# The load collapse for elastic plastic trusses

# The Load Collapse for Elastic Plastic Trusses

La charge limite pour un treillis élasto-plastique

Traglast elasto-plastischer Fachwerke

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Abstract - The collapse load of a truss is investigated taking into consideration the way the bars actually behave, namely the effects of the strain hardening and the buckling respectively for the bars under tension and for those under compression.

During the buckling process the diagram which represents load versus axial deflection, on account of yelding of mid section, due to the bending, takes the form of a hiperbola branch (fig.1) [1] [2] [3]. At this stage, the bar, whose characteristic is a negative strain hardening - softening - becomes unstable. If, however, it is within a hyperstatic system, its buckling does not necessarily cause the collapse of the structure. Especially for multi-hyperstatic trusses, the collapse load may be found to be higher by far than the load generating the buckling condition of the first bar.

The problem has been put up with the restrictions as described in the following: The bars are pin hinged bars; the stress-strain relationship, as indipendent from the temperature and time, follows Prandtl's model [4] ; the deflections are assumed to be infinitesimal, that is finite but small, just that the geometry of the system and thereby the internal condition of the stresses are not affected at all; both localized and global bifurcation phenomena are ruled out. Of this structure are discussed the stability conditions in the classical meaning, that is for infinitesimal perturbances.

This problem has already been dealt with by other authors [5] , [6] [7] . From the stability postulate of Drucker's [8] [9] the sufficient conditions for stability and uniqueness of the solution have been deduced. In the discussion which follows only the first aspect of the question has been examined closely: By an original procedure, the necessary and sufficient stability conditions have been formulated.

The problem has been traced back to analysing the development to which is subjected the structural yield locus, which varies with the varying loads, under the action of incremental plastic deformations. Upon the external load reaching its critical value, to the increment of the plastic deformations corresponds a contraction into the yield locus which make it impossible to balance the original

load. From the discussion is possible to elaborate a graph which enables making a stability verification immediately, which can be made, hoverer, for pratical purposes, in the only case of two variables.

In the general case the problem is transferred into algebraic form: The parameter which indirectly furnishes the answer of the yield locus to an increase in the plastic deformations is determined by the energy irreversibly stored into the system: the elastic constrained energy and the energy dissipated through the plastic phenomena. If,to an increment whatever in the plastic deformation, the corresponding variation in the stored energy is still positive then the equilibrium is stable; if of no value or negative then at least in one case the equilibrium is neutral or unstable. The question is restricted to researching the sign of a quadratic form, associated with the matrix of rigidities, function of the plastic deformations and constrained thus by the signs of the latter.

These conditions can be brought to some other form as function of such parameters as are typical of the stability problems, that is the work done by the disturbing forces or the total energy of the system. It is demonstrable that if the variation occurring in the stored energy is either negative or zero the variation of the total energy of the resulting work done by the disturbing forces will likewise be either negative or zero. So we again come to a formulation which, though less praticable because of the further difficulty encontered in assessing the free elastic energy, connects directly to a principle which is as a rule normal within the elastic range or Drucker's postulate.

The problem is susceptible of generalizations.At this time the preference has been given to focussing the attention on the concepts rather than going deep into a more complex program.

<u>The behaviour of the bars</u> - The assumption is made that the bars,eithe in tension or compression, follow Prandtl's model [4] , indifferently.

In fig.1 is shown the curve relative to the relationship existing between axial force S, elongation or shrinkage $\delta$ for any bar in general The bar behaves elastically according to Hooke's law up to stress $S_e$; Past this point,plastic deformations take place, such that the linear trend of the line is changed. Upon relieving the load the representative point of the stress condition moves along the line parallel to O-A.

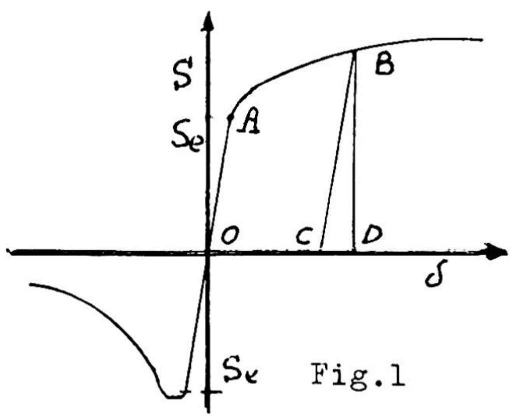



Fig.1

Segment O-C indicates the plastic deformation $\bar{\delta}$,at B,which at the time the load is relieved remains unaltered;segment C-D represents the elastic deformation $\delta_e$. If the bar is isolated for $S=0, \delta=\bar{\delta}$;if it is within a hyperstatic system,for $S=0, \delta=\bar{\delta}+\bar{\delta}_e$,where $\bar{\delta}_e$ indicates the elastic deformation constrained whithin the system and recoverably only through cutting the bar.

Area OABD represents the total

work performed by the external forces which is necessary to achieve pattern B. In particular OABC is the graphical representation of as much amount of energy as is absorbed by the system and is dissipated through the plastic phenomena;the area CBD is the elastic energy which can be returned only if the bar it is isolated or part of an isostatic system.

Unlike the currently adopted convention on the signs for the axial forces S, a different one is being introduced here. The starting axial force S is assumed to be positive in all cases; increments are either positive or negative whether or not they are in accord with the starting force.

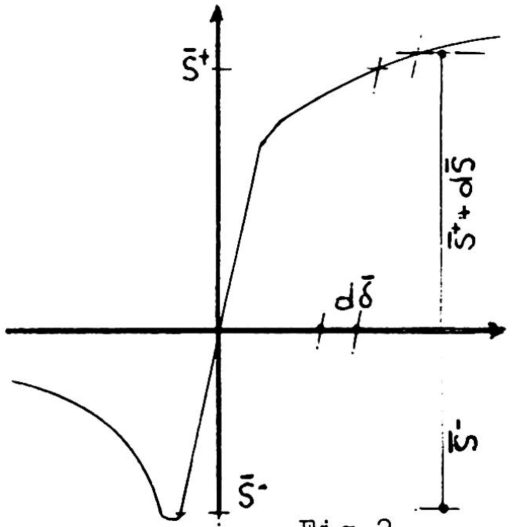


Fig.2

For assigned plastic deformation $\bar\delta$ (fig.2), $\bar S^+\bar S^-$ are meant to be indicative of the limiting values within whose range the axial force can oscillate performing elastically.Therefore the yield locus shall be as established by the relation:

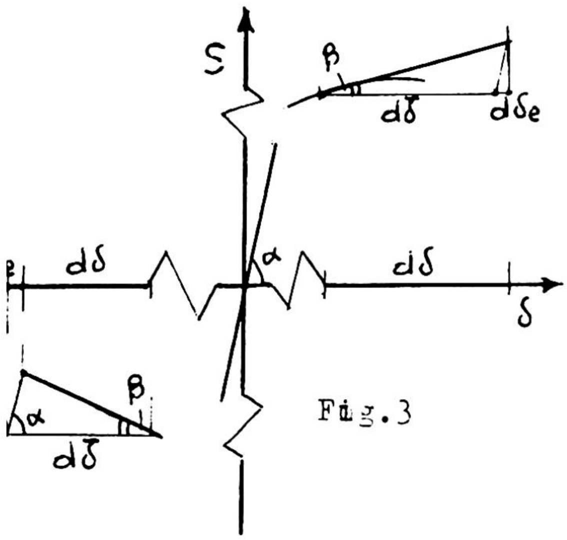
$$(1) \qquad S \le \bar S$$

where $\bar S$, generically,indicates the $\bar S^+$, $\bar S^-$ limiting values according to whether is correspondingly a traction or compression. If the verification yelds a disequality, the bar under test is in the elastic range, whereas the equality proves it is in the plastic range.

Where the bar is in the plastic range, i.e.if $S = \bar S$,the stress-strain relationship is linear, when the increments are infinitesimal: Curve $S(\delta)$ is substituted whith its tangential line at $\bar S$. Then by differentiating (1) in relation to $\delta$ or $\bar\delta$ :

$$(2) \qquad dS \le \frac{dS}{d\delta}\, d\delta = W d\delta = \frac{dS}{d\bar\delta}\, d\bar\delta = \bar W d\bar\delta = d\bar S$$

a limitation to the incremental relationship $\bar S-\delta$ is obtained. Owing to a $d\bar\delta$ increment in the plastic deformation the bar, initially stressed under $\bar S$, is now capable of taking a stress increment, at the limit, $d\bar S$ : Therefore $d\bar S$ determines the dislocation of the yield locus (fig.2).



In the eq (2) W represents the differential rigidity, $\bar W$ the plastic differential rigidity (fig.3): the following is the correlation of the above rigidities to the elastic rigidity We:

$$\bar W = \frac{W \quad W_e}{W - W_e}$$

the result is that where $W \gtrsim 0$ , $\bar W$ is likewise $\gtrsim 0$.The plastic deformation $d\bar\delta$ is restricted in sign by the relationship sign $d\bar\delta$ = sign $\bar S$,which, for the position of on the forces signs, is reduced to condition:

$$(3) \qquad d\bar\delta \gtrsim 0$$

The interval within which rigidity W is included is so defined:

$$-\infty < W < We$$

By combining eq. (2) with limitations:

$$d\bar{\delta} > 0 \qquad dS = dS = Wd\delta \quad (-\infty < W < We)$$

$$d\bar{\delta} = 0 \qquad dS = Wed\delta \qquad (W = We)$$

the stress-strain incremental relationship is thus obtained. The eq. (2) covers the (4) and in a more general sense may be intended as relating to a cycle. At first, the incremental force dS verifies the equality with the bar being in the plastic range, subsequently is subjected to a reversal and thus verifies the disequality.

The behaviour of the system - As a reference, let it be taken a general type of reticular pin-hinged, made up by n bars, times r hyperstatic truss and let it be subjected to a loading pattern F: for an Fo load let Co be the corresponding in equilibrium and compatible pattern, typified by k number of bars ($K \gtrless r$) in plastic range, $\bar{\delta}_1 \dots \bar{\delta}_K$ being the corresponding elogations.

Let the displacements of the system be assumed as being infinitesimal, or finite, but such that they cannot affect the originary geometry of the system and, hence, indirectly, the stressed condition. This supposes that the strain condition which corresponds to Co can be regarded as borne by the plastic deformations $\bar{\delta}$, intended as distorsions, and by loads Fo, as applied to the elastic structure.

This as a reference Sei indicates the stress exercised by load Fo into bar "i"; Sij the stress transmitted to bar "i" through distorsion $\bar{\delta}_j = 1$ at "j". Then the resulting stress in bar "i" is:

$$(5)\ \bar{S}i = Sei + \sum_K Sij\,\bar{\delta}j \qquad (i = 1 \dots n)$$

Eq.(5) is substituted in (1) by transferring to the right hand side the term relative to the distorsions:

$$(6)\ Sei \leq \bar{S}i + \sum_K Sij\,\bar{\delta}j = \bar{\bar{S}}i$$

on the assumption that:

$$\bar{\bar{S}}i = \bar{S}i + \sum Sij\,\bar{\delta}j$$

The $\bar{\bar{S}}i$, different, whether tractive or compressive, are a generalization of the $\bar{S}i$ referred in (1) and define, within the space of the plastic deformations, the yield locus for pattern Co. If stresses Sei verify the inequality, the point representative of the stress condition falls inside yield locus. On the contrary, if for some of the bars the equality has been verified the representative point falls onto the edge of the yield locus and the structure is in the plastic range.

A variation is assigned to pattern Co by attributing to the bars in the plastic range a $d\bar{\delta}$ increment to the initial plastic deformations. on the assumption that the bars in the elastic range will stay such. The resulting C'o pattern is described as "perturbed" pattern. By differentiating (6) for the $d\bar{\delta}$ increments assigned and consistent with (3) we obtain the stress increments which C'o can absorb:

$$(7)\ dSe_i \leq \bar{W}_i d\bar{\delta}\,i + \sum_K S_{ij} d\bar{\delta}_j = d\bar{\bar{S}}_i$$

Eq.(7) is a generalization of eq.(2). The dislocation of the initial yield locus $\bar{S}$, consequent to the assigned plastic deformations $d\delta j$ is just supplied by the $d\bar{\bar{S}}$. If the representative point of a new stress condition comes to fall inside of or into the edge of the yield locus, the equilibrium between the stresses and the strength of the bars is verified for pattern $C_o'$ ; if outside, that is if for a certain number of bars: (8) $dSe_i > d\bar{\bar{S}}_i$
the equilibrium is impossible: the plastic deformations continue their pursuance to a new pattern $C_o''$ which may still verify eq.(7).

Stability of the system - A graphical method for the verification of the stability, in which the above indicated concepts ore expounded,is illustrated the problem being dealt with is limited to the case involving two placicized bars only. It will not be difficult but rather easy to extend, conceptually at least, the representation to the more generalized case.

As a reference let us consider a Cartesian system having as many axes as are the plasticized bars. Let us mark on the axes plastic deformations $d\delta$ : The origin of the axes thus defines the pattern Co.As is conventional for the signs on plastic deformations (3),all $C_o'$ patterns are comprehended within the quadrant of the positive $d\delta$. Chosing this as reference frame,we now draw as many straight lines $d\bar{\bar{S}}_i = 0$ as are the bars in the plastic range: the enveloping line defines the boundary of the plasticity field for that part which influences the stability of the system; on the perpendiculars are marked the stresses $Se_i$ and the corresponding increments $dSe_i$. Therefore point Co sets also the initial stress condition in which $Se_i = \bar{\bar{S}}_i$.

Fixed the perturbed pattern C'o, the sides of the yield locus translate: according to $d\bar{\bar{S}}_i \gtrless 0$ it will correspondingly expand or contract: the new yield locus, so obtained, is defined "perturbed". The equilibrium in this stage is assuredly verified if the transposition to $C_o'$ is considered as effected by forcing a set of supplemental restraints, non efficient in $C_o$. Point $C_o'$ moreover establishes the elastic stresses $dSe_i$, relative to the reactions dF of the additional restraints constituting the,so called,perturbing forces.

The supplemental restraints are then removed and, hence, $dF \to 0$: Where $dSe_i \to 0$ the elastic stress condition $C_o'$ has a tendency to resuming the initial position $C_o$. If $C_o$ is found to fall inside the area of the perturbad yield locus, that is, if:

$$0 \leq d\bar{\bar{S}}_i$$

eq.(7) is verified: the pattern settles in C'o and the system behaves elastically again. If, on the contrary, for some of the bars eq. (8) is verified, that is

$$0 > d\bar{\bar{S}}_i$$

$C_o$ comes to fall outside the perturbed field and there are no possibilities for an equilibrium. These bars keep being subjected to the plastic phenomenon with the field paralleley evolving in pursuance of a new pattern $C_o''$ which comprehend $C_o$. More forces are supposed to be interfering ot this stage such that a point-by-point equilibrium is assured.

For example, in the case illustrated in figura 4, what C'o might be, the resulting system is in any case that of equilibrium. Being tha at all times $d\bar{\bar{S}}_1 > 0$, $d\bar{\bar{S}}_2 > 0$, eq.(7) is verified, even where $dSe \rightarrow 0$: The perturbed yield locus shall alwais comprehend the originating pattern $C_0$. In this case the equilibrium of pattern Co is stable.

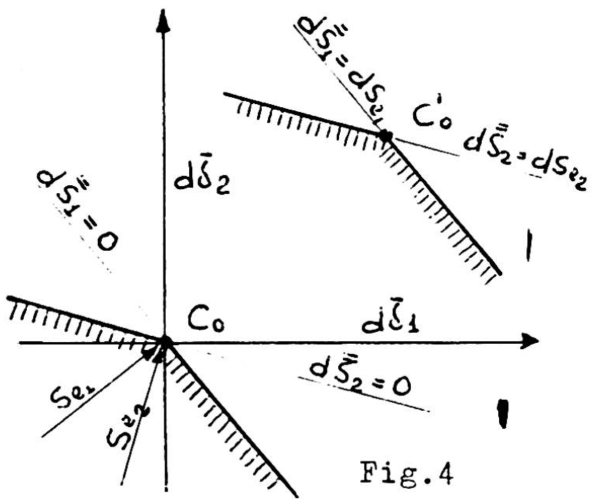


Fig.4

A diametrally opposed case, is that shown in figura 5. Whatever $C'_0$ the resul is alwais $d\bar{\bar{S}}_i < 0$ $d\bar{\bar{S}}_2 < 0$. Hence by eliminating the perturbing forces eq.(8) is verified: within the two bars the plastic deformations increase. However, whatever the $C''_0$ pattern which one can come to, during the unloading stage, the situation repeats itself again: the plastic deformations have a tendency to become infinitely great. Parallelely the edge of the yield locus, originally $\bar{\bar{S}}$, moves to $\bar{\bar{S}} - d\bar{\bar{S}}$: for $d\bar{\delta} \rightarrow \infty$, $\bar{\bar{S}} - d\bar{\bar{S}} \rightarrow 0$: the plasticity field for at least one its sides shrinks gradually up to becoming null. At $C_0$ the equilibrium is therefore unstable.

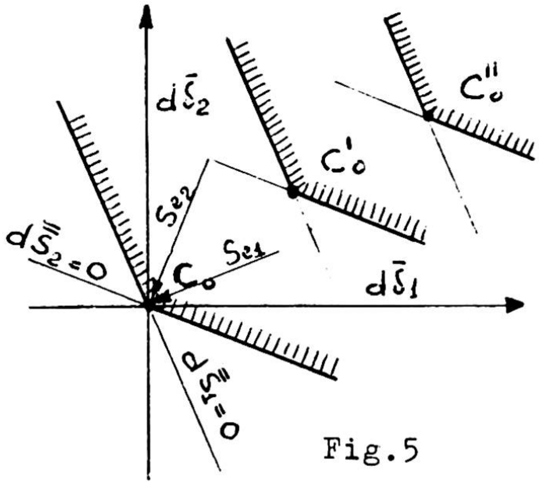


Fig.5

Figures (6) and (7) report some intermediate situations. The first shows a se of stable equilibrium, the second one a case of instability.

In fig. 8 is then illustrated a situation of neutral equilibrium. Whatever $C'_0$ the system is apt to assuming an equilibrium pattern $C''_0$ coincident or not whith the former. From this viewpoint the system is apparently stable. On the other hand, though, all patterns $C'_0$ falling on straight line $d\bar{\bar{S}}_1 = d\bar{\bar{S}}_2 = 0$ are also corresponded by $dSe_1 = dSe_2 = 0$. All these patterns and, to the limit, the infinity one, are then attainable without the aid of a perturbing set-up for forcing the system, ond hence without any energy dissipation.

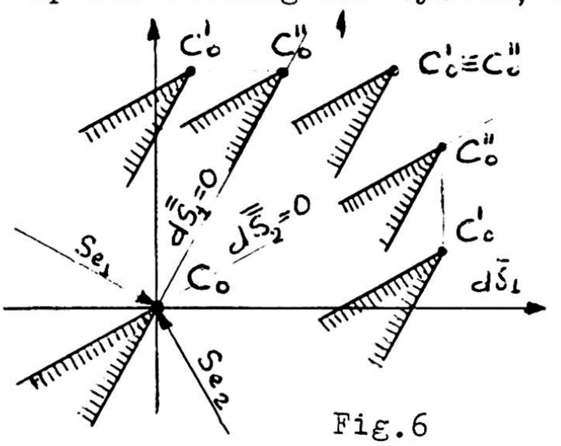


Fig.6

Along this directrix the system is seemingly worn out, unfit to counteract the modification of the original pattern $C_0$. The situation as illustrated in fig.7 is unstable although still presenting an indifference directrix.

Even if hardly usable, owing to the unpratical possibility of extending it to an n dimension system, this graphical representation helps to clarify the problem and affords a comparison whith

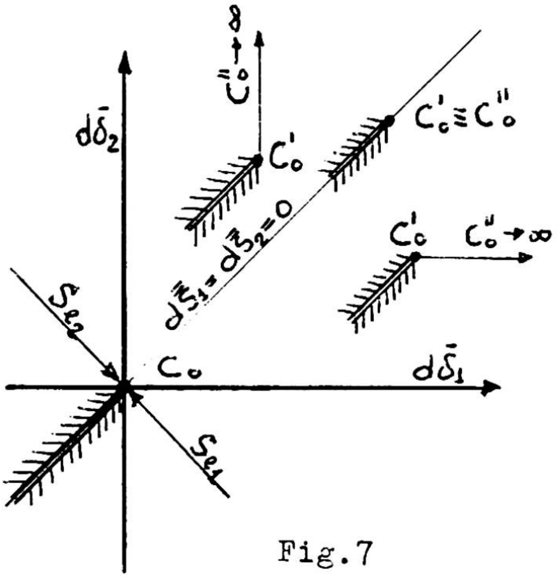


Fig.7

the analogus elastic problem.

In the elastic range, if the equilibrium is stable, $C'_0 \to C_0$ once eliminated the perturbation. In the elastic-plastic range we find that $C'_0$, apart from not returning in $C_0$ at all, may furtherly move away from it and reach $C''_0$, which alike $C'_0$, is very close to $C_0$. It follows that lacks the clear differentiation between a stable and a neutral equilibrium, as is found in the elastic range. The distinguishing point that differentiates the latter from the former lies only in the fact that, for translating the system from one pattern to another along the indifference directrix, there is no need of any external work.

<u>The system energy</u> - The stability conditions are algebraically expressed as functions of the energy. As an introduction some hint is therefore made about the energy stored in the system and its variations.

In an intermediate stage of the loading process $0-F_0$, the work done by forces F in equilibrium with the internal stresses S, under the action of a d increment in the dispacements associated with an increment in the bar deformazion d is:

$$(9)\quad dL = \sum F d\eta = \sum_n S_i d\delta_i = \sum_n (Se_i - \sum_k S_{ij}\bar\delta_j)(d\delta e_i + d\bar\delta e_i + d\bar{\bar\delta}_i) = \sum_n Se_i\, d\delta e_i + \sum_k (\sum_k S_{ij}\bar\delta_j + \bar{\bar S}_i)\, d\bar\delta_i = \sum_n Se_i\, d\delta e_i + \sum_k Se_i\, d\bar\delta_i$$

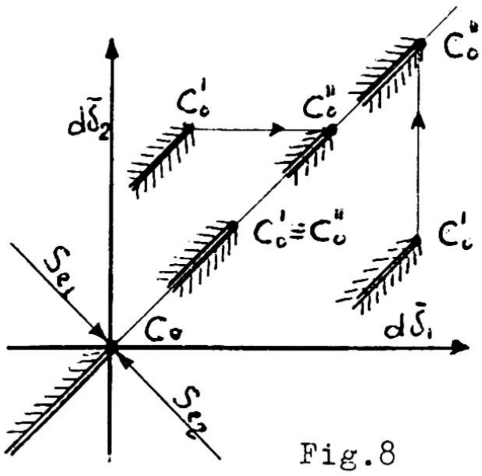


Fig.8

the assumption having been made that in this stage too, K bars are plasticized.

The total work L, spent by the external forces for the development of pattern $C_0$ is:

$$(10)\quad L = \int F d\eta = \frac{1}{2}\sum_n Se_i\, \delta e_i + \sum_k (\sum_k S_{ij}\, \bar\delta_j)\,\bar\delta_i + \sum \int_0^{\bar\delta_i} S_i\, d\bar\delta_i = E_e + E_v + E_p$$

The right hand side indicating the energy absorbed by the structure. In detail the first term, $E_e$, signifies the free elastic energy, in other words that quantity of energy which totally returns to the external forces at the unloading stage. The second term, $E_v$, the elastic ener_ gy constrained within the system by the plastic deformations which can be released to the outside only by making cuts in such a way that the structure becomes isostatic. The third term, Ep, the irreversible energy absorbed by the system, used to produce those alterations in the internal structure of the material which give origin to the plastic dislocations.

For translating the system from pattern $C_0$ to $C_0'$ the work, $d_2L$, of the second order, done by the perturbing forces, taking into account the linearity of the stress-strain relationship, is

$$(11)\quad d_2L = \frac{1}{2}\sum dF d\eta = \frac{1}{2}\sum_n dSe_i d\delta e_i + \frac{1}{2}\sum_\kappa(\sum_\kappa S_{ij} d\bar{\delta}_j + \bar{W}_i d\bar{\delta}_i)\, d\bar{\delta}_i$$

$$= \sum_n dS_{ei}\, d\delta_{ei} + \sum_\kappa dSe_i\, d\bar{\delta}_i = d_2E_1 + d_2Ev + d_2Ep = d_2E_1 + d_2\bar{E}$$

$d_2\bar{E}$ being the global constrained energy of the system both elastic and plastic.

The constrained energy $d_2E$ is expressed by a homogeneous quadratic polynomial whose variables, however, are conditioned, in sign, by eq (3). For that part relative to the hiperquadrant 0 this polynomial coincides with the quadrantic form, associated to the matrix of the rigidities (7) and may result positive, null or negative: the last circumstance being possible in the sole case that, at least one bar be characterized by softening. The $E_1$ and $E_v$ polinomials are instead always positive.

Generalizing the notion of the total energy of the system 10 by adding, in addition to the positional energy of the external agencies, and the free elastic energy, also the constrained energy, eq.(11),after transferring to the right hand side the external work, defines the variation prime, $dE_t$, of the total energy, stationary for the $C_0$ equilibrium pattern. Variation second $d_2E_t$ is furnished instead by the right hand side of eq.(11).

<u>Stability conditions</u> - Let us suppose that the quadratic form $d_2\bar{E}$, devised for pattern $C_0$, is always positive for all the $d\bar{\delta}$ consistent with (3), but not simultaneonaly nought, that is:

$$(12)\quad d_2\bar{E} = \sum d\bar{S}_i\, d\bar{\delta}_i = \sum(\sum S_{ij}\, d\bar{\delta}_j + \bar{W}_i\, d\bar{\delta}_i)\, d\bar{\delta}_i > 0$$

In particular let for $C_0'$ be:

$$dS_i = \left[\frac{d}{d\bar{\delta}_i}(d_2E)\right]_{C_0'} \geq 0$$

Eq (7) verified at the beginning in respect to the interference of the perturbing forces still rests verified for $dSe_i \to 0$: through the unloading stage the system behaves in an elastic way. In the space of the $d\bar{\delta}$ the pattern settles in $C_0'$.

Its supposed, instead, that for $C_0'$ :

$$dS_i = \left[\frac{d}{d\bar{\delta}_i}(d_2E)\right]_{C_0'} \gtrless 0$$

In this case, although as a whole eq. (12) is verified, same of the addenda result as being negative. Whith the elimination of the perturbing forces for some of the bars eq.(8) is verified. For such bars the plastic phenomenon then progresses spontaneously and the system moves away passing from $C'_o$ to $C''_o$. The second principle of the thermodynamics, as formulated by Lewis, [11] affirms that any spontaneaus phenomenon is corresponded by a decrease in the system energy which is transformed into the work of the balancing forces, that is dF, in the present case. Thus, if with $d_2\bar{E}_{c'_i}$ we designate the energy corresponding to travel $C_o$-$C'_o$, and $d_2\bar{E}_{c''_i}$ that relative to Co-C'o-C"o, the result will always yield:

$$(13) \qquad d_2E_{c'_i} > d_2\bar{E}_{c''_i}$$

But, for the supposition made in eq. (12), the verification of this relationship can only be ascertained where C"o within the space of the $\bar{\delta}$ - comes to falling around $C'_o$ and, hence $C_o$. The pattern $C''_o$ defines a relative extreme (minimum) of function $d_2\bar{E}$, conditioned by eq (3) and therefore:

$$d\bar{\bar{S}}_i = \left[\frac{d}{d\bar{\delta}_i} \quad (d_2E)\right]_{c''_i} \geq 0$$

Hence at $C''_o$, also for $dSe_i \to 0$, eq (7) is verified. So eq (12) represents a condition sufficient for Co being a pattern of stable equilibrium.

As a substitute of (12) let us assume:

$$(12') \qquad d_2\bar{E} \geqslant 0$$

In particular then let, for C'o, be $d_2\bar{E} = 0$: In the other case we come to fall again within the preceding situation.

Allowing for eq.(12') the risult will alvays yield:

$$d\bar{\bar{S}}_i = \left[\frac{d}{d\bar{\delta}_i} \quad ( d_2E )\right]_{c'_i} \geq 0$$

Thus $C'_o$ is a pattern of equilibrium with no interference of perturbing forces and as such are all those other patterns which fall into directrix Co-C'o which is justly typified by $d_2\bar{E}=0$. The system moves along this direction with no external work being done. Then the following is particularly to be verified:

$$d\ \bar{\bar{S}}_i > 0 \quad \text{for} \quad d\bar{\delta}_i = 0$$
$$d\ \bar{\bar{S}}_i = 0 \quad \text{for} \quad d\bar{\delta}_i > 0$$

Pattern $C_o$, which is corresponded by (12'), is then a pattern of neutral equilibrium.

For (12) let us assume as substitute:

$$(12'') \quad d_2\ \bar{E} \gtrless 0$$

In particular is assumed as the assigned pattern $C'_o$ that for which $d_2\bar{E} < 0$. In this case for some of the bars:

$$d\bar{\bar{S}}_i = \left[\frac{d}{d\bar{\delta}_i} \quad (d_2E)\right]_{c'_o} < 0$$

The perturbing forces eliminated, the plastic phenomenom then progress: the energy relative to a successive pattern $C''_o$ is related to the energy at $C'_o$ by eq.(13). In $C''_o$, and so for the successive patterns, is thus repeated the like situation as is found in $C'_o$. The plastic phenomenon keeps continuing indefinitely with the system never reaching a pattern of equilibrium with load Fo. Therefore if the pattern Co is associated to eq.(12") the equilibrium is unstable.

The considerations on the eq.(12'), (12") follows that eq.(12) represents also a condition necessary for the stability of the system.

Drucker's second stability postulate [8] [9], as applied in the "small", fully confirms this result. In order that the system is stable the closed cycle work accomplished by the perturbing forces, applied at first and removed afterward, is to be positive. As this cycle terminates this work is found again under the form of stored energy: thus if $d_2\bar{E} > 0$ the equilibrium is stable. On the contrary, if $d_2\bar{E} < 0$ the result is that the cycle cannot be closed, that is the equilibrium is not verifiable without the introduction of an equilibrating system dF: then the equilibrium is unstable.

From the above it can be easy to deduce that, where the bars behav in an ideally plastic way (W=0), under the collapse load the equilibriu is neutral. True, in general $d_2\bar{E} \geq 0$ ($d_2Ep = 0$), particularly it nullifies for that $d\delta$ set which is corrisponded by the collapse mechanism. If the bars are instead strain hardened (W > 0), $d_2\bar{E} > 0$ as $d_2Ep > 0$: In this case the equilibrium is stable.

The stability according to Drucher's postulate - The first postulate of Drucher's states that a system is stable, in the "small", if the work accomplished by whatever forces dF yields always a positive result. If these forces are supposed as acting in a proportional way, the work accomplished by forces dF is coincident with the energy stored by the system, (11), that is the total energy variation. In the following is the demonstration that this principle and the one expounded in the preceding paragraph match perfectly at least as far as concerns the specific case under consideration. It is demonstrated particulary that if $d_2\bar{E} > 0$ or $d_2\bar{E} = 0$, parallely, always does exist at least one perturbing pattern dF for which $d_2Et > 0$ or $d_2Et = 0$.

Let us assume that $d_2\bar{E} > 0$ and as dF a system of forces proportionate to load Fo acting in Co, characterized, thus, by a proporzionality factor $d\lambda$, infinitesimal. Since the system results being unstable for a given number of bars $d\bar{S}_i < 0$. In order that $C'_0$ be an equilibrium pattern, eq.(7) must be verified and the result $dSe_i < 0$ must thus be yielded. Since, for convention, stresses $Se_i$ are positive, factor $d\lambda$ must be negative, or:

$$d\lambda\, Se_i = -\, dSe_i$$

The perturbing pattern dF must then result opposite to that Fo. In these conditions, at all times, eq.(7) is verified, even if plastic deformations are absent, in which case $dSe_i$ 0. Among the $C'_0$ solutions which verify eq.(7) there exists at least one, $C'_0$ which verifies also eq.(4) in its generalized form, or:

$$(14)\quad \begin{array}{ll} -\, dSe_i = d\bar{\bar{S}}_i & d\bar{\delta}_i > 0 \\ -\, dSe_i < d\bar{\bar{S}}_i & d\bar{\delta}_i = 0 \end{array}$$

This solution defines one extreme of function $d_2\bar{E}$ [12] [13] [14] conditioned by eq. (7) and in particular for the assumption adopted on the sign, (12"), it defines a maximum. The work accomplished by forces dF, in moving the system from pattern Co to that $C'_0$, is then supplied by eq.(11) agrees with eq.(9) multiplied by the $\frac{1}{2}\, d\lambda$ negative factor.

Since is always: $dL > 0$

$$-\tfrac{1}{2}\, d\lambda \ dL = d_2L < 0$$

Obviously, if $d_2 L < 0$, such is also the right hand side of eq. (11) that is the variation $d_2E_t$ of the total energy. This implies that in $C_o$ if (12") is verified, Ft definies a maximum and there exists, at least, one perturbed pattern $C'_o$ for which $d_2L < 0$.

On the contrary if $d_2E \geqslant 0$, for the patterns $C'_o$ falling on the indifference directrix:

$$dSe_i = \alpha_i dF = 0$$

then:

$$dF = 0$$
$$d_2E_t = 0$$

If finally:

$$d_2E > 0$$

since $d_2F_1 > 0$, also $d_2E_t > 0$. In $C_o$ the function $E_t$ defines a minimum.

In the following a very simple example has been evolved. The structure is that as shown in fig.9. In figg.9, 9-a, 9-b, 9-c the graph shows plotted, in the upper part, the Ft force versus the slope $\delta$, at C, for the beam, whose behaviour is supposed to be infinitely elastic; in the lower part of the same graph for the stanchion subjected to a buckling at A, assuming three different values for rigidity $\overline{W}a$. Starting from pattern $C_o$, to which corresponds load $F_o = F_t + F_a$, an increment $d\delta$ is attributed to the plastic deformation and pattern $C'_o$ is reached. Addenda $d_2 E_1$, $d_2E_V$, $d_2E_p$, all coming within the energy balance, hold as follows.

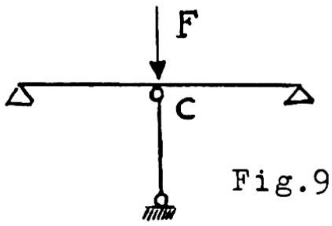
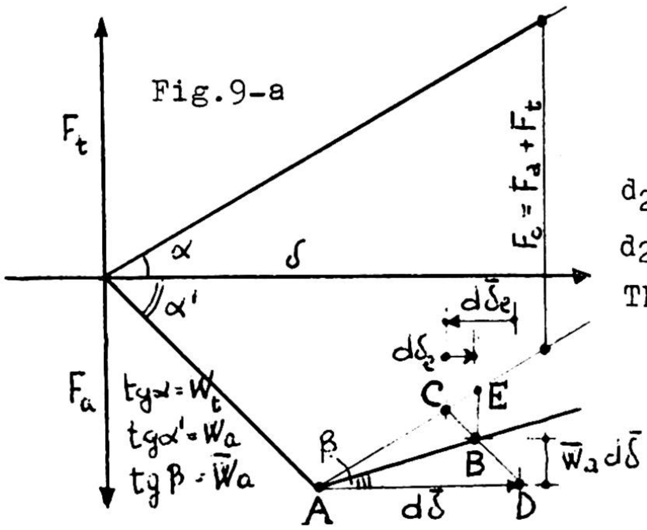



Fig.9

$$d_2E_e = \tfrac{1}{2}\ (W_a + W_t)d\,\delta\, e = \text{CBE area}$$
$$d_2E_V = \tfrac{1}{2}\ (W_a\ d\bar{\delta}ed\bar{\delta} = \text{ACD area}$$
$$d_2E_p = \tfrac{1}{2}\ \overline{W}_a\ d\ \bar{\delta}^2 = \text{ABD area}$$

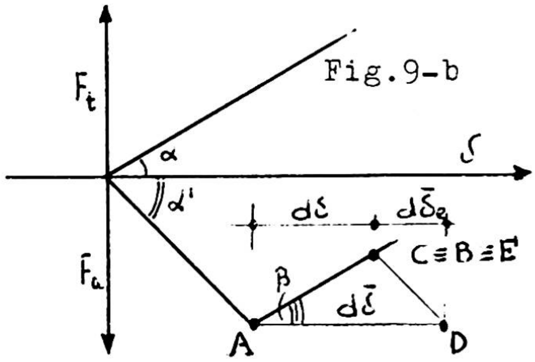
In particular, for.chart in fig.(9-a):




$$d_2\bar{F} = \text{ACD} - \text{ABD} = \text{ACB} > 0$$
$$d_2E_t = \text{ACB} + \text{CBE} = \text{ABE} > 0$$

The equilibrium is stable.

For chart in fig.(9-b):

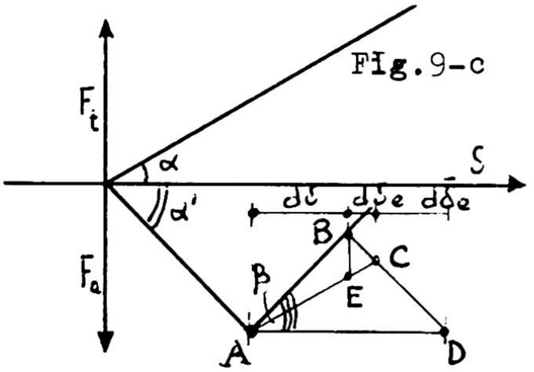



$d_2\bar{E}$ = ACD-ABD = 0

$d_2E_t$= 0

The equilibrium is neuter .

For chart in fig. (9-c)




$d_2\bar{E}$ = ACD-ABD = - ABC < 0

$d_2E_t$= ABC-CBE = - ABE < 0

The equilibrium is unstable.

## Conclusions

The stability analyis of an olonomous system, whose components are stressed axially and are typified by positive and negative rigidities is led back to the study of function $d_2\,\bar{E}$, that is the quadratic form associated to the matrix of the differential rigidities within the hyperquadrant of the positive $d\bar{\delta}$ . If, within this boundary, $d_2\bar{E} > 0$ then the equilibrium is stable: on the contrary it is neutral or unstable.

By the avail of the matrices theory [14] some conclusions can be drawn. If the quadratic form, associated to the matrix of the rigidities, is definite positive, such it will be also in the hyperquadrant $d\bar{\delta} > 0$: therefore the result is $d_2\,\bar{E} > 0$. Hence the equilibrium is stable. Instead if the quadratic form is definite negative, in like manner, $d_2\bar{E} < 0$: the equilibrium is then unstable. The same holds true if the quadratic form is semi-definite negative: the range of the matrix can never be less than one, and thus the indifference direction, at the limit, can only occupy a subspace of the positive hyperquadrant, the quadratic form in the complementary subspace remaining negative.

More complicated the question presents itself where the quadratic form is semidefinite positive or indefinite: In the first case $d_2\,\bar{E} > 0$ or $d_2\,\bar{E} = 0$, in the second case $d_2E \gtrless 0$ or the intermediate cases. The research of an algorism for the solution of this problem will be the subyect of a forthcoming information.

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SUMMARY

The stability analysis of an olonomous system, whose components are stressed axially and are typified by positive and negative rigidities is led back to the matrix of function $d_2 \bar{E}$, that is the quadratic form associated to the matrix of the differential rigidities within the hyperquadrant of the positive $d\sigma$. If, within this boundary, $d_2\bar{E} > 0$ then the equilibrium is stable: on the contrary it is neutral or unstable.

RÉSUMÉ

L'analyse de la stabilité d'un treillis, dont les barres ne subissent que des efforts axiaux, est déduite à l'étude de la fonction $d_2 \bar{E}$. Si $d_2 \bar{E} > 0$ le système est stable, sinon, il est neutre ou instable. Avec l'aide de la théorie des matrices [14] on peut tirer des conclusions sur la forme quadratique associée à la matrice. Le problème est plus ou moins simple, selon que cette forme quadratique est définie positive ou négative, ou semi-définie négative, ou alors si elle est semi-définie positive ou indéfinite. Ces derniers cas seront traités dans une information ultérieure.

ZUSAMMENFASSUNG

In diesem Beitrag wird die Stabilität unter Berücksichtigung der Traglast an einem Fachwerk, deren Stäbe achsialer Kräfte unterworfen sind, untersucht und mit Hilfe der Matrizenrechnung die Fälle des stabilen, labilen oder instabilen Gleichgewichts beschrieben.