Rheological Theory of Membranes Undergoing Large Deformations (Physical, Geometrical and Engineering Aspects)

Théorie rhéologique des membranes soumises aux grandes déformations (Ses aspects physiques, géometriques et techniques)

Rheologisch-theoretische Untersuchung von Membranen unter Berücksichtigung grosser Deformationen (Deren physikalische, geometrische und ingenieur-technische Aspekte)

Z. BYCHAWSKI
Assoc. Prof. Dr.

W. OLSZAK
Prof. Dr., Dr. h.c.

Poland

Introduction

In the last years, a rapid development of the theory of large deformations can be observed. It is due to the need of obtaining a more powerful tool in investigating the modern structural materials and their mechanical properties. However, the research work in this field is mostly concentrated on the elastic behaviour of rubberlike materials under certain specific conditions. A systematic development of this direction is given in the books of Green and Zerna 1 and Green and Adkins 2 . In the latter, we also find some indications which may lead to further generalizations of the theory as far as materials with rheological properties are concerned. It should be mentioned, however, that because of the generality of considerations, lack of physical aspects and applications they only point out clearly the difficulties encountered in formulating the problem.

In recent structural mechanics and the design of engineering structures, we are oftne faced with the necessity of considering nonlinearities of different kinds, even within the classical concepts of strain and stress states. However, all of them cumulate, if large displacements and, especially, large deformations of flat or spatial modern constructions have to be taken into account. Then the problem is that of a double nonlinearity: physical and geometrical. A typical example of such a problem in engineering are the deformation and stress states in a pneumatic structure. Except some particular cases, the theory of pneumatic structures must necessarily be based on that of nonlinear membranes. Furthermore, since the materials used, such as, for example, plastics and textiles, are very extensible, the constructions undergo large deformations.

Depending on the physical properties of the applied material and loading conditions, it is then necessary to take into account not only the instantaneous effects which occur at the instant of pressure application, but also time-dependent phenomena. These are of rheological nature and may considerably influence the resulting states of strain and stress.

Although the rheological aspects of the theory of nonlinear membranes are of great practical importance in different fields of applications, the available information to be found in literature is rather scarce. It is evident that one of the reasons of such a situation is the lack of an appropriate theoretical approach to interpreting experimental data for real materials in question. On the other hand, it seems to be clear that without such data concerned particularly with large deformations in multiaxial states of strain and stress, the rheological theory of nonlinear membranes may less be determined in an explicit way as could be expected on the basis of its mathematical strictness.

Even if a physical nonlinear theory is founded on proper assumptions, the problem of solving the resulting nonlinear integral or differential equations for the considered concrete cases of practical significance still remains. It is evident that solutions can be found only by applying approximate methods. If these methods are appropriate and carefully chosen, we may, in some particular cases, even expect to obtain analytical results. It would then be possible to have a more general basis for discussions than in the case of a numerical solution.

The main difficulty in establishing a physical theory of large rheological deformations, besides that mentioned above, lies in a proper choice of the form of constitutive equations and physical variables which we want to expose as those of outstanding importance. In the theory of nonlinear membranes it is preferable to have stresses expressed through strains or strain rates. Therefore, all theories which are founded on strain superposing rather than stress are not very suitable in application. This is due to the fact that usually the inversion of a constitutive equation, if at all possible, leads to complicated expressions for stresses, particularly in high nonlinear cases.

According to our opinion, the most convenient approach in founding a physical theory for our purposes, especially concerned with nonlinear membranes of rotational symmetry, is that based on energy considerations. Since a nonlinear rheological process is mainly associated with dissipation of mechanical energy, it seems to be reasonable to introduce into investigation the form of dissipation power.

On the other hand, in order to obtain a more clear physical significance of constitutive relations, it is possible to make use of the known concepts of thermodynamical potentials of deformation states. Independently of the fact that thermodynamical equalities find, in principle, application to stationary reversible processes, they can also be utilized under certain conditions by analogy in investigating quasi-stationary irreversible ones. Thus, the stresses can be found as derivatives of the corresponding energy forms which are functions of the strain state invariants.

Expecially, use can be made of dissipative potentials (often introduced in analogy to elastic potentials) when solving plastic and creep problems by applying variational theorems and methods.

It is the main aim of our paper to give a comprehensive discussion of the problem of setting up a physical theory of nonlinear viscoelastic materials which can be adapted directly to membranes exhibiting large deformations . Because of specific features of the problem in the case of rotational symmetry , which we want to study exclusively , characterized by symmetry of strain and stress states , it is possible to base our considerations on purely homogeneous deformation . Therefore , we do not intend to go deeper into generalities than necessary for our purposes . We shall touch these questions only which , according to our opinion, are fundamental and can lead us directly to effective results . In realizing this aim we bear in mind the possible applications .

In our further investigation we shall assume that the materials considered are isotropic , homogeneous and incompressible .

1. Geometrical aspects of the theory

We consider geometry of deformation of a nonlinear membrane the middle surface of which at time $t = t_o$ (neutral state) is denoted by S_o and represented by dotted lines in Fig.1 . S_o is generated by the revolution of a plane curve f through a full angle about x_3-axis in its plane . The curve f has no multiple points and is smooth . All kinds of singularities are excluded from our considerations .

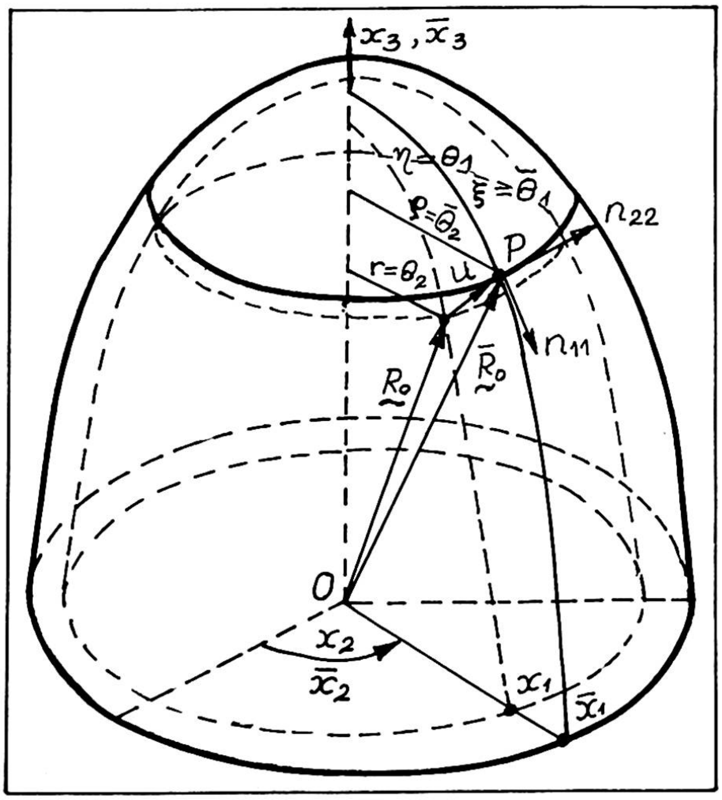

Fig.1

The membrane is of very small thickness $2h_o$ which is constant in the neutral state .

S_o is given in a system of cylindrical coordinates x_i (i=1,2,3) .

At $t=t_o$ the membrane is loaded by a uniform pressure $p = p(t)$ and at an arbitrary instant t ,in consequence of deformation process , we obtain a different rotational membrane with initial axis of symmetry . Its middle surface is now S and thickness 2h . The latter varies with surface coordinates and time .

We assume for S a system of cylindrical coordinates $\bar{x}_i$. Both the systems assumed satisfy the relations

$$x_i = x_i(\bar{x}_j,t) \ , \quad \bar{x}_i = \bar{x}_i(x_j,t) \ , \tag{1.1}$$

where j = 1,2,3 for every i .

On the other hand , we introduce a system of curvilinear coordinates θ_i in which θ_α (α =1,2) coincide with lines of main curvatures on S_o . θ_1 varies within the values $\pm h_o$ on the direction of outward normal to S_o and $\theta_3 = 0$ defines S_o . This system deforms in time together with the membrane and the set of θ_i related to a fixed (at t_o) point P_o remains attached to it as it moves to a new position P at t . Thus , we may write

$$x_i = x_i(\theta_j) \quad , \quad \bar{x}_i = \bar{x}_i(\theta_j, t) \quad , \tag{1.2}$$

where j = 1,2,3 for every i .

The displacement vector of P_o to P is given as the difference of corresponding radius vectors of these positions from the origin O

$$\underset{\sim}{u}(\theta_i, t) = \bar{\underset{\sim}{R}}{}^*_o(\theta_i, t) - \underset{\sim}{R}{}^*_o(\theta_i) = \bar{\underset{\sim}{R}}_o(\theta_\alpha, t) - \underset{\sim}{R}_o(\theta_\alpha) + \theta_3[\underset{\sim}{A}_3(\theta_\alpha, t) - \underset{\sim}{a}_3(\theta_\alpha)] \; , \tag{1.3}$$

where the vectors $\underset{\sim}{R}_o$ and $\bar{\underset{\sim}{R}}_o$ are related to S_o and S , respectively , and $\underset{\sim}{a}_3$ and $\underset{\sim}{A}_3$ are vectors normal to S_o and S . Thus , for a point on the middle surface we find from (1.3)

$$\underset{\sim}{u}_o(\theta_\alpha, t) = \underset{\sim}{R}_o(\theta_\alpha, t) - \underset{\sim}{R}_o(\theta_\alpha) . \tag{1.4}$$

The line elements on S and S_o , corresponding to the vectors $\bar{R}_o$ and R_o are , respectively ,

$$ds^2 = A_{\alpha\beta}\, d\theta_\alpha d\theta_\beta \quad , \tag{1.5}$$

$$ds_o^2 = a_{\alpha\beta}\, d\theta_\alpha d\theta_\beta \quad , \tag{1.6}$$

where $A_{\alpha\beta}$ and $a_{\alpha\beta}$ are covariant base tensors of S and S_o , respectively .

According to the existing symmetry we define the principal extension ratios λ_i (i=1,2,3) in meridional , latitudinal and normal directions . These coincide with the principal directions of strain . Denoting by $\bar{\theta}_1$ and θ_1 the arc lengths measured along meridians from A to P and A_o to P_o , respectively , we have

$$\lambda_1 = d_{\theta_1}\bar{\theta}_1 \quad , \quad \lambda_2 = \bar{x}_1/x_1 \quad , \quad \lambda_3 = \lambda = (\lambda_1\lambda_2)^{-1} \; , \tag{1.7}$$

where $d_{\theta_1} = d/d\theta_1$; the value of λ (from the condition of incompressibility) being dependent on both the remaining ratios .

On the basis of Eq. (1.7) we may write Eqs. (1.5) and (1.6) as follows

$$ds^2 = \lambda_1^2(d\theta_1^1)^2 + \lambda_2^2 x_1^2 (d\theta^2)^2 \quad , \tag{1.8}$$

$$ds_o^2 = (d\theta^1)^2 + x_1^2(d\theta^2)^2 \quad , \tag{1.9}$$

and thus define the surface strain tensor

$$\gamma_{\alpha\beta} = \frac{1}{2}(A_{\alpha\beta} - a_{\alpha\beta}) \quad , \tag{1.10}$$

given by the difference of Eqs. (1.8) and (1.9)

$$2\gamma_{\alpha\beta} d\theta_\alpha d\theta_\beta = ds^2 - ds_o^2 = (\lambda_1^2 - 1)(d\theta^1)^2 + x_1^2(\lambda_2^2 - 1)(d\theta^2)^2 \; . \tag{1.11}$$

The relative extension of the line element ds is then found to be

$$\gamma = (ds-ds_0)/ds_0 \ , \quad (1+\gamma)^2 - 1 = 2\gamma_{\alpha\beta}d\theta_\alpha d\theta_\beta (ds_0)^{-2} \ . \tag{1.12}$$

Let us consider now the strain rate tensor which plays an important part in what follows . If the rheological process of deformation is such that at t_0 the line element ds_0 is given by Eq.(1.6) and at t by Eq.(1.5) , then at t+dt its length becomes equal to

$$ds + d(ds) = (1+\gamma)\, ds_0 + d(ds) \ , \tag{1.13}$$

the rate of extension being d(ds)/dt . Thus , the deformation rate is obtained as the ratio[d(ds)/dt]/ds

$$\omega = \dot{\gamma}(1+\gamma)^{-1}, \quad \dot{\gamma} = d\gamma/dt \ . \tag{1.14}$$

On the other hand , by differentiating Eq.(1.12) at fixed t we find

$$\dot{\gamma}(1+\gamma) = \dot{\gamma}_{\alpha\beta}d\theta_\alpha d\theta_\beta (ds)^{-2} \ , \tag{1.15}$$

and from Eq.(1.14) follows that

$$\omega = \dot{\gamma}_{\alpha\beta}d\theta_\alpha d\theta_\beta[(1+\gamma)ds]^{-2} \ , \tag{1.16}$$

the counterpart of Eq.(1.11) being

$$\dot{\gamma}_{\alpha\beta}d\theta_\alpha d\theta_\beta = \dot{\lambda}_1\lambda_1(d\theta^1)^2 + x_1^2\,\dot{\lambda}_2\lambda_2(d\theta^2)^2 \ . \tag{1.17}$$

Since on the basis of Eq.(1.11) we have main strains at the middle surface

$$\gamma_{11} = \lambda_1^2 - 1 \ , \quad \gamma_{22} = (\lambda_2^2 - 1)\,x_1^2 \ , \tag{1.18}$$

the main strain rates are

$$\dot{\gamma}_{11} = \dot{\lambda}_1\lambda_1 \ , \quad \dot{\gamma}_{22} = \dot{\lambda}_2\lambda_2\, x_1^2 \ . \tag{1.19}$$

The nondimensional strain tensor components are obtained by means of transformation

$$e_{\alpha\beta} = \gamma_{\alpha\beta}(a_{\alpha\alpha}\, a_{\beta\beta})^{-\frac{1}{2}} \ , \tag{1.20}$$

which instead of Eq.(1.18) gives

$$e_{11} = e_1 = \lambda_1^2 - 1 \ , \quad e_{22} = e_2 = \lambda_2^2 - 1 \ . \tag{1.21}$$

Thus , the nondimensional strain rate tensor components are

$$\dot{e}_1 = \dot{\lambda}_1\lambda_1 \ , \quad \dot{e}_2 = \dot{\lambda}_2\lambda_2 \ . \tag{1.22}$$

The remaining components of the general strain tensor γ_{ij}

are found on the basis of general metric tensors for S and S_o ,denoted G_{ij} and g_{ij} , respectively .

For an arbitrary point of the membrane we may write , respectively ,

$$G_{\alpha\beta} = A_{\alpha\beta} - 2\theta_3 B_{\alpha\beta} \quad , \quad G_{\alpha 3} = 0 \quad , \quad G_{33} = \lambda^2 \quad , \qquad (1.23)$$

where

$$A_{11} = \lambda_1^2 \; , \; A_{22} = x_1^2\lambda_2^2 \; , \; A_{\alpha\beta} = 0 \; (\alpha \neq \beta) \; , \; B_{11} = -k_{11} \; , \qquad (1.24)$$

$$B_{22} = -k_{22} \quad , \quad B_{\alpha\beta} = 0 \; (\alpha \neq \beta) \quad ,$$

and

$$g_{\alpha\beta} = a_{\alpha\beta} - 2\theta_3 b_{\alpha\beta} \quad , \quad g_{\alpha 3} = 0 \quad , \quad g_{33} = 1 \quad , \qquad (1.25)$$

where

$$a_{11} = 1 \; , \; a_{22} = x_1^2 \quad , \; a_{\alpha\beta} = 0 \; (\alpha \neq \beta) \quad , \; b_{11} = -k_{11}^{o} \; , \qquad (1.26)$$

$$b_{22} = -k_{22}^{o} \quad , \quad b_{\alpha\beta} = 0 \; (\alpha \neq \beta) \quad .$$

Here $B_{\alpha\beta}$, $b_{\alpha\beta}$ are tensors associated with the second fundamental form of the surfaces S and S_o , respectively , and $k_{\alpha\beta}$ and $k_{\alpha\beta}^{o}$ are corresponding curvatures .

From Eq.(1.23) and Eq.(1.25) it follows that

$$\gamma_{\alpha 3} = 0 \quad , \quad \gamma_{33} = \tfrac{1}{2}(\lambda^2 - 1) = \tfrac{1}{2}[(\lambda_1\lambda_2)^{-1} - 1] \quad , \qquad (1.27)$$

and its rate is

$$\dot{\gamma}_{33} = \dot{\lambda}\lambda = -\tfrac{1}{2}(\lambda_1\lambda_2)^{-2}(\dot{\lambda}_1\lambda_2 - \lambda_1\dot{\lambda}_2) \quad . \qquad (1.28)$$

Furthermore , from Eq.(1.2o) we conclude that

$$e_{33} = e_3 = \tfrac{1}{2}(\lambda^2 - 1) \quad , \quad \dot{e}_3 = \dot{\lambda}\lambda \quad . \qquad (1.29)$$

We shall express now the components of strain tensor through the components of displacement vector given by Eq.(1.3)(for $\theta_3=0$), or Eq.(1.4). In order to do that, we represent the displacement vector in the form

$$\underset{\sim}{u} = u_i \underset{\sim}{g}^i = u^i \underset{\sim}{g}_i \quad , \qquad (1.3o)$$

where $\underset{\sim}{g}_i$ and $\underset{\sim}{g}^i$ are covariant and contravariant vectors of the base , respectively ,

$$\underset{\sim}{g}_i \underset{\sim}{g}_j = g_{ij} \quad , \quad \underset{\sim}{g}^i \underset{\sim}{g}^j = g^{ij} \quad . \qquad (1.31)$$

Taking into account the relation between base vectors

$$\underset{\sim}{G}_i = \underset{\sim}{g}_i + \underset{\sim}{u}_{,i} \quad , \qquad (1.32)$$

and , hence ,

$$G_{ij} = g_{ij} + \underset{\sim}{g}_i \underset{\sim}{u}_{,j} + \underset{\sim}{g}_j \underset{\sim}{u}_{,i} + \underset{\sim}{u}_{,i} \underset{\sim}{u}_{,j} \quad , \qquad (1.33)$$

where

$$u_{,i} = u_k|_i g^k \quad , \quad u_k|_i = u_{k,i} - \Gamma^{l}_{ki} u_l \quad , \tag{1.34}$$

we find strain tensor γ_{ij} by introducing Eq. (1.34) into Eq. (1.33). Thus , we obtain

$$\gamma_{ij} = \frac{1}{2}(u_i|_j + u_j|_i + u^k|_i u_{k\ j}) , \quad u^k|_i = u^k{}_{,i} + \Gamma^{k}_{li} u^l \quad , \tag{1.35}$$

where Christoffel's symbols Γ are calculated for S_o from the metric tensors g_{ij}, g^{ij} of S_o .

By performing the indicated in Eq. (1.35) operations on displacement components and having in mind Eq. (1.20) we find physical components of strain state (for S_o , $\theta_3 = 0$)

$$e_1 = \partial_{\theta^1} u_1 + \frac{1}{2}(\partial_{\theta^1} u_1 + k_{11}^{\ o} u_3)^2 + \frac{1}{2}(\partial_{\theta^1} u_3 - k_{11}^{\ o} u_1)^2 \quad , \tag{1.36}$$

$$e_2 = k_{22}^{\ o} u_3 + \frac{1}{2}(k_{22}^{\ o} u_3)^2 \quad , \tag{1.37}$$

$$e_3 = -\frac{1}{2}(k_{11}^{\ o} u_1)^2 \quad . \tag{1.38}$$

These are geometrical formulas expressing strain state components through displacement components . As it is seen , the first equation depends on Eqs. (1.37) and (1.38). By solving the latter with respect to u_3 and u_1 , respectively , and introducing the results into the former , we thus obtain the condition of compatibility of strain state .

From Eqs. (1.37) and (1.38) we find , respectively ,

$$u_3 = -R_2^o\left(1 + \sqrt{1+2e_2}\right) , \quad u_1 = R_1^o\sqrt{-2e_3} \quad , \quad e_3 < 0 \quad , \tag{1.39}$$

and the said condition gives

$$\partial_\eta\left(R_1^o\sqrt{2\bar{e}_3}\right) + \frac{1}{2}\left[\partial_\eta\left(R_1^o\sqrt{2\bar{e}_3}\right) - R_2^o R_1^{o^{-1}}\left(1+\sqrt{1+2e_2}\right)\right]^2 + \frac{1}{2}\left\{\partial_\eta\left[R_2^o\left(1+\sqrt{2e_2+1}\right)\right] + \right.$$

$$\left. + \sqrt{2\bar{e}_3}\right\}^2 - e_1 = 0 \quad , \quad \bar{e}_3 = -e_3 \; , \quad \eta = \theta_1 \; . \tag{1.40}$$

Here , R_1^o, R_2^o denote main curvature radii

$$k_{11}^{\ o} = R_1^{o^{-1}} \; , \quad k_{22}^{\ o} = R_2^{o^{-1}} \quad . \tag{1.41}$$

The curvatures of Eq. (1.41) satisfy the equation

$$d_\eta\left(r k_{22}^{\ o}\right) = k_{11}^{\ o}\, d_\eta r \quad , \quad r = x_1 \quad , \tag{1.42}$$

where

$$k_{11}^{\ o} = -\, d_\eta^2 r\left[1 - (d_\eta r)^2\right]^{-\frac{1}{2}} \quad . \tag{1.43}$$

Substituting Eq.(1.43) into Eq.(1.42)and taking into account the fact that k_{22}^{o} is finite for r=0 and $d_\eta r = 1$ for r=0 , through integration we obtain

$$k_{22}^{o} = r^{-1}\left[1 - (d_\eta r)^2\right]^{\frac{1}{2}} . \tag{1.44}$$

Analogous formulae are valid for the curvatures k_{11} and k_{22} at arbitrary instant t of the deforming membrane . Thus , we have

$$k_{11} = - d_\xi^2 \varrho \left[1 - (d_\xi \varrho)^2\right]^{-\frac{1}{2}} , \quad k_{22} = \varrho^{-1}\left[1 - (d_\xi \varrho)^2\right]^{\frac{1}{2}} , \tag{1.45}$$

where we put $\varrho = \bar{x}_1$ and $\xi = \bar{\theta}_1$. Thus , Eqs.(1.43)and (1.44) can be considered as initial conditions for Eqs.(1.45) describing the continuous change of curvatures during deformation process

$$k_{11}(t) = \left[R_1(t)\right]^{-1} , \quad k_{22}(t) = \left[R_2(t)\right]^{-1} , \tag{1.46}$$

where R_1, R_2 are the radii of main curvatures at instant t .

Finally , it should be mentioned that the corresponding components of strain rate state may be found by differentiating Eq.(1.35) or Eqs.(1.36)-(1.38) with respect to time .

2.Statical aspects of the theory

According to the membrane theory, we neglect all moments and shearing forces in our considerations of quasi static equilibrium of rheological process . In what follows we refer all results to the undeformed membrane .

The physical stress resultants per unit length are given by the relation

$$n_{\alpha\beta} = n^{\alpha\beta}\left(a_{\beta\beta}/a^{\alpha\alpha}\right)^{\frac{1}{2}} , \tag{2.1}$$

where $n^{\alpha\beta}$ satisfy the conditions of equilibrium

$$n^{\alpha\beta}|_\alpha = 0 , \quad n^{\alpha\beta} b_{\alpha\beta} + p = 0 , \quad p = p_1 - p_2 , \tag{2.2}$$

Here, p is the resultant pressure in the direction of the outward normal to the middle surface and

$$n^{\alpha\beta}|_\alpha = n^{\alpha\beta}{}_{,\alpha} + \Gamma^\beta_{\alpha\gamma} n^{\alpha\gamma} + \Gamma^\alpha_{\alpha\gamma} n^{\gamma\beta} . \tag{2.3}$$

In our particular case we have only two stress resultant components n_{11} and n_{22} and two non-vanishing components of the Christoffel tensor , and Eq.(2.2) furnishes

$$n^{11}{}_{,1} + \Gamma^2_{21} n^{11} + \Gamma^1_{22} n^{22} = 0 , \tag{2.4}$$

or

$$d_\eta(r n_{11}) = r^2 n_{22} d_\eta r , \tag{2.5}$$

if Eq.(2.1) is taken into account . On the other hand , the second of Eq.(2.2) gives

$$k_{11}^{o} n_{11} + r^2 k_{22}^{o} n_{22} = p \tag{2.6}$$