however utterly dissimilar from ours. — The impossibility of sliding in certain positions of the shearing planes, has been pointed out by the author in connection with the resistance of weakened oblique sections (De Ingenieur, No. 8, 1931).

Zusammenfassung.

Ausgehend von quasi-isotroper plastischer Deformation und von der Fließbedingung der begrenzten Gestaltänderungsarbeit stellt der Spannungszustand in einem abgeschwächten Schnitt einer breiten Platte sich so ein, daß die schiefe Spannung μ anwächst bis zur Umhüllenden der kritischen Spannungskreise $\varrho_1\varrho_2$. Bei symmetrisch angeordneten Schnitten wächst die effektive schiefe Spannung μ_e an bis an die Pedalkurve der Umhüllenden, weil ein Fließen in der sogenannten Fließrichtung dort verhindert wird. Aus der experimentellen Bestimmung der Fließrichtung wird geschlossen auf die Richtigkeit der Deformationshypothese der Quasi-Isotropie und daraus auf die Richtigkeit der Fließbedingung der begrenzten Gestaltänderungsarbeit von Huber-von Mises-Hencky. Die den Berührungspunkten der Kreise $\varrho_1\varrho_2$ auf der Umhüllenden zugeordneten Schnitte, deren Schnittlinien mit der breiten Fläche der Platte die plastische Dehnung Null zeigen, die sogenannten gefährlichen Schnitte, sind die Ebenen mit dem geringsten Widerstand gegen örtliche plastische Deformation. Darin erscheinen auch die Fließlinien und die Einschnürung. Auch bei dicken Stäben treten die Fließlinien in Ebenen auf, welche, der Fließbedingung der begrenzten Gestaltänderungsarbeit nach, den geringsten Widerstand gegen örtliche plastische Deformation haben. Die Differenz zwischen oberer und unterer Fließgrenze wird erläutert. Die scheinbare Erhöhung der Fließgrenze bei nichthomogenen Spannungszuständen wird auch mit der Fließbedingung der begrenzten Gestaltänderungsarbeit erklärt. Die Theorie bildet eine Einführung zu unserer gleichfalls auf dieser Plastizitätsbedingung beruhenden Theorie der plastischen Beulung von Platten.

Résumé.

Partant d'une déformation plastique quasi-isotrope et de la condition pour la déformation plastique du travail de transfiguration limité, le système de contraintes dans une section oblique affaiblie d'une plaque mince, avant que la section cède, s'établit de sorte que la contrainte oblique μ croît jusqu'à l'enveloppe des cercles de contraintes $\varrho_1\varrho_2$ critiques. Pour des sections situées symétriquement, la contrainte effective oblique μ_e croît jusqu'à la courbe pédale de l'enveloppe, parce qu'une déformation plastique dans la direction de la déformation plastique est empêchée. De la détermination expérimentale de la direction de la déformation plastique nous concluons à la justesse de la loi de déformation quasi-isotrope et de celle-ci à la justesse de la condition de déformation plastique du travail de transfiguration limité de Huber-von Mises-Hencky. Les plans indiqués par les points de tangence des cercles $\varrho_1\varrho_2$ à l'enveloppe, dont les lignes d'intersection avec la surface large de la plaque montrent un allongement plastique égal à zéro, c'est-à-dire les plans dangereux, sont les plans dont la résistance est minimum par rapport à la déformation plastique et locale. C'est dans ces plans que les lignes de Hartmann et la striction apparaissent. Aussi pour des barres épaisses les lignes de Hartmann paraissent dans les plans qui, suivant la condition du travail de transfiguration limité montrent une résistance minimum par rapport à la déformation plastique et locale. La différence entre les limites d'écoulement inférieures et supérieures est expliquée. L'élévation apparente du palier en cas de systèmes de contraintes non-homogènes peut également être élucidée par la condition du travail de transfiguration limité. Cette théorie forme une introduction à notre théorie du flambage plastique de plaques, qui a aussi comme base la même condition de déformation plastique.