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Safety of Structures as a Problem of Time

Sécurité des constructions en fonction du temps

Sicherheit der Bauten als ein Zeitproblem

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First attempts of probabilistic approach to safety of structures were made as early as 1936 (W.Wierzbicki /1/, M.Prot /2/) . The probabilistic philosophy has been discussed, for a long time, and gained its devotees and its skeptics, the problem being looked upon mainly from the point of view of a direct applicability to design and calculation. By the present writer's opinion, too little emphasis has ever been laid on the explanation of actual phenomena and interrelations that, in fact, has been dimmed by traditional methods, and this is the fundamental purpose of every theory. Once we realize we operate quantities affected with random scatter, we are induced, necessarily and at the same time, to the notion of safety and to probabilistic considerations, irrespective of whether we intend to establish a pure probabilistic theory of safety or to explain precisely the meaning of conventional coefficients of safety. Similar development can be noticed in those branches of technical activity where problems of reliability are of importance.

The present contribution aims at explaining the role of time in safety which, from the mathematical point of view, means making a step from random variables to random stochastic processes. So far, the basic end of the theory consisted in finding the probability of the strength criterion to be fulfilled, i.e. of the inequality $P < R$, where, loosely, P is the load and R the strength (carrying capacity) . Since every structure is to be reliable during a limited period of time (called, in what follows, life time or period of exploitation), P denotes the maximum load that can occur in the course of this period and R is assumed to be independent of time and of previous history of loading. The former assumption presents serious difficulties as, in general, statistics containing long periods of time, within a more or less homogeneous population of structures, are not available. The second assumption is only a crude approximation, e.g. it disregards phenomena connected with rheologic strength or fatigue. An attempt to avoid the first of the above assumpions is given by A.M.Freudenthal /3/. The author considers a sequence of load applications, the probability distribution of P in a single application being known. However, it is not always easy or even possible to say what is a single load application as the loading is a continuous process. Basides, in order to "locate" the process in time the intervals between those applications must be assumed. Thus, in general, the whole of the problem is to be discussed in the language of stochastic processes, the approximation with different discrete models being of course possible and valuable in view of effective calculations.

1. Measures of safety in time

A fundamental merit of the probabilistic approach to safety is the introduction of a unique and universal probabilistic measure of safety. We shall descuss here some basic notions following the very clear exposition of the subject in /3/. The generalization depends on passing from the discrete model to a continuous time process.

On the assumption that the carrying capacity R is independent of time (which will hold in this point) we define the probability of safety or the reliability, L, as the probability that the time to failure t_R, i.e. the effective life time of a structure exceeds the period of exploitation $t = T$, $T < t_R$. This is equivalent to the condition $P_{max} < R$ if P_{max} denotes the maximum load during T; hence we have

$$L(t) = Pr\,(t < t_R) = Pr\,(P_{max} < R), \quad t = T \tag{1.1}$$

The probability of failure within that period equals

$$F\,(t) = 1 - L(t) = Pr\,(t > t_R)\;. \tag{1.2}$$

The a priori probability density of failure at the instant t is

$$f\,(t) = \frac{dF(t)}{dt}\;. \tag{1.3}$$

The failure rate, accordingly to /3/, is the probability that a structure that has survived t will fail in a time unit at t,

$$h(t) = \frac{f(t)}{L(t)} = -\frac{d}{dt} \ln L(t). \tag{1.4}$$

Obviously, the above formulae correspond to (2.1)$\div$(2.7) in /3/. Here, t_R is a random variable and denotes the time to first surpassing the value R by the load.

The load P, being a continuous time process which we denote by g(t), the results of measurements can, depending on the type of measuring devices, be obtained in threefold form: (1) as a continuous graph (self-recording instruments), Fig.1, (2) as periodic readings at time intervals δt (points denoted by small circles), (3) as maximum values at fixed time intervals Δt, usually related to cyclicity of load accurrence, e.g. in 24 hours, a year, etc. (devices recording maximum values denoted by little crosses).

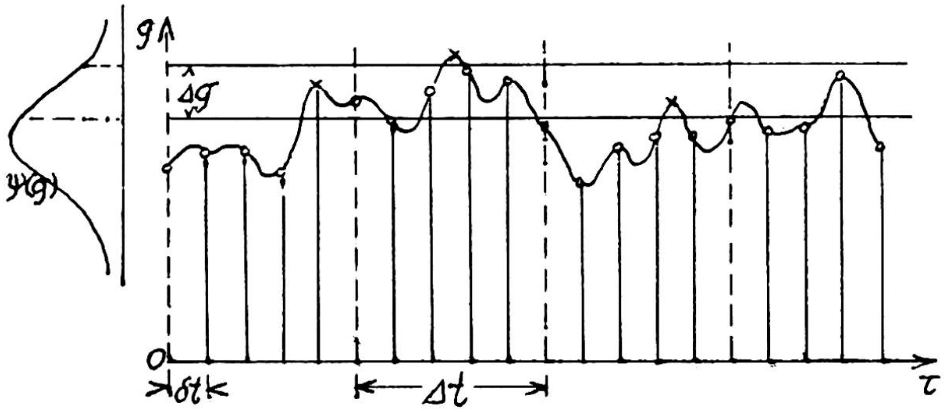

Fig.1

By taking the ratios of the number of points (marked with little circles) in the consecutive intervals Δg i.e. in consecutive horizontal bands to their total amount for a sufficiently long time interval t we obtain the frequency distribution and for $t \to \infty$ the probability density of load at a given instant, $\psi(g)$, as shown on the left hand side of Fig.1. When the recording is continuous one can take arbitrary time intervals δt. On "matching" this probability density to that of R we arrive at the probability of failure at a single load application (p_F in /3/).

By considering the time interval Δt in which we are interested and a sufficiently long interval $n\,\Delta t$, and on establishing a constant value of g, we find the number m of intervals Δt in which the latter has not been surpassed. The ratio m/n provides an approximate measure (exact for $n \to \infty$) of the probability of not surpassing g in Δt. This probability is a function of the load, $p = p(g)$ that can be found empirically by repeating the procedure for subsequent values of g. At the same time, it represents the distribution function of maximum load in Δt, since non-surpassing of g is equivalent to non-exceeding it by the maximum load. The probability density $\pi(g)$ of maximum load in Δt can be obtained as the derivative $dp(g)/dg$ or else directly from the graph, from the occurrences of "maximum" points in the consecutive intervals Δg (a procedure similar to the one already used for $\psi(g)$).

If Δt were equal to the period of exploitation, the function $p(g)$ would represent directly the distribution required. For practical purpose, however, it is important to arrive at some conclusions as to the distribution of maximum load in the period of exploitation T from the distribution of max g during an interval Δt that is, as a rule, shorter or, directly, from the density function $\psi(g)$, which implies two possible procedures, discussed in what follows.

In the first of them we find the probability of not exceeding g in n intervals Δt during the time $T = n\Delta t$,

$$\Phi(g) = p^n(g), \tag{1.5}$$

valid under the restriction of independence of those events. Here, $\Phi(g)$ represents the distribution of the maximum load. The probability density of this load is obtained by taking the derivative of 1.5,

$$\varphi(g) = n\, p^{n-1}(g)\, \pi(g). \tag{1.6}$$

The above formula can also be obtained directly on taking into account that np^{n-1} represents the probability of not surpassing g in n – 1 intervals Δt, whereas $\pi(g)\,dg$ is the probability of the maximum load amounting to g in the remaining one interval Δt.

On establishing the load max g at a sufficiently high level so that higher values of g will occur but rarely, e.g. once in several months or even years, the interval Δt can be so far reduced that – without encroaching on the assumption of independent loads in consecutive intervals – very high values of p (near to 1) are attained. The distribution (1.5) now tends to the Poisson distribution

$$\Phi(g) = \exp\left[-\nu(g)\,T\right], \qquad (1.7)$$

with $\nu(g)$ denoting here the average number of events when g is surpassed per unit time ($\nu = 1/t_0$, where t_0 is the average time interval between such events), this number being dependent on the fixed value of g. The function $\nu(g)$ is found experimentally, e.g. by computing t_0 for consecutive g (on a graph of the type of Fig.1).

The second procedure we mentioned above does not require determining of the function $\nu(g)$ or $p(g)$ and is based on Fisher--Tippett asymptotic extremal distribution representing the distribution of the highest (or lowest) value in a test, where the number of particular test readings increases infinitely. Thus, it is a matter of finding the limiting disturbation, for $n \to \infty$ of the largest of n randomly chosen ordinates of points marked by little circles in Fig.1, at a fixed value of δt (so as to satisfy the requirement of independent loads). It is this form to which the distribution of max g tends for $t \to \infty$, since a test of infinite number of test readings tends to become strictly representative. For finite n we obtain here Eqs. (1.5 and 1.6); albeit, $p(g)$ and $\pi(g)$ have to be replaced by $\Psi(g)$ and $\psi(g)$, respectively (cf. Fig.1), i.e. by the probabilities of g at a given moment (in the experiment under consideration).

On introducing the new variables

$$z = n\left[1 - \Psi(g)\right],$$

$$-u = \ln z = \ln n + \ln \int_g^{\infty} \psi(g)\, dg,$$

we obtain

$$\varphi(g)\, dg = n\, \Psi^{n-1} \psi\, dg = n \left(1 - \frac{z}{n}\right)^{n-1} \psi\, dg \xrightarrow[n \to \infty]{}$$

$$= n\, e^{-z} \psi \frac{1}{dz/dg}\, dz = -\, e^{-z}\, dz = e^{-e^{-u}}\left(-e^{-u}\right) du,$$

whence the variable u is seens to possess the asymptotic density distribution

$$\omega(u) = \exp\left(-u - e^{-u}\right). \qquad (1.8)$$

The variable u is seen to be related linearly to g if $\psi(g)$ is exponential. Consequently, $\varphi(g)$ can be obtained directly from Eq.

(1.8). It is proved in mathematical statistics that the asymptotic extremal distribution of Eq. (1.8) holds for normal distribution of $\psi(g)$, too. Obviously, this is but an approximate calculation, since the centre of the distribution is shifted proportionally to ln n, at finite $n = T/\delta t$ (to be calculated from the records), whereas the form of the distribution (1.8) itself is exact only for $n \to \infty$ and does not depend on n.

Further calculations depend on the particular form of the distribution $\psi(g)$ and, for different theoretical assumptions, are developed in the theory of extreme value distributions (cf. for instance /4/). Once we have found the extremal probability density $\varphi(g)$ [from (1.6) or the derivative of (1.7) or else (1.8)] we insert it into the integral

$$L = \iint_{P<R} \varphi(g)\,\psi(R)\ dg\ dR, \qquad (1.9)$$

$\psi(R)$ being the probability density distrubation for the strength R, where the integral is taken over the part of the plane (g,R) determined by the inequality $g < R$. Since $\varphi(g)$ is a function of time, so is $L = L(t)$ and our problem is solved.

2. Concept of damage and outline of a general theory

Precedent considerations were based on the assumption that R is constant which is but a crude approximation. We know that it depends, for instance, on the number of repeating load cycles in fatigue tests or on the time of loading if rheologic phenomena are involved. In order to describe this behavior the notion of "cumulative damage" is introduced in the theory of fatique of materials and similar notions are also known from the general theory of reliability.

Let us generalize this notion and assume that the actual state of a structure (or a material) at a given instant (from the view-point of its carrying capacity) is defined by a unique positive number, δ , $0 \leq \delta \leq 1$, called damage, where zero damage ($\delta = 0$) describes a perfect state and $\delta = 1$ a complete failure. In general, δ increases in the course of time, particularly, in the course of the loading precess, which means that the ruin sets in progressively and results in the "death" of the structure when δ attains 1. For a simple example (not to be directly extended), δ can represent the reduction of the cross- section of an axially loaded rod because of an expanding crack. Thus, our strength condition $P < R$ is to be replaced by a more general one

$$\delta < 1. \tag{2.1}$$

Now, the problem consists in the prediction of the time t at which the damage becomes 1 or, in a probabilistic approach (δ being a random variable), in the determination of the probability

$$L(t) = Pr(\delta < 1) \tag{2.2}$$

for a given period of exploitation $t = T$.

For the classical case δ remains 0 as long as $P < R$, R being the carrying capacity. On surpassing R for the first time δ suddenly increases to 1 and the structure fails (Fig.2). It is seen that δ is defined to be a Heaviside function

$$\delta(t) = H(t - t_R),$$

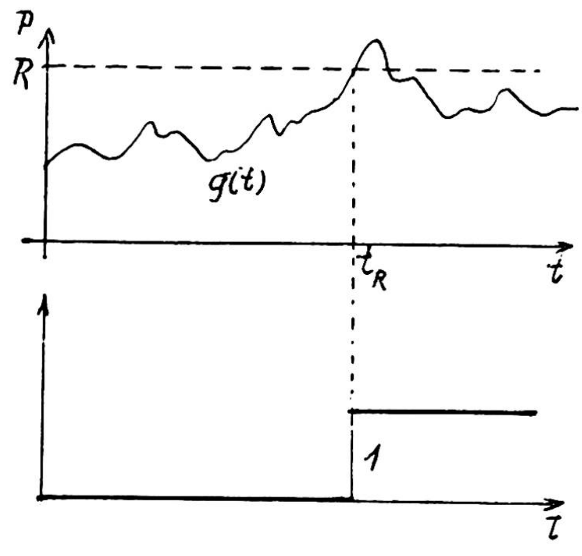

Fig.2

t_R denoting the time to first surpassing R by $P = g(t)$. The probability (2.2) reduces to (1.1) and exactly the theory in point 1 provides the solution.

In general, the hypothesis that the physical state of a structure can be determined by a unique parameter, δ, is a considerable simplification of actual conditions, albeit it results in a far reaching generalization of the former theories of safety. In fact, δ can depend on the whole previous history of loading and, therefore, is a functional defined on the class of all possible functions $P = g(t)$. Depending on what phenomena are to be included (e.g. fatigue, rheology, etc.) and for the sake of effective calculation further restricting hypotheses have to be introduced. First of all, we shall assume that δ is cumulative so that we only examine the increments $d\delta(t)$ which simply integrate in time. If, for the time being, we abstract from rheologic phenomena, we are able to make the assumption that $d\delta$ depends only: (1) on the instantaneous internal state described by δ, (2) on the external state described by $P = g(t)$, (3) on the change of the external state given by $dP = d\,g(t)$, (4) directly on time. Taking the increments in a time unit, i.e. replacing them by velocities (denoted with dots) we obtain

$$\dot{\delta} = f(\delta,\ g,\ \dot{g},\ t). \tag{2.3}$$

The direct dependence on time reflects corrosion-like phenomena affecting δ and will be neglected in further consideration. If we assume that damage is irreversible, the function f will be non-negative with respect to all arguments. If g approaches the limit strength R the velocity $\dot{\delta}$ rapidly increases; if, furthermore, R depends on δ and is independent of $\dot{g}$, we have $f \to \infty$ for $g \to R(\delta)$ Further simplifying assumptions may state that $\dot{\delta}$ does not depend on the sign of dg (internal friction-like phenomena at fatigue) - resulting in f symmetric respectively to $\dot{g}$, and that it is proportional to $\dot{g}$ which gives the form

$$\dot{\delta} = f(\delta,\ g)\,|\dot{g}| \tag{2.4}$$

or, equivalently

$$d\delta = f(\delta,\ P)\ |dP|$$

Formulae of similar form, where instead of dP appears dn (n - number of cycles), can be found in the theory of fatigue (cf., for instance, /5/); however, those do not include any hypothesis as to the mechanism of failure and hold only within the above theory, for symmetric oscillations).

The simplest possible assumption for (2.4) is

$$f(\delta,\ g) = \beta = \text{const} \tag{2.5}$$

within the admissible region (Fig.3, shaded area) and $f \to \infty$ for $g \to R(\delta)$, that is the actual smooth passage of the surface $f(\delta, g)$ is replaced by a singularity. If, in particular, $\beta = 0$, we have the classical case, with the additional assumption that initial damage is possible and makes R lower (we are moving along vertical lines in Fig.3). On integrating (2.4) for (2.5) we get $\delta = \beta \Sigma |\Delta g|$

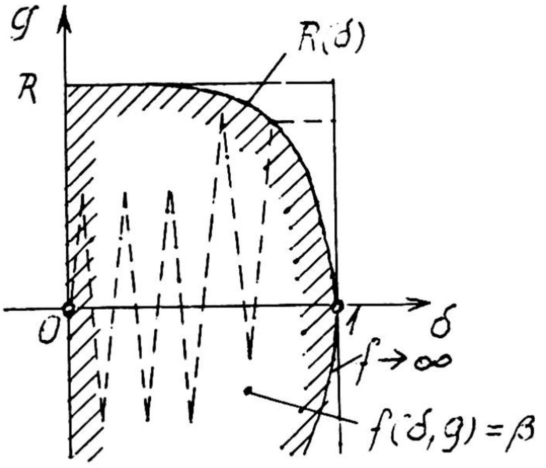

Fig.3

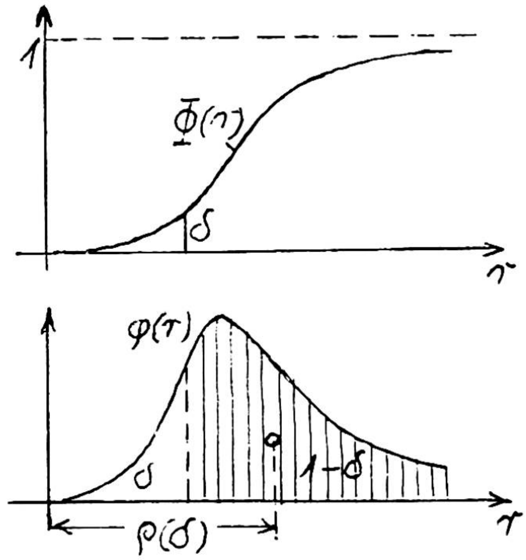

Fig.4

i.e. δ is proportional to the sum of amplitudes of all load cycles, irrespective of the mean value (in Fig.5, below, the sum of segments $\overline{01} + \overline{12} + \overline{23} + \dots$). In the case of simple (symmetric) oscillations it is proportional to the number of cycles, n,

$$\delta = 4 n \beta g_m ,$$

where g_m is the maximum load at one cycle. The path of loading is composed of straight segments with constant slope $|d\delta/dg| = \beta$, independent of the forms of "waves" in time, and on intersection of the curve $R(\delta)$ it jumps horizontally till $\delta = 1$ (Fig.3). This assumption is equivalent to the well-known Miner's hypothesis in the theory of fatigue of materials about a constant damage in a single oscillation with given amplitude. If, in particular, the curve $R(\delta)$ coincides with the bounding straight segments, R = const and $\delta = 1$ respectively, the equation of Wöhler's curve will result directly from the above formula for $\delta = 1$,

$$g_m = \frac{1}{4 \beta N}$$

which is the equation of a hyperbola. More generally, if the equation of Wöhler s curve $g_m = W(N)$ is available, we obtain the curve $R(\delta)$ solving for R the equation

$$R = W\left(\frac{\delta}{4 \beta R}\right) .$$

However, it must be pointed out that this curve might eventually not fit other non- zero values of the mean force, the assumption (2.5) being too simple.

The equation of the curve $g = R(\delta)$ can also be argued theoretically, for instance, in the following way. Imagine a body composed of grains with variable strength properties and a process of damage that consists in consecutive failing of weaker grains. The volume proportion of elements at different levels of local strength, r, can be represented by an integral or differential probability distrubution (Fig.4) . Define the damage δ as the part of the area (normalised to 1) under the curve $\varphi(r)$ or else as an ordinate of the curve $\Phi(r)$. The shaded part of the area, $1-\delta$, represents the actual carrying capacity (due to stronger grains) . The equation sought for is

$$R(\delta) = (1 - \delta)\rho(\delta)$$

where $\rho(\delta)$ is the abscissa of the centre of gravity of the shaded area.

The analysis of safety can be performed similarly to what has been said in point 1 (Fig.5). For a stationary process of loading

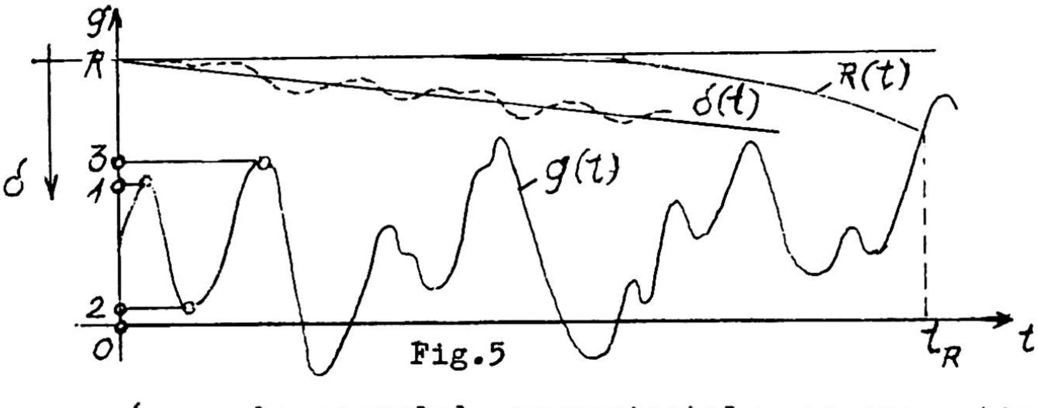

Fig.5

the damage δ can be regarded, approximately, as proportional to time and assimilated to a straight line $\delta = \delta_0 t$, where $\delta_0 = \beta \sum_{t=0}^{1} |\Delta g|$ is the average damage during a time unit (obtained from the load curve by averaging over a sufficiently long time period). The strength curve is expressed in new units

$$g = R(\delta_0 t) \qquad (2.6)$$

and failure appears at first intersection of this curve with $P = g(t)$, the problem being reduced to the one of a material with decreassing strength.

If strength properties (represented by (2.6)) are not affected with scatter, we can use, for instance, the same reasoning as for the formula (1.5) . Assume, we have got records for a fixed period Δt and determined for this period the probability destribution p(g) (similarly as for (1.5)). Since the strength changes in time, we obtain

$$\Phi = p(R_1)\ p(R_2)\ p(R_3)\ldots \qquad (2.7)$$

where, according to (2.6), R_k refers to the k- th sector Δt (cf. Fig.1). Taking logarithms of both sides

$$\ln \Phi = \sum_k \ln p(R_k)$$

and taking for $\ln p(R_k)$ its average value in the respective sector

$$\ln p(R_k) = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} \ln p[R(t)]\ dt$$

we have

$$\ln \Phi = \frac{1}{\Delta t} \int_0^{t} \ln p[R(t)]\ dt$$

and, finally,

$$\Phi = \exp \frac{1}{\Delta t} \int_0^{t} \ln p[R(t)]\ dt. \qquad (2.8)$$

This is an explicit function of parameters describing the function R(t). Of course, this is but an approximate calculation, those parameters and, what more, the curve R(t) by itself being random (cf. Fig.5) .

So far, the analysis was based on the assumptions (2.4) and (2.5) which is only a first step towards a theory including time - dependent phenomena. One of serious difficulties to be surmounted is connected with specifying the functions (2.3), (2.4) . In general, if Wöhler - type curves for different non-zero mean stresses were available, we could come at a result on comparing them with respective solutions of the differential equation (2.4) for sinusoidal forms of load curves and for $\delta = 1$, $n = N$, $\omega t = n$, a,b - constants,

$$\frac{d\delta}{dt} = f(\delta\ ,\ b + a \sin\omega t)\ |a\omega \cos\omega t|\ .$$

Further generalizations could take into account rheologic phenomena and the formulae of the type (2.3) would be replaced by functionals, e.g. in an integral or an operational form. The simplified assumptions would, possibly, retain formulae of the type (2.3), introducing, however, some characteristic values of the load from the precedent history (e.g. the next local or the absolute maximum and minimum values of g). The analysis, however, would be much more complex and is beyond the scope of this article.

In the present contribution we did not consider conventional measures of safety (e.g. coefficients of safety), as the methods of derivation of such measures have been discussed many times (cf. for instance, /3/, /5/) and a "pure" theory of safety can (and ought to) do without them.

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SUMMARY

The contribution is concerned with the problem of safety of structures on the basis of the theory of cumulative damage. The actual state of a structure (from the point of view of its carrying capacity) is described with a parameter, variable in time, depending on the previous course of loads. The latter is regarded as a stochastic process and a probabilistic measure of safety is derived.

RÉSUMÉ

L'auteur a examiné le problème de la sécurité de constructions du point de vue de la théorie du dommage cumulé. L'état actuel d'une construction est caractérisé par un paramètre unique (le dommage) variable avec le temps, dépendant des charges préalables. Celui-ci est considéré comme un processus stochastique et une mesure probabiliste de la sécurité est dérivée.

ZUSAMMENFASSUNG

Im vorliegenden Beitrag wird die Frage der Sicherheit einer Konstruktion auf Grund der Theorie der Anhäufung der Beschädigungen behandelt. Der Zustand der Konstruktion vom Standpunkt seiner Tragfähigkeit wird durch einen Parameter beschrieben, der die Beschädigung charakterisiert und von dem vorigen Verlauf der Belastung abhängig ist. Der obenerwähnte Verlauf wird als ein zufälliger Prozess aufgefasst und ein wahrscheinliches Mass der Sicherheit wird abgeleitet.