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ELASTIC STABILITY OF A PONY TRUSS

STABILITÉ ÉLASTIQUE DES PETITS PONTS À POUTRES EN TREILLIS

ELASTISCHE STABILITÄT VON KLEINEN FACHWERKBRÜCKEN

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General presentation of the problem.

The strength of engineering structures and parts of structure is sometimes governed not by the stresses in the members, but by their elastic stability. Thus, columns, struts and webs of compressed members can fail not on account of an excessive unit stress but on account of their slenderness, causing collapse of the member when the stress is comparatively low.

The problem of elastic stability of a half-through truss belongs to the same class. It has been treated by several authors, and though not completely solved theoretically, some definite advance has been made toward its solution. Professor Timoshenko in Volume 94 of the "Transactions of the American Society of Civil Engineers" develops formulas and presents tables of coefficients to be used in actual design of a pony truss with parallel chords, similar to the one shown on fig. 1. The author of this paper attempts to solve the same problem, but in the course of his discussion he takes into consideration certain factors left out of question in the paper just mentioned, a circumstance due to which the results of this analysis are generally somewhat different from those of Professor Timoshenko's.

The critical load producing collapse of the structure will be thought of as applied all at the bottom chord, and may be either in the form of uniform load covering the whole span or in the form of concentrated weights. Material will be considered as perfectly elastic in all parts of the structure under the critical load. The structure studied will be like the one on fig. 1; its top chord has constant cross section, the end verticals are absolutely rigid, and all the intermediate verticals are of the same cross section constant along their length, and are rigidly fixed at the bottom ends. Some additional assumptions will be made later in the course of the discussion.

Let fig. 2 represent the plan view of a deformed pony truss. Here ADB is the straight bottom chord, and ACB the buckled top chord; the intermediate verticals appear on this view as HG, FE etc., and the diagonals as GF, ED etc. The ordinates of the buckled top chord, such as δ, are considered infinitesimals of the first order, then the longitudinal displacements of the points of the top chord become infinitesimals of the second order and are not indicated on this sketch. (Thus, the points A and B represent the extremities of both the top and the bottom chords.)

The forces acting on the top chord at each panel point, such as the point E, are two in number, the force of the diagonal and the force of the vertical;

the first can be resolved into P and P_1 acting horizontally, and P_2 acting vertically; the second resolves into the horizontal force S (the bending resistance of the vertical), and the vertical force V, which cancels P_2. This leaves only P, P_1 and S to act on the top chord at the panel point E. P_1 is the force producing buckling, and the sum $P + S = F$ is the total resisting force, opposing the buckling. In most cases in addition to the forces discussed

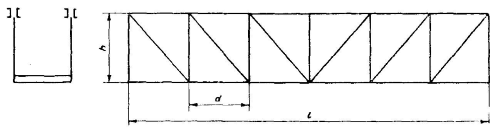


Fig. 1.

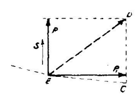


Fig. 2a.

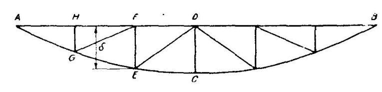
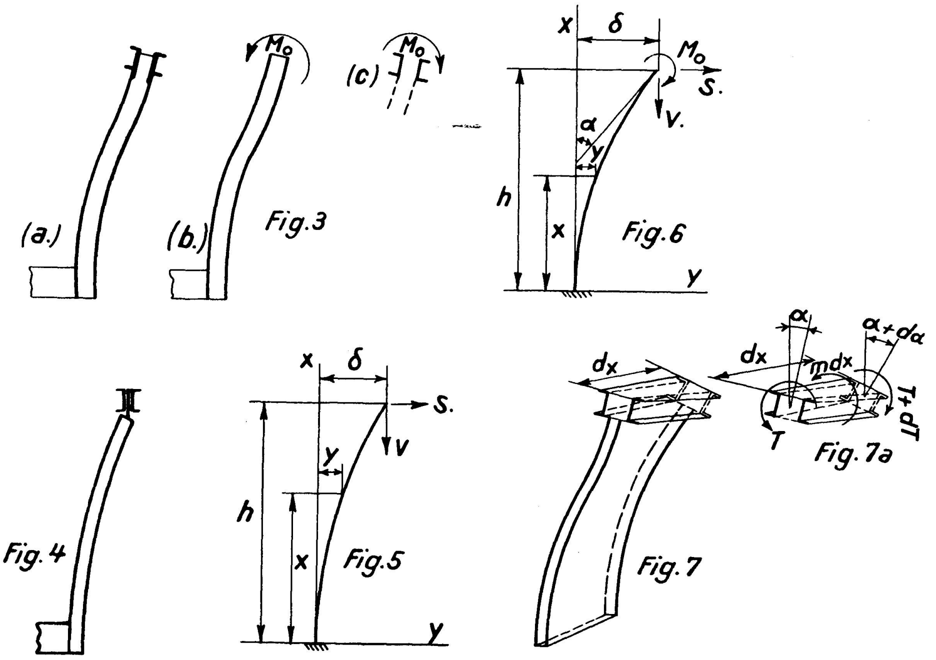


Fig. 2.

already, there is a moment M_0, applied to the top chord at each panel point by the deflected vertical, as is evident from fig. 3; this moment affects bending of the vertical and produces torsion of the top chord. Angular continuity between the vertical and the top chord has great stabilizing effect, both, on the structure as a whole, and on the individual vertical as a strut, and cannot be neglected without a substantial error. Only in rare cases of chords with single webs and no provision for continuity (fig. 4), the moment at the top end of the vertical is absent. It is needless to say that the moment M_0 affects



the magnitude of the resisting force of the vertical S, and that both M_0 and S are affected by the direct stress in the vertical. In the process of buckling of the top chord, the forces here mentioned S, P, P_1 and the moment M_0 do work on the top chord, which is stored in the latter as potential energy of bending (buckling) and twist. The value of the critical load is found from the equation expressing mathematically this statement, after assumption of a suitable shape for the buckled chord.

With this qualitative discussion in view, the quantitative study of the subject will be undertaken.

Lateral resistance of a pony truss vertical deflected a distance δ, with no bending moment at the top end.

Such a vertical (fig. 5) satisfies the conditions of fig. 4, where there is no angular continuity between the top chord and the vertical. The distance δ is determined by buckling of the top chord; the direct stress in the vertical V is supposed to be known, and it is required to find the force S.

The bottom end of the vertical will be considered fixed in vertical position. The errors involved in this assumption will be discussed later.

Differential equation of the elastic curve is:

$$-V(\delta - y) - S(h - x) = -EI_v \frac{d^2 y}{dx^2} \tag{1}$$

where E is the Young's modulus and I_v is the constant moment of inertia of the vertical for bending out of the plane of the truss. Calling $\sqrt{\frac{V}{EI_v}}$ by u the equation (1) reduces to

$$\frac{d^2 y}{dx^2} + u^2 y = u^2 \frac{S}{V}(h - x) + u^2 \delta \tag{2}$$

whose general solution is

$$y = A \operatorname{Cos} ux + B \operatorname{Sin} ux + \frac{S}{V}(h - x) + \delta \tag{3}$$

The constants of integration A and B, as well as the force S, are found from the following three conditions:

$$x = 0, \quad y = 0,$$
$$x = 0, \quad \frac{dy}{dx} = 0,$$
$$x = h, \quad y = \delta,$$

These give for S the following expression:

$$S = \frac{uV}{\tan uh - uh}\delta. \tag{4}$$

This relation shows proportionality between S and δ, the coefficient of proportionality varying inversely with V (u being a function of V). When $uh = \sqrt{\frac{V}{EI_v}}\,h$ increases to $\frac{\pi}{2}$, S becomes zero, i. e. no force is required to deflect the strut, and still further increase in V makes S negative, which means that the strut not only does not resist deflection, but requires some support on the part of the top chord and diagonal to prevent its collapse.

Collapsing load $V_{cr.}$ for lateral buckling corresponds to zero value of the denominator in eq. (4)

$$\tan \cdot uh - uh = 0, \quad \text{which gives}$$

$$(uh) = 4.49, \quad \text{and} \quad V_{cr} = u_{cr}^2 EI_v = \frac{20.2\, EI_v}{h^2}. \tag{5}$$

It is worthy of notice that the critical value of V is independent of the amount of deflection of the top end δ. It is true that δ determines the magnitude of the lateral force S, but the value of V for which S becomes infinite, and the whole system unstable, bears no relation to δ.

Returning to fig. 2 a, the total resisting force $F = S + P$, where P is the lateral horizontal component of the stress in the diagonal. Since the vertical component of the stress in the diagonal is V,

$$P = \frac{\delta}{h} V, \tag{6}$$

and
$$F = \frac{V}{h}\delta + \frac{u V}{\tan uh - uh}\delta = \frac{\tan uh}{\tan uh - uh}\frac{V}{h}\delta. \tag{7}$$

This equation shows that F is proportional to δ, and decreases with increase of V.

For $V = 0$, $\quad F = \frac{3EI_v}{h^3}\delta$

for $(uh) = \frac{\pi}{2}$, or $V = \frac{\pi^2}{4}\frac{EI_v}{h^2}$, $\quad F = \frac{\pi^2}{4}\frac{EI_v}{h^3}\delta = 2.47\frac{EI_v}{h^3}\delta$

for $(uh) = \pi$, or $V' = \frac{\pi^2 EI_v}{h^2}$ (7a), $\quad F = 0$.

As V increases above this value F becomes negative, i. e. the diagonal in combination with the vertical cease their resisting action on the chord and begin to exert a deflecting force on it.

It must be mentioned at this point that the ability of the vertical to withstand a compressive stress up to the magnitude $V_{cr.}$, given by the equation (5), is predicated on adequate support on the part of the top chord, otherwise the collapse of the vertical will occur for the value of V somewhere between the expressions (7 a) and (5). Thus, should the end verticals be absolutely rigid, any of the intermediate ones can stand a compression up to $V_{cr.} = \frac{20.2 EI_v}{h^2}$; on the other hand, with the end verticals not absolutely rigid, failure will inevitably result should all the verticals, including the end ones, be stressed to the values a little above their respective $V' = \frac{\pi^2 EI_v}{h^2}$.

This brings up for consideration one of the possible ways in which failure of a pony truss on account of elastic instability may occur. When some of the verticals are stressed below their V' and others above it, there is a possibility of such failure.

However, this possibility is purely theoretical. All the practical column formulas used in design of truss verticals are based on the value of collapsing stress not over $\frac{\pi^2 EI_v}{h^2}$, allowing a suitable factor of safety, and since it is not the object of this paper to discuss the adequacy of compression formulas, and since in actual pony trusses it is the verticals that support the top chord and not vice versa, the mode of elastic failure owing to failure of individual verticals, as outlined above, is dismissed as practically impossible.

As is evident from the examples given, it is possible to write generally

$$F = b_1 \frac{EI_v}{h^3}\delta, \tag{8}$$

where b_1 depends solely on uh and can be calculated from the eq. (7). The diagram 1 of the factor b_1 in terms of uh has been plotted to facilitate the calculations.

Though the type of the vertical considered in this chapter may be treated as a special case of the vertical with bending moment on the end, it was thought advisable to bring this case up independently in order to point out the difference in individual stability of the two kinds of verticals and to explain on a more simple case the supporting influence of the top chord. In the following developments, however, this type will be referred to only occasionally, and the main discussion will be concerned with the kind of the vertical of the next chapter.

Lateral resistance of a pony truss vertical deflected a distance δ, with bending moment at the top end.

The vertical referred to in this section satisfies the conditions of fig. 3. The buckling and twist of the top chord determine the amounts of angular and linear deflections at the top end α and δ; V is known, and it is required to find S and M_0 for the given values of α and δ. The positive direction of M_0 in the following discussion will be taken as indicated on fig. 6. The bottom end will be again assumed vertically fixed.

The differential equation of the elastic curve is:

$$-M_0 - S(h-x) - V(\delta - y) = EI_v \frac{d^2y}{dx^2}, \tag{10}$$

which reduces to

$$\frac{d^2y}{dx^2} + u^2 y = u^2\left[\delta + \frac{M_0}{V} + \frac{S}{V}(h-x)\right], \tag{11}$$

where again
$$u = \sqrt{\frac{V}{EI_v}} \tag{12}$$

The general solution of (11) is

$$y = A_1 \operatorname{Cos} ux + B_1 \operatorname{Sin} ux + \delta + \frac{M_0}{V} + \frac{S}{V}(h-x). \tag{13}$$

The constants of integration A_1 and B_1, as well as the unknowns M_0 and S, are found from the following four conditions:

1. $x = 0, \quad y = 0,$
2. $x = 0, \quad \dfrac{dy}{dx} = 0,$
3. $x = h, \quad y = \delta,$
4. $x = h, \quad \dfrac{dy}{dx} = \alpha,$

which determine the unknowns:

$$S = \frac{uV}{2\tan\frac{uh}{2} - uh}\,\delta - \frac{\tan\frac{uh}{2}}{2\tan\frac{uh}{2} - uh}\,V\alpha \tag{14}$$

$$M_0 = -\frac{\tan\frac{uh}{2}\,V}{2\tan\frac{uh}{2} - uh}\,\delta + \frac{\frac{1}{u} + \frac{h}{2}\left(\tan\frac{uh}{2} - \cot\frac{uh}{2}\right)}{2\tan\frac{uh}{2} - uh}\,V\alpha \tag{15}$$

Adding to S the lateral horizontal component P of the stress in the diagonal, the total resisting force F is found as explained before:

$$F = P + S = \frac{V}{h}\delta + \frac{uV}{2\tan\frac{uh}{2} - uh}\delta - \frac{\tan\frac{uh}{2}V}{2\tan\frac{uh}{2} - uh}\alpha =$$

$$= \frac{2\tan\frac{uh}{2}}{2\tan\frac{uh}{2} - uh}\frac{V}{h}\delta - \frac{\tan\frac{uh}{2}V}{2\tan\frac{uh}{2} - uh}\alpha \tag{16}$$

The lowest value of V making either M_0 or S infinite, corresponds to $\frac{uh}{2} = \pi$, which gives

$$V'_{cr} = \frac{4\pi^2 EI_v}{h^2} = 39.5\frac{EI_v}{h^2}, \tag{17}$$

an expression almost twice greater than the expression (5) for a vertical free to rotate at the top. It may be repeated here, that although the individual verticals can stand a stress up to $V'_{cr.}$ according to eq. (17), the others, stressed considerably lower, must come to their aid. To anticipate danger to stability of the truss from this source is again quite unnecessary.

Returning to equations (15) and (16), the following formulas are obtained after substitution for V of its expression in terms of (uh):

$$F = \frac{2\tan\frac{uh}{2}}{2\tan\frac{uh}{2} - uh}(uh)^2\frac{EI_v}{h^3}\delta - \frac{\tan\frac{uh}{2}}{2\tan\frac{uh}{2} - uh}(uh)^2\frac{EI_v}{h^2}\alpha =$$

$$= b_2\frac{EI_v}{h^3}\delta - \frac{b_2}{2}\frac{EI_v}{h^2}\alpha. \tag{18}$$

where
$$b_2 = \frac{2\tan\frac{uh}{2}}{2\tan\frac{uh}{2} - uh}(uh)^2; \tag{19}$$

also
$$M_0 = -\frac{b_2}{2}\frac{EI_v}{h^2}\delta + b_3\frac{EI_v}{h}\alpha, \tag{21}$$

where
$$b_3 = \frac{1 + \frac{uh}{2}\left(\tan\frac{uh}{2} - \cot\frac{uh}{2}\right)}{2\tan\frac{uh}{2} - uh}(uh). \tag{22}$$

The coefficients b_2 and b_3 are plotted on the diagram 2.

The equations (18) and (21) solve the preliminary problem of the force resisting the buckling, and of the moment producing torsion of the top chord. It may be noticed, that the force has been determined as caused by the combined action of bending in the vertical and direct stress in the diagonal; while the moment — as caused only by bending in the vertical. Actually, this is not quite true: the diagonal has some bending resistance; and the fact of application of the lateral horizontal component of its stress P (fig. 2a) within the depth of the gusset plate, that is below the centre of gravity of the chord section, results in some additional twisting moment augmenting that produced by the vertical. These influences have been disregarded in the analysis as of minor importance, and the effect of ignoring them tends to make the results on the safe side.

Since the factors b_1, b_2 and b_3 in equations (8), (18) and (21) depend on V and through that on the loading of the bridge, they are not known at the beginning, and the "trial and error" method must be resorted to. Some expected value of V must be assumed, that will determine u and the b coefficients. When the critical load is found, the stresses V are calculated, checked against the assumed and, if necessary, the procedure should be repeated with new values of the coefficients.

To facilitate the mathematical treatment of the subject the concentrated action of the verticals and of the diagonals on the top chord at the panel points will be replaced with continuous action all along the chord of the intensity per unit length of the chord equal to the force at the panel point divided by the length of the panel. Then for truss with verticals free to rotate at the top ends

$$f = \frac{F}{d} = b_1 \frac{EI_v}{h^3 d} \delta = k_1 \delta; \tag{23}$$

and for truss with verticals having moments at the top ends

$$f = \frac{F}{d} = b_2 \frac{EI_v}{h^3 d} \delta - \frac{b_2}{2} \frac{EI_v}{h^2 d} \alpha = k_2 \delta - \frac{k_2 h}{2} \alpha, \tag{24}$$

and $$m = \frac{M_0}{d} = -\frac{b_2}{2} \frac{EI_v}{h^2 d} \delta + b_3 \frac{EI_v}{hd} \alpha = -\frac{k_2 h}{2} \delta + k_3 \alpha. \tag{25}$$

In these formulas d is the panel length of the truss,

$$k_1 = b_1 \frac{EI_v}{h^3 d}, \tag{26}$$

$$k_2 = b_2 \frac{EI_v}{h^3 d}, \tag{27}$$

$$k_3 = b_3 \frac{EI_v}{hd}, \tag{28}$$

and f and m are, of course, the intensities of force and moment acting on the chord per unit length of it. It is needless to say that the positive direction of f and m should be regarded as opposite to what is indicated by arrows for S and M_0 on figures 5 and 6, since it is the action of the web members on the chord, that is considered now.

In further development the coefficients k_1, k_2 and k_3 will be considered as constant along the bridge; actually they vary for different verticals, but ordinarily not greatly, so that their mean value should give satisfactory results.

Torsional deformation of the buckled top chord.

The definite relation in which the linear and angular deformations of the top chord δ and α (fig. 6) stand to each other, will be found now. Figures 7 and 7a represent a small length dx of the top chord with corresponding continuous wall of verticals. Directing the attention to torsion of this elemental length, the equation of equilibrium will be

$$T + m\,dx = T + dT,$$

where m is the moment at the end per unit length of the wall of verticals, and T is the twisting torque at the point of the top chord with an abscissa along the length of the truss x. This gives

$$m = \frac{dT}{dx} \tag{29}$$

The torque T produces on the length dx a change in the angle of twist $d\alpha$. If the constant torsional rigidity of the top chord is called C, then $d\alpha = \frac{T}{C}\,dx$

or

$$T = C\frac{d\alpha}{dx}. \tag{30}$$

Substituting into (29) for m and T their expressions from (25) and (30), the following differential equation results:

$$-\frac{k_2 h}{2}\delta + k_3\alpha = C\frac{d^2\alpha}{dx^2} \tag{31}$$

When a suitable expression of δ in terms of x is decided on, α is determined without difficulty from this equation. Then f, m and T are easily

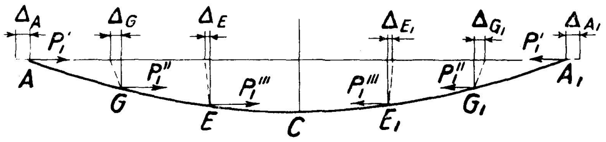


Fig. 8.

found in terms of x from (24), (25) and (30). Naturally, when the verticals are connected to the top chord according to fig. 4, there is no moment at the top end of the verticals and no torsion in the top chord. Consequently, all that is required to know in that case is the force f, which is given by the equation (23).

With preliminary work thus completed the main problem of the elastic stability of the pony truss will be attacked.

The energy method as applied to the question of elastic stability of the pony truss.

In the process of buckling, the ends of the top chord are not permitted by the absolutely rigid end verticals to have either linear or angular deformation laterally, but, of course, the longitudinal displacements are taking place.

The key to the solution is the "Energy Method", extensively used by Professor Timoshenko, and it is felt, that its brief explanation in relation to the present problem will not be out of place here.

Suppose, that the load is placed on the bridge, and that the truss members become stressed with the primary direct stresses and undergo certain deformations, as result of which a definite amount of elastic energy is stored in the deformed structure. Considering now the possibility of collapse due to elastic instability, imagine the top chord buckled by a small amount, its axis assuming certain appropriate curved shape in horizontal plane. This buckling automatically brings into play the resistances f and m on the part of the verticals and diagonals, and the moments m cause some torsion of the chord. The primary stresses in the truss members, and the corresponding original amount of the elastic energy of the structure before buckling are not affected by this process, and thus, the agency producing the buckling is called upon to supply all the additional elastic energy W brought about by buckling, which comes under the following four items:

W_B — energy required for buckling of the top chord proper,

W_T — energy required for torsion of the top chord,

W_f and W_m — energy required to overcome f and m, the web resistances to buckling of the top chord.

Thus,
$$W = W_B + W_T + W_f + W_m. \tag{32}$$

As the buckling deformation takes place, the two ends of the top chord come closer together, and the horizontal longitudinal components of the stresses in the diagonals, called P_1 on fig. 2a, do some positive work U, as may be seen from fig. 8.

The elastic energy W depends on the amount of buckling and on the sizes and shapes of the cross sections of the truss members; as to the load on the truss, it affects W only as far as the coefficients k_1, k_2 and k_3 in the

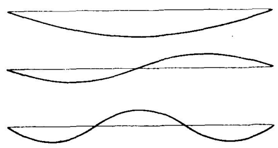


Fig. 9.

equations (23), (24) and (25) are affected; and these coefficients, as well as the corresponding amount of energy, decrease with an increase in load. The work U, on the other hand, is directly proportional to the load on the bridge, and also depends on the amount of buckling, but not on the sizes of the truss members. It is evident, that as long as $W > U$, the assumed buckling cannot be brought about by the load alone, without the aid of some outside agency. As the load increases, U increases and for certain value of the load $W = U$

or
$$W_B + W_T + W_f + W_m = U \tag{33}$$

The load of this magnitude is quite sufficient alone to bring about the buckling; this load is the required critical load.

The curve of buckling is a sine like curve, whose number of waves is determined by the relative stiffnesses of the chord and of the verticals. When stiffness of the verticals is small compared to that of the chord, the curve is one wave curve, but as the stiffness of the verticals increases, the number of waves will increase to two, three or even more (see fig. 9). The curve of any shape can be represented as a harmonic series of sine curves. It is sufficiently accurate to think of the curve of buckling as the sum of two sine curves, the primary one, roughly outlining the shape of the curve, and the secondary one, whose addition modifies the shape of the primary bringing it into a greater conformity with the actual buckling curve. When the load on the bridge is symmetrical both curves must be either with odd or even number of waves. Thus, fig. 10a represents the sum of one wave primary and three wave secondary sine curves, and fig. 10 b the sum of two wave primary and four wave secondary. A combination of even and odd sine curves, like the one on fig. 10 c, would be considered impossible for symmetrical loading of the truss, as making the two halves of the resultant curve dissimilar. In the following discussion the equation of the buckling curve (fig. 10 a and 10b) will be assumed:

$$\delta = a_p \operatorname{Sin} \frac{p\pi}{l} x + a_n \operatorname{Sin} \frac{n\pi}{l} x, \tag{34}$$

where the number of waves in the primary p and in the secondary n are two consecutive odd or even numbers. That p and n are consecutive numbers, and that n in the larger of the two, are naturally only assumptions.

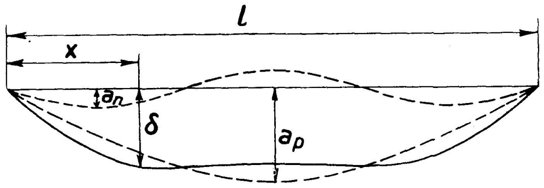


Fig. 10 a.

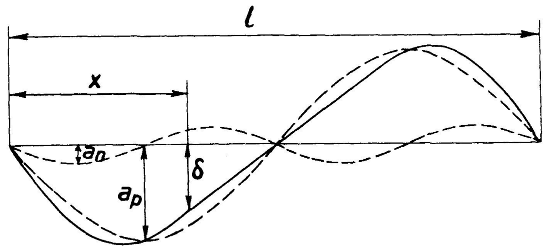


Fig. 10 b.

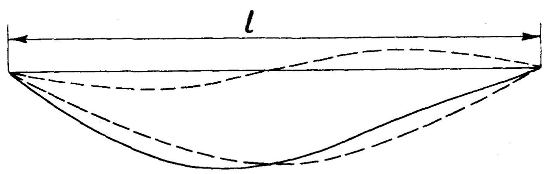


Fig. 10 c.

With the shape of the buckling curve decided on, the various terms in the equations (32) and (33) will be determined.

Expression for the work done by the external forces.

The direct method to find the work U done by the buckling forces P_1 (fig. 8) would be to express these forces in terms of the load and to multiply them by the decreases of respective distances EE_1, GG_1 and AA_1, caused by buckling; however, there is another method believed to be more instructive, which will be followed here.

Thinking of the top chord as separate from the rest of the structure, it is the forces P_1, that do the work U, but considering the whole truss, the forces P_1, now internal forces between the chord and the diagonals, do no work, and all the work done naturally comes from the load on the bridge. As the top chord buckles by an infinitesimal amount in horizontal plane, the bottom chord deflects in the vertical plane, the ordinates of deflection being infinitesimals of the second order, and the load applied at the bottom chord does on its lowering the same amount of work U. It may be necessary to mention, that the deflection just referred to is the one due exclusively to buckling of the top chord, and has nothing to do with the deflection caused by the direct stresses in the truss, nor with the additional deflection caused by bending of the verticals.

By the way of explanation of the nature of the additional deflection referred to in the previous paragraph, it may be said, that as the vertical bends, its top end lowers pulling down the diagonal member connected to the top of it. Since the diagonal is assumed to remain straight, its bottom end also goes down, lowering with it the next vertical. This causes an additional deflection, over and above that produced by buckling of the top chord proper, and, consequently, an additional work done by the load on the bridge. However, in the method of approach adopted in this paper, the influence of this factor is to be taken into account by the reduction in the resistance of verticals caused by direct stresses; hence, this additional deflection and its work should not be considered here.

Considering the influence of buckling of the top chord alone, as this takes place, the plane of the truss deforms into curved surface of such geometrical nature, that the magnitude of any angle in the plane of the truss is preserved, and all the verticals remain perpendicular to the bottom and to the top chords, as they were before the buckling (Deflection due to the direct stresses need not be considered here). The differential equation of the deflected bottom chord is

$$\frac{dz}{dx} = \frac{\Delta}{h}, \tag{35}$$

as may be seen from fig. 11 a and 11 b representing diagrammatically the plan and the elevation of a deformed truss. Here Δ is the longitudinal displacement toward the centre of the truss of any point A of the top chord caused by buckling.

$$\Delta = \int_x^{\frac{l}{2}} \left(\frac{1}{\operatorname{Cos} \Phi} - 1\right) dx = \frac{1}{2} \int_x^{\frac{l}{2}} \left(\frac{d\delta}{dx}\right)^2 dx \tag{36}$$

from (34) $$\frac{d\delta}{dx} = \frac{\pi}{l}\left(p\, a_p \operatorname{Cos} \frac{p\pi}{l} x + n\, a_n \operatorname{Cos} \frac{n\pi}{l} x\right).$$

This is substituted into (36), and in the integration the following formulas are made use of, remembering that $(p+n)$ and $(p-n)$ are both even.

$$\int_x^{\frac{l}{2}} \operatorname{Cos}^2 \frac{p\pi}{l} x\, dx = \frac{l}{4} - \frac{x}{2} - \frac{l}{4p\pi} \operatorname{Sin} \frac{2p\pi}{l} x;$$

and $$\int_x^{\frac{l}{2}} 2 \operatorname{Cos} \frac{p\pi}{l} x \operatorname{Cos} \frac{n\pi}{l} x\, dx = -\frac{l}{\pi}\left(\frac{\operatorname{Sin} \frac{p+n}{l} \pi x}{p+n} + \frac{\operatorname{Sin} \frac{n-p}{l} \pi x}{n-p}\right);$$

Then $$\Delta = \frac{\pi^2}{2l^2}\Bigg[p^2 a_p^2\left(\frac{l}{4} - \frac{x}{2} - \frac{l}{4p\pi} \operatorname{Sin} \frac{2p\pi}{l} x\right) +$$

$$+ n^2 a_n^2\left(\frac{l}{4} - \frac{x}{2} - \frac{l}{4n\pi} \operatorname{Sin} \frac{2n\pi}{l} x\right) - \frac{p\, n\, a_p a_n l}{\pi}\left(\frac{\operatorname{Sin} \frac{p+n}{l} \pi x}{p+n} + \frac{\operatorname{Sin} \frac{n-p}{l} \pi x}{n-p}\right)\Bigg] \tag{37}$$

This expression for Δ is substituted into (35), and on integration of the resultant equation, and after substitution of the initial conditions $z = 0$, when

$x = 0$ and $x = l$, the following expression for the downward deflection of the bottom chord, due to buckling of the top chord, results:

$$z = \frac{\pi^2}{8hl^2}(p^2 a_p^2 + n^2 a_n^2)(lx - x^2) - \frac{1}{8h}\left(a_p^2 \operatorname{Sin}^2 \frac{p\pi}{l} x + a_n^2 \operatorname{Sin}^2 \frac{n\pi}{l} x\right)$$

$$- \frac{p n a_p a_n}{h}\left[\frac{\operatorname{Sin}^2 \frac{\pi(p+n)}{2l} x}{(p+n)^2} + \frac{\operatorname{Sin}^2 \frac{\pi(n-p)}{2l} x}{(n-p)^2}\right]. \tag{38}$$

The work U done by the load applied at the bottom chord can now be easily found.

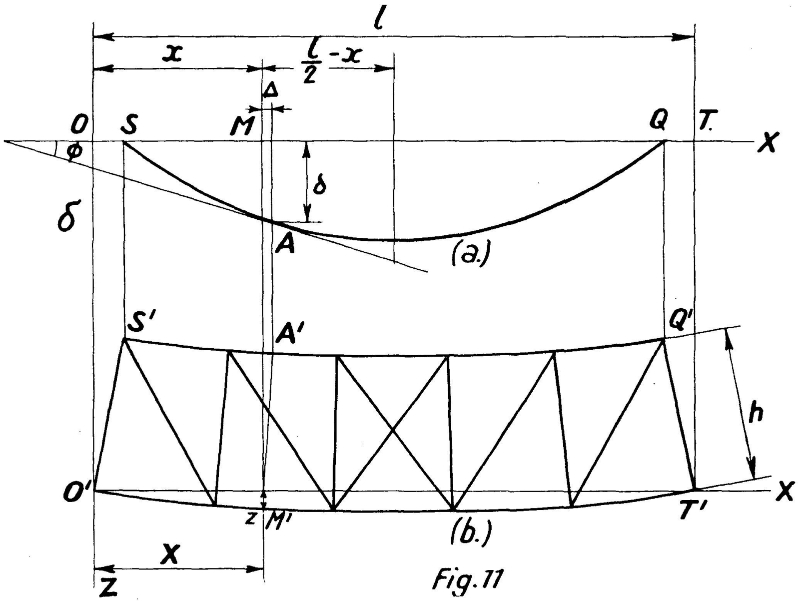


Fig. 11

When uniform load q covers the whole span

$$U_u = q\int_0^l z\,dx. \tag{38a}$$

When the load is a series of concentrated weights Q, the work

$$U_c = \sum(Qz), \tag{38b}$$

where the ordinates z are taken under the weights Q.

Let
$$Q = NQ_1, \tag{38c}$$

where N is a number defining the unknown critical intensity of a set of weights, having a definite ratio among themselves, like, for example, a Cooper's loading; and Q_1 are the known values of weights corresponding to some arbitrary unit intensity of the set, say, Cooper's E. 10. Then the equation (38b) takes the form:

$$U_c = N\sum(Q_1 \cdot z). \tag{38d}$$

When the integration in (38a) is performed, the following expression for the work done by the uniform load results:

$$U_u = \frac{ql}{8h}\left[\frac{1}{2}\left(\frac{\pi^2}{3}p^2 - 1\right)a_p^2 + \frac{1}{2}\left(\frac{\pi^2}{3}n^2 - 1\right)a_n^2 - 8a_p a_n \frac{pn(p^2+n^2)}{(n^2-p^2)^2}\right] \tag{39}$$

Determining the ordinate z at the centre of the bridge from (38), the expressions for the work U_1, done by a single concentrated load Q at the centre, are obtained from (38 b). When both p and n are odd:

$$U_1 = \frac{Q}{8h}\left[\left(\frac{\pi^2}{4}p^2 - 1\right)a_p^2 + \left(\frac{\pi^2}{4}n^2 - 1\right)a_n^2 - 2pna_pa_n\right]. \tag{39a}$$

When both p and n are even:

$$U_1 = \frac{Q}{8h}\left\{\frac{\pi^2}{4}p^2a_p^2 + \frac{\pi^2}{4}n^2a_n^2 - pna_pa_n\left[\frac{8}{(p+n)^2} + 2\right]\right\} \tag{39b}$$

With U known, the terms on the left hand side of the equation (33) will now be determined.

Elastic energy of the deformed structure.

Calling the constant moment of inertia of the top chord for bending in the horizontal plane I_c, the elastic energy of bending (buckling) of the top chord W_B is found from

$$W_B = \frac{EI_c}{2}\int_0^l\left(\frac{d^2\delta}{dx^2}\right)^2 dx. \tag{40}$$

$$\frac{d^2\delta}{dx^2} = -\frac{p^2\pi^2}{l^2}a_p \operatorname{Sin}\frac{p\pi}{l}x - \frac{n^2\pi^2}{l^2}a_n \operatorname{Sin}\frac{n\pi}{l}x.$$

This is substituted into (40).

Since $$\int_0^l \operatorname{Sin}^2\frac{p\pi}{l}x\,dx = \frac{l}{2},$$

and $$\int_0^l \operatorname{Sin}\frac{p\pi}{l}x \operatorname{Sin}\frac{n\pi}{l}x\,dx = 0,$$

after simplification $$W_B = \frac{\pi^4EI_c}{4l^3}(a_p^2p^4 + a_n^2n^4). \tag{41}$$

In expressing the elastic energy of twist of the top chord W_T, combine it with W_m, the energy required to overcome the resistance of bending moments on the ends of the idealized verticals.

$$W_T + W_m = \int_0^l\frac{T^2}{2C}dx + \frac{1}{2}\int_0^l m\,\alpha\,dx. \tag{42}$$

From (29) and (30) $$T^2 = C^2\left(\frac{d\alpha}{dx}\right)^2,$$

and $$m = \frac{dT}{dx} = C\frac{d^2\alpha}{dx^2}.$$

This gives $$W_T + W_m = \frac{C}{2}\int_0^l\left[\left(\frac{d\alpha}{dx}\right)^2 + \alpha\frac{d^2\alpha}{dx^2}\right]dx =$$

$$= \frac{C}{2}\left[\left(\alpha\frac{d\alpha}{dx}\right)_{\substack{at\\x=l}} - \left(\alpha\frac{d\alpha}{dx}\right)_{\substack{at\\x=0}}\right]. \tag{43}$$

As the end verticals are absolutely rigid, $\alpha = 0$ for both $x = l$ and $x = 0$, and, consequently, the expression in the square brackets and the sum $(W_T + W_m)$ are both equal to zero. In understanding this important result one must realize that as the buckling of the top chord progresses, and the angle α increases its positive value, the moment m also increases numerically, remaining negative, i. e. in the direction opposite to α, and, consequently, the work of overcoming m is negative. It is interesting to note that it is for the whole top chord and not for any small portion of it, that the work of torsion and the work of overcoming the moment m balance completely. Thus for the element dx at the centre of the wave $T = 0$, and the elementary energy of twist $dW_T = 0$, while the negative elementary work of bending dW_m is the greatest here; on the other hand, the positive dW_T at the end of the wave is the greatest due to the greatest T, but dW_m is here zero.

Though the sum $(W_T + W_m)$ is zero, it would be wrong to think that the torsional resistance of the top chord is immaterial to the stability of the structure. The fact is that the magnitudes f of the resisting forces on the ends of the verticals are considerably affected by torsional rigidity of the top chord. The following is the expression for W_f, the work of overcoming the resisting forces f:

$$W_f = \frac{1}{2}\int_0^l f\,\delta\,dx. \tag{44}$$

The equation (24) gives f in terms of δ and α; and α must be determined from the differential equation (31). Substitution into (31) of the expression for δ from (34) brings this equation into form:

$$\frac{d^2\alpha}{dx^2} - \frac{k_3}{C}\alpha = -\frac{k_2 h}{2C}\left(a_p \operatorname{Sin}\frac{p\pi}{l}x + a_n \operatorname{Sin}\frac{n\pi}{l}x\right). \tag{45}$$

General solution of this equation is:

$$\alpha = A_2 \operatorname{Cosh}\left(\sqrt{\frac{k_3}{C}}x\right) + B_2 \operatorname{Sinh}\left(\sqrt{\frac{k_3}{C}}x\right) +$$
$$+ \frac{k_2 h l^2}{2(Cp^2\pi^2 + k_3 l^2)} a_p \operatorname{Sin}\frac{p\pi}{l}x + \frac{k_2 h l^2}{2(Cn^2\pi^2 + k_3 l^2)} a_n \operatorname{Sin}\frac{n\pi}{l}x. \tag{46}$$

The conditions at the ends are such that when $x = 0$ or $x = l$, $\alpha = 0$; this makes the constants of integration $A_2 = B_2 = 0$.

Introduce new symbol μ, so that

$$\mu = \frac{k_2 l^2 h^2}{4\pi^2 C} = \frac{b_2}{4\pi^2}\,\frac{l^2}{hd}\,\frac{EI_v}{C}; \tag{47}$$

then

$$\frac{k_3 l^2}{C\pi^2} = 4\frac{b_3}{b_2}\mu, \tag{48}$$

as follows from (27) and (28). Substituting these into (46), the following expression for α is obtained

$$\alpha = \frac{2\mu a_p}{h\left(p^2 + 4\frac{b_3}{b_2}\mu\right)} \operatorname{Sin}\frac{p\pi}{l}x + \frac{2\mu a_n}{h\left(n^2 + 4\frac{b_3}{b_2}\mu\right)} \operatorname{Sin}\frac{n\pi}{l}x. \tag{49}$$

then from (24)

$$f = k_2 a_p \left(1 - \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right) \operatorname{Sin}\frac{p\pi}{l}x + k_2 a_n \left(1 - \frac{\mu}{n^2 + 4\frac{b_3}{b_2}\mu}\right) \operatorname{Sin}\frac{n\pi}{l}x. \tag{50}$$

and $$W_f = \frac{1}{2}\int_0^l f\,\delta\,dx = \frac{k_2 l}{4} a_p^2 \left(1 - \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right) + \frac{k_2 l}{4} a_n^2 \left(1 - \frac{\mu}{n^2 + 4\frac{b_3}{b_2}\mu}\right). \tag{51}$$

The corresponding expression for W_f when no torsion of the top chord is possible, is found from (23):

$$W_f = \frac{k_1 l}{4}(a_p^2 + a_n^2)\,. \tag{51a}$$

In the following discussion, however, only the expression (51) will be used, because (51 a) is a special case of it corresponding to the value of torsional rigidity of the top chord $C = 0$ and $\mu = \infty$.

Critical value of the uniform load covering the whole span.

Both sides of the equation (33) being known, the energy equation for the uniform load q will be:

$$\frac{\pi^4 E I_c}{4 l^3}\left(a_p^2 p^4 + a_n^2 n^4\right) + \frac{k_2 l}{4}\left[a_p^2\left(1 - \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right) + a_n^2\left(1 - \frac{\mu}{n^2 + 4\frac{b_3}{b_2}\mu}\right)\right] =$$

$$= \frac{q l}{h}\left[\frac{1}{16}\left(\frac{\pi^2}{3}p^2 - 1\right)a_p^2 + \frac{1}{16}\left(\frac{\pi^2}{3}n^2 - 1\right)a_n^2 - a_p a_n \frac{p n (p^2 + n^2)}{(n^2 - p^2)^2}\right] \tag{52}$$

Let $a_n = y a_p$ (53), where y is an unknown. Substituting this expression for y into (52), cancelling a_p^2 and making the necessary transformations, the following expression results for the compressive stress at the centre of the top chord corresponding to the critical uniform load over the whole span of the intensity q per unit length of one truss:

$$\frac{q l^2}{8 h} = \frac{\frac{\pi^2}{4}\frac{E I_c}{l^2}(p^4 + y^2 n^4) + \frac{k_2 l^2}{4}\left[\left(1 - \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right) + \left(1 - \frac{\mu}{n^2 + 4\frac{b_3}{b_2}\mu}\right)y^2\right]}{\frac{1}{2}\left(\frac{\pi^2}{3}p^2 - 1\right) + \frac{1}{2}\left(\frac{\pi^2}{3}n^2 - 1\right)y^2 - 8y\frac{p n (p^2 + n^2)}{(n^2 - p^2)^2}} \tag{53}$$

This can be represented in the form similar to the Euler's formula

$$\frac{q l^2}{8 h} = \gamma \frac{\pi^2 E I_c}{l^2}\,. \tag{54}$$

Calling

$$\eta = \frac{k_2 l^4}{4 \pi^2 E I_c} = \frac{b_2}{4 \pi^2}\frac{l^4}{h^3 d}\frac{I_v}{I_c}, \tag{55}$$

the following is the expression for γ from (53):

$$\gamma = \frac{\frac{\pi^2}{4}(p^4 + y^2 n^4) + \eta\left[\left(1 + \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right) + \left(1 - \frac{\mu}{n^2 + 4\frac{b_3}{b_2}\mu}\right)y^2\right]}{\frac{1}{2}\left(\frac{\pi^2}{3}p^2 - 1\right) + \frac{1}{2}\left(\frac{\pi^2}{3}n^2 - 1\right)y^2 - 8y\frac{p n (p^2 + n^2)}{(n^2 - p^2)^2}}\,. \tag{56}$$

The first term of the numerator in this expression comes from the bending resistance of the top chord proper; the second — from the web resistance to bending and from the torsional resistance of the chord, and it contains the coefficients η and μ, and the ratio $\frac{b_3}{b_2}$. As may be seen from (55) and (47) η is a factor involving the ratio of flexural rigidities of the verticals and of the top chord, and μ — a factor depending on the ratio of the flexural rigidity of verticals to the torsional rigidity of the chord; both, as much as they depend on b_2, depend on the loading of the verticals.

The unknown y (the ratio of the amplitudes of the secondary and of the primary sine waves of the buckled top chord) is found so as to make γ a minimum. This requirement involves solution of a quadratic equation, whose general form is complicated, but each case taken individually presents no difficulty.

A circumstance of great significance is the cancelling of the amplitude of buckling out of the equations (53) and (56). This fact demonstrates independence of the critical value of the load q from the value of the amplitude. As long as the intensity of the load q is below that given by the equation (53), the top chord remains straight; when the critical value is reached there exists a state of neutral equilibrium, and when the critical value of q is exceeded the collapse of the structure ensues.

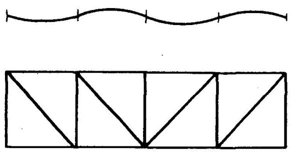


Fig. 12

The first approximation of γ, sufficiently accurate for the preliminary estimation of the critical load, corresponds to $y = 0$, i. e. to a case when the secondary sine wave is neglected.

Then
$$\gamma = \frac{\dfrac{\pi^2}{4} p^4 + \eta \left[1 - \dfrac{\mu}{p^2 + 4 \dfrac{b_3}{b_2} \mu} \right]}{\dfrac{1}{2} \left(\dfrac{\pi^2}{3} p^2 - 1 \right)}. \tag{57}$$

The number of waves p in the primary curve is unknown at the start, and is determined by trials from (57) so that the corresponding γ is the smallest; after that the second approximation of γ is found from (56) for the same p.

To facilitate the analysis the diagrams 3 and 4 giving the first approximations of γ in term of μ and η have been plotted for $4 \dfrac{b_3}{b_2} = 1.333$ (the maximum possible value) and for $4 \dfrac{b_3}{b_2} = 1.25$.

As may be seen from the diagrams, the $\gamma - \eta$ curve for every μ consists of several straight portions. The first from the left part corresponds to a single wave of buckling, the second — to a double wave etc. Of course, the boundary between any two types of the buckling curve, as determined by a corner point on $\gamma - \eta$ curve, is only approximate, insofar as the curves of the diagrams give only the first approximations of γ, hence, when calculating the second approximation for a combination of η and μ near a corner point on the diagram the possibility of either type of the buckling curve must be investigated.

It may be mentioned here, that in addition to the primary and to the secondary a tertiary wave might have been introduced into the curve of buckling by some simple modification of the equation (56), resulting in a seemingly more accurate expression for γ. However, the accuracy would have been imaginary. As the number of waves increases, the replacement of the actual truss verticals with an equivalent continuous wall becomes generally incorrect. In an extreme case, when the number of waves in the primary, secondary or tertiary, as the case may be, equals the number of panels (fig. 12), the work of overcoming the corresponding resistance of verticals W_f is actually zero, since the verticals coincide with the nodal points. As a result, the part of the second term in the numerator in (56) and (57), corresponding to this wave, must be cancelled out, — a circumstance discouraging the refinement of introduction of higher harmonics as too theoretical. On the other hand, as long as the number of panels is an exact multiple (but not equal) of the number of waves, the replacement discussed here holds exactly true. Judging from few examples solved, the theoretical error introduced by neglecting the tertiary is very small.

Critical value of a concentrated load at the centre.

Turning to the question of a single concentrated load at the centre, the energy equation (33) is again the basis of the analysis, but while the left hand side of it, the elastic energy of the deformed structure, has exactly the same expression as for the uniform load, the right hand side, the work done by the load, must be represented by the equations (39a) and (39b) instead of (39). Retracing the steps taken in connection with determination of γ for the uniform load, the resultant equations for single concentrated load Q at the centre, corresponding to equations (54), (56) and (57) are:

$$\frac{Ql}{8h} = \gamma_1 \frac{\pi^2 E I_c}{l^2}. \tag{54a}$$

For p and n odd

$$\gamma_1 = \frac{\frac{\pi^2}{4}(p^4 + y^2 n^4) + \eta\left[\left(1 - \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right) + \left(1 - \frac{\mu}{n^2 + 4\frac{b_3}{b_2}\mu}\right)y^2\right]}{\left(\frac{\pi^2}{4}p^2 - 1\right) + \left(\frac{\pi^2}{4}n^2 - 1\right)y^2 - 2pny} \tag{56a}$$

For p and n even

$$\gamma_1 = \frac{\frac{\pi^2}{4}(p^4 + y^2 n^4) + \eta\left[\left(1 - \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right) + \left(1 - \frac{\mu}{n^2 + 4\frac{b_3}{b_2}\mu}\right)y^2\right]}{\frac{\pi^2}{4}p^2 + \frac{\pi^2}{4}n^2 y^2 - pn\left[\frac{1}{(p+n)^2} + \frac{1}{4}\right]y} \tag{56b}$$

First approximations of γ_1 corresponding to $y = 0$ are as follows:

for p odd:

$$\gamma_1 = \frac{\frac{\pi^2}{4}p^4 + \eta\left[1 - \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right]}{\left(\frac{\pi^2}{4}p^2 - 1\right)} \tag{57a}$$

for p even:

$$\gamma_1 = \frac{\frac{\pi^2}{4}p^4 + \eta\left[1 - \frac{\mu}{p^2 + 4\frac{b_3}{b_2}\mu}\right]}{\frac{\pi^2}{4}p^2} \tag{57b}$$

These first approximations of γ_1 have been plotted on the diagram 5 for different μ and η, and for the value of $4\frac{b_3}{b_2} = 1.333$.

Previous discussion of the significance of various terms in the expression for γ and of the shape of the buckling curve holds equally true in this case.

Effect of inclination of bottom ends of verticals.

The formulas for the resistance of verticals were derived on the assumption of fixation of the bottom ends in vertical positions. This assumption is not absolutely correct in view of flexibility of floor beams. Bending of floor beams is firstly the result of buckling of top chord, and secondly, it is due to the loads resting on them. Since ordinarily the floor beams are much more rigid than verticals, disregard of the influence of the first factor on the stability of the bridge is probably justified in many cases.

As to the influence of the second factor, two conditions may obtain under various kinds of loading. The ends of all floor beams, including the end ones, may get equal inclinations, in that case the top chord will be straight before buckling, and the formulas of this paper will not be affected. On the other

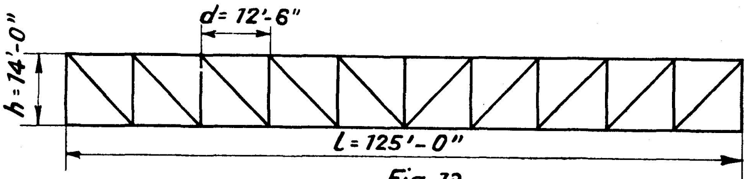


Fig. 13

hand, unequal slopes of the ends of floor beams will cause some additional bending of verticals and of the top chord. Detailed study of this question shows that unequal inclination of the ends of floor beams does not affect the magnitude of the load under which the structure becomes unstable, but while with the floor beams bent equally the top chord remains straight up to the critical value of the load, and then collapses suddenly; with unequally bent floor beams the buckling of the top chord increases with increase in load, reaching very large value under the load of the same critical intensity.

Reciprocal influence line of the critical load.

If the position on the span of single concentrated load varies, the work U done in lowering of the load on account of buckling of the top chord also varies in proportion to the ordinate z of the deflection curve of the bottom chord. The shape of the buckling curve and the elastic energy of the deformed structure W, as well as the shape of the deflection curve of the bottom chord, change only slightly, as will be seen from the numerical example, and as an approximation may be considered constant.

This circumstance is very important, and it suggests interesting use for the curve (38), as a curve whose ordinates are approximately inversely proportional to the load concentrations producing collapsing effect on the bridge, if placed at the points of their abscissae. The principle if developed a little further leads to the idea of a curve, which may be appropriately termed "the reciprocal influence line of the collapsing load".

Using notation of the earlier part of the paper, let the top chord buckle to certain suitable shape, first, under the action of uniform load of critical intensity q over the whole span, and secondly, under a set of concentrated loads of critical intensity N, in some definite position on the bridge. Then the energy equation (33), combined with the expressions for the work done by the loads (38 d) and (39), gives the following relations:

$$q\,l = \frac{W}{\frac{a_p^2}{8h}\left[\frac{1}{2}\left(\frac{\pi^2}{3}p^2-1\right)+\frac{1}{2}\left(\frac{\pi^2}{3}n^2-1\right)y^2-8\,\frac{p\,n(p^2+n^2)}{(n^2-p^2)^2}y\right]}, \tag{58}$$

and $$N = \frac{W}{\Sigma(Q_1 z)}. \tag{59}$$

Dividing one by the other

$$N = \frac{p\,l}{\Sigma(Q_1\,\varepsilon)} \tag{60}$$

where the variable ε is proportional to z, and stands to it in the following relation:

$$\varepsilon = \frac{8h}{a_p^2}\,\frac{z}{\left[\frac{1}{2}\left(\frac{\pi^2}{3}p^2-1\right)+\frac{1}{2}\left(\frac{\pi^2}{3}n^2-1\right)y^2-8\,\frac{p\,n(p^2+n^2)}{(n^2-p^2)^2}y\right]}.$$

Substituting z from (38)

$$\varepsilon = \frac{\pi^2\frac{(lx-x^2)}{l^2}(p^2+n^2y^2)-\left(\mathrm{Sin}^2\frac{p\pi}{l}x+y^2\,\mathrm{Sin}^2\frac{n\pi}{l}x\right)-8pny\left[\frac{\mathrm{Sin}^2\frac{\pi(n+p)}{2l}x}{(p+n)^2}+\frac{\mathrm{Sin}^2\frac{\pi}{l}x}{4}\right]}{\frac{1}{2}\left(\frac{\pi^2}{3}p^2-1\right)+\frac{1}{2}\left(\frac{\pi^2}{3}n^2-1\right)y^2-8\frac{p\,n(p^2+n^2)}{(n^2-p^2)^2}y} \tag{61}$$

The ε curve is quite easy to visualize in view of its physical meaning, the deflection curve of the bottom chord. Since its main part, the first term in the numerator, is parabola, the curve is somewhat parabolic in form, with maximum ordinate at the centre. The ratio of the amplitudes of the secondary and the primary sine waves, designated by letter y, must be considered as constant for the whole span and having the value corresponding to the uniform critical load on the bridge. The shape of ε curve and the magnitudes of its ordinates are thus quite determined by two variables y and p, of which the second is of major importance (n is the next after p consecutive number of the same kind, i. e. odd if p is odd, and even when p is even).

With ε curve constructed, and the critical value of the uniform load ql previously determined, the critical intensity N of any group of concentrated weights can be easily found from the equation (60). The critical value of single concentrated load Q is found from

$$Q = \frac{q\,l}{\varepsilon}, \tag{62}$$

which is a special case of (60). The equation (60) can also be extended to a combination of uniform and concentrated loads.

The equation (62) provides simple interpretation of ε curve. According to it, ε is an abstract number whose reciprocal shows how many times the critical value of a concentrated load at any point on the span is greater than the critical value of the uniform load covering the whole span. The average ε for the whole span is naturally unity; near the centre ε is greater than unity, and near the end — smaller than unity, indicating that in order to cause collapse of a bridge the concentrated load, if placed near the centre, may be lighter than the critical uniform load, and if placed near the ends — must be heavier.

Being an influence line, the curve ε provides also the means of finding the most unfavourable position of the moving load, for which N becomes a minimum.

However, it must be remembered that these simple relations based on the principle of superposition hold only approximately true, since the bending resistance of the verticals (coefficients b_2 and b_3) and the ratio y depend somewhat on the position and amount of loading. The relations are more nearly true when the concentrated weights causing collapse of the truss are present in pairs symmetrical about the centre of the bridge, otherwise the assumption that the two sine waves composing the buckling curve are both either odd or even, will be incorrect, and the error in using the reciprocal influence line of the critical load will be greater.

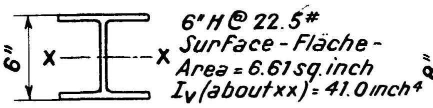
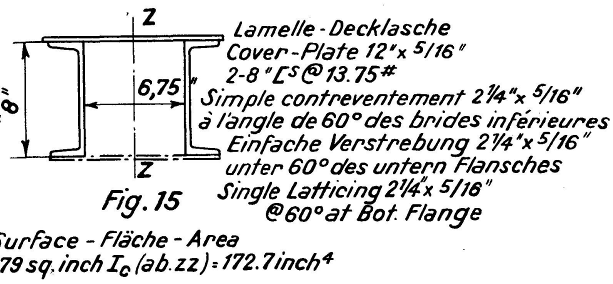


Fig. 14

Sections de toutes les verticales intermédiaires.
Querschnitt aller Zwischenpfosten.
Section of all intermediate Verticals.



Section constante de la membrure supérieure.
Konstanter Querschnitt des Obergurtes.
Constant Section of Top Chord.

Example.

In order to illustrate how the developed formulas are used in checking design of a pony truss for elastic stability, an example will be given here.

Let figure 13 represent the pony truss required to be checked. The end verticals are supposed to be absolutely rigid, and the constant sections of the top chord and of the intermediate verticals are given on figures 14 and 15. Sections of the bottom chord and of the diagonals are immaterial. Such a truss is good for a total equivalent uniform load of about 1 kip/ft of one truss, which corresponds to an ordinary highway bridge with wooden deck and one lane of traffic.

The torsional rigidity of the top chord C has nothing to do with the polar moment of inertia of the section, and since the method by which it is determined is not a matter of common knowledge, it will be taken here in full detail. It is based on so-called hydrodynamic analogy, according to which the problem of torsion of a shaft is reduced to a mathematically equivalent problem of steady circulating motion of frictionless fluid in a vessel in the shape of the shaft[1]). The velocity of fluid at any point is proportional to and in the same direction as the torsional stress. The motion of fluid inside the shaft is easy to visualize. Referring to fig. 16, representing a somewhat

[1]) For the explanation of hydrodynamic analogy see for example S. TIMOSHENKO, Strength of Materials. Van Nostrand, New York.

simplified section of the member in question, under the action of a torque T, the velocities (or stresses) in different parts of the section will be parallel to the walls, as shown by arrows, and inversely proportional to the respective thicknesses t. Thus, fluid in the web of the channel on reaching the flange will swerve parallel to it and will proceed to the gauge line of rivets, through which it will enter the cover plate and move along it in the direction opposite to that in the flange. The stresses τ, τ_1 and τ_2 will bear the relation:

$$\tau t = \tau_1 t_1 = \tau_2 t_2 \tag{68}$$

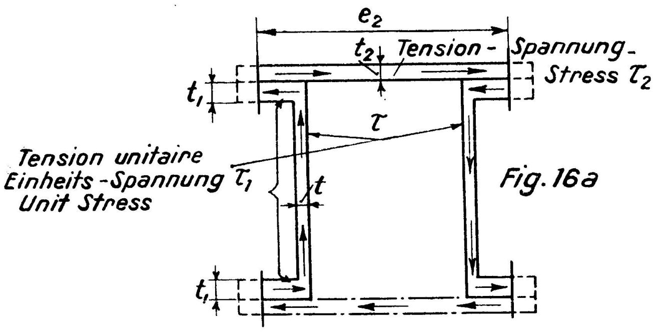


Fig. 16a

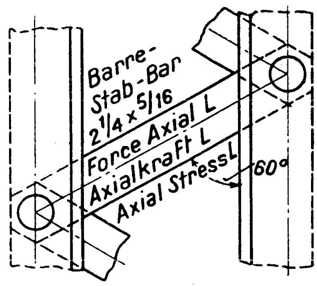


Fig. 16b

Portions of the channel flanges and of the cover plate outside of the gauge lines of rivets, shown dotted, do not contribute to the torsional rigidity of the section.

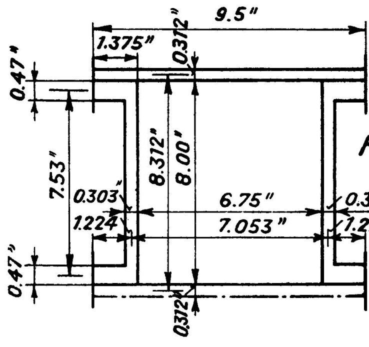


Fig. 17

Latticing at the bottom flange introduces no special complications. Its duty is similar to that of the cover plate, namely to bind the flanges of the channel together, and in performing it each lattice bar develops a force L, whose lateral component is equal to the total shearing force in the cover plate:

$$L \operatorname{Sin} \cdot 60^0 = \tau_2 t_2 e_2 \tag{69}$$

Referring to fig. 17 the equation of moments is:

$$T = \tau (0.303)(7.53)(7.053) + \tau_1 (0.47)(1.224)(2)(7.53) + \\ + \tau_2 (0.312)(9.5)(8.312). \tag{70}$$

Equating external work to resilience

$$\frac{T^2}{2C} = \frac{\tau^2}{2G}(7.53)(0.303)(2) + \frac{\tau_1^2}{2G}(0.47)(1.224)(4) + \\ + \frac{\tau_2^2}{2G}(9.5)(0.312) + \frac{L^2(2)}{2E(2.25)(0.312)}. \tag{72}$$

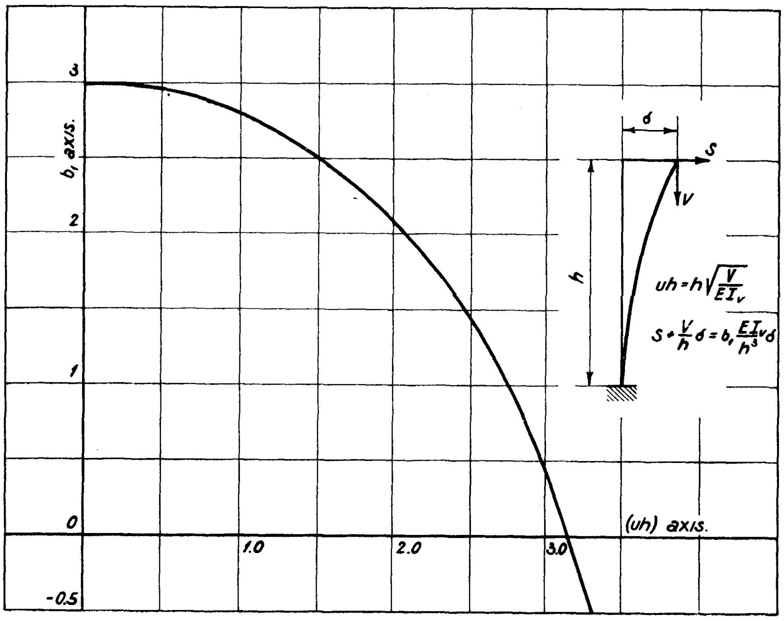
Expressing τ, τ_1, τ_2 and L in terms of T from (68), (69) and (70), and using the ratio between the modulii of elasticity and shear $\frac{E}{G} = 2.5$, the following value for the torsional rigidity C is obtained from equation (72):

$$C = 100.4\, G \tag{73}$$

In this relation C is expressed in kip. inch^2, and G — in kip./sq. inch [2]).

It may be mentioned in passing that the greatest part of the elastic energy of torsion of the member is stored in its weakest segment, the lattice, and should the latter be replaced with a cover plate, the amount of internal energy would greatly decrease, and, consequently, C would rise in proportion.

Strictly speaking, C is affected by the direct stress in the member. Compression of the top chord makes it less rigid in torsion. However, this influence is very slight.



Diagr. 1. Diagramme du coefficient b en fonction de (uh). Représentation graphique de la réduction de résistance latérale du système lorsque V augmente.

Diagramm des Koeffizienten b, gezeichnet in Funktion von (uh). Darstellung der Verminderung des Widerstandes gegen seitliches Ausbiegen des Fachwerkes mit zunehmendem V.

Diagram of Coefficient b. Plotted on the Base of (uh). Showing Decrease in Lateral Resistance of Combination of Truss Vertical and Diagonal as V increases.

Knowing C and assuming, as a first approximation, maximum values for the coefficients b_2 and b_3 (see diagram 2), $b_2 = 12$ and $b_3 = 4$, so that $4\frac{b_3}{b_2} = \frac{4}{3}$, the coefficients μ and η will be found from (47) and (55).

$$\mu = \frac{b_2}{4\pi^2}\frac{l^2}{hd}\frac{EI_v}{C} = \frac{12}{39.5}\frac{125^2}{(14)(12.5)}2.5\frac{41}{100.4} = 27.69.$$

$$\eta = \frac{b_2}{4\pi^2}\frac{l^4}{h^3 d}\frac{I_v}{I_c} = \frac{12}{39.5}\frac{125^4}{(14)^3(12.5)}\frac{41}{172.7} = 514.$$

First approximation of γ is found from (57), making the number of waves $p = 1$, 2, 3 and 4.

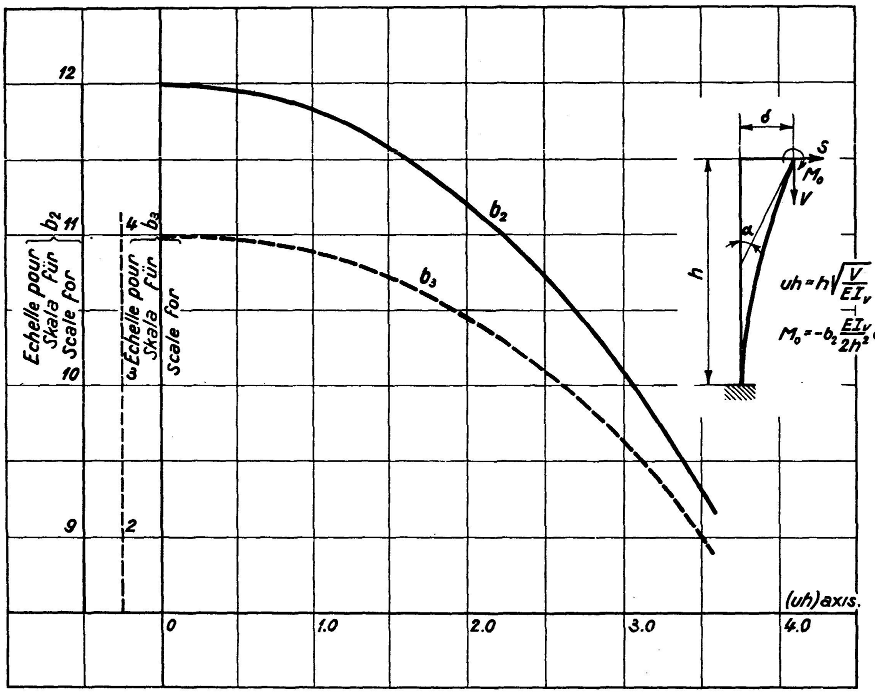
[2]) kip = 1000 lbs.

For $p = 1$ $\gamma = 123.35$,
$p = 2$ $\gamma = 33.81$,
$p = 3$ $\gamma = 28{,}23$,
and $p = 4$ $\gamma = 33.96$.

These figures show that the curve of buckling will be a triple wave, and that approximately $\gamma = 28.23$, which value can also be read off the diagram of γ.

For the second approximation substitute into (56) $p = 3$ and $n = 5$. The resulting expression is

$$\gamma = \frac{1823\,y^2 + 404}{40.6\,y^2 - 15.93\,y + 14.3} \tag{74}$$



Diagr. 2. Diagramme des coefficients b_2 et b_3 en fonction de (uh). Représentation graphique de la diminution de résistance latérale de la contre-fiche lorsque V augmente.

Diagramm der Koeffizienten b_2 und b_3, gezeichnet in Funktion von (uh). Darstellung der Abnahme des Widerstandes des Pfostens gegen seitliches Ausbiegen mit zunehmendem V.

Diagram of Coefficients b_2 and b_3. Plotted on the Base of (uh). Showing Decrease in Resistance of Strut with Increase in V.

The value of y making this expression minimum is

$$y = -0.242, \tag{75}$$

and the second approximation of γ from (74) is

$$\gamma = 24.9, \tag{76}$$

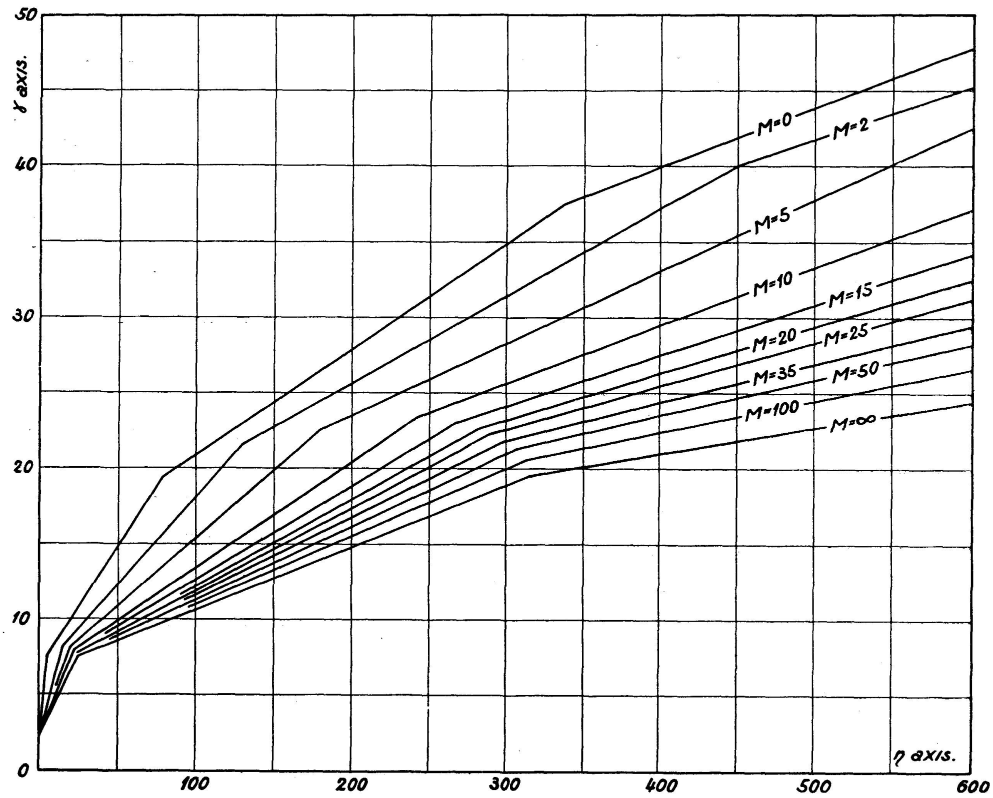
about 12 % less than the first one.

Knowing γ, the value of the compression stress in the central panel of the top chord, corresponding to the collapsing load on the bridge is found from (54):

$$\frac{q l^2}{8h} = \gamma \frac{\pi^2 E I_c}{l^2} = 24.9 \frac{\pi^2 \times 29000 \times 172.7}{(125 \times 12)^2} = 546.5 \text{ kips.} \tag{77}$$

and the corresponding intensity of the critical uniform load per unit length of one truss

$$q = 3.92 \text{ kip/ft.} \tag{78}$$



Diagr. 3. Diagramme des coefficients γ (première approximation) pour une charge uniformément répartie sur toute la portée, en fonction de η pour différentes valeurs de μ $4\frac{b_3}{b_2} = 1{,}333$

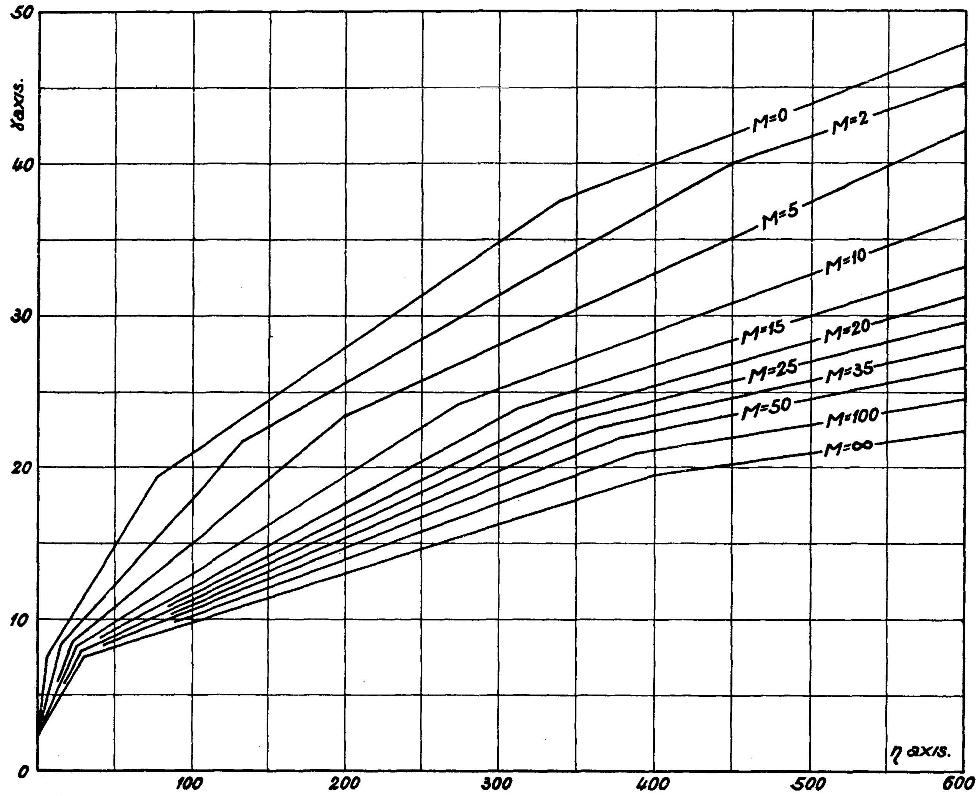
Diagramm der Koeffizienten γ (Erste Annäherung) für gleichmäßige Belastung über die ganze Spannweite. Gezeichnet in Funktion von η für verschiedene μ $4\frac{b_3}{b_2} = 1{,}333$

Diagram of Coefficients γ (First Approximation). For Uniform Load Covering Whole Span-Plotted on Base of η for different μ $4\frac{b_3}{b_2} = 1.333$

This figure has been obtained on the basis of the maximum values for the coefficients of resistance of verticals b_2 and b_3; actually as the stress in the verticals increases their resisting capacity decreases, and it is well to inquire at this point, how much stability of the bridge will be affected by the direct stresses in the verticals, corresponding to the uniform load $q = 3.92$ kip/ft.

The intermediate verticals are numbered from the outside of the truss, and the results of calculations involving formulas (12), (19) and (22), or the diagram 2, are tabulated below.

Intermediate Verticals	Stress kips	$u\ \frac{1}{\text{inch}}$	uh	b_2	b_3
1	171.5	.01200	2.018	11.18	3.42
2	122.5	.01015	1.707	11.41	3.60
3	73.5	.00786	1.321	11.66	3.78
4	24.5	.00454	0.762	11.90	3.95
5	0	0	0	12.00	4.00
Average of 9 Verticals				11.59	3.72



Diagr. 4. Diagramme des coefficients γ (première approximation) pour une charge uniformément répartie sur toute la portée, en fonction de η pour différentes valeurs de μ $\quad 4\frac{b_3}{b_2} = 1{,}25$

Diagramm der Koeffizienten γ (Erste Annäherung) für gleichmäßige Belastung über die ganze Spannweite. Gezeichnet in Funktion von η für verschiedene μ $\quad 4\frac{b_3}{b_2} = 1{,}25$

Diagramm of Coefficients γ (First Approximation). For Uniform Load Covering Whole Span. Plotted on Base of η for different μ $\quad 4\frac{b_3}{b_2} = 1.25$

Using $\quad b_2 = 11.59$

and $\quad 4\frac{b_3}{b_2} = 1.285$ $\qquad$ (79)

the values obtained for μ and η are:

$\mu = 26.72$

and $\quad \eta = 496.$ $\qquad$ (80)

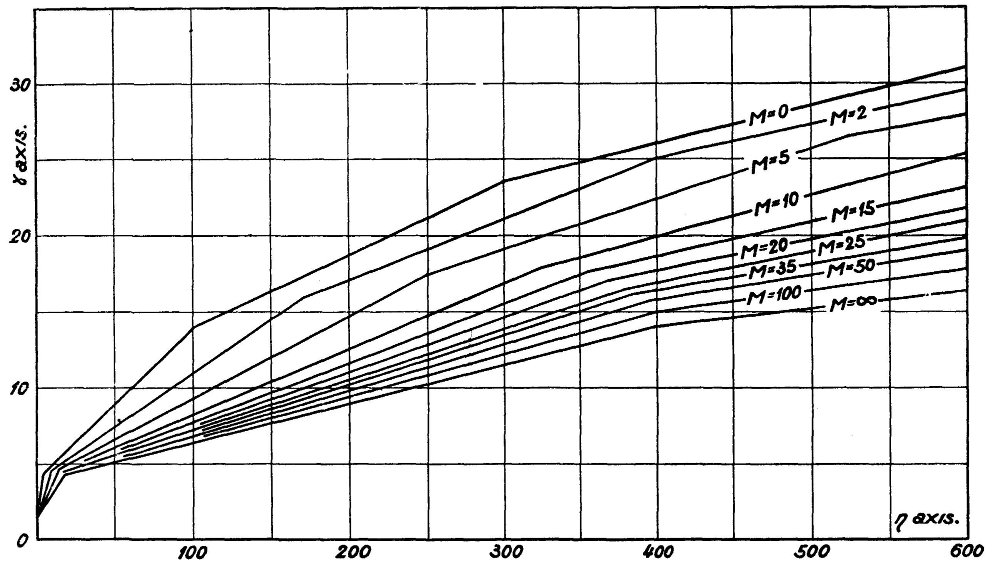
With these figures for μ and η following the same procedure, the ratio

$$y = -0.23, \tag{81}$$

the second approximation of γ comes to $\gamma = 24.2$ (82)

and $q = 3.81$ kip/ft. (83)

These values of γ and q may be considered as final, and they are only about 3 % less than those calculated on the basis of maximum resistance of verticals. It is believed, that such small difference justifying the use of (76), to save all the labour of calculating the stresses and resistances of verticals, is general in most cases.



Diagr. 5. Diagramme des coefficients γ_1 (première approximation) par une charge concentrée au milieu, en fonction de η pour différentes valeur de μ $4\frac{b_3}{b_2} = 1{,}333$

Diagramm der Koeffizienten γ_1 (Erste Annäherung). Für Einzellast in der Mitte. Gezeichnet in Funktion von η für verschiedene μ $4\frac{b_3}{b_2} = 1{,}333$

Diagram of Coefficients γ_1 (First Approximation). For Concentrated Load at the Centre. Plotted on Base of η for different μ $4\frac{b_3}{b_2} = 1{,}333$

It is interesting to notice, that if the influence of torsion of the top chord is disregarded, the error is considerable. When $C = 0$, $\mu = \infty$. Substituting this value of μ together with $\eta = 514$ and $4\frac{b_3}{b_2} = \frac{4}{3}$ into (56), the second approximation of γ comes out

$$\gamma = 20.7, \tag{84}$$

which is 17 % less than (76).

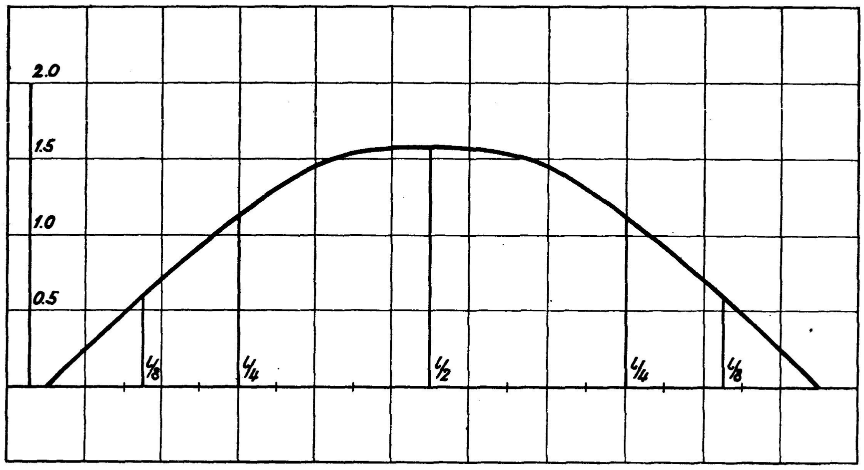
Coming now to the study of stability of the truss under the action of concentrated loads, and the accuracy of the influence line method, the curve ε (diagram 6) is constructed for the span under consideration, allowing $y = -0.23$, as was determined for the uniform load. The following three cases of concentrated loads are studied:

1. Single concentrated load Q at the centre.
2. Two concentrated loads $\frac{Q}{2}$ each, at the quarter points.
3. Two concentrated loads $\frac{Q}{2}$ each, at one eighth points.

The critical values Q are found in two ways: by the influence line method, formula (62); and by the direct method, using formulas (54 a) and (56 a) for the central load, and modifying the denominator in (56 a) for the other two cases, with the value of y in each case corresponding to γ_1 minimum. Various constants used in calculations are as follows:

$$\mu = 26.72; \ \eta = 496; \ q = 3.81 \text{ kip/ft}; \ ql = 476.2 \text{ kips}.$$

Results are presented in the table below.



Diagr. 6.

Ligne d'influence réciproque de la charge critique. Formule 61.
Gegenseitige Einflußlinie der kritischen Belastung. Formel 61. $\Bigg\}$ $p = 3$; $n = 5$; $y = -0{,}23$.
Reciprocal Influence Line of Critical Load. Formula 61.

	Influence Line Method		Direct Method		
	ε	Q kips	y	γ_1	Q kips
Load Q at the Centre	1.556	306.3	−0.265	15.50	305.2
2 Loads $\frac{Q}{2}$ at the $\frac{1}{4}$ Points	1.094	435.5	−0.205	22.10	435.0
2 Loads $\frac{Q}{2}$ at $\frac{1}{8}$ Points	0.581	820.0	−0.178	41.62	819.0

The table shows, that although the shape of the buckling curve varies somewhat as the position of the concentrated load changes, which is manifested by the variation in the value of y, the error of using the influence line method based on constant y, is so small as to be barely detectable in slide rule work. The accuracy for several concentrated loads will evidently be even greater. The other error inherent in the influence line method, when it fails to take into account the change in resistance of verticals, is also a small matter, as can be easily shown by calculation.

Some final remarks on the method and on the approximations and assumptions involved.

Analysis of a pony truss for stability can be greatly assisted by plotting two sets of curves: first, γ curves (second approximations) for different μ, η and $4\frac{b_3}{b_2}$, similar to the first approximations plotted on the diagrams 3 and 4, and secondly, ε curves for different p and y. Determination of the critical uniform load will then be reduced to simple calculation of η and μ, and reading the corresponding γ off the diagram. The critical value of the movable load will be found by proper placing of the load on ε curve with suitable p and y.

It should not be lost sight of, that various stability formulas brought up in this paper, are applicable only as long as the material in any part of the structure, loaded with the critical load, remains below the elastic limit, and as soon as this point is exceeded, the critical values obtained become too high. This limits the applicability of the above formulas only to very slender structures. Thus, in the truss of fig. 13, just considered, the critical uniform load $q = 3.81$ kip/ft causes unit compression stresses: in the first intermediate vertical 25.2 kip/inch^2, and in the central panels of the top chord 45 kip/inch^2. For ordinary structural steel the latter figure is above the elastic limit, and, consequently, the actual value of q will be below 3.81 kip/ft. However, as Professor Timoshenko rightly points out, with present tendency of introduction construction materials of higher strength and of higher elastic limit, the lateral dimensions of the members decrease, and with that the field of application of theoretical formulas, based on perfect elasticity of the material, increases.

It must be pointed out, that although the development of the stability formulas was purely mathematical, a due cognizance was taken of most of the physical factors of importance. It is true that difficulty of the problem required various idealizations of conditions, such as substitution of a continuous wall for actual verticals, and the assumption of constancy of the sections of intermediate verticals and of the top chord, but in those assumptions reality was not idealized out of existance; furthermore, even if the verticals of an actual truss may not have the same cross-section, and the top chord may not be constant on all its length, the formulas, nevertheless, can be used judiciously, as is the case with many other engineering problems.

Of lesser factors left out of consideration may be mentioned on the safe side:

a) Bending resistance of diagonals and their stabilizing effect on the top chord in its tendency to twist.
b) Strengthening with brackets of the bottom end connections of verticals to the floor beams, resulting in increased resistance of verticals to bending.
c) Torsional resistance of verticals, which however is small for *H* or *I* sections.

On the other side may be mentioned the effect of deflection of floor beams caused by bending of the verticals, when the top chord buckles.

The methods used in this paper can be applied to trusses with the end posts of the same rigidity as that of intermediate verticals, and also to the types of trusses different from the one on fig. 1.

Summary.

The author of this paper endeavours to find an expression for the value of the load which causes collapse of certain type of bridge without the top lateral bracing. A qualitative study of the question reveals that the resisting action of the web members is affected by the magnitudes of their axial stresses, and this conclusion leads to the necessity of determination of web resistances in terms of loading. The energy method, consisting in comparison of the elastic energy of structure with the work done by the loading during buckling, is used as the method of attack of the main problem. After evaluation of various terms in the energy equation, expressions for the critical values of different types of loadings are found. It is noticed that the shape of buckling curve is affected by the type of loading only very slightly, and this leads to the idea of the reciprocal influence line of critical loading, allowing an easy treatment of the questions involving movable load. The application of the formulas developed is demonstrated on an example, and the paper is concluded with some final remarks on the method and its assumptions.

Résumé.

L'auteur s'efforce de trouver une expression pour la valeur de la charge de rupture d'un type de pont déterminé sans contreventement supérieur. Une étude qualitative de cette question montre que la résistance des barres du treillis est influencée par la valeur de leurs contraintes axiales et cette constatation conduit à la nécessité d'exprimer cette résistance des barres du treillis en fonction de la charge. L'application du principe de l'énergie, qui consiste à comparer l'énergie d'élasticité du treillis avec le travail que fournit la charge pendant le flambage, est utilisée comme méthode de résolution du problème principal.

Après établissement de différentes expressions pour l'équation de l'énergie, l'auteur donne d'autres expressions pour les valeurs critiques correspondant à différents cas de charge. Il est à remarquer que la forme de la courbe de flambage n'est que très faiblement influencée par la nature de la charge et l'on en arrive ainsi à la notion de la courbe d'influence réciproque pour la charge critique, notion qui permet de traiter facilement les problèmes que posent les charges mobiles.

L'auteur donne un exemple d'application pratique des formules établies et termine son rapport par quelques conclusions sur cette méthode et sur les hypothèses qu'elle implique.

Zusammenfassung.

Der Autor versucht, einen Ausdruck zu finden für den Wert der Bruchlast eines bestimmten Brückentypes ohne oberen Windverband. Eine qualitative Studie dieser Frage ergibt, daß der Widerstand der Fachwerkstäbe durch die Größen ihrer Axialbeanspruchungen beeinflußt wird und diese Folgerung führt zur Notwendigkeit, die Widerstände der Fachwerkstäbe in Ausdrücken der Belastung zu bestimmen. Die Energiemethode, bestehend im Vergleich der elastischen Energie des Fachwerks mit der Arbeit, die die Last während des Ausknickens leistet, wird als Lösungsmethode des Hauptproblems gebraucht. Nach der Bestimmung von verschiedenen Ausdrücken der Energiegleichung werden Ausdrücke für die kritischen Werte für verschiedene Belastungsfälle gefunden. Es wird bemerkt, daß die Form der Knickkurve durch die Art der Belastung nur sehr schwach beeinflußt wird und das führt zu der Idee von der gegenseitigen Einflußlinie für kritische Belastung, die eine leichte Behandlung der Fragen betreffend bewegliche Lasten erlaubt. Die Anwendung der entwickelten Formeln wird an einem Beispiel erläutert und der Beitrag endet mit einigen Schlußfolgerungen über die Methode und ihre Voraussetzungen.