Thick-walled reinforced concrete pipes: proposals for increasing their statical efficiency

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Thick-Walled Reinforced Concrete Pipes: Proposals for Increasing their Statical Efficiency.

Dickwandige Eisenbetonleitungen. Vorschläge zur Verbesserung ihres statischen Wirkungsgrades.

Tuyaux de béton armé à parois épaisses. Propositions en vue d'améliorer leur rendement statique.

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It is a characteristic of thick walled structures loaded as in Fig. 1 by normal forces uniformally distributed over the circumference that (whether reinforcement is provided or not) only very ineffective use is made of the material. This fact is made clearly apparent by the well known formulae of Lamé in which the conditions of stress arising in thick walled structures of this kind are defined on the basis of their elastic, isotropic behaviour. If the cross section of a thick pipe line, or of a tunnel shaft or other mining structure analogous thereto is designed by keeping the maximum stresses below certain values which are looked upon as "permissible" — the method which continues to serve the basis of most of the official rules for statical calculations — then it is straightaway obvious that long before the remainder of the cross sectional area is stressed up to its permissible limit the concentration of stress at the inner face will bring the tangential stress σ'_t up to its limit σ_{zul} (see Fig. 2).[1]

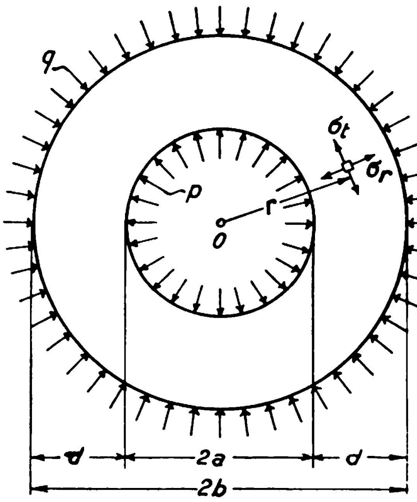

Fig. 1.
General arrangement.

If, on the other hand, the design is based on the observance not of permissible stresses but on permissible strain — a procedure which is preferable to the first mentioned on statical grounds — then the conditions which arise in the case here contemplated may in some circumstances be still more unfavourable. The magnitude of the strain σ'_{red} (see Fig. 2) in structures subject to an internal hydrostatic pressure p follows an even more steeply rising curve than does the distribution of peripheral stresses σ'_t, as is at

[1] Here, and frequently in what follows below, the radial stresses σ'_r are ignored as being of smaller importance.

once apparent on plotting the radial stresses σ'_r. The heavy concentration of strain $\sigma'_{red\ max}$ which thus occurs on the inner face is not only very undesirable as regards the statical stability of the structure but also (especially in the case of pressure pipe lines) very dangerous if faultless operation is to be ensured, since it is precisely at this point that cracks and other defects are apt to originate in the event of casual overloading produced by water-shocks.

These conditions obtain even in unreinforced pipes and in similar thick walled concrete structures which may properly be treated as isotropic, displaying the same elastic behaviour in all directions so that the Lamé formulae, already

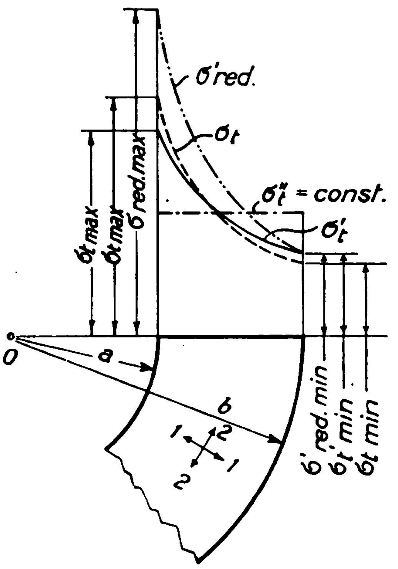

Fig. 2.

Distribution of tangential (circumferential) stresses.

——— σ'_t in material of isotropic texture ($E_1 = E_2$).

- - - - σ_t in material of (polar or cylindrical) orthotropic texture ($E_2 > E_1$).

—·—·— σ''_t constant, with improved construction (non-homogeneous orthotropic texture of material). (E_2 variable.)

—··—··— σ'_{red} strain in material of isotropic texture.

mentioned, can be directly applied. When the cross section is strengthened by the provision of steel reinforcement the result becomes even more objectionable; such reinforcement is usually provided, to make sure of coping with any tensile stresses liable to arise in pipe lines, and it becomes indispensable when either the internal pressure p or the external pressure q attains any considerable value, for the thickness of the wall cannot on purely practical grounds be increased beyond certain limits, where this would make the structural elements clumsy and difficult to handle. Moreover, the strength effect does not increase in direct proportion to the thickness of the wall; the rate of increase of strength (in relation to the diameter of the pipe) becomes slower as the thickness becomes greater. The necessary increase in resistance against internal or external normal pressures would be proportional to the thickness of the pipe only if the distribution of the stresses were such that the average circumferential stresses σ''_t assumed to exist over the cross sections were the same for any thickness ($\alpha = \frac{a}{b}$ = any chosen value, using the notation shown in Fig. 1); or, assuming a constant value of $\sigma'_{t\,max}$, roughly $\sigma'_{t\,max} = \sigma_{zul}$.

We know, however, that in fact σ''_t rapidly decreases as the thickness of the wall increases, and in order to determine this numerically it is sufficient to relate the circumferential stress assumed te be evenly distributed over the cross section to the maximum stress arising therein.

$$\sigma''_t = \eta' \sigma'_{t\,max}. \tag{1}$$

For isotropic thick walled pipes

$$\eta' = \frac{\sigma''_t}{\sigma'_{t\,max}} = \left(\frac{ap}{b-a}\right) : \left(\frac{a^2+b^2}{b^2-a^2}\,p\right) = \alpha\,\frac{1+\alpha}{1+\alpha^2}, \tag{2}$$

when for simplicity the internal normal pressure p is taken as the only loading, so that $q = 0$.

The "coefficient of utilisation" η' — or "statical efficiency" of the structure as this characteristic figure will be called in what follows below — may be found from Table 1 in relation to different ratios of $\alpha = \frac{a}{b}$:

Table I.

α	0.0	0.2	0.4	0.5	0.6	0.8	1.0
η'	0.00	0.23	0.48	0.60	0.71	0.88	1.00

This table shows that as the thickness of the wall increases the material is less and less effectively utilised, and consequently the structure becomes less and less economical. Even with a quite moderate thickness of wall the material is very wastefully employed.

It should also be noticed that as soon as $p > \sigma_{zul}$ $\left(\text{or } q > \frac{\sigma_{zul}}{2}\right)$ an unreinforced form of construction becomes altogether impossible, because even if the cross section were infinitely thick $(\alpha = \frac{a}{b} = 0)$ the maximum stress would never fall below the value $\sigma'_{t\,max} = p$ (or $\sigma'_{t\,max} = -2q$).

In practice, therefore, recourse is had to the proved expedient of steel reinforcement, divided between several (usually two) concentric rings. In this way the tensile or compressive stress in the concrete is reduced approximately in the proportion $\frac{100}{100 + nF_2}$ wherein F_2 represents the percentage of circumferential reinforcement and $n = \frac{E_s}{E_c}$ the ratio between Young's elastic moduli for steel and concrete respectively. In this way, however, the elasticity of the structural element around the periphery is reduced; that is to say it becomes unisotropic, and behaves in a way directly opposite to that previously explained, so that the lack of uniformity in distribution of stress through the thickness of the wall which is so undesirable an occurrence becomes further emphasised, or in other words the σ_t line (see Fig. 2) becomes still steeper, and the dangerous concentration of stress on the inner face of the pipe partly counteracts the favourable effect of the reinforcement provided.

There is no particular difficulty in representing this effect mathematically. The unisotropic character of the compound structure due to the reinforcing steel is expressible by saying that in thick walled pipes and in similar reinforced concrete structures the elastic behaviour differs along each of three principal directions which are respectively given by (1) the direction of the radius vector r (percentage of reinforcement F_1, elastic constants E_1 and m_1) (2); the tangent t to the concentric circles drawn with the axis of the pipe as origin F_2, E_2, m_2; and (3) the longitudinal deformation z of the cylindrical structure (F_3, E_3, m_3). Hence, by contrast with the rectilinear orthogonal anisotropy (or "orthotropy" as it is sometimes called), we now have a condition of curvilinear anisotropy for which the author proposes the designation *"polar orthotropy"* in the case of thin slices or *"cylindrical orthotropy"* in that of pipes or cylinders of finite or infinite length.[2]

Under certain conditions the moduli of elasticity in these three directions (E_1, E_2, E_3) may differ greatly from one another, and the same is true of the Poisson's ratios (or, more properly, the coefficents of transverse deformation m_1, m_2, m_3). These constants are, however, not independent of one another, for the following simple relationship exists between them:

$$m_1 E_1 = m_2 E_2 = m_3 E_3 \tag{3}$$

The value of this constant product will be denoted henceforth by M. The shear moduli G are without significance for the present argument.

The different values of the elastic constants along the three principal directions are influenced by the method of manufacturing the pipe (as, for instance, by the centrifugal process), but they are primarily due to the differing percentages of steel reinforcement in each direction.

If the percentage of reinforcement in one of the directions i ($i = 1, 2, 3$) is denoted by F_i, then the modulus of elasticity of the compound body along this direction may be made to agree with the formula of Professor *M. T. Huber*[3]:

$$E_i = E_b \lambda_i \quad \text{where } \lambda_i = 1 + (n - 1)\frac{F_i}{100} \tag{4}$$

using the ratio $n = E_s/E_c$ as already defined. It is true that this assumes the substitution of the unstable unisotropic compound body by an ideal stable orthotropic model — a simplification which is necessary for the clear understanding and numerical expression of what follows below, and is the more justifiable the greater the closeness of the structural element (steel reinforcement) in proportion to the remaining dimensions. Later, when applying the results obtained by this method in practice account must be taken of this fact.[4]

[2] It was originally intended to discuss the whole complex of problems of polar-orthotropic slices and cylindrical-orthotropic pipes in the present paper, but the limitations later imposed upon the length of individual contributions led the author to relegate the more exhaustive treatment of these to a publication of his own, „Beiträge zur Statik von polarorthotropen Scheiben und zylinderorthotropen Rohren", in Der Bauingenieur, 1936, N° 31/32. In the present paper frequent reference will be made to this using the abbreviation WO 17.

[3] *M. T. Huber:* Probleme der Statik technisch wichtiger orthotroper Platten. (Statical problems of orthotropical shells of technical importance.) Academy of Technical Sciences, Warsaw, 1929 (in German), p. 13.

[4] An accurate account of this discontinuity, in terms of elastic theory, would be possible only by abandoning the simpler case of a homogeneous anisotropic model and considering the problem spatially (with axial symmetry) as that of a non-homogeneous complex of two isotropic components, somewhat after the manner that Dr. Ing. *A. Freudenthal* has dealt with hooped columns („Verbundstützen für hohe Lasten", Columns of compound nature and heavy loads, Berlin, 1933). To anyone sufficiently interested who is not afraid of the time and trouble involved this offers the certainty of a profitable and illuminating task.

The determination of the transverse extension coefficient m_i ($i = 1, 2, 3$) is more difficult because only a very limited amount of experimental data are at present available on this matter. The determination of these values (which, as will presently appear, is necessary in order to arrive at the conditions of stress and strain arising in the anisotropic compound construction) will be explained by the writer elsewhere (WO 17).

In a cylindrical structure subject to uniform internal or external pressure the conditions of strain and stress do not depend upon the co-ordinate z, in other words it is a case of the "plane problem" in elastic theory. As a result of the circular symmetry here assumed, and of the geometrical form of the cross sections and external loads, the conditions of strain and stress are also independent of the central angle φ, so that the whole field of stress is influenced only by a single variable, namely the radius vector r. Instead of partial differentiations, total differential quotients will arise.

Equal emphasis should be laid on the remarkable fact that in an unisotropic structure of material the uniplanar condition of *strain* (deformation) and the uniplanar condition of stress are not interchangeable, this being directly contrary to what is true of isotropic materials. In the latter case, as is well known, it is frequently of no importance whether a piece be cut out of the inside of an (indefinitely) long cylinder so that the restraint of the portions adjoining its front and back surfaces prevent it from deforming except internally, or whether a similar slice be taken from the free end of a cylinder so that it possesses complete freedom of deformation. In these two cases the conditions of stress in the coordinated planes of an isotropic slice are identical (apart from considerations into which the constants of the material enter). In isotropic media these two cases can always be handled in the same way.[5]

This is not true for an anisotropic structure. Here not only is the distinction between a uniplanar condition of strain and a uniplanar condition of stress essential in principle, but the fields of stress deviate *effectively* from one another. This difference, numerically considered, can however be ignored in practical engineering, because in the group of equations for radial and tangential stresses σ_r and σ_t the difference is given by the "structural figures" s and t which differ only slightly from one another (as will be shown below). The plane anisotropic problem must further be divided into the study of an annular disc of an infinitely long pipe and that of infinitely long but developed pipe. The numerical solution to these problems must also be separately approached.

There is insufficient space here to work out this procedure of calculation for all the possible variants and cases that can arise, and reference must be made to the work "WO 17" cited in footnote,[2] but the line of thought to be followed in dealing with the simplest case, namely a circular annular slice of limited thickness (or depth) may be indicated here briefly to serve as a guide towards the final result.

[5] See, for instance, the author's paper „Der ebene Formänderungs- und Spannungszustand der Elastizitätstheorie" (The plane stress and strain conditions of the elastic theory) in Zeitschrift des Österr. Ingenieur- und Architektenvereines, 1936, N° 15/16, where it is shown that in an isotropic structure both the limiting cases mentioned above will *always* admit of a formal treatment in common, even where there is a complex of interrelated zones, or where displacements (instead of stresses) are laid down as the limiting conditions.

The starting point is found in the equation of equilibrium for an annular element bounded by two neighbouring radial sections and by two neighbouring concentric circles within the annulus:

$$\sigma_t = \frac{d}{dr}[r\sigma_r] \tag{5}$$

Since, in any case likely to arise in practice, the annular slice will be reinforced with steel mainly in the direction of the periphery, the modulus of elasticity E_2 in that direction will always be greater than the modulus of elasticity E_1 measured in the direction of the radius vector r, that is $E_2 > E_1$, use being made of Equation (4) for calculating E_2. (If, however, the conditions differ from these, as for instance in the case of a fly-wheel, where radial reinforcement may have to be taken in account, the line of thought will still be the same in principle though the quantitative results will be different.) The influence of special methods of making and placing concrete (such as the use of vibration or centrifugal action) on the ratio $\frac{E_2}{E_1}$ must for the present be left out of account in the absence of any experimental data on the subject.

Taking account of the difference in the elastic properties of the material in the two opposed curvilinear principal directions at right angles to one another, "1" and "2", the fundamental relationship between the deformation and stress components to be considered here, which result in the statical indeterminacy of the elastic problem for any continuous medium, may be written as follows,

$$\left.\begin{aligned} \varepsilon_r &= \frac{du}{dr} = +\frac{1}{E_1}\sigma_r - \frac{1}{m_2}\frac{1}{E_2}\sigma_t, \\ \varepsilon_t &= \frac{u}{r} = -\frac{1}{m_1}\frac{1}{E_1}\sigma_r + \frac{1}{E_2}\sigma_t, \end{aligned}\right\} \tag{6}$$

wherein u represents the increase in radius.

It will be noticed that here the stress component σ_z is completely eliminated ($\sigma_z = 0$) this being characteristic of such a case of the uniplanar stress condition as has just been dealt with (and also of what many authors[6] call the quasi-planar stress condition) when the plate (or slice) is of small thickness (depth). Actually the thinner the disc in proportion to its other dimensions the more nearly will this condition be approached. It should also be noticed that all the stress values (σ, τ) and also the strain deformation components (ε) are averages taken through the thickness of the disc.

The solution to the group of equations (6) for the stress component leads to the following stress-strain equations:

$$\left.\begin{aligned} \sigma_r &= \frac{M}{m_1 m_2 - 1}\left[\frac{u}{r} + m_2\frac{du}{dr}\right], \\ \sigma_t &= \frac{M}{m_1 m_2 - 1}\left[m_1\frac{u}{r} + \frac{du}{dr}\right]. \end{aligned}\right\} \tag{7}$$

[6] See, for instance, *H. Reißner* and *F. Strauch:* Ringplatte und Augenstab (Annular slab and eye-bars). Ing.-Archiv, 1933, p. 483.

There remains only one further step, namely that of substituting the values obtained from (7) in the equations of equilibrium (5), giving rise to the differential equation

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{m_1}{m_2}\frac{u}{r^2}=0 \tag{8}$$

from which the radial displacement u is obtainable. The integral of this

$$u=Ar^s+Br^{-s}, \tag{9}$$

wherein

$$s=\sqrt{\frac{m_1}{m_2}}=\sqrt{\frac{E_2}{E_1}} \tag{10}$$

must be made to comply with the marginal conditions prescribed:

$$\sigma_r=\begin{cases}-p\\-q\end{cases}\quad \text{for } r=\begin{cases}a\\b\end{cases} \tag{11}$$

After a somewhat lengthy process of calculation and the introduction of the unnamed (dimensionless) "reduced" radial co-ordinates $\rho=\frac{r}{b}$ the desired stress values are finally obtained in the form[7]:

$$\left.\begin{aligned}\sigma_r&=\frac{1}{1-\alpha^{2s}}\left\{\rho^{s-1}[p\alpha^{s+1}-q]-\left(\frac{\alpha}{\rho}\right)^{s+1}[p-q\alpha^{s-1}]\right\},\\ \sigma_t&=\frac{s}{1-\alpha^{2s}}\left\{\rho^{s-1}[p\alpha^{s+1}-q]+\left(\frac{\alpha}{\rho}\right)^{s+1}[p-q\alpha^{s-1}]\right\},\end{aligned}\right\} \tag{12}$$

wherein α denotes the ratio of internal to external diameter (as already used), $\alpha=\frac{a}{b}$, which definitely fixes the form of cross section.

It may easily be shown that by putting $E_1 = E_2$ (so that $s = 1$) the transition to the Lamé formulae which hold good of isotropic bodies is obtained, this being a special case of the argument given here.

The case, which is of special interest here, of a uniformly distributed internal pressure p (simultaneously with $q = 0$) imposed alone, leads to the expressions

$$\left.\begin{aligned}\sigma_r&=\frac{1}{1-\alpha^{2s}}\left(\frac{\alpha}{\rho}\right)^{s+1}[\rho^{2s}-1],\\ \sigma_t&=\frac{s}{1-\alpha^{2s}}\left(\frac{\alpha}{\rho}\right)^{s+1}[\rho^{2s}+1],\end{aligned}\right\} \tag{13}$$

which, when $s = 1$ (and therefore $E_1 = E_2$), also of course lead to the generally well known formulae for thick-walled isotropic pipes.

The extreme fibre stresses of the circumferential stresses may be taken as especially characteristic values of the stress components derived above. For the

[7] The same results may be confirmed in a rather different way by introducing what is called the "stress function", out of which the stress components are obtained by derivation.

inner face where r = a we obtain with $\rho = \alpha$ the maximum stress that can arise anywhere in the cross section, amounting to

$$\sigma_{t,r=a} = s\frac{1+\alpha^{2s}}{1-\alpha^{2s}}p, \qquad \left(\sigma'_{t,r=a} = \frac{1+\alpha^2}{1-\alpha^2}p\right); \tag{14}$$

which is always positive, and therefore tensile. For the outermost fibre r = b, and with $\rho = 1$ we obtain

$$\sigma_{t,r=b} = 2s\frac{\alpha^{s+1}}{1-\alpha^{2s}}p, \qquad \left(\sigma'_{t,r=b} = 2\frac{\alpha^2}{1-\alpha^2}p\right). \tag{15}$$

If the results obtained in this way are compared with those for the extreme fibre stresses existing in an isotropic form of body[8] (as given here in brackets on the corresponding lines in order to facilitate comparison) it will be seen that in the reinforced (and, therefore, orthotropic) pipe the outer fibres and the material close to them are relieved of load relatively to the unreinforced form of construction ($\sigma_{t,r=b} < \sigma'_{t,r=b}$). The result is that the concentration of stress existing at the inner face — already dangerous with the kind of anisotropy here considered ($E_2 > E_1$; $s > 1$), which is the very one that most often arises in practice — is thereby considerably worsened ($\sigma_{t,r=a} > \sigma'_{t,r=a}$).

This is true of the annular slice of limited thickness (depth) which we have just examined, that is to say for the case of uniplanar *stress* conditions. When the case of a pipe of finite but great length (E_z = k = const. $\neq$ 0) or infinite (E_z = k = 0) length is considered, the conditions become even more unfavourable. Without entering on the mathematical consideration of the conditions of stress and strain arising in these latter cases (which for lack of space must be relegated to sections III and IV of the work "WO 17" cited in the footnote[2]) the results for these two cases may be indicated here. By a process of reasoning similar to that indicated above, the remarkable fact is established that the stress components σ_r and σ_t are exactly the same in form as those determined by the group of equations (12), with the sole difference that the term s which occurs as a factor and exponent, instead of being evaluated from (10) now has the value

$$t = \sqrt{\frac{m_1 m_3 - 1}{m_2 m_3 - 1}} \tag{16}$$

Instead of the characteristic $\sigma_z = 0$, which arose above, we now have the function

$$\sigma_r = \frac{1}{m_3}(\sigma_r + \sigma_t + kM) \tag{17}$$

which determines the stresses acting in the direction of the length of the cylinder; stresses which now cause purely plane deformations in the individual lamellae.

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[8] The un-dashed values of functions relate to orthotropic structure while those with one dash refer to isotropic structure, in order to distinguish the results. The double dash used already in Equations (1) and (2), and often in what follows below, relates to the ideal case G where the circumferential stresses are uniformly distributed over each radial section, and to the characteristic values of this.

In engineering practice the conditions indicated by

$$t \geqq s \geqq 1 \tag{18}$$

always obtain, and it will at once be apparent that in the case of cylindrical elements of structure (of whatever length) — that is to say in the case where plane *deformations* occur — the absence of uniformity in the distribution of stress becomes even more marked (even if not appreciable in practice) than in the case of a thin annular slice — namely in the case of a plane *stress* condition.[9]

Tables II and III contain a numerical summary of the results indicated above. Table II gives the determining values s and t for polar and cylindrical orthotropic texture wherein it is assumed that $n = \frac{E_s}{E_c} = 10$. (As regards the numerical

Table II.

F_2 en %	λ_2	s	t
0	1.00	1.000	1.000
1	1.09	1.043	1.044
2	1.18	1.086	1.088
3	1.27	1.127	1.130
4	1.36	1.166	1.170
5	1.45	1.204	1.210
7	1.63	1.277	1.286
10	1.90	1.378	1.393

Table III.

s, t		1.00				1.20				1.50				G	
α	ρ	S′	D %	A %	U %	S	D %	A %	U %	S	D %	A %	U %	S″	U %
0.00	α	1.00	±0	+ ∞	∞	1.20	+ 20	+ ∞	∞	1.50	+ 50	+ ∞	∞	0.00	±0
	1	0.00		± 0		0.00	± 0	± 0		0.00	± 0	± 0			
0.25	α	1.13	±0	+ 239	300	1.29	+ 14	+ 287	351	1.55	+ 37	+ 365	435	0.33	±0
	1	0.13		− 61		0.12	− 8	− 64		0.10	− 23	− 70			
0.50	α	1.67	±0	+ 67	100	1.76	+ 5	+ 76	112	1.93	+ 16	+ 93	133	1.00	±0
	1	0.67		− 33		0.64	− 4	− 36		0.60	− 10	− 40			
0.75	α	3.57	±0	+ 19	33	3.62	+ 1	+ 21	36	3.69	+ 3	+ 23	39	3.00	±0
	1	2.57		− 14		2.56	− 1	− 15		2.53	− 2	− 16			
0.90	α	9.54	±0	+ 6	11	9.56	+ 0.2	+ 6	11	9.58	+ 0.4	+ 7	12	9.00	±0
	1	8.54		− 5		8.54	∼ 0.0	− 5		8.51	− 0.4	− 5			
1.00	α / 1	∞	±0	±0	0	∞	±0	±0	0	∞	±0	±0	0	∞	±0

[9] The special cases where, exceptionally, the signs of equality hold good in (18) are further discussed in WO 17. It is apparent in any case that the values s or t (which would be unity in a material of isotropic structure) may be regarded as a measure of the extent of deviation from isotropy; they are, as it were, characteristic structural figures.

values of m_1, m_2, m_3 comparison should be made with "WO 17"). It is evident that the difference between s and t is inappreciable *in practice* (though, as already explained, it is essential to distinguish between the two values *in principle*).

The functional relationships indicated in Table III are more important, and this summary enables the stress values in isotropic texture ($S' = \sigma_t/_p$) to be compared with those in the orthotropic texture ($S = \sigma_t/_p$) contemplated here. The last two columns enable consideration of the case to which we are now leading up, characterised by an entirely uniform distribution of the circumferential stresses

$$S'' = \sigma''_t/p = \frac{a}{b-a} = \frac{\alpha}{1-\alpha}$$

over each radial section. It is assumed that only an internal hydrostatic pressure p occurs, so that $q = 0$.

The upper figures in Table III give the stress values at the inner face ($\rho = \alpha$) and the lower figures those at the outermost fibre ($\rho = 1$), for differently proportioned cross sections ($\alpha = \frac{a}{b} = 0$, $^1/_4$, $^1/_2$, $^3/_4$, $^9/_{10}$, 1). The columns D give the percentage differences from the values for isotropic material as calculated by means of Lamé's equations, and the columns A give the positive (+) or negative (—) percentage deviations from the ideal case of a completely uniform distribution of stress S''. The algebraic sum of these deviations A serves as a measure of the lack of uniformity, U, which can be read off in the appropriate columns and which affords a good indication of how unfavourable the stress is distributed in all thick walled structures constructed in the way hitherto customary. This undesirable effect becomes particularly bad when the thickness of wall is increased (that is to say when α decreases) or when heavy reinforcement is used (that is to say when s or t become large).

As the thickness of the wall and the heaviness of the reinforcement increases, the flow of stress becomes more and more concentrated within an ever narrower inner zone, the internal stresses being forced more and more towards the inside face and the outer portions of the cross section being correspondingly withdrawn from participation in the stress.

In the limiting case where the reinforcement is very strong and inelastic $\left(\frac{E_2}{E_1} \longrightarrow \infty\right)$ only the innermost fibre would be left to perform the whole of the statical function by offering resistance under an infinite amount of stress.

It will be seen from the columns D how great is the error involved in calculating orthotropic compound structures by the simple Lamé formulae and in designing them accordingly, as has hitherto always been the practice. In structures intended for special purposes, as for instance the reinforced concrete pipe described by the present author for hydraulic erosion mining[10] (using an operating pressure equal to 20 or more atmospheres) in which the thickness of the wall has necessarily to be great and the reinforcement heavy, the error on this

[10] *W. Olszak:* Eisenbetonrohre für Spülversatzzwecke (R.C. pipes for erosion mining). Zement, 1935, Nos 14, 15, 16.

account may be as much as 15, 20, 30 or more per cent. In such cases simplified assumptions no longer hold good and it is not permissible to make use of them in the calculations, especially if this is done (as here) at the expense of safety against breakage and cracking.

The statical efficiencies η of reinforced pipelines and of similar reinforced concrete structures are notably smaller than that of unreinforced pipes. The values of η are always smaller than the values of η' indicated in Table I.

The foregoing considerations suggested the idea of trying to construct thick walled pipes (and other cylindrical structures similarly loaded), in which the uneven distribution of stress with its attendant evils (poor utilisation of the material, increased risk of cracking and fracture starting from the inner face, unavoidable wastage of material, etc.) might be eliminated.

It is, in fact, possible by a simple method to impose beforehand on the compound construction an elastic behaviour whereby the desired purpose is completely attained in practice, and in this way to substitute for the accepted method of building thick walled structures a new method which is superior in every respect.

At first sight the proper thing to do might seem to be to make the reinforcement heavier close to the inside face where the flow of forces is concentrated, but this would be altogether wrong. It is true, indeed, that the increase in the maximum stress at the inner face implies a corresponding relief of stress in the outer portions of the thick walled pipe, and that the increased steepness of the line of distribution of the circumferential stresses has the effect of displacing the centre of gravity of the stress diagram inwards; moreover, as pointed out above, this phenomenon becomes particularly marked when the reinforcement is especially heavy and the wall especially thick.

But to attempt to combat this undesirable disposition of the stress by altering the uniform arrangement of the reinforcement (making it heavier towards the inside of the annulus), so that the centre of gravity of the steel shall approximately coincide with that of the stress diagram, is merely to encourage still further the concentration of stress at the *inner* face of the pipe, and to provoke a further increase in a maximum stress which is already harmful: — for it is a characteristic of any statically indeterminate system to act in such a way that its "stronger" (less yielding) portions tend to participate more fully in the internal flow of stress.

This circumstance suggests a directly contrary procedure to the above: namely that of increasing the heaviness of the steel reinforcement *outward* and of thereby gradually increasing the modulus of elasticity E_2 towards the outer portion of the cross section, so as to cause the latter to participate more fully in carrying the stress, and, in the ideal case, ensure that every fibre of the pipe shall be uniformly stramed (and in the case of an internal hydrostatic pressure p shall carry a uniform tensile stress).

The usual treatment of the problem is, therefore, completely reversed. Hitherto we have sought to define the statical values of stress and strain in accordance with the known elastic properties of compound bodies; now we lay down *a priori* a certain condition of stress and seek to define elastic properties, not yet known, which shall correspond with the desired distribution of stress.

In reinforced concrete structures it is relatively easy to vary the elastic properties of the compound body at will, for one is dealing with two structural components — concrete and steel — which differ greatly from one another in this respect, and by suitable adjustment of the effective cross sections of these two components it is possible in a simple way to attain the desired objective, namely the improved construction of the compounded material.

Once again it is impossible to go into details of the calculation — more especially as there is more than one way of reaching the goal, for the solution depends less on the absolute elastic values than on their mutual relationship, that is to say on varying ratio $\frac{E_2}{E_1}$. Here either the value E_2 may be altered in relation to the fixed value E_1 or the desired relationship may be obtained by varying E_1 while E_2 is kept constant, or again by varying both values in opposite directions. The case previously considered of a (homogeneous) polar or cylindrical orthotropy will now be replaced by that of a curvilinear but movable and therefore non-homogeneous condition of orthotropy.

Further details will be given in a paper now ready and shortly to be published, which will be cited below by the abbreviation "WO 20".[11] Here reference will be made only to the results obtained in the simplest case of all which can be carried out in practice without any difficulty; the circumferential reinforcement is suitable increased outward while the value of E_1 is kept constant at E_b, in order to obtain the desired effect.

Starting from the fundamental requirement that the circumferential stresses are to be entirely uniform through-out every radial section, we may write

$$\sigma''_t = \frac{d}{dr}[r \cdot \sigma''_r] = \text{const.} = C = \frac{ap - bq}{b - a}. \tag{19}$$

From this, by simple integration, we obtain the law governing the radial stresses:

$$\sigma''_r = C + \frac{D}{r}, \tag{20}$$

wherein the constant of integration D has to be chosen to suit the marginal conditions (11). By putting

$$D = ab\,\frac{q - p}{b - a} \tag{21}$$

we satisfy (11) at either face.

We will now consider the relationship between the conditions of stress and those of strain. If, for the sake of simplicity, we take the case of an annular slice of limited thickness or depth (that is to say the case of uniplanar *stress* condition) we may apply the group of equations (6), provided that we now take E_2 not as a constant but as a function of the radius vector, $E_2 = E_2(r)$.

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[11] Meanwhile published in the Polish journal Czasopismo Techniczne, 1937, N^os 1, 2, 3, 4, 5, 6.

If, further, we utilise the first of the relationships (3) (an assumption which *is strictly* fulfilled for the case of polar or cylindrical orthotropy as hitherto taken, and the validity of which still would remain to be checked in the present case, but which may provisionally be accepted as adequate) then nothing more is wanted for arriving at the desired function E_2 which remains unknown.

The result is simply

$$E_2 = E_b \cdot \lambda''_P, \tag{22}$$

wherein

$$\lambda''_P = \frac{\rho}{A_P + \rho + \frac{1}{b}\frac{D}{C}\ln\rho}. \tag{23}$$

If, on the other hand, an (infinitely) long pipe is to be considered, so that the condition of plane *strain* is used as a basis for the calculation, then the solution takes the form

$$E_2 = E_b \cdot \lambda''_R, \tag{24}$$

where

$$\lambda''_R = \frac{\rho}{A_R + \rho + \frac{1}{b}\frac{D}{C}\left(1 - \frac{1}{m_1 m_3}\right)\ln\rho}. \tag{25}$$

In order to avoid having to use an endless double series expression for determining A_P and A_R these may be arrived at from the following approximate expression which gives very good results (see WO 20):

$$A_P = -\frac{(n-1)F''_2}{100+(n-1)F''_2}\frac{1+\alpha}{2} - \frac{1}{b}\frac{D}{C}\ln\frac{1+\alpha}{2}, \tag{26}$$

$$A_R = -\frac{(n-1)F''_2}{100+(n-1)F''_2}\frac{1+\alpha}{2} - \frac{1}{b}\frac{D}{C}\left(1 - \frac{1}{m_1 m_3}\right)\ln\frac{1+\alpha}{2}. \tag{27}$$

The varying circular reinforcement f''_2 must be

$$f''_2 = \frac{100}{n-1}(\lambda'' - 1) \tag{28}$$

wherein λ'' has the value (23) or (25) according to the conditions of the problem. Of course the relationship

$$\frac{1}{b-a}\int_a^b f''_2 \cdot dr = F''_2. \tag{29}$$

must always hold good.

Naturally the function $E_2 = E_e \cdot \lambda''$ cannot actually be made smaller than E_c (that is to say λ'' cannot be less than unity), for we are only able to increase the modulus of elasticity of the compound body by suitably *increasing* the amount of the steel by comparison with E_c. We can, in other words, render the compound body "denser", but we cannot make "voids" in it. There is thus a certain practical limitation to the method, which as a guide may be put at

$\alpha = \frac{a}{b} \geqq$ (0.6 to 0.8) according to the amount of reinforcement F''_2. But forms of construction thicker than those corresponding to the inequality just stated — i. e. with thicknesses greater than (0.4 to 0.2) b — will rarely occur in practice, and when they do occur recourse may be had to the more difficult method, that of altering E_1. In the upshot, therefore, it may be said that in all cases likely to arise in practice we can obtain the favourable uniform distribution of stresses.

As an example let us take a pressure pipe of the proportions $\alpha = \frac{a}{b} = 0.6$ capable of being influenced in the desired direction by the simpler method of suitably arranging the circumferential reinforcement.

We have

$$E_2 \cong \frac{\rho}{0{,}510 + \rho - \ln \rho},$$

$$f''_2 \cong \frac{100}{n-1}\left(\frac{\rho}{0{,}510 + \rho - \ln \rho} - 1\right).$$

As will be seen from Fig. 3, the increase in E_2 is almost rectilinear, its value at the inner face being $E_{2,\, r=a} \cong E_b$ and its maximum value, at the outer face, $E_{2,\, r=b} = 2.04\, E_b$. The line f''_2 also is almost straight, giving the corresponding values $f_{2,\, r=a} = 0$ and $f''_{2,\, r=b} = 7.4\,\%$ (for $n = 15$). The thickness of the pipe is divided into practically equal parts by the centre of gravity of the diagram, and the centre of gravity of the reinforcement comes at a distance of $^2/_3\, d = {}^2/_3\, (b - a)$ from the internal face of the pipe.

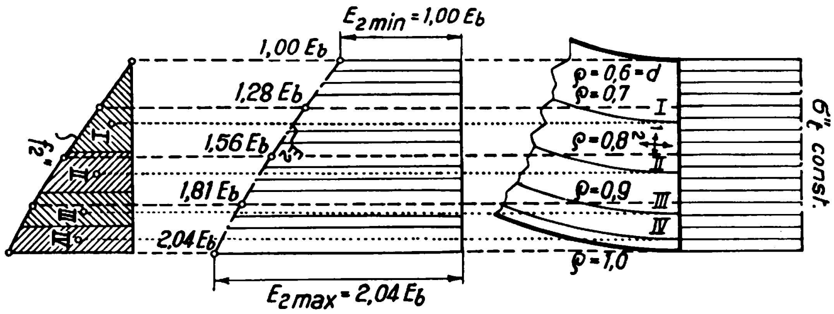

Fig. 3.

The uniformly distributed tangential stresses σ''_t = const. are ensured by suitably adjusting the modulus E_2. This also gives the reinforcement diagram f''_2 whence the position of the reinforcing rings I, II, III, IV is determined.

The distribution of the steel reinforcement may advantageously be carried out by using a similar method to that commonly adopted in arranging the shear reinforcement in reinforced concrete beams, by reference to the shear diagram. It will be best to use as many and as thin reinforcing bars as possible in order to obtain as uniform a compounding effect as possible, and the reinforcing rings should then be placed at the centres of gravity of the corresponding portions of the areas f''_2, as illustrated in Fig. 3 for four bars of equal diameter. Further details as to the most favourable arrangement of the reinforcement *as a whole* F''_2 (29) may be found in WO 20.

It is possible now to go one step further. As is well known, the permissible stress in the compound body depends among other factors on the percentage of reinforcement, a quantity which increases outward if the improved solution here suggested is adopted. Matters may be so arranged that the permissible stress increases outward at a rate corresponding to the percentage of reinforcement, the case being analogous to that already considered, and the fundamental equation (19) beeing replaced by

$$\sigma_t''' = \sigma_{b\,zul}\left[1 + n\frac{f_2'''}{100}\right] = \sigma_{b\,zul}\,\frac{1}{n-1}\left[n\frac{E_2}{E_1} - 1\right], \tag{30}$$

wherein

$$E_2 = E_b \cdot \lambda''', \qquad f_2''' = \frac{100}{n-1}(\lambda''' - 1).$$

We will go no further into this special problem. By contrast with the procedure hitherto in question (wherein certain permissible stresses are not exceeded) it is more important to ensure that the amount of strain in the structure shall be the same at every point; that is to say to ensure that the degree of safety against breaking or cracking shall remain the same at every point in the thick walled structure. This cannot however be obtained either through the requirement (19) or through the more rigorous requirement (30), but must be derived from a special hypothesis of strain.

The considerations which are relevant here are either the *Guest-Mohr* hypothesis of shear stress or the *Huber* hypothesis of strain energy. Instead of (19) or (30) the requirement to be imposed is that the "reduced" stresses shall remain unchanged:

$$\sigma_{red}'''' = \sigma_t'''' - \sigma_r'''' = \text{const}, \tag{31}$$

or

$$\sigma_{red}'''' = \sqrt{(\sigma_r'''')^2 + (\sigma_t'''')^2 + (\sigma_z'''')^2 - \sigma_r''''\,\sigma_t'''' - \sigma_t''''\,\sigma_z'''' - \sigma_z''''\,\sigma_r''''} = \text{const}. \tag{32}$$

This, however, ensures an ideal solution from every point of view — statically (as regards safety from fracture), operationally (as regards freedom from cracking), and economically (as regards saving in material).[12] How the law $\lambda'''' = \frac{E_2}{E_1}$ reads in this case, and how the reinforcement is to be distributed (function f''''_2), cannot, for lack of space, be further discussed here and must be left for a separate later publication.

The statical efficiency η'' obtained by way of the improved constructions as here suggested is always $\eta'' = 1 = 100\,\%$ (by contrast with the rather meagre values of η and η'), quite independently of the wall thickness, and the notable saving in material indicated in Footnote 11 is straightaway realised.

[12] For comparison, the amount of material required in pipes made by the old method hitherto in use is 20, 50, 100 or more per-cent greater (according to the dimensional ratio α and the amount of reinforcement F_2) than by the improved method. See WO 19.