DISCUSSION PRÉPARÉE / VORBEREITETE DISKUSSION / PREPARED DISCUSSION

Model Analysis for Structural Safety and Optimization

Analyse sur modèles de la sécurité et de l'optimisation des structures

Modelluntersuchung der Bausicherheit und -optimierung

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1.- Foreword.

a) In a short note presented at the Rio de Janeiro Congress of the IABSE in 1964 I stated that the possibility of analyzing on models, even to failure, of large structures, particularly plain or reinforced concrete structures, has long been proved by me in a great number of cases.

In fact, a model study under elastic conditions furnishes the values of the prototype stresses under working load, which is important for several reasons. Firstly, the results obtained, unaffected by the assumptions and limitations which impair the classical methods of calculation, can profitably be compared with those supplied by these methods. Secondly, it is not hard to solve on models unusual three-dimensional problems, contrary to what is the case with the conventional analytical procedures both because of extreme complexity (only partly reduced by the finite element method) and difficult mathematical schematization of accurate boundary conditions.

Extension beyond the elastic range is still always invaluable to the structural engineer as it may enable him to locate possible weak points in the design and thus assist in securing greater safety and optimization.

Models may be classified in elastic (tested within the elastic range only), structural (carried to failure) and geomechanical (when the foundation influences the structural performance).

Present trends, based on experience at ISMES, are:

- increasing emphasis on structural models;
- constant improvement of model materials to better suit the aims pursued;

- growing interest in thermal stress investigation, especially for concrete dams and reinforced (prestressed or not) concrete vessels of nuclear reactors;
- dynamic testing on large shake tables and marked concern for earthquake effects.

b) Theme I has been treated by the general reporters prof. A. M. Freudenthal and prof. J.Courbon.

In a first theoretical and critical paper regarding topic 1a, prof. Freudenthal deals with the evaluation of overall structural safety based on probabilistic criteria related to the operating loads, which seems fit for statically determinate structures only. In a second paper concerning topic 1b and also of a theoretical probabilistic nature, the same author discusses the possibility of predicting ultimate safety based on the physical properties of the materials and their influence at failure. None of the papers mentions structural model analysis.

Prof. Courbon's paper treats with topic 1c of Theme I. It concludes by mentioning, all too briefly, the great services rendered by model studies in the design of dams, thin shells and shields of nuclear reactors.

I, therefore, believe it of use to outline, the present-day possibilities of model analysis in evaluating the safety degree of large statically indeterminate structures.

c) Model investigation primarily concerns <u>statically highly indeterminate structures</u> and may be regarded as:

I) a modern method of stress analysis;

II) a tool for failure load evaluation.

In any case, it is possible to consider or predict the <u>statistical dispersion</u> of the operating loads and of the structural resistance of the prototype material.

In case II), when several models are tested, it is possible to evaluate the ultimate carrying capacity R of the structure for each type of load S, so that the model functions as a tool for determining the overall safety factor ν.

This factor may vary for each type of structure, depending on the probabilistic possibility assumed for the operating loads and the structural resistance, associated with a definite risk of failure.

Thus, for concrete dams, the loads are practically known (excepting those for earthquake-resistant design), and the uncertainties about the concrete resistance are quite small. The highly redundant type of these structures generally reduces the importance of the concrete strength dispersion. The safety factor in this case, therefore, serves rather as a coefficient of security against the insecurity of the analytical results, especially in relation to the real properties of the rock foundation.

2.- Actual possibilities of model analysis.

a) Elastic models are based on linear elasticity (Hooke's law) and, hence, a superposition of effects is allowed. They also permit to proportionately modify the loadings so as to obtain the most suitable testing conditions. In particular, it is possible to operate at strains that are amplified with respect to those required by similitude (which demands that the strains in the prototype and in the model be the same).

Elastic models, widely used in "stress-analysis", may be divided in two groups.

The first group concerns plane elastic structures,and for them the deformeter, photoelastic and Moiré methods are predominant.

Deformeters are based on the well-known reciprocal theorems (Maxwell, Betti, Müller-Breslau). Photoelasticity is a first-rate research method, most used in structural engineering laboratories. The Moiré method is primarily used in flat slab investigations.

However, it should be observed that the importance of these methods has lately decreased due to the use of computers in solving problems relating to plane elastic structures.

The secondgroup deals with three-dimensional models. In statical tests the loading equipment is usually made up of calibrated weights or hydraulic jacks, the pistons of which react against an external rigid frame; the loads are applied to the model through wooden cork-soled pads. Strain gages, ordinarily applied to the surface of the model, are used for measuring the direction and magnitude of the principal strains.

Young's modulus and Poisson's ratio of the model material are determined as usual, the former by tensile and flexural tests and the latter by torsional tests. The material may quite differ from that of the prototype, provided it obeys Hooke's law and its Poisson's ratio is similar to that of the structure. The model then functions as a "stress computer",and its results may be compared with the theoretical ones.

For elastic models, ISMES has recently succeeded in using epoxy resins mixed with various aggregates.They permit obtaining a wide range of elastic moduli in accordance with the requirements of each case, and stress-strain relationships that are similar even when the stresses are high.

b) Structural models are best made of the same material as the prototype. This is generally possible for steel or prestressed concrete structures when suitable scale (1:4-1:20) models are used. But for very large structures, such as concrete dams, we are forced, also for economic reasons, to adopt greatly reduced scales (1:30-1:100) and hence to use model materials whose mechanical characteristics are reduced compared with those of concrete in accordance with similitude requirements.

For structural models, I have long since used special materials simulating the mechanical properties of concrete, by introducing the technique of "wet models" (with a waterproof coating) practically free of internal stresses.

The tests are then divided into two successive stages. In the first stage, called "normal load tests", the deformations are investigated for values close to similitude conditions (that is, $\varepsilon = \varepsilon'$) under loads corresponding to those of the structure in operation(°).

The second stage concerns ultimate load tests and the transition to them is gradual. The ratio of the highest actually supported load to that of the design load is generally assumed as overall "factor of safety".

This ratio can easily be referred to all the operating loads equally or differently increased following a probabilistic coefficient applied to each independent load. In the case of statically high indeterminate structures it differs from the classical ratio of ultimate to working stress, and its meaning is greater since it takes into account the bi- and triaxial strength of the material under stresses in different directions and the plastic adjustment.

One can by expedients increase on the model solely the loads which in the prototype may rise through extraordinary action. Such are wind load for skyscrapers and water pressure for dams. The horizontal loads alone may undergo increases of consequence for the stability of these structures. In setting up a model study it is, therefore, of basic importance that the factor of safety shall be evaluated as simultaneously affected by:

- loads having a fixed value (dead load);
- loads which may increase with respect to their ordinary value (wind effect);
- actions the occurrence of which is only probable (earthquake).

In practice, when the so-called "weight" of each of the above phenomena has been established, one can obtain

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(°) It is advisable to secure, through repeated loading cycles, non-elastic displacements (settlement of the foundation, adjustment and opening of joints, localized plasticity) which are likely to occur since the first loading in order to obtain an elastic and uniform model performance fit for repeated measurements and controls. This permits obtaining the stresses, displacements and structural behavior of the prototype under working conditions.

the factor of safety by experimentally increasing all the loads up to the failure of the model, considering the "weight" corresponding to each type of load

Therefore, not only one but a number of factors of safety can be secured, each of which corresponds to a given set of phenomena the influence of which is to be analyzed.

c) Geomechanical models investigate structures on foundations whose equilibrium conditions may affect the safety of the structures, as is the case of dams, large bridges and power or highway tunnels.

The stability of block foundations has lately beer simulated and studied on geomechanical models the characteristics of which had conveniently been schematized on the basis of geognostic tests.

It may also be pointed out that in-situ and laboratory investigations of the geomechanical features of the rock and soil mass are increasingly used and recommended as an aid to model studies.

The models, therefore, must faithfully simulate the rock and soil conditions and its mechanical properties. The tests are usually carried to failure.

These investigations are to be considered as basic when extensive discontinuities (faults, cavities) or a pronounced anisotropy (stratifications and diaclases) are present in the rock mass, especially when sliding or least-resistance planes may develop or, more generally, when large low-strength block formations are involved.

In these models, cohesion and angle of friction must also be faithfully reproduced. The difficulty encountered in establishing the true values of the angles of friction makes it in the modeling conservative to assume reduced values which are still within the approximation allowed by field tests.

The prototype and model strains have to be the same and, therefore, the scale ratio must be reduced. The model materials then shall have high densities and low mechanical properties (i.e., very low moduli of elasticity, yield-point and ultimate loads) in order to comply with similitude.

3.- Assessment of (structural) safety at the design stage.

The adoption of model techniques is firstly of considerable importance at the design stage of structures, especially if these are statically highly indeterminate.

Structural safety can then be evaluated by modern probabilistic criteria as suggested in the Freudenthal report when:

- the expected statistical dispersion of the loads, i.e. of the external or operating forces S, is taken into consideration by determining the dimensions of the prototype on the basis of a force $\gamma_s \cdot S$ (where γ_s, the load safety factor, is >1) and adopting equivalent working forces in the model;

- the statistical dispersion of the strength σ_R of the prototype material is taken into account by assuming an ultimate load, or a yield point, equal to σ_R / γ_R (with γ_R, the rupture safety factor, >1) and comparing the highest internal stresses furnished by the model at that value.

The model then becomes a very efficient tool for a "structural analysis".

As typical examples I shall mention:

- the statical and dynamic investigations carried out on an elastic model of the Polcevera viaduct, of the Maracaibo bridge type, designed by prof. R. Morandi (fig. 1);
- the far more elaborate analysis, made particularly on a structural model carried to failure, of the new San Francisco Cathedral designed by prof. P.L.Nervi(figs.2,3);
- the study of the safety degree of the Kurobe IV Dam and its foundation (figs. 4, 5, 6).

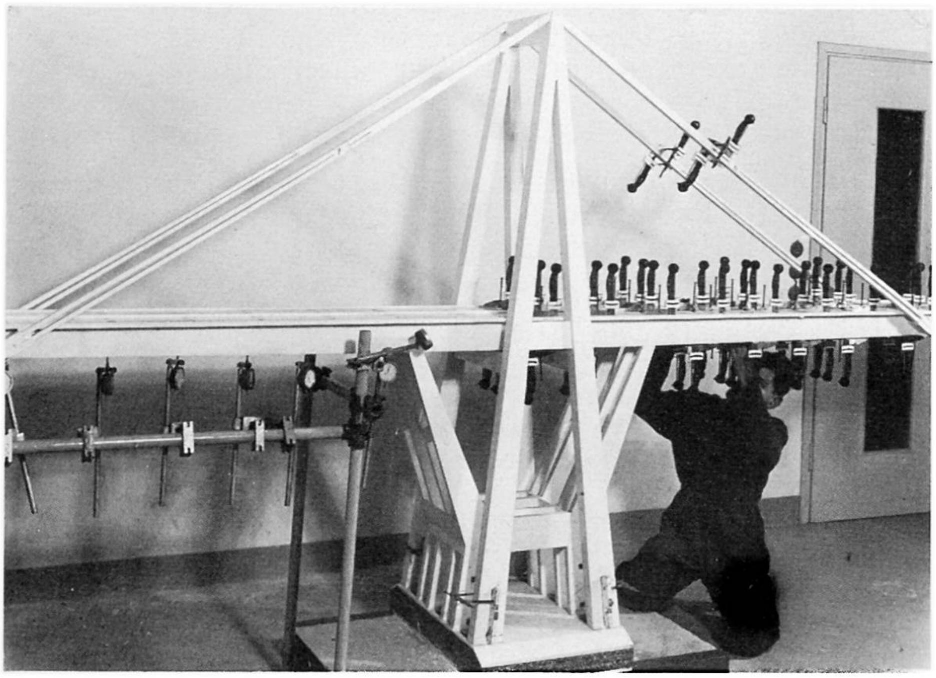


Fig. 1 Polcevera viaduct, Italy. Elastic model: Scale 1:50

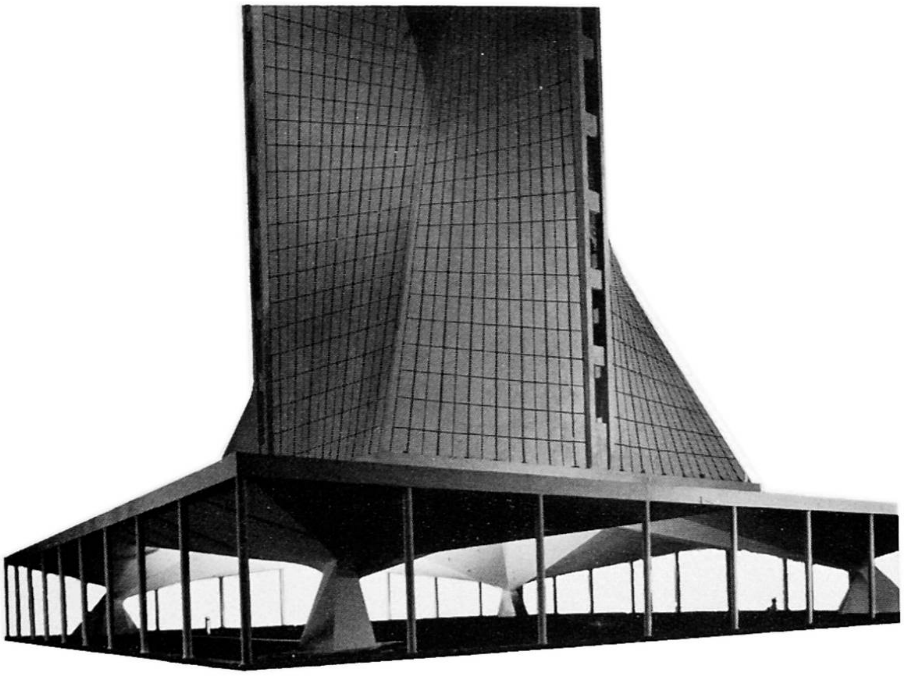


Fig. 2 San Francisco Cathedral, U.S.A.
General view of model. Scale 1:15.

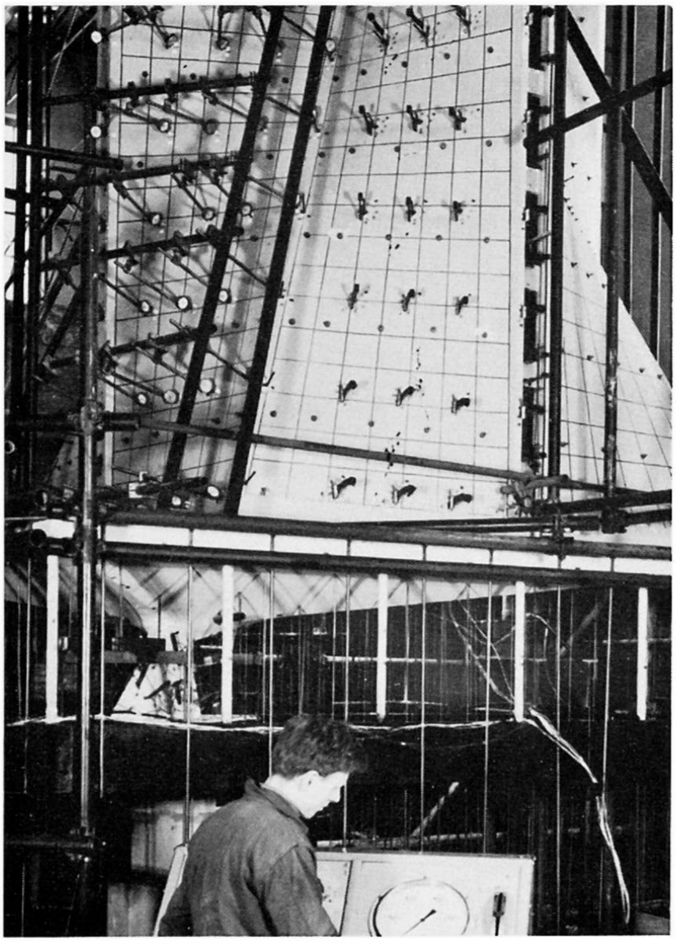
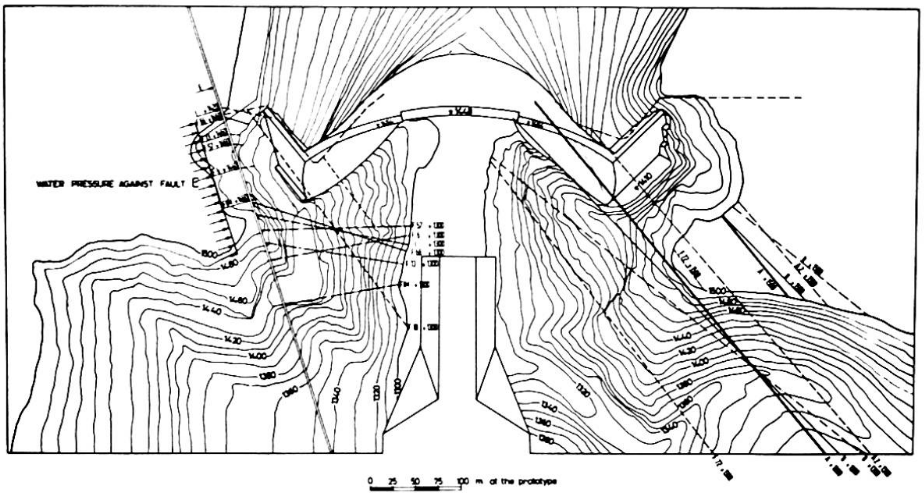


Fig. 3 San Francisco Cathedral.
Structural model under failure tests.



CHARACTERISTIC DATA FOR ACTUAL FAULTS

RIGHT BANK				
FAULT	THICKNESS m	COHESION kg cm⁻²	FRICTION ANGLE φ	MODULUS OF ELASTICITY
E	2.00	2.00	40°	4000 kg cm⁻²
F 6	0.40	2.00	25°	1000 "
F1M	0.40	2.00	25°	1000 "
F 57	0.40	2.00	25°	1000 "
F 18	0.25	2.00	25°	1000 "
I	0.25	2.00	25°	1000 "
F 13	0.25	2.00	25°	1000 "
F 85	0.25	2.00	25°	1000 "

LEFT BANK				
FAULT	THICKNESS m	COHESION kg cm⁻²	FRICTION ANGLE φ	MODULUS OF ELASTICITY
A	1.00	0.15	8°	1000 kg cm⁻²
A 2	0.40	0.40	10°	1000 "
B	0.40	0.40	10°	1000 "
D	0.40	0.40	10°	1000 "
F 72	0.40	0.40	10°	1000 "

Fig.4 Kurobe IV Dam, Japan.
General Layout of geomechanical model.

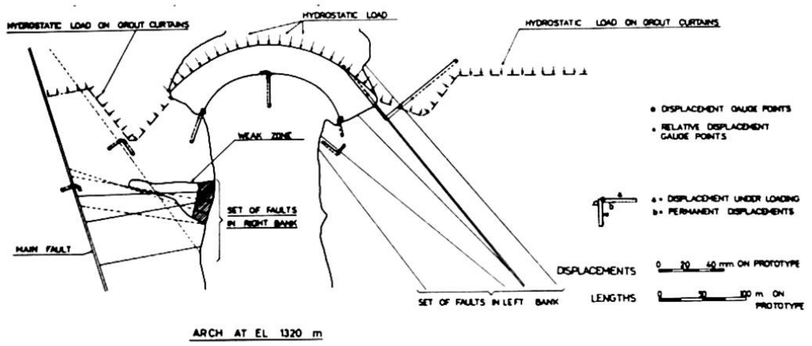


Fig.5 Kurobe IV Dam. Displacements recorded at a horizontal section (el. 1320 m).

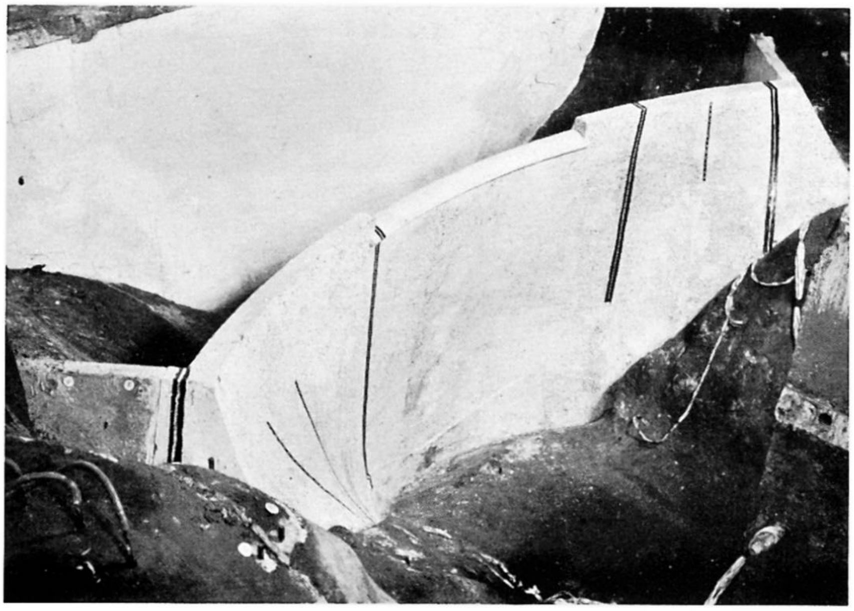


Fig.6 Kurobe IV Dal. Structural model at failure, showing the opening of joints.

4.- Evaluation of the safety degree of an already existing structure.

Models may be of great assistance not only when structures are being designed but also when the stability and safety degree of erected works are being checked. This is particularly true when verifying large structures which have undergone statical conditions unpredicted or unpredictable at the design stage.

Here, too, I shall briefly illustrate some examples in which testing on models has yielded highly significant and conclusive results, with particular regard to the safety degree of the structure.

After a few years of operation, extensive subhorizontal microcracks were found at the upstream face of a large arch-gravity dam completed in 1958. The influence of these cracks on the structural performance and safety of the dam at full reservoir has been investigated on a large structural model in which the number and pattern of the microcracks had faithfully been reproduced (figs. 7, 8).

Interesting tests were also conducted on a 1:4 scale model to verify the compression safety degree of the main columns of the Cathedral in Milan (fig. 9). The two materials (Candoglio and Serizzo marbles), of highly different moduli of elasticity, and the geometry of the individual blocks were identical with those of the prototype (fig. 10). The pattern of the stresses in the masonry dome carrying the main spire of the Cathedral has then been analyzed on a large elastic model (fig. 11).

The effect of the horizontally stratified bedrock anisotropy on the stability of a recently constructed dam was investigated by means of geomechanical models. The various expedients devised to raise the safety degree of the dam-foundation unit were also examined (fig. 12).

Finally, the model tests carried out for the double-curvature arch Vajont Dam should be mentioned. As is known, this dam has brilliantly withstood the extraordinary sliding of Mount Toc into the partly filled reservoir and is now sustaining the enormous asymmetric mass of slide material (fig. 13). After the disaster, model studies were conducted to determine the safety degree of this imposing structure under the present exceptional live load (fig. 14).

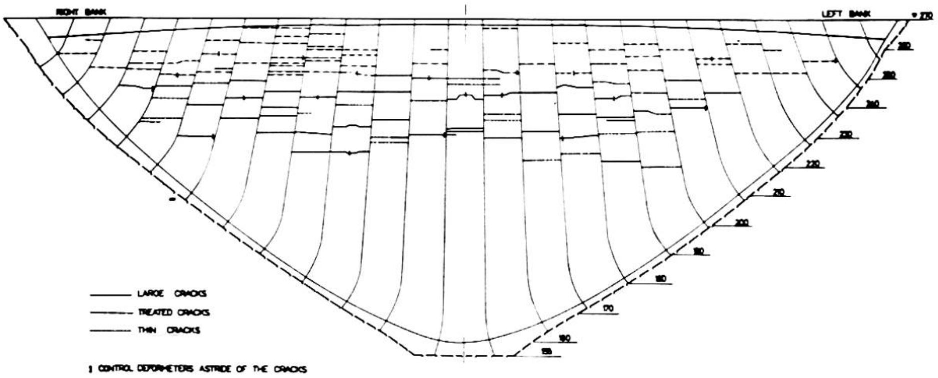


Fig. 7 Flumendosa Dam, Italy. Microcracks on upstream face of model.

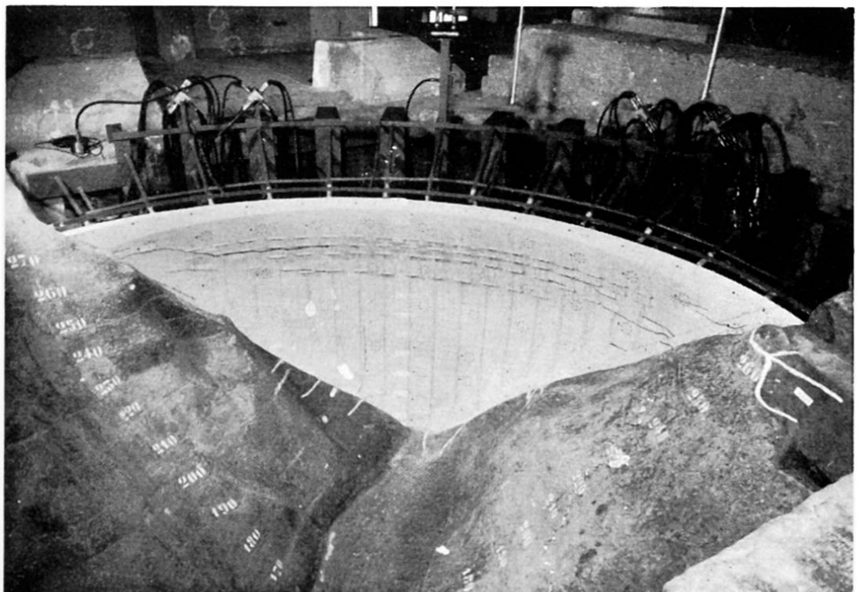


Fig. 8 Flumendosa Dam. Downstream view of model under test.

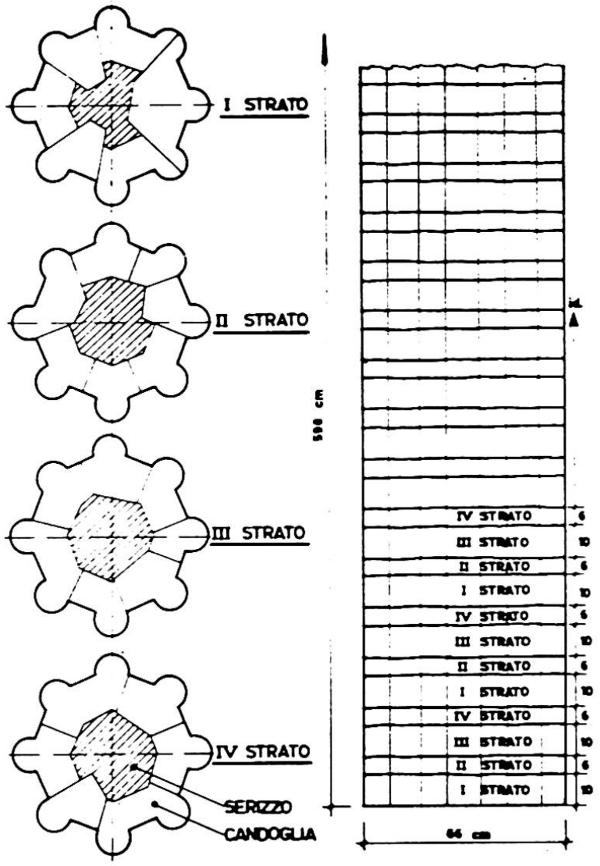


Fig. 10 Milan Cathedral.Cross-sections of columns.

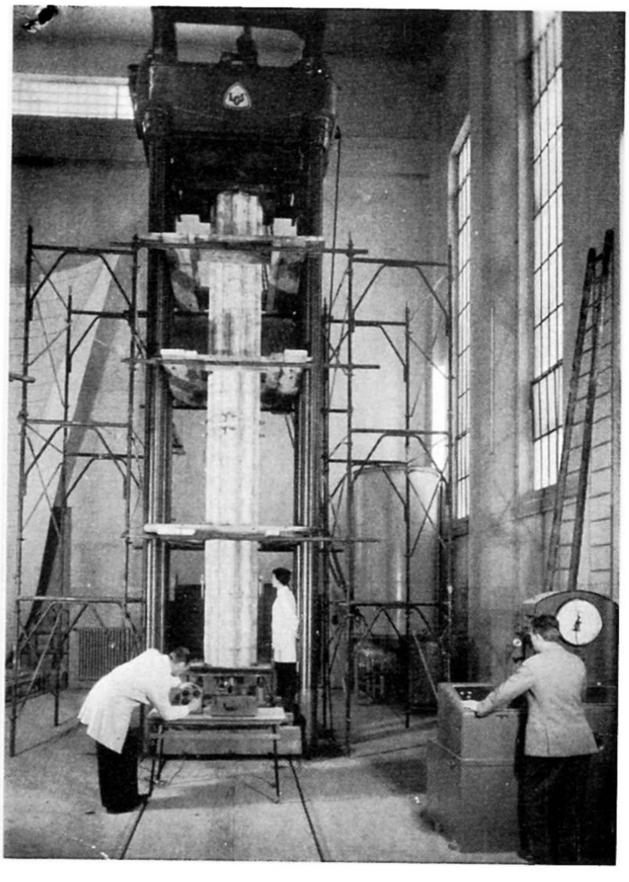


Fig. 9 Milan Cathedral. Model of column under test. Scale 1:4.

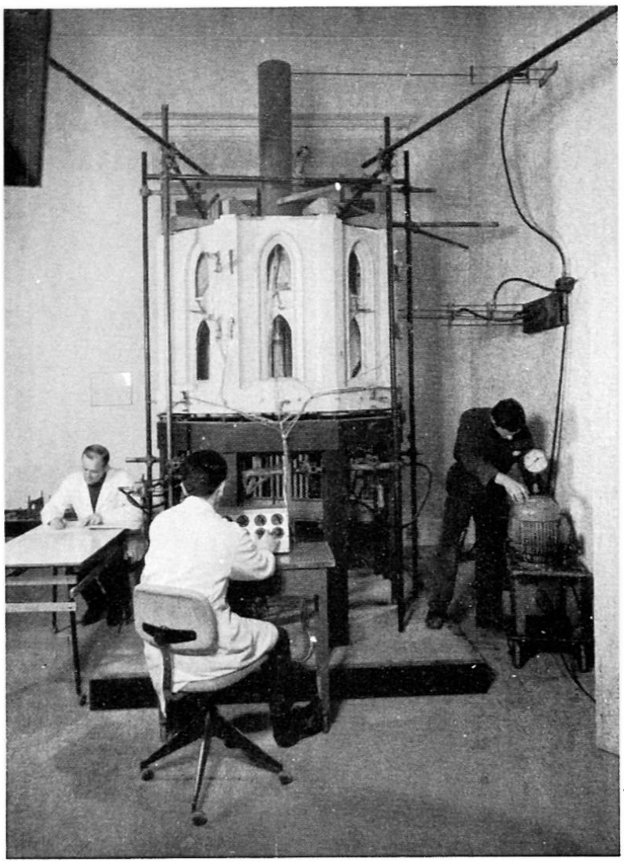


Fig. 11 Milan Cathedral. Elastic model of masonry dome.

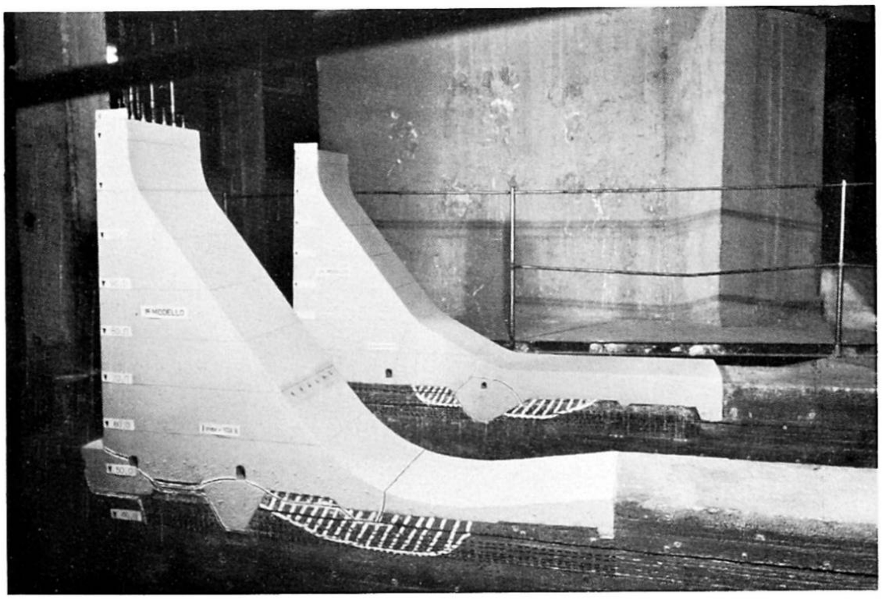


Fig. 12 Mequinenza Dam, Spain. Plane model on geomechanical foundation.

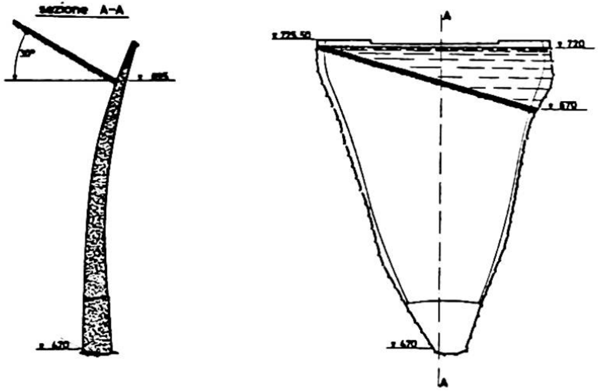


Fig. 13 Vajont Dam. Asymmetric slide material acting on upstream face.

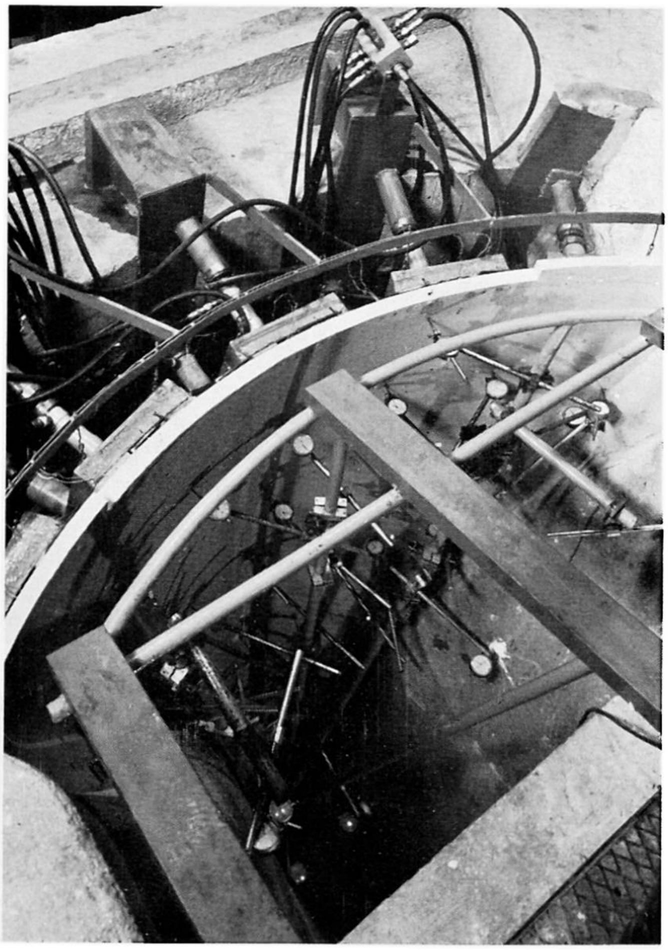


Fig. 14 Vajont Dam. Model under asymmetric load test.

5. - Conclusions.

It is believed that the above brief outline has clearly shown the contribution given, and which can be given, by testing on models when evaluating the safety degree of a structure at its design stage and after construction.

When it is assumed, in accordance with modern trends, that a rational determination of safety involves the adoption of an acceptable risk of failure, a design procedure for uniform safety, and hence optimization, can be based on structural model investigation.

Finally, the arduous problem of structural reliability of statically indeterminate structures, related to the failure mechanisms depending on the consecutive loads mentioned also by prof. Freudenthal at the end of his paper, can satisfactorily be solved through a judicious adoption of the present model test technique.

SUMMARY

After a short introduction the paper outlines the actual possibilities of evaluating the safety degree of a structure by testing elastic, structural and geomechanical models.

The evaluation may concern: 1)structural safety at the design stage; 2) safety degree of an existent structure and of one operating under extraordinary conditions.

The importance of model investigation particularly for the optimization of statically highly indeterminate structures is then emphasized.

RÉSUMÉ

Après quelques mots d'introduction le rapport souligne les possibilités actuelles données par les differents types de modèles(elastiques,structureaux,géomechaniques)pour l'analyse de la sécurité dea grandes structures.

On considère après:1) l'examen du coefficient de securité dans la phase du projet de l'ouvrage; 2)l'evaluatior du dégré de securité d'un ouvrage dejà achevé ou soumis à des actions exceptionelles.

Le rapport termine en soulignant les possibilités des modèles surtout pour l'étude et l'optimization des structures hautement hyperstatiques.

ZUSAMMENFASSUNG

Nach einer kurzen Einleitung werden die wirklichen Möglichkeiten einer Untersuchung des Sicherheitskoeffizienten eines Bauwerkes an elastischen, strukturellen und geomechanischen Modellen beschrieben.

Der untersuchte Sicherheitskoeffizient kann sich auf den Entwurf, ein bestehendes oder ein unter ausserordentlichen Verhältnissen befindliches Werk beziehen.

Die Wichtigkeit der Modelluntersuchungen für die Optimisierung statisch hochunbestimmter Werke wird nachdem besonders unterstrichen.

Critical Appraisal of Safety Criteria and their Basic Concepts

Etude critique des critères de sécurité et de leurs fondements conceptuels

Kritische Betrachtung der Sicherheitskriterien und ihrer grundsätzlichen Auffassungen

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The subject of structural safety is primarily a matter of common sense and not of mathematics. This does not mean that mathematics should be excluded when safety standards are being established, but it means that its role must be subservient. The conclusions of a most erudite mathematical derivation are only as valid as the underlying assumptions. With this thought in view the writer intends to examine closely some of the propositions forming the basis of the author's mathematical development.

The author associates safety of structures with the concept of probability of failure and he outlines the method of derivation of the necessary relations based on this principle. He is careful however to point out that his formulae are not suitable for practical use for the reason of absence of the pertinent statistical data characterising the random variation of the relevant factors.

Furthermore he freely admits the presence of causes of failure unrelated to random factors and even holds mistakes in design of details as the usual cause of failure. In the light of these admissions one cannot see the virtue of the formulae associating failures solely with the random factors, seldom if ever responsible for the actual failure, and leaving out of consideration, of necessity, the really significant non-random causes.

The author's reference to the alleged use of the failure oriented probabilistic concept of factor of safety in the design of aeroplanes poses an interesting question as to the relevancy of this concept in the design of bridges and buildings. Once a person steps into an aeroplane the risk of failure and death, however remote, is tacitly accepted, and so it is not illogical to associate the design of the aeroplane structure with a probability of failure. The situation is however different in case of buildings and bridges. With his probabilistic approach the author in effect proposes an intentional reduction of safety, however small, compared to the one implied in the conventional design. Neither the society in general nor the engineering profession in particular would accept this idea. The present practice is, and hopefully will always remain, that the building should be designed as safe as humanly possible. This does not insure an absolute safety, because life is full of hazards. Factors responsible for these hazards are mostly of a non-random nature and unpredictable, although some of them, such as tornadoes and earthquakes, excessively severe for a given region, are akin to the phenomena normally incorporated in design. It is no more rational to provide for these overviolent actions than for the acts of war, riots, collision with aircraft, gas and chemical explosions and other factors always left out of consideration.

The kind of reliability required for the design of structures seems to be provided adequately by the commonly used factor of safety covering the uncertainties and faults of all types, i.e. of design, construction, loads, materials and operation. This factor expresses the best collective judgment of engineering profession, and its value is subject to revision with improvement of all aspects of engineering practice.

The concept of failure as an integral part of the probabilistic theory, and several aspects of it, as used by the author, warrant close examination. A natural question is, how to analyze a particular structure for failure. The theory of ultimate or limit design gives in some cases an answer to this question. But this theory is highly controversial (50), and the acceptance of its answer means the endorsement of the theory. In other words, an expert on probability, and normally not an expert on structural theory, makes a decision for the designer, that of the two conflicting theories the elastic and the plastic, he must accept the latter.

Limit design procedure, right or wrong, is available only for low flexural frames. What should one do for the multitude of structures of other kinds? Wait until such solutions by ultimate theory become available, even if one has no confidence that they may be forthcoming?

No distinction is made in the author's theory between the actual physical failure and the functional failure, i.e. an excessively large deformation. This implies that in the author's view it does not matter whether people get killed in the collapse of a probabilistically designed structure or are merely inconvenienced by a large deformation,–a proposition, which is not likely to meet a ready acceptance.

A reader would find difficulty in following the author's argument that failure of a single member signifies failure of the whole structure irrespective of whether the latter is statically determinate or indeterminate.

A major impression which one gathers from the discussion of the probabilistic theory of failure is apparent lack of appreciation by its supporters of a bewildering multiplicity of causes affecting vitally the reliability of a structure. The writer wishes to illustrate this point by two examples.

Comparative stress analyses were made by the writer and his colleague (51) of a reinforced concrete barrel roof by two different methods: firstly, the theory of finite element, a new and highly effective tool of structural analysis, and secondly, by the equations of elasticity given in the Manual of Engineering Practice 31 of the American Society of Civil Engineers. Some significant stresses determined by the two methods differed greatly. How then should the choice between different discordant but still admissible methods be made by a probabilistic designer? By the way of explanation it may be pointed out, that in the present design practice, once the disagreement of the existing methods is recognized, concensus is reached in a course of time leading to the acceptance of one method in preference to the other. In the meantime the factor of safety covers the uncertainty.

The situation in the example considered is however much more complicated than mere disagreement of the two sets of numerical results. Both methods of analysis were based on constant moment of inertia (i.e. an uncracked section), constant values of the modulus of elasticity and Poisson's ratio and the absence of creep and shrinkage. These assumptions are obviously not true. The designer would allow for these unknowable factors by judgment based on experience. Design is an art as well as a science, and is more than a mere substitution of numerical values into complicated probability formulae.

The other example is borrowed from the writer's discussion of a recent paper on probabilistic theory by the same author (52) .

"A collapse of an important bridge in the course of erection several years ago (accompanied by loss of life) was found to have been caused by the wrong design of a detail of the erection structure, accentuated by the contributing factors, including an unfortunate and destructive combination of the yielding of steel and crushing of plywood (a phenomenon neither described nor even recognized before), an inadequacy of prescribed allowable stress in the significant area, and two elementary blunders in calculation. Such nondescript errors would baffle any classification, yet they are real and not infrequent, although they are usually less drastic and seldom lead to failure",

In conclusion the writer recapitulates the reasons for his unqualified rejection of the probabilistic theory of safety of structures involving human occupancy.

1. The concept of the probability oriented factor of safety is inacceptable in principle.

2. The factors which usually cause failure are not of a random type.

3. The data for evaluation of parameters characterizing the random type factors are mostly unavailable.

4. The failure causing factors are so numerous and varied that they defy any classification and codification.

5. The value of the intensity of a given load pattern causing failure of a given structure is usually unknowable by a method of structural analysis and is questionable when such analysis is available.

6. Distinction between physical and functional failures and between determinate and redundant structures results in further difficulties for a probabilistic designer.

7. **The usual concept of the factor of safety of the conventional elastic design is the best one available.**

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(50). A. Hrennikoff. Plastic and Elastic Designs Compared. Preliminary Publication. Seventh Congress, Rio de Janeiro, 1964. International Association for Bridge and Structural Engineering.

(51). A. Hrennikoff and S. Tezcan. Analysis of Cylindrical Shells by the Finite Element Method. International Association on Shell Structures. Symposium. Leningrad, U.S.S.R. 1966.

(52). A. Hrennikoff. Discussion. Analysis of Structural Safety by A. Freudenthal, J. Garrelts and M. Shinozuka. Journal of the Structural Division of A.S.C.E.

SUMMARY

The writer rejects the probabilistic method of design of structures involving human occupancy, because (1) it is inacceptable in principle, (2) leaves out of consideration the really significant non-random causes of failure, (3) is based only on a few random factors whose characteristic parameters incidentally are mostly unavailable and (4) for most structures, the condition of failure may not be identified by any existing method of analysis.

RÉSUMÉ

L'auteur rejette la méthode de projection de constructions qui se base sur la probabilité et tient compte de l'occupation humaine.

1 Le principe même de la méthode est inadmissible

2 Elle néglige les causes de ruine non-accidentelles vraiment importantes

3 Elle se base uniquement sur quelques facteurs aléatoires dont les paramètres caractéristiques sont le plus souvent inutiles

4 Pour la plupart des constructions, les conditions de ruine ne peuvent être déterminées par aucune méthode de calcul existante

ZUSAMMENFASSUNG

Der Autor verwirft die wahrscheinlichkeitstheoretische Entwurfsmethode für Gebäude, die von Menschen bewohnt werden, weil sie

erstens im Prinzip unannehmbar ist,

zweitens die tatsächlich wichtigen, nicht zufälligen Bruchursachen auslässt,

drittens auf wenigen zufälligen Grössen gegründet ist, deren charakteristischen Parameter übrigens meist unbrauchbar sind,

und schliesslich viertens, weil für die meisten Bauwerke die Bruchlast mit keiner bestehenden analytischen Methode bestimmt werden kann.

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Critical Appraisal of Safety Criteria and their Basic Concepts

Etude critique des critères de sécurité et de leurs fondements conceptuels

Kritische Betrachtung der Sicherheitskriterien und ihrer grundsätzlichen Auffassungen

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In his brilliant survey of the present status of structural safety problems Professor Freudenthal makes clear that engineers are not designing structures close enough to the "state of art" limit, that the rational approach to the problem of safety has to be a probabilistic one, and that absolute safety is no more than a convenient fiction.

The key to a rational approach to structural safety is in his own words the concept of an "acceptable risk of failure". But most engineers, because they believe they can or they have to design absolutely safe structures, are reluctant to accept such a concept.

The difference between the attitude of accepting or not accepting a risk of failure, be it a small one, is not an academic question, because structures will be designed quite differently depending on whether one does or does not recognize the impracticability of building absolutely safe structures. The consequences of these two opposite attitudes seem worthwhile emphasizing.

If the existence of risks is to be recognized, accepted and taken into consideration in the design of engineering structures, instead of trying to have uniform safety in all elements of a structure - which is an ideal recommended by several authors - one has to reduce the strength of the elements of which the failure will have less costly consequences, with a view to reinforcing those elements of which the failure would have costlier consequences. Such criteria will enable the design to be improved without increase in cost.

If the existence of risks is to be taken into consideration, one has to adjust the strength of the whole structure to the consequences of possible failures.

This implies building a dam, if located upstream of a town, stronger than one located downstream, even if both dams could otherwise be built perfectly alike.

If the existence of risks is recognized and accepted, the structure should be designed so as to reduce as much as possible the consequences of accidents. There will even be instances where it may be convenient to increase the probability of failure of a structure so as to reduce its cost, the savings being used to minimize the consequences of a possible failure. This will be the case with dykes against floods and sea invasions, where transverse dykes are built using the money that could otherwise be used to increase the height and reinforce the main dyke. The function of the transverse dykes is the reduction of the area flooded in case of failure of the main dyke rather than direct protection against sea invasion.

If the existence of risks is to be recognized and taken into consideration, structures will have to be designed so as to fail in the less inconvenient way. In some cases this will imply the use of devices similar in function to fuses, for instance when a lighter and lower dam is built on a secondary valley as a protection to a big earth or rock-fill dam on the main valley.

In spite of Professor Freudenthal's well presented arguments against redundant elements, the presence of such elements can, in some particular cases, contribute to increase the safety of the structure. Not only can the failure of redundant elements give warning to halt operation and avoid serious consequences of accidents, but in some other cases the presence of redundant elements will avoid complete collapse, that would, otherwise, have catastrophic consequences.

Some structures are intended to absorb energy rather than to hold forces.The amount of energy consumed in the destruction of redundant elements can, at least in some cases, be sufficient to save the structure. This is apparently the main reason why ships are always moored with a large number of redundant cables, instead of with a few strong cables.

SUMMARY

The need to design structures accepting the existence of risks and taking into account the possible consequences of accidents may, or may not, be recognized. The practical implications of one or the other attitude are quite different.

If the existence of risks is recognized, the adoption of an uniform safety factor for all the elements of a structure, and the adoption of the same safety factor for structures submitted to the same loads but whose failure can have different consequences, should be discontinued.

It is also pointed out how redundant elements can contribute to increase the overall safety of a structure.

RÉSUMÉ

Reconnaître ou ne pas reconnaître l'impossibilité de sécurité absolue quand on projècte une structure peut avoir des conséquences pratiques très différentes.

Si cette impossibilité est reconnue, on doit choisir le coefficient de sécurité de chaque élément d'une structure, et de chaque structure en elle-même, d'après les conséquences des possibles accidents.

L'influence des éléments superabondants sur la sécurité d'une structure hyperstatique est aussi discutée.

ZUSAMMENFASSUNG

Sehr verschiedene praktische Folgerungen beruhen auf dem Erkennen bzw. Nichterkennen der Möglichkeit, Tragwerke mit Ausschluss aller Risiken zu entwerfen.

Wenn das Bestehen von Risiken erkannt wird, muss der Sicherheitskoeffizient für jedes Element des Tragwerkes sowie das Tragwerk an sich nach dem Umfang der Folgen eventueller Unfälle gewählt werden, anstatt der üblichen Wahl von genormten Sicherheitskoeffizienten.

Der Einfluss der Verwendung zusätzlicher konstruktiver Elemente zur Erhöhung der Sicherheit von statisch unbestimmten Tragwerkssystemen wird ebenfalls beleuchtet.

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Some Safety Problems

Quelques questions de la sécurité

Einige Fragen zur Sicherheit

E. MISTÉTH
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1./ Designations

R	Internal breaking forces and moments
S	Internal forces provoked by load
$Y=R-S$	The Basler reserve
$\xi,\ \eta$	Probability variables
$\xi(t),\ \eta(t)$	Stohastic processes
$a=E(\xi)$	Expectation value
$s=D(\xi)$	Deviation
$v=\frac{s}{a}$	Relative deviation, variation coefficient
$\mu_r=E[(\xi-a)^r]$	Central moment of the r^{th} order
$f=\frac{\mu_3}{s^3}$	Asymmetry
$c=\frac{\mu_4}{s^4}-3$	Excess
$h=\frac{\mu_5}{s^5}$	Asymmetry of the fifth order
$\frac{1}{k}$	Risk
t	Time
T	Lifetime of the construction
q	Interest factor
W	Cross sectional quantity corresponding to the nature of internal forces and moments
σ	Stress corresponding to the nature of internal forces and moments

$\eta = G(\xi_1, \xi_2, \ldots \xi_n)$	Functional relation of independent probability variables
m	Independent variable of the standardised distribution function
$C(k)$	Cost of rebuilding /the bearing element/
$L(k)$	Annual maintenance cost of the construction /the bearing element/
Q	Sum of the damage caused by the ruin of the construction /the bearing element/, profil lost included

2./ Raising of the problem. Methods applied so far in calculating of dimensions

In dimensioning engineering structures for stability it is most essential to determine safety. The first question to be raised is wether an objective standard of safety can be found and what is the most economical magnitude thereof. Thus the general question of dimensioning is this: In what dimensions should be designed the bearing structure of an engineering construction at a time t = 0, if the construction is being designed for a lifetime t = T, with rebuilding cost of the bearing structure being C, and the annual maintenance cost of the bearing structure being L, sum of damages incurred by the ruin of the bearing structure, profit lost included, being Q.

The classical dimensioning specifications present safety in terms of the magnitude of allowable stress. Allowable stress is an empirical value: it is a quotient of breaking strength and safety. Present time specifications are threefold.

Into the first group come those specifications in which safety manifests itself in the measure of allowable stresses and the grouping of loadings. These specifications show, e.g. three groupings as to the combinations of loading forces: operational loading forces, extraordinary loads, catastrophal loads and influences. To each of the three groupings pertains a different allowable stress.

The second group comprises those specifications in which safety is divided in the grouping of loading forces, the dimensioning stresses and the cross section. These specifications proceed from the ruin of the construction and take every uncertainty, with a divided safety sector, at its proper place into consideration; to a greater relative deviation pertains a higher safety sector, to a smaller one a lower factor. The theoretical basis of these specifications was elucidated by Basler [3].

Specifications that come into the third group calculate safety on the basis of probability theory, with consideration given to loading forces and their deviation, rupture stress and its deviation. These specifications calculate with an undivided safety factor.

Safety factor is determined by a probability of rupture assumed in advance, and probability distribution. The assumption of the probability of rupture /10^{-3}, 10^{-4}, etc./ is a result of subjective evaluation, though it is much more perceptible than saying, that, e.g., a twofold safety is required. The function of the selected distribution is also based on individual judgement. The difference between, e.g., the Weibull and lognormal distribution in the rate of the safety factor can be 10-100 per cent [2].

By means of this procedure are calculated airplanes and solved the dimensioning problems connected with space travel. Theoretical considerations were set forth by Freudenthal [4] [5].

All of the three procedures, though at different places and in various ways, give the rate of safety on the basis of individual deliberation. This rate, the expression of safety, will be further on considered to include failures of an accidental character only.

Safety only provides an objective rate of measurement, as it will be demonstrated further below, together with economical considerations.

3./ A new procedure for dimensioning

The known basic relation for the calculation of dimensions, based on probability theory is, if the time parameter is also considered:

$$\lim_{t \to T} \min \left\{ \left[R(t) - S(t) \right] \geqq 0 \right\} \geqq 1 - \frac{1}{k} \qquad \ldots 1./$$

Expression 1./ says so much in words that, during the lifetime of the construction, the Basler reserve [3] Y/t/ = R/t/–S/t/ must be greater by a probability given in advance /1 – $\frac{1}{k}$/ than zero.

The basic relation is not unequivocal without the time variable t /fig.1./

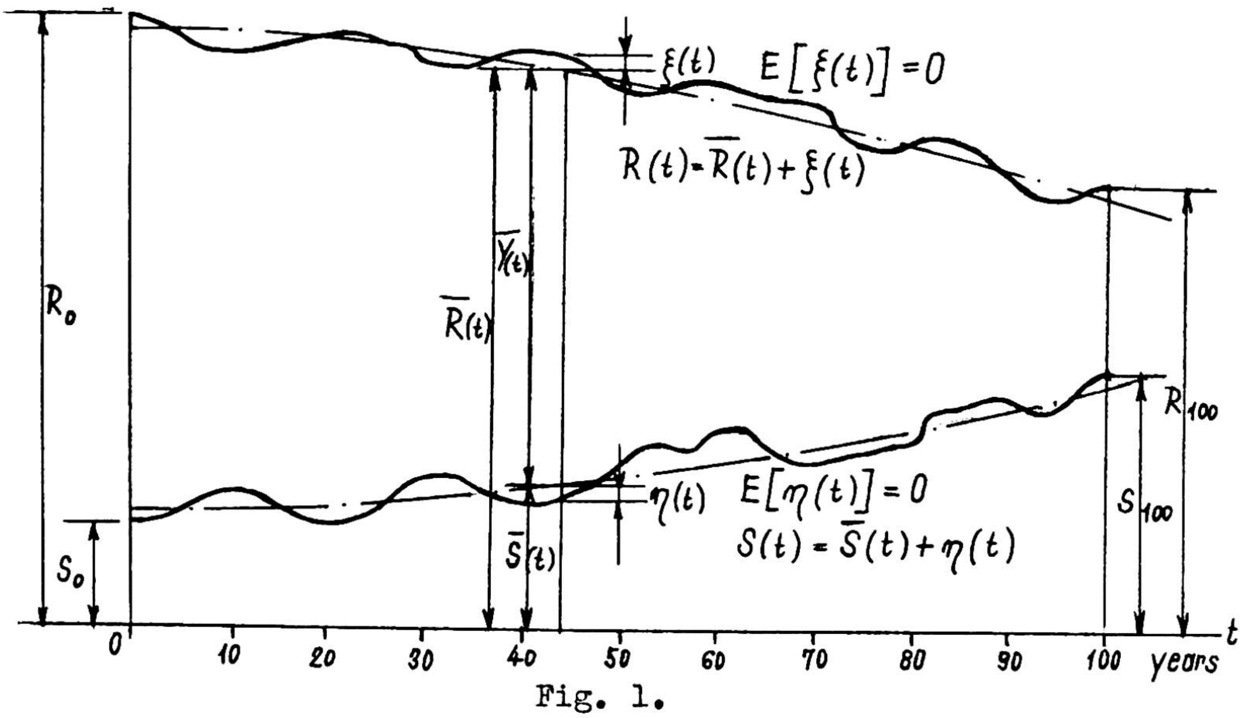

Fig. 1.

4./ Lifetime of the engineering structure

Safety of engineering structures can be related only to a certain lifetime. If T = ∞, the ruin of the structure is considered a certain event. The lifetime of engineering structures, therefore, has to be determined in advance. In respect of lifetime permanent and temporary structures can be dealt with.

For permanent structures lifetime has to be stated in T = 10 years, if it is a vehicle, in T = 100 years if it is an earthwork.

For temporary structures, when an earthwork is concerned, T = 10 years, for structures of locomotion it may be that T = 1. Largely speaking, for permanent structures it is reasonable to count with T = 50 years, for temporary ones with T = 5. From expression 1./ it is perceivable that if T is small, the difference, R/t/-S/t/, is greater than it is for a long T time. This holds particularly if R/t/ and S/t/ are stohastic processes with a notable trend /fig.1/.

5./ Loading forces, loading movements

Loading forces which are constant within time /dead load, earth pressure/, constitute a stohastic multitude, loading forces and movements which vary within time /useful load, water pressure, snow or wind pressure, variation of temperature, creeping, etv./ constitute a stohastic process. The periodical /e.g.annual/ maxima of these latter stohastic processes only form stohastic multitudes. With the processing of technical data it is reasonable to calculate four probability characteristics: the expectation value /a/, deviation /s/, asymmetry /f/ and excess /c/. The processing of the data must be performed on the basis of some textbook of mathematical statistics 1 . In order to provide that the relative deviation of excess itself should not exceed 15 per cent, the number of elements of the multitude has to be selected $\sim$500 at the least. For the determination of the probability characteristics of the useful load it must be proceeded, with consideration given also to future development, from the loading spectre. Forces of a meteorological character must be processed from statistical data.

6./ Rupture stress, geometrical dimensions

Rupture stress within a t time, which constitutes a stohastic process with a trend, has to be determined in principle through precessing a stohastic multitude of rupture tests of the material taken at different times. In want of data it is supposed, in first approximation, that at the end of lifetime rupture stress can be taken as equal to the longtime stress limit and its probability characteristics are the following:

$$\left.\begin{aligned} &\bar{\sigma}_R(T) = \frac{\bar{\sigma}_{Ro}}{\delta} \qquad (\text{e.g. } \delta = 1{,}15) \\ &s^{\sigma}(T) = s_o^{\sigma}\sqrt{\delta}\,;\quad f^{\sigma}(T) = f_o^{\sigma}\,;\quad c^{\sigma}(T) = c_o^{\sigma} \end{aligned}\right\} \qquad \ldots 2./$$

Geometrical dimensions always display a normal distribution. Because of the corrosion effect the geometrical dimensions have to be diminished at the end of the lifetime by a value Δ , which may be, in absence of statistical data, 1 to 5 per cent of the dimension. Deviation of corrosion is taken equal to deviation of dimension:

$$\left.\begin{aligned} &\bar{\mu}(T) = \bar{\mu}_o - \Delta \\ &s^{\mu}(T) = s_o^{\mu}\sqrt{2}\,;\quad f^{\mu}(T) = f_o^{\mu} = 0\,;\quad c^{\mu}(T) = c_o^{\mu} = 0 \end{aligned}\right\} \qquad \ldots 3./$$

7./ Probability characteristics of the function distribution

Probability characteristics of internal forces which cause rupture and internal forces which are the result of loading cannot be determined in a direct way, it is therefore necessary to determine from the probability characteristics of the components the probability characteristics of a quantity characterised by a functional relation. Exact formulation of the problem is this: the independent probability variables $\xi_1, \xi_2, \ldots, \xi_n$ are given, the probability characteristics of these, a_i, s_i, f_i, c_i

/i= 1, 2, ..., n/ are also independent, and $v_i < 0{,}5$; what are the amounts of the probability characteristics of the functional value η, characterised by the functional relation $\eta = G(\xi_1, \xi_2, \dots, \xi_n)$?

If the function η is expanded in a Taylor series and also members of the third order are considered, the probability characteristics of a rational whole function can be determined according to the rule

$$\left.\begin{aligned}
a_\eta &= E(\eta) = G(a_1, a_2, \dots, a_n) + \frac{1}{2}\sum_{i=1}^{n} G_{ii}\, s_i^2 + \dots\dots \\
s_\eta^2 &= D^2(\eta) = \sum_{i=1}^{n} [G_i s_i]^2 + \sum_{i=1}^{n} G_i G_{ii} f_i s_i^3 + \dots\dots \\
f_\eta &= \frac{1}{s_\eta^3}\Big[\sum_{i=1}^{n} G_i^3 f_i s_i^3 + \frac{3}{2}\sum_{i=1}^{n} G_i^2 G_{ii}(c_i+2) s_i^4 + \sum_{\substack{i=1\\ j=1}}^{n} G_i G_j G_{ij} s_i^2 s_j^2 + \dots\dots\Big] \\
c_\eta &= \frac{1}{s_\eta^4}\Big[\sum_{i=1}^{n} G_i^4 (c_i+3) s_i^4 + 6\sum_{\substack{i=1\\ j=1\\ i\neq j}}^{n} (G_i G_j s_i s_j)^2 + \\
&\quad + 2\sum_{i=1}^{n} G_i^3 G_{ii}(h_i - f_i) s_i^5 + 6\sum_{\substack{i=1\\ j=1\\ i\neq j}}^{n} (2 G_i G_{ij} + G_j G_{ii}) G_{ij} f_i s_i^3 s_j^3 + \dots\dots\Big] - 3
\end{aligned}\right\}\quad \dots 4./$$

In expression 4./ e.g.

$$G_{ij} = \left[\frac{\partial^2 G}{\partial \xi_i\, \partial \xi_j}\right]_{\substack{\xi_i = a_i\\ \xi_j = a_j}} \qquad \begin{matrix} i = j = 1, 2, \dots, n \\ i \neq j \end{matrix}$$

Generally if $v_\eta \leq 0{,}15$, it suffices to calculate with the first, in the case of excess, with the first two members. If $0{,}15 < v_\eta \leq 0{,}35$, all of the members written here must be counted with.

If $0{,}35 < v_\eta \leq 0{,}5$ it is necessary to calculate accordingly further members which are not written here. If the derivatives of function G are not limited derivatives but the function can be expanded to a Taylor series at the places a_1, a_2, ..., a_n, also members or an order hidher than the third may be required [6].

If $|f_\eta| < 0{,}1$ and $|c_\eta| \leq 0{,}2$, the resultant distribution may be considered normal.

8./ Internal forces that cause rupture

They depend in general on rupture stress and cross section quantity and constitute a stohastical series

$$R(t) = \sigma_R(t) \cdot W(t) \qquad \dots 5./$$

or

$$R(t) = H\left[\sigma_R^{(1)}(t),\ \sigma_R^{(2)}(t),\ \mu^{(3)}(t),\ \dots\dots,\ \mu^{(n)}(t)\right] \qquad \dots 6./$$

In expression 5./ internal forces causing rupture can generally be established as a product of rupture stress and cross section quantity. Expression 6./ refers to cases in which the bearing structure is not made of a homogenous material and the type of bearing is such as cannot be separated from the geometrical dimensions of the cross section /e.g. excentrical internal forces within a r.-c, bearer/.

Probability characteristics of internal forces causing a rupture, if calculation has to be made on grounds of expression 5./, are

$$E\left[R(t)\right]=\overline{G}_R(t).\overline{W}(t)$$
$$\left[v^R(t)\right]^2=\left[v^G(t)\right]^2+\left[v^W(t)\right]^2+\left[v^G(t).\,v^W(t)\right]^2$$
$$f^R(t)=f^G(t)\left[\frac{v^G(t)}{v^R(t)}\right]^3 \qquad \ldots 7./$$
$$c^R(t)=\frac{1}{\left[v^R(t)\right]^4}\left\{\left[c^G(t)+3\right]\left[v^G(t)\right]^4+3\left[v^W(t)\right]^4+6\left[v^G(t).\,v^W(t)\right]^2\right\}-3$$

If the internal forces producing a rupture are to be calculated by expression 6./, probability characteristics must be determined with the use of formula 4./.

9./ Determination of the cross section quantity

If probability characteristics of the Basler reserve Y/t/ are to be determined from expression 1./, then, on the basis of expression 4./

$$a^Y(t)=\overline{Y}(t)=\overline{R}(t)-\overline{S}(t)$$
$$s^Y(t)=\sqrt{\left[s^R(t)\right]^2+\left[s^S(t)\right]^2}$$
$$f^Y(t)=\frac{1}{\left[s^Y(t)\right]^3}\left\{f^R(t)\left[s^R(t)\right]^3-f^S(t)\left[s^S(t)\right]^3\right\} \qquad \ldots 8./$$
$$c^Y(t)=\frac{1}{\left[s^Y(t)\right]^4}\left\{\left[c^R(t)+3\right]\left[s^R(t)\right]^4+\left[c^S(t)+3\right]\left[s^S(t)\right]^4+\right.$$
$$\left.+6\left[s^R(t)\cdot s^S(t)\right]^2\right\}-3$$

From expression 8./ it is to be seen that the dimensioning will be correct if

$$\left.\begin{aligned}R(t)&\geq S(t)+m\,s^Y(t)\\ m&=m(f,c,k)\end{aligned}\right\} \qquad \ldots 9./$$

where

The value of m depends on the selected distribution and the risk given in advance. Before proposing a type of distribution for the determination of the value of m, a simple relation can be given for the cross section quantity at a time t = 0, if internal forces that provoke rupture are such as according to expression 5./

$$W_0=\Delta W_k+\frac{\overline{S}(T)}{\overline{G}_R(T)}\;\frac{1+m\sqrt{\left[v^S(T)\right]^2+\left\{1-\left[m v^S(T)\right]^2\right\}\left\{\left[v^G(T)\right]^2+\left[v^W(t)\right]^2\right\}}}{1-m^2\left\{\left[v^G(T)\right]^2+\left[v^W(T)\right]^2\right\}} \qquad \ldots 10./$$

In expression 10./ the surplus cross section quantity, being a result of corrosion, ΔW_k, depends on the value m = m/f,c,k/. S/T/ is the expectation value of the sum of internal forces provoked by loadings, at the end of the service time, $\overline{G}_R$/T/ is the expectation value of the rupture stress of the structural element in question, v/T/ are the final values of the variation factors of the variable quantities.

If internal forces provoking rupture can be calculated by expression 6./, for the determination of the dimensions expression 5./ must be satisfied by way of the trial and error method.

10./ The selected type of distribution

The problem is what kind of a distribution function should be selected for the stohastical process Y/t/ at a time t = T at the end of lifetime or any time t. From among the internal forces caused by loading meteorological forces and movements /internal forces caused by wind, snow, modifications of temperature/ can best be described theoretically with the use of the Weibull distribution [7]. A great part of useful loads and internal forces provoked by dead load do not follow the Weibull distribution pattern. The distribution of cross section dimensions is normal. The type of distribution drawn upon the rupture stress can be treated as though from among a homogenous multitude of bearers a discretional one were selected and given to rupture. This problem is, in its essential conception, an urn-model to which one of the Pearson distributions will best apply. Since in the resultant distribution it is the rupture stress that generally has the greatest part and meteorological forces generally play but a slight part, for a resultant distribution the four parameter Pearson distribution, Pearson IV, can be recommended.

$$\gamma \int_{-\infty}^{m} \frac{e^{-\alpha \operatorname{arctg} \frac{x-d}{g}}}{\left[1+\left(\frac{x-d}{g}\right)^2\right]^{\beta}} dx = 1 - \frac{1}{k} \qquad -\infty < x < +\infty \quad g>0 \quad \beta > \frac{1}{2} \qquad \ldots 11./$$

The Pearson IV. distribution which is interpreted between $-\infty < x < +\infty$ is not suitable because effective distribution is no clear urn model and, because distribution is dimensioned only for $x > 0$. If the four probability characteristics of distribution, $a^Y/t/$, $s^Y/t/$, $f^Y/t/$, $c^Y/t/$ are given, β, α, d and g can be determined. The value of γ can be determined from the condition that the integral of expression 11./ between $-\infty$ and $+\infty$ is [1].

$$\left.\begin{aligned}
\nu &= 6\,\frac{c^Y + 2 - (f^Y)^2}{2c^Y - 3(f^Y)^2} \quad \text{auxiliary quantity} \\
\beta &= \frac{\nu}{2} + 1 \\
\alpha &= f^Y \cdot \nu(\nu-2)\sqrt{\frac{1}{16(\nu-1) - (f^Y)^2(\nu-2)^2}} \\
g &= \sqrt{\frac{\nu-1}{\alpha^2+\nu^2}}\;\nu s^Y; \quad d = a^Y - \frac{g\alpha}{\nu} \\
\gamma &= \frac{1}{\int_{-\pi/2}^{+\pi/2} e^{-\alpha z}\cos^{\nu} z\,dz}
\end{aligned}\right\} \qquad \ldots 12./$$

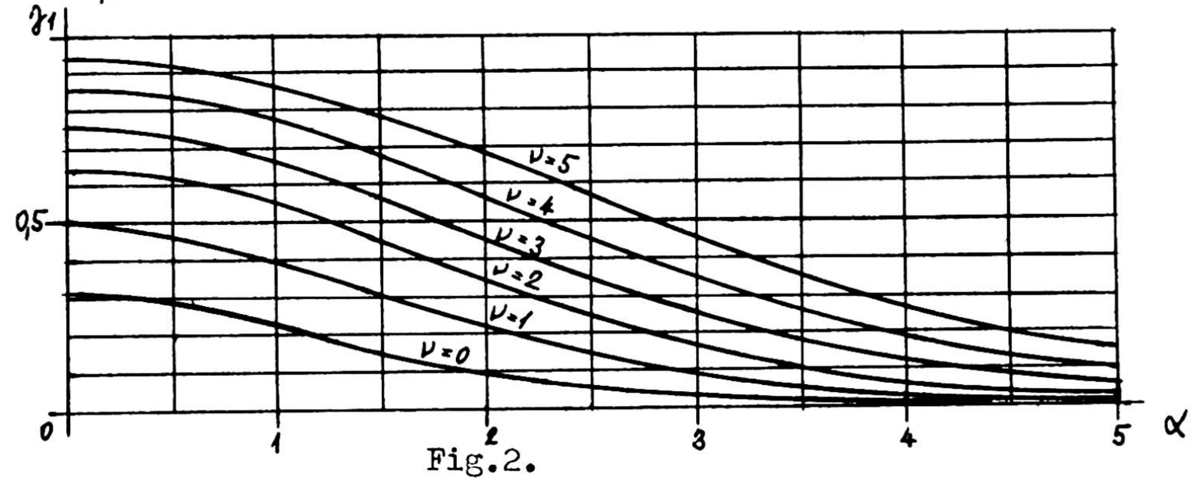

Fig.2.

From expression 12./ numerous conditions present themselves for the distribution characteristics which are not dealt with in this paper. The values of γ for whole number values of α and ν are shown in figur 2.

11./ Optimum risk

The value of m = m/f,c,k/ has furthermore to be determined in expression 10./ in order that the cross section quantity could be determined. The determination from expression 11./ is unequivocal if the value of k is given. For the determination of k two conditions can be offered.

The first condition is that the total cost of the installation should be a minimum. Supposing the interest factor to be q, the cost of rebuilding of the installation, C/k/, must be written off, during a service time T, and it is assumed that the installation will be ruined after a time t<T and the part of the construction not yet written off at that time will be capitalised to the date of the ruin. Then the total cost will be

$$K_{(k)} = C_{(k)} + \frac{1}{k}\left\{C_{(k)}\left[1+\frac{q^T-q^t}{q^T-1}\right]+Q\right\} \qquad \ldots 13./$$

A minimum cost can occur where the first derivative by k of expression 13./ is zero.

The costs of the bearing structure increase in a linear way with the cross section quantity, C/k/ = A+BW . This linearity will hold if the cross section quantity represents an area or if the ratio of cross section modulus to radius of inertia is constant. If expression 10./ is expanded to a series, it will hold, with a good approximation, that $W_0 \cong W' + wm$.

If f, c and s are constant, for the thinkable values of k, $m = \lambda_0 + \lambda_1 \log k$. A deeper reason for linearity is to be found in the fact that distribution functions generally are exponential /For a normal distribution e.g. between $2 < \log k < 6$, it is true with a 3 per cent accuracy that m = 1,22+0,6 log k/. Substituting the above expressions into eachother, $C/k/ = /A + BW' + B\lambda_0 w/ + B\lambda_1 w \log k$ that is

$$C_{(k)} = C_0\left(1 + b_1 \log k\right) \qquad \ldots 14./$$

Expression 14./ reflects a stohastic relation which can be verified for a series of numerical examples with a difference less than 2 per cent, doing regressional calculations. The minimum that results from expressions 13./ and 14./ is

$$k(t) = \frac{1 + b_1 \log \frac{k}{e}}{b_1 \log e}\left[\frac{Q}{C\left(\frac{k}{e}\right)}\left(1 + \frac{q^T - q^t}{q^T - 1}\right)\right] \qquad \ldots 15./$$

The maximum for expression 15./ produces itself at a time t=0. On an average, if the cost of the bearing structure of a building based on a 3 per cent risk, is C,

$$k_{max} \cong \frac{2,6}{b_1}\left[\frac{Q}{C} + 2\right] \qquad \ldots 16./$$

It is apparent from expression 16./ that the more damage will be caused by the installation when it is ruined, the less will be the risk permitted to be taken. E.g. if T = 1 and Q = 0, $k_{min} \sim 50$. If T = 50 and $\frac{Q}{C}$ = 100, $k_{max} \cong 5000$. Here the risk that is taken, varying within a range of 2 per cent and 0,02 per cent, corresponds to the percentage wastes in non damaging industrial production, which is 1-2 per cent [8].

Another condition is that the annual quotient to write off for the installation is $\frac{q^T/q-1/}{q^T-1}$ C/k/ and the total of annual maintenance, L/k/ is the minimum of what is referred to as entire cost. The concept of entire cost was defined by a Congress held on "Perspective of the user and reliability of the system" in the United States in 1962. It is assumed that maintenance costs take the sum of

$$L(k)=L_0\left(1+\frac{b_2}{\log k}\right) \qquad \ldots 17./$$

Expression 17./ is not proved, it merely appears to be logícal upon the analogy of expression 14./ [9]. The minimum of all anual costs is secured by the expression

$$k=10^{\sqrt{\frac{L_0 b_2 (q^T-1)}{C_0 b_1 q^T (q-1)}}} \qquad \ldots 18./$$

being satisfied. Expression 17./, if $T\sim 50$ and $q-1=\frac{p}{100}$ where p, in percentage, is the interest rate, will be

$$k\cong 10^{9,5\sqrt{\frac{L_0 b_2}{C_0 b_1 p}}} \qquad \ldots 19./$$

From the two expressions /15./ and 18./ that one must be satisfied which gives a larger value for k. From the comparison of expressions 14./ and 19./ results that when the condition

$$\frac{Q}{C}\geqq\frac{b_1}{2,6}\,10^{9,5\sqrt{\frac{L_0 b_2}{C_0 b_1 p}}}-2 \qquad \ldots 20./$$

is satified, that is, if the ratio of the damage incurred and the rebuilding costs is greater than the right side of expression 20./, the value of k must be determined on the basis of expression 15./ and /or 16./, if it is smaller, expressions 18./ and/ or 19./ will give the value of k.

It should be noted that in the vicinity of the optimum value for extremely small differences will result in expression 13./, therefore the value of k must be determined with a rounding up and on a rather large scale.

12./ <u>Other probabilities that may be considered</u>

Beside what has been said above it is also essential, how many uniform structural elements are going to be built in. If n number of bearing elements are to be fabricated and a risk of $\frac{1}{k}$ has to be taken for each of them, for one piece a risk of $\frac{1}{k'}<\frac{1}{k}$ must only be taken. The solution of the problem is

$$1-\frac{1}{k}=\binom{n}{0}\left(\frac{1}{k'}\right)^0\left(1-\frac{1}{k'}\right)^n \qquad \ldots 21./$$

Eypression 21./ displays a binomial distribution.

$$\left.\begin{aligned} k' &= \frac{1}{1-\sqrt[n]{1-\frac{1}{k}}} \\ \text{and/or} \quad k &= \frac{1}{1-\left(1-\frac{1}{k'}\right)^n} \end{aligned}\right\} \qquad \ldots 22./$$

If k>50, then, with a good approximation

$$k' \cong nk \qquad \ldots 23./$$

Thus, if it is required that no one of the n pieces should get ruined with a probability of $\frac{1}{k}$, every piece must be fabricated with a risk of $\frac{1}{nk}$ being taken.

13./ Conclusions

1./ The following answer can be given to the problem raised: The correct dimensions are given in expressions 9./ or 10./, the value of m = m /f, c, k/ can be determined through the Pearson IV. distribution according to expression 11./, the most economical measure of risk taken against failure, 1/k, is given by expressions 16/ or 19./. As to the measure of the assumed risk it must furthermore be considered, how many uniform elements in question are going to be fabricated.

The suggested method of calculation does not contain any subjective factors, all dimensions and safety can - on the basis of mathematical statistics - be determined solely upon economical considerations.

2./ Safety, consequently, is a mutual and unequivocal function of the probabilities for the installation that during its scheduled service time all circumstances provoking ruin can occur simultaneously in the most possibly unfavourable arrangement [3].

The lower this probability /the risk taken/ is, the greater is the safety. We suggest the acceptance, as a measuring value, of log k the logarythm of the reciprocal value of the risk taken. This expression has proved suitable in information theory [10], as a quantity proportional to the measuring number of the information quantity. For a great safety a large amount of information is required about the given bearer. Investment expenditures increase with safety, maintenance costs decrease with it.

3./ For our bearing structures the principle of equal safety is in appropriate. The more damage is caused by the structure with its ruin, with so much more safety must it be designed. Secondary bearers, the ruin of which causes no damage, must be designed with a lesser safety. Installation with a short designed service time can be of smaller divisions, because within a shorter period the rupture strength of the load-bearing building material shows a lesser decrease and the probability for the occurrence of the loads, particularly meteorological ones, is lower within a shorter period.Consequently, if it is to proceed from the safety of the primary system of bearers of definite installations, the primary system of bearers of temporary installations can be fabricated with a lesser safety and so can the secondary bearers of the definite installations. Still lesser safety is required for secondary bearers of temporary installations.

4./ General rules for dimensioning are provided by deterministic interrelations in technical mechanics. By reason of a deviation of parameters in the functional interrelations the economical dimensions have to be determined with the aid of stohastic interrelations based on probability theory.

5./ A processing of statistical data is required. The described method, however, can be applied even without processing the data, in that case it has to be proceeded from loading given in the rules and from nominal geometrical and strength measures. For deviations there is to be taken, in the absence of data, one half of the tolerance. Tolerance, then, is based on statistical experience.

6./ Author considers the application of this procedure absolutely necessary: in setting up rules for dimensioning, stability calculations for elements of serial production and structures of high cost.

7./ Rules in operation at present are over-dimensioned even today. The degree of over-dimensionedness is, with various rules, in terms of costs 8-12 per cent for primary bearers at definite installations. Over-dimensionedness for secondary bearers and for temporary installations is 11-17 per cent. This can best be helped if calculations will be made, instead of the minimum values as specified in the rules, with their expectation values. If this proportion is considered, there will result economical dimensions /E.g. the expectable value of the yield point of St. 37 is, on the basis of statistical data, [11] ~ 2800 kp per square cm, whereas the rule specifies 2400 kp per square cm.

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Summary.

From all factors influencing the dimensions /loading, geometrical dimensions, crushing stress, etc./ the value and the probability variables of the load capacity reserve, Y/t/, can be determined. By means of the Pearson IV. distribution the geometrical dimensions can be determined for a risk arbitrarily undertaken.The undertaken risk can be determined through only economical conside-

rations. The greater the damage caused, the lesser risk should be undertaken. If the damages are not significant, dimensions are influenced by the annual maintenance costs, /L/. For a safety rate the logarithm of the reciprocal value of risk /log k/ is suggested.

RÉSUMÉ

La valeur de la réserve de limite admissible de la charge, Y/t/, et ses variables de probabilité peuvent être déterminés à l'aide de tous les facteurs qui sont à même d'influencer les dimensions - tels la charge, les dimensions géométriques, tensions de rupture, etc. Par moyen de la distribution Pearson IV. les dimensions géométriques peuvent s'établir pour un risque arbitrairement entrepris. Or, le risque entrepris peut être déterminé moyennant des seules considérations économiques. Plus le dégât est grand, moins de risque doit être entrepris. Si le dégât n'est pas important, les dimensions seront influencées par les frais annuels d'entretien. Proposition est faite d'employer, pour mesure de sécurité, le logarithme de la valeur réciproque de la risque entreprise (log k).

ZUSAMMENFASSUNG

Aus sämtlichen, die Abmessungen beeinflussenden Faktoren - wie Belastung, geometrische Abmessungen, Bruchspannungen usw. - können der Wert Y/t/ der Belastungsfähigkeitsreserve und seine Wahrscheinlichkeitsveränderlichen berechnet werden. Mit Hilfe der Verteilung Pearson IV. können die geometrischen Abmessungen zu einem beliebig unternommenen Risiko bestimmt werden. Das unternommene Risiko kann wieder auf Grund allein wirtschaftlicher Betrachtungen festgelegt werden. Je grösser der Schaden, ein um so niedrigeres Risiko darf unternommen werden. Bei nicht bedeutendem Schaden beeinflussen die jährlichen Wartungskosten, /L/, die Abmessungen. Als Sicherheitsmass wird der Logarithmus des Kehrwertes des unternommenen Risikos vorgeschlagen (log k).

Safety of Structures as a Problem of Time

Sécurité des constructions en fonction du temps

Sicherheit der Bauten als ein Zeitproblem

C. EIMER
Poland

First attempts of probabilistic approach to safety of structures were made as early as 1936 (W.Wierzbicki /1/, M.Prot /2/) . The probabilistic philosophy has been discussed, for a long time, and gained its devotees and its skeptics, the problem being looked upon mainly from the point of view of a direct applicability to design and calculation. By the present writer's opinion, too little emphasis has ever been laid on the explanation of actual phenomena and interrelations that, in fact, has been dimmed by traditional methods, and this is the fundamental purpose of every theory. Once we realize we operate quantities affected with random scatter, we are induced, necessarily and at the same time, to the notion of safety and to probabilistic considerations, irrespective of whether we intend to establish a pure probabilistic theory of safety or to explain precisely the meaning of conventional coefficients of safety. Similar development can be noticed in those branches of technical activity where problems of reliability are of importance.

The present contribution aims at explaining the role of time in safety which, from the mathematical point of view, means making a step from random variables to random stochastic processes. So far, the basic end of the theory consisted in finding the probability of the strength criterion to be fulfilled, i.e. of the inequality $P < R$, where, loosely, P is the load and R the strength (carrying capacity) . Since every structure is to be reliable during a limited period of time (called, in what follows, life time or period of exploitation), P denotes the maximum load that can occur in the course of this period and R is assumed to be independent of time and of previous history of loading. The former assumption presents serious difficulties as, in general, statistics containing long periods of time, within a more or less homogeneous population of structures, are not available. The second assumption is only a crude approximation, e.g. it disregards phenomena connected with rheologic strength or fatigue. An attempt to avoid the first of the above assumpions is given by A.M.Freudenthal /3/. The author considers a sequence of load applications, the probability distribution of P in a single application being known. However, it is not always easy or even possible to say what is a single load application as the loading is a continuous process. Basides, in order to "locate" the process in time the intervals between those applications must be assumed. Thus, in general, the whole of the problem is to be discussed in the language of stochastic processes, the approximation with different discrete models being of course possible and valuable in view of effective calculations.

1. Measures of safety in time

A fundamental merit of the probabilistic approach to safety is the introduction of a unique and universal probabilistic measure of safety. We shall descuss here some basic notions following the very clear exposition of the subject in /3/. The generalization depends on passing from the discrete model to a continuous time process.

On the assumption that the carrying capacity R is independent of time (which will hold in this point) we define the probability of safety or the reliability, L, as the probability that the time to failure t_R, i.e. the effective life time of a structure exceeds the period of exploitation $t = T$, $T < t_R$. This is equivalent to the condition $P_{max} < R$ if P_{max} denotes the maximum load during T; hence we have

$$L(t) = Pr\left(t < t_R\right) = Pr\left(P_{max} < R\right), \quad t = T \tag{1.1}$$

The probability of failure within that period equals

$$F(t) = 1 - L(t) = Pr\left(t > t_R\right) . \tag{1.2}$$

The a priori probability density of failure at the instant t is

$$f(t) = \frac{dF(t)}{dt} . \tag{1.3}$$

The failure rate, accordingly to /3/, is the probability that a structure that has survived t will fail in a time unit at t,

$$h(t) = \frac{f(t)}{L(t)} = -\frac{d}{dt} \ln L(t). \tag{1.4}$$

Obviously, the above formulae correspond to (2.1) ÷ (2.7) in /3/. Here, t_R is a random variable and denotes the time to first surpassing the value R by the load.

The load P, being a continuous time process which we denote by g(t), the results of measurements can, depending on the type of measuring devices, be obtained in threefold form: (1) as a continuous graph (self-recording instruments), Fig.1, (2) as periodic readings at time intervals δt (points denoted by small circles), (3) as maximum values at fixed time intervals Δt, usually related to cyclicity of load accurrence, e.g. in 24 hours, a year, etc. (devices recording maximum values denoted by little crosses).

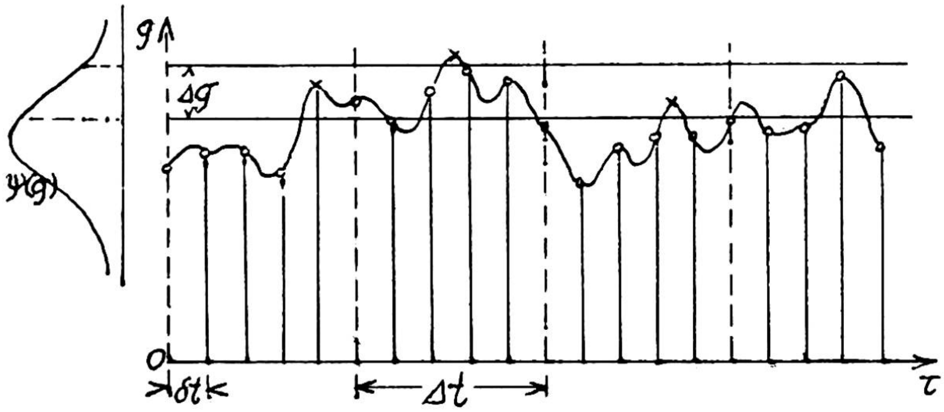

Fig.1

By taking the ratios of the number of points (marked with little circles) in the consecutive intervals Δg i.e. in consecutive horizontal bands to their total amount for a sufficiently long time interval t we obtain the frequency distribution and for $t \to \infty$ the probability density of load at a given instant, $\psi(g)$, as shown on the left hand side of Fig.1. When the recording is continuous one can take arbitrary time intervals δt. On "matching" this probability density to that of R we arrive at the probability of failure at a single load application (p_F in /3/).

By considering the time interval Δt in which we are interested and a sufficiently long interval $n \Delta t$, and on establishing a constant value of g, we find the number m of intervals Δt in which the latter has not been surpassed. The ratio m/n provides an approximate measure (exact for $n \to \infty$) of the probability of not surpassing g in Δt. This probability is a function of the load, $p = p(g)$ that can be found empirically by repeating the procedure for subsequent values of g. At the same time, it represents the distribution function of maximum load in Δt, since non-surpassing of g is equivalent to non-exceeding it by the maximum load. The probability density $\pi(g)$ of maximum load in Δt can be obtained as the derivative $dp(g)/dg$ or else directly from the graph, from the occurrences of "maximum" points in the consecutive intervals Δg (a procedure similar to the one already used for $\psi(g)$).

If Δt were equal to the period of exploitation, the function $p(g)$ would represent directly the distribution required. For practical purpose, however, it is important to arrive at some conclusions as to the distribution of maximum load in the period of exploitation T from the distribution of max g during an interval Δt that is, as a rule, shorter or, directly, from the density function $\psi(g)$, which implies two possible procedures, discussed in what follows.

In the first of them we find the probability of not exceeding g in n intervals Δt during the time $T = n\,\Delta t$,

$$\Phi(g) = p^n(g), \tag{1.5}$$

valid under the restriction of independence of those events. Here, $\Phi(g)$ represents the distribution of the maximum load. The probability density of this load is obtained by taking the derivative of 1.5,

$$\varphi(g) = n\,p^{n-1}(g)\,\pi(g). \tag{1.6}$$

The above formula can also be obtained directly on taking into account that np^{n-1} represents the probability of not surpassing g in $n - 1$ intervals Δt, whereas $\pi(g)\,dg$ is the probability of the maximum load amounting to g in the remaining one interval Δt.

On establishing the load max g at a sufficiently high level so that higher values of g will occur but rarely, e.g. once in several months or even years, the interval Δt can be so far reduced that – without encroaching on the assumption of independent loads in consecutive intervals – very high values of p (near to 1) are attained. The distribution (1.5) now tends to the Poisson distribution

$$\Phi(g) = \exp\left[-\nu(g)T\right], \tag{1.7}$$

with $\nu(g)$ denoting here the average number of events when g is surpassed per unit time ($\nu = 1/t_0$, where t_0 is the average time interval between such events), this number being dependent on the fixed value of g. The function $\nu(g)$ is found experimentally, e.g. by computing t_0 for consecutive g (on a graph of the type of Fig.1).

The second procedure we mentioned above does not require determining of the function $\nu(g)$ or $p(g)$ and is based on Fisher--Tippett asymptotic extremal distribution representing the distribution of the highest (or lowest) value in a test, where the number of particular test readings increases infinitely. Thus, it is a matter of finding the limiting disturbation, for $n \to \infty$ of the largest of n randomly chosen ordinates of points marked by little circles in Fig.1, at a fixed value of δt (so as to satisfy the requirement of independent loads). It is this form to which the distribution of max g tends for $t \to \infty$, since a test of infinite number of test readings tends to become strictly representative. For finite n we obtain here Eqs. (1.5 and 1.6); albeit, $p(g)$ and $\pi(g)$ have to be replaced by $\Psi(g)$ and $\psi(g)$, respectively (cf. Fig.1), i.e. by the probabilities of g at a given moment (in the experiment under consideration).

On introducing the new variables

$$z = n\left[1 - \Psi(g)\right],$$

$$-u = \ln z = \ln n + \ln \int_g^\infty \psi(g)\, dg,$$

we obtain

$$\varphi(g)\, dg = n\, \Psi^{n-1}\psi\, dg = n\left(1 - \frac{z}{n}\right)^{n-1} \psi\, dg \xrightarrow[n\to\infty]{}$$

$$= n\, e^{-z}\psi \frac{1}{dz/dg}\, dz = -\, e^{-z}\, dz = e^{-e^{-u}}\left(-e^{-u}\right) du,$$

whence the variable u is seens to possess the asymptotic density distribution

$$\omega(u) = \exp\left(-u - e^{-u}\right). \tag{1.8}$$

The variable u is seen to be related linearly to g if $\psi(g)$ is exponential. Consequently, $\varphi(g)$ can be obtained directly from Eq.

(1.8). It is proved in mathematical statistics that the asymptotic extremal distribution of Eq. (1.8) holds for normal distribution of $\psi(g)$, too. Obviously, this is but an approximate calculation, since the centre of the distribution is shifted proportionally to ln n, at finite $n = T/\delta t$ (to be calculated from the records), whereas the form of the distribution (1.8) itself is exact only for $n \to \infty$ and does not depend on n.

Further calculations depend on the particular form of the distribution $\psi(g)$ and, for different theoretical assumptions, are developed in the theory of extreme value distributions (cf. for instance /4/). Once we have found the extremal probability density $\varphi(g)$ [from (1.6) or the derivative of (1.7) or else (1.8)] we insert it into the integral

$$L = \iint_{P<R} \varphi(g)\,\psi(R)\ dg\ dR, \tag{1.9}$$

$\psi(R)$ being the probability density distrubation for the strength R, where the integral is taken over the part of the plane (g,R) determined by the inequality $g < R$. Since $\varphi(g)$ is a function of time, so is $L = L(t)$ and our problem is solved.

2. Concept of damage and outline of a general theory

Precedent considerations were based on the assumption that R is constant which is but a crude approximation. We know that it depends, for instance, on the number of repeating load cycles in fatigue tests or on the time of loading if rheologic phenomena are involved. In order to describe this behavior the notion of "cumulative damage" is introduced in the theory of fatique of materials and similar notions are also known from the general theory of reliability.

Let us generalize this notion and assume that the actual state of a structure (or a material) at a given instant (from the view-point of its carrying capacity) is defined by a unique positive number, δ, $0 \leq \delta \leq 1$, called damage, where zero damage ($\delta = 0$) describes a perfect state and $\delta = 1$ a complete failure. In general, δ increases in the course of time, particularly, in the course of the loading precess, which means that the ruin sets in progressively and results in the "death" of the structure when δ attains 1. For a simple example (not to be directly extended), δ can represent the reduction of the cross- section of an axially loaded rod because of an expanding crack. Thus, our strength condition $P < R$ is to be replaced by a more general one

$$\delta < 1. \tag{2.1}$$

Now, the problem consists in the prediction of the time t at which the damage becomes 1 or, in a probabilistic approach (δ being a random variable), in the determination of the probability

$$L(t) = \Pr(\delta < 1) \tag{2.2}$$

for a given period of exploitation $t = T$.

For the classical case δ remains 0 as long as $P < R$, R being the carrying capacity. On surpassing R for the first time δ suddenly increases to 1 and the structure fails (Fig.2). It is seen that δ is defined to be a Heaviside function

$$\delta(t) = H(t - t_R),$$

t_R denoting the time to first surpassing R by $P = g(t)$. The probability (2.2) reduces to (1.1) and exactly the theory in point 1 provides the solution.

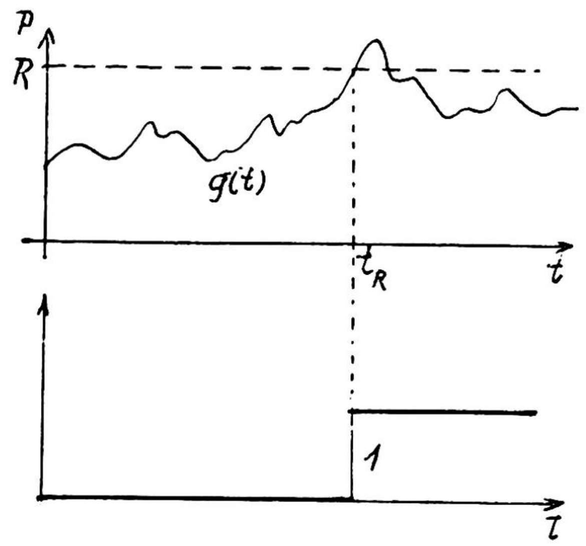

Fig.2

In general, the hypothesis that the physical state of a structure can be determined by a unique parameter, δ, is a considerable simplification of actual conditions, albeit it results in a far reaching generalization of the former theories of safety. In fact, δ can depend on the whole previous history of loading and, therefore, is a functional defined on the class of all possible functions $P = g(t)$. Depending on what phenomena are to be included (e.g. fatigue, rheology, etc.) and for the sake of effective calculation further restricting hypotheses have to be introduced. First of all, we shall assume that δ is cumulative so that we only examine the increments $d\delta(t)$ which simply integrate in time. If, for the time being, we abstract from rheologic phenomena, we are able to make the assumption that $d\delta$ depends only: (1) on the instantaneous internal state described by δ , (2) on the external state described by $P = g(t)$, (3) on the change of the external state given by $dP = d\,g(t)$, (4) directly on time. Taking the increments in a time unit, i.e. replacing them by velocities (denoted with dots) we obtain

$$\dot{\delta} = f(\delta, g, \dot{g}, t). \tag{2.3}$$

The direct dependence on time reflects corrosion-like phenomena affecting δ and will be neglected in further consideration. If we assume that damage is irreversible, the function f will be non-negative with respect to all arguments. If g approaches the limit strength R the velocity $\dot{\delta}$ rapidly increases; if, furthermore, R depends on δ and is independent of $\dot{g}$, we have $f \to \infty$ for $g \to R(\delta)$ Further simplifying assumptions may state that $\dot{\delta}$ does not depend on the sign of dg (internal friction-like phenomena at fatigue) - resulting in f symmetric respectively to $\dot{g}$, and that it is proportional to $\dot{g}$ which gives the form

$$\dot{\delta} = f(\delta, g)\,|\dot{g}| \tag{2.4}$$

or, equivalently

$$d\delta = f(\delta, P)\,|dP|$$

Formulae of similar form, where instead of dP appears dn (n - number of cycles), can be found in the theory of fatigue (cf., for instance, /5/); however, those do not include any hypothesis as to the mechanism of failure and hold only within the above theory, for symmetric oscillations).

The simplest possible assumption for (2.4) is

$$f(\delta, g) = \beta = \text{const} \tag{2.5}$$

within the admissible region (Fig.3, shaded area) and $f \to \infty$ for $g \to R(\delta)$, that is the actual smooth passage of the surface $f(\delta, g)$ is replaced by a singularity. If, in particular, $\beta = 0$, we have the classical case, with the additional assumption that initial damage is possible and makes R lower (we are moving along vertical lines in Fig.3). On integrating (2.4) for (2.5) we get $\delta = \beta\Sigma|\Delta g|$

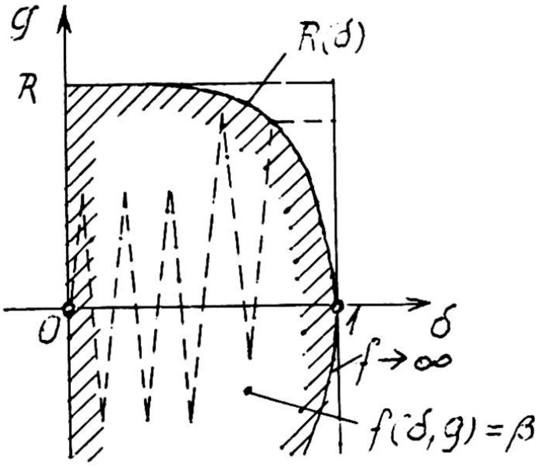

Fig.3

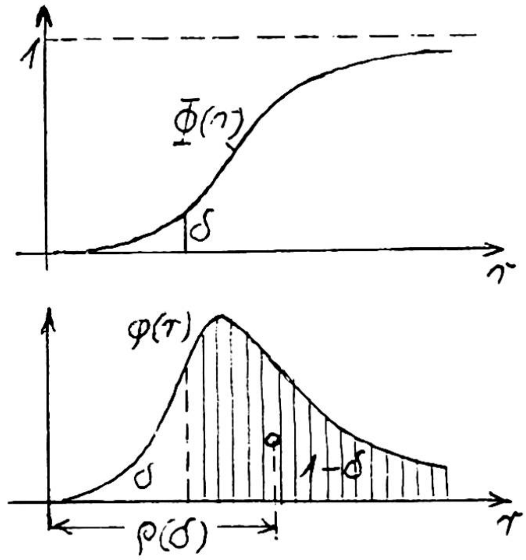

Fig.4

i.e. δ is proportional to the sum of amplitudes of all load cycles, irrespective of the mean value (in Fig.5, below, the sum of segments $\overline{01} + \overline{12} + \overline{23} + \ldots$). In the case of simple (symmetric) oscillations it is proportional to the number of cycles, n,

$$\delta = 4\, n \beta g_m,$$

where g_m is the maximum load at one cycle. The path of loading is composed of straight segments with constant slope $|d\delta/dg| = \beta$, independent of the forms of "waves" in time, and on intersection of the curve $R(\delta)$ it jumps horizontally till $\delta = 1$ (Fig.3). This assumption is equivalent to the well-known Miner's hypothesis in the theory of fatigue of materials about a constant damage in a single oscillation with given amplitude. If, in particular, the curve $R(\delta)$ coincides with the bounding straight segments, R = const and $\delta = 1$ respectively, the equation of Wöhler's curve will result directly from the above formula for $\delta = 1$,

$$g_m = \frac{1}{4\beta N}$$

which is the equation of a hyperbola. More generally, if the equation of Wöhler s curve $g_m = W(N)$ is available, we obtain the curve $R(\delta)$ solving for R the equation

$$R = W\left(\frac{\delta}{4\beta R}\right).$$

However, it must be pointed out that this curve might eventually not fit other non- zero values of the mean force, the assumption (2.5) being too simple.

The equation of the curve $g = R(\delta)$ can also be argued theoretically, for instance, in the following way. Imagine a body composed of grains with variable strength properties and a process of damage that consists in consecutive failing of weaker grains. The volume proportion of elements at different levels of local strength, r, can be represented by an integral or differential probability distrubution (Fig.4). Define the damage δ as the part of the area (normalised to 1) under the curve $\varphi(r)$ or else as an ordinate of the curve $\Phi(r)$. The shaded part of the area, $1-\delta$, represents the actual carrying capacity (due to stronger grains). The equation sought for is

$$R(\delta) = (1 - \delta)\rho(\delta)$$

where $\rho(\delta)$ is the abscissa of the centre of gravity of the shaded area.

The analysis of safety can be performed similarly to what has been said in point 1 (Fig.5). For a stationary process of loading

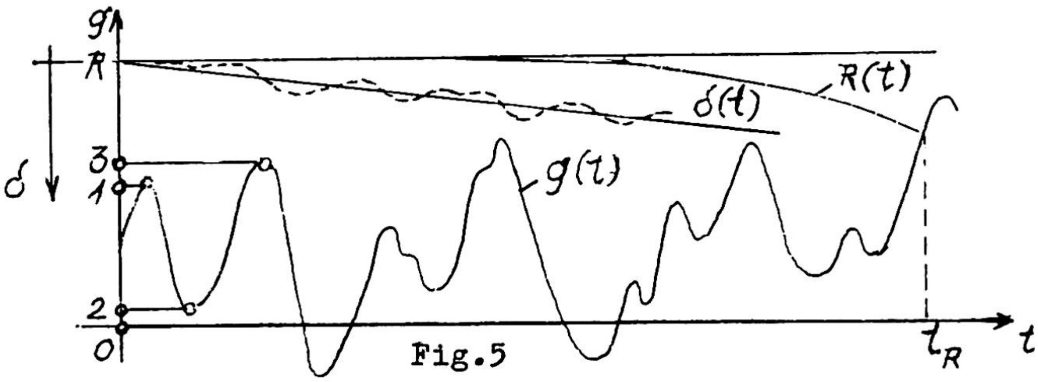

Fig.5

the damage δ can be regarded, approximately, as proportional to time and assimilated to a straight line $\delta = \delta_0 t$, where $\delta_0 = \beta\sum_{t=0}^{1}|\Delta g|$ is the average damage during a time unit (obtained from the load curve by averaging over a sufficiently long time period). The strength curve is expressed in new units

$$g = R(\delta_0 t) \tag{2.6}$$

and failure appears at first intersection of this curve with $P = g(t)$, the problem being reduced to the one of a material with decreassing strength.

If strength properties (represented by (2.6)) are not affected with scatter, we can use, for instance, the same reasoning as for the formula (1.5) . Assume, we have got records for a fixed period Δt and determined for this period the probability destribution p(g) (similarly as for (1.5)). Since the strength changes in time, we obtain

$$\Phi = p(R_1)\ p(R_2)\ p(R_3)\ldots \qquad (2.7)$$

where, according to (2.6), R_k refers to the k- th sector Δt (cf. Fig.1). Taking logarithms of both sides

$$\ln \Phi = \sum_k \ln p(R_k)$$

and taking for $\ln p(R_k)$ its average value in the respective sector

$$\ln p(R_k) = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \ln p[R(t)]\ dt$$

we have

$$\ln \Phi = \frac{1}{\Delta t} \int_0^t \ln p[R(t)]\ dt$$

and, finally,

$$\Phi = \exp \frac{1}{\Delta t} \int_0^t \ln p[R(t)]\ dt. \qquad (2.8)$$

This is an explicit function of parameters describing the function R(t). Of course, this is but an approximate calculation, those parameters and, what more, the curve R(t) by itself being random (cf. Fig.5) .

So far, the analysis was based on the assumptions (2.4) and (2.5) which is only a first step towards a theory including time - dependent phenomena. One of serious difficulties to be surmounted is connected with specifying the functions (2.3), (2.4) . In general, if Wöhler - type curves for different non-zero mean stresses were available, we could come at a result on comparing them with respective solutions of the differential equation (2.4) for sinusoidal forms of load curves and for $\delta = 1$, $n = N$, $\omega t = n$, a,b - constants,

$$\frac{d\delta}{dt} = f(\delta\ ,\ b + a \sin\omega t)\ |a\omega \cos\omega t|\ .$$

Further generalizations could take into account rheologic phenomena and the formulae of the type (2.3) would be replaced by functionals, e.g. in an integral or an operational form. The simplified assumptions would, possibly, retain formulae of the type (2.3), introducing, however, some characteristic values of the load from the precedent history (e.g. the next local or the absolute maximum and minimum values of g). The analysis, however, would be much more complex and is beyond the scope of this article.

In the present contribution we did not consider conventional measures of safety (e.g. coefficients of safety), as the methods of derivation of such measures have been discussed many times (cf. for instance, /3/, /5/) and a "pure" theory of safety can (and ought to) do without them.

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SUMMARY

The contribution is concerned with the problem of safety of structures on the basis of the theory of cumulative damage. The actual state of a structure (from the point of view of its carrying capacity) is described with a parameter, variable in time, depending on the previous course of loads. The latter is regarded as a stochastic process and a probabilistic measure of safety is derived.

RÉSUMÉ

L'auteur a examiné le problème de la sécurité de constructions du point de vue de la théorie du dommage cumulé. L'état actuel d'une construction est caractérisé par un paramètre unique (le dommage) variable avec le temps, dépendant des charges préalables. Celui-ci est considéré comme un processus stochastique et une mesure probabiliste de la sécurité est dérivée.

ZUSAMMENFASSUNG

Im vorliegenden Beitrag wird die Frage der Sicherheit einer Konstruktion auf Grund der Theorie der Anhäufung der Beschädigungen behandelt. Der Zustand der Konstruktion vom Standpunkt seiner Tragfähigkeit wird durch einen Parameter beschrieben, der die Beschädigung charakterisiert und von dem vorigen Verlauf der Belastung abhängig ist. Der obenerwähnte Verlauf wird als ein zufälliger Prozess aufgefasst und ein wahrscheinliches Mass der Sicherheit wird abgeleitet.

Zur Schätzung der Bruchwahrscheinlichkeiten der Tragwerke

Estimation of the Probability of Failure of Structures

L'estimation de la probabilité de rupture des structures

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1. Einleitung

Seit längerer Zeit laufen Bemühungen, die Sicherheitsuntersuchungen der Tragwerke aussagefähiger zu machen. Hierzu bieten sich die Methoden der Wahrscheinlichkeitsrechnung und mathematischen Statistik an [1]. Trotz zahlreicher Untersuchungen auf diesem Gebiet, die im Endergebnis auf eine Bestimmung der Bruch- bzw. Überlebenswahrscheinlichkeit der Tragwerke an Stelle der klassischen Sicherheitsberechnung hinzielen, haben diese modernen Methoden bisher noch keine oder nur sehr zögernde praktische Anwendung gefunden.

Der Grund dafür dürfte in den noch recht erheblichen Schwierigkeiten bei der Anwendung dieser Methoden liegen und in den dadurch bedingten bedeutenden Umstellungen für den Konstrukteur.

Solchen weitgreifenden Umstellungen werden erfahrungsgemäß berechtigte und unberechtigte Widerstände entgegengestellt, und es ergibt sich daraus die Aufgabe, Methoden und Möglichkeiten der Bestimmung von Bruch- bzw. Überlebenswahrscheinlichkeiten zu finden, die dem Konstrukteur für die Anwendung zumutbar, übersichtlich und in ihren Auswirkungen durchschaubar sind.

Diese Arbeit befaßt sich daher mit einer Möglichkeit, die Bruchwahrscheinlichkeit eines Tragwerkes mit möglichst einfachen Mitteln zu bestimmen. Die akzeptablen Bruchwahrscheinlichkeiten eines Tragwerkes sind relativ gering, so daß es genügt diese in der Größenordnung richtig zu schätzen. Eine größere Genauigkeit ist schon wegen der Schwierigkeit in der Präzisierung der Ausgangswerte kaum zu erhalten und für praktische Belange sicher auch von weniger Interesse.

Demgegenüber bleiben die weiteren Probleme, die hiermit im Zusammenhang stehen und die z. B. FREUDENTHAL [1] ausführlich behandelt hat, unberührt.

2. Grundsätzlicher Lösungsweg und Schwierigkeiten

Bezeichnet y die Tragfähigkeit und x die Belastung, so ist die bisherige Sicherheitskonzeption durch das Verhältnis

$$\nu = \frac{y}{x} > 1 \tag{1}$$

definiert. Ohne auf die Vor- oder Nachteile der einen oder anderen Darstellungsweise einzugehen, könnte auch

$$z = y - x > 0 \tag{2}$$

als ein solches Kriterium betrachtet werden.

Werden, was ihrem tatsächlichen Charakter besser entspricht, die Tragfähigkeit als Zufallsgröße Y und die Belastung als Zufallsgröße X betrachtet, so ist z selbst eine Zufallsgröße Z. Die Gl. (2) stellt hierbei für alle Zustände $z = y - x \leqq 0$ den Bruchbereich und für $z = y - x > 0$ den Überlebensbereich dar. Gelingt es, die Wahrscheinlichkeitsverteilung für die Zufallsgröße Z zu formulieren und über $-\infty < z \leqq 0$ bzw. $0 < z < +\infty$ zu integrieren, so ist die Bruch- bzw. Überlebenswahrscheinlichkeit bestimmt und das Problem gelöst.

Ist die Wahrscheinlichkeitsverteilung der Tragfähigkeit G(y) und der Belastung F(x) bekannt, so wird die Wahrscheinlichkeitsverteilung der neuen Zufallsvariablen Z durch Faltung [2] gefunden

$$H(z) = \iint\limits_{x-y<z} df(z - y)\ dG(y). \tag{3}$$

Diesem formal einfachen Lösungsweg stellen sich praktische Schwierigkeiten entgegen, die vor allem folgende Gründe haben:

a) Die mathematischen Modelle für die Wahrscheinlichkeitsverteilungen der Tragfähigkeit Y und der Belastung X sind häufig nichtsymmetrische und oft ein- oder zweiseitig begrenzte Verteilungsfunktionen.

b) Die Belastung auf ein Tragwerk besteht in der Regel aus einer Summe von Zufallsvariablen

$$X = \sum_{i=1}^{k} X_i \tag{4}$$

und die Bruchbedingung lautet daher

$$z = y - \sum_{i=1}^{k} x_i \leqq 0 \tag{5}$$

c) Die Belastung auf ein Tragwerk wird im allgemeinen wiederholt, d. h. häufig eingetragen, so daß nicht die Ausgangsverteilung der Belastung $F(x)$, sondern die Extremwertverteilungen $F_n(x^{(n)})$ für n Belastungen maßgebend sind, wobei n auch durch die Zeit t ausgedrückt sein kann.

d) Durch die wiederholte Belastung auf das Tragwerk wird oberhalb einer Anzahl n_o von Belastungen eine Minderung der Tragfähigkeit des Tragwerkes auftreten. Daher wird $G(y)$ von der Zeit t abhängig und geht in den stochastischen Prozeß $G(y,t)$ über.

Hierzu ist folgendes zu bemerken:

zu a) Eine geschlossene Lösung des Faltungsintegrals (3) gelingt außer für Normalverteilungen bisher nur für Sonderfälle, die jedoch für das Bemessungsproblem wenig Bedeutung haben. Auch Potenzreihenentwicklungen führten bisher nicht zum Erfolg.

zu b) Die Berücksichtigung der k Zufallsvariablen X_i der Belastung führt zu einer Mehrfachfaltung entsprechend Gl. (3), wodurch das mathematische Problem noch wesentlich komplizierter wird.

zu c) Die Notwendigkeit, bei wiederholten Belastungen statt einer Ausgangsverteilung $F(x)$ eine Extremwertverteilung $F_n(x^{(n)})$ zu verwenden, ist leicht einzusehen, wenn die auftretenden Belastungen x_i nach Ranggrößen $x^{(n)}$ geordnet werden, so daß

$$x^{(1)} < \ldots < x^{(i)} < \ldots < x_t^{(n)}$$

wobei die $x^{(i)}$ der Ausgangsverteilung $F(x)$ entsprechen und $x_t^{(n)}$ die im Zeitraum t aufgetretene größte Belastung $x^{(n)}$ bedeutet. Unter der Voraussetzung, daß die Tragfähigkeit zeitlich konstant ist, kann nur die größte Belastung zum Bruch führen und es ist die Extremwertverteilung des größten Wertes $x_t^{(n)}$ maßgebend. Diese Extremwertverteilung ist abhängig von der Ausgangsverteilung und der Anzahl n der Belastungen [3, 4] .

Wirken mehrere zufällige Belastungsgrößen X_i, so ist nur die maßgebende Extremwertverteilung mit den übrigen Ausgangsverteilungen zu falten, da sonst vorausgesetzt würde, daß mehrere Extremwerte gleichzeitig auftreten.

Mit der Annahme, daß der gesamte Belastungsablauf als diskreter stationärer Prozeß aufgefaßt werden kann, läßt sich die Richtigkeit und die Notwendigkeit der Verwendung von Extremwertverteilungen in der hier aufgeführten Art mit Hilfe der Übergangswahrscheinlichkeiten beweisen.

3. Die vier Fälle der Zuverlässigkeitsuntersuchung

Die Bestimmung einer Überlebenswahrscheinlichkeit unterscheidet sich von der Sicherheitsuntersuchung und wird als Zuverlässigkeitsuntersuchung bezeichnet.

Aus dem unter 2. Genannten ergeben sich vier Fälle der Zuverlässigkeitsuntersuchung:

1. Statischer Fall

   Die Tragfähigkeit $G(y)$ ist zeitlich unbeeinflußt; für die einmalige Belastung ist die Ausgangsverteilung $F(x)$ maßgebend.

2. Quasi-statischer Fall

   Die Tragfähigkeit $G(y)$ ist zeitlich unbeeinflußt; für die n-malige Belastung ist die Extremwertverteilung $F_n(x^{(n)})$ maßgebend.

3. Betriebsfestigkeitsfall

   Die Tragfähigkeit $G(y)$ ist zeitlich beeinflußt und geht über in $G(y,t)$; für die Tragwerksschädigung ist das gesamte Belastungskollektiv eines stochastischen Belastungsvorganges und für die Brucheinleitung die Extremwertverteilung $F_n(x^{(n)})$ maßgebend.

4. Zeit- oder Dauerfestigkeitsfall

Hierunter wird ein Belastungsvorgang mit konstanter Mittelspannung, Spannungsamplitude und Frequenz verstanden. Dieser Fall kann unter bestimmten Voraussetzungen auf den statischen Fall zurückgeführt werden.

In der angegebenen Form

$$H_1(u) = \int_{-\infty}^{+\infty} F(u-y) \; dG(y)$$

gilt Gl. (3) für den statischen Fall. Für den quasi-statischen Fall erhält sie die Form

$$H_2(u) = \int_{-\infty}^{+\infty} F_n(u-y) \; dG(y) \tag{3 a}$$

und für den Betriebsfestigkeitsfall

$$H_3(u) = \int_{-\infty}^{+\infty} F_n(u-y) \; dG(y,t) \tag{3 b}$$

Daraus folgt, daß der statische, der quasi-statische und unter gewissen Voraussetzungen auch der Zeit- bzw. Dauerfestigkeitsfall, ausgehend von Gl. (3) und Gl. (3 a) lösbar sind. Dagegen kann der Betriebsfestigkeitsfall wegen seines Charakters eines stochastischen Prozesses mit diesen Mitteln nicht gelöst werden, weshalb er hier zunächst nicht weiter behandelt wird.

4. Lösung mit Hilfe der Edgeworth-Reihe

Zur näherungsweisen Bestimmung der Bruchwahrscheinlichkeit bietet sich eine Reihenentwicklung aus der Wahrscheinlichkeitsrechnung an, die unter dem Namen Edgeworth-Reihe bekannt ist. In der Form für die Wahrscheinlichkeitsdichte wird sie auch als Gram-Charlier-Reihe bezeichnet.

Wahrscheinlichkeitsverteilungen

$$F(x) = \int_{-\infty}^{x} f(t)dt \tag{6}$$

können auch durch die Gesamtheit ihrer Momente

$$m_k = \int_{-\infty}^{+\infty} x^k f(x)dx \ , \quad k = 1,2,3 \ldots. \ , \tag{7}$$

vollständig beschrieben werden.

Aus Gl. (7) lassen sich die Momente häufig nur schwer berechnen. Wendet man auf Gl. (6) eine Fourier-Stieltjes-Transformation an, so erhält man die sogenannte charakteristische Funktion der Verteilungsfunktion F(x):

$$\varrho\,(t) = \int_{-\infty}^{+\infty} e^{itx} f(x)dx \ . \tag{8}$$

Die k-te Ableitung der charakteristischen Funktion nach t ergibt den Ausdruck

$$\varrho^{(k)}(t) = \int_{-\infty}^{+\infty} i^k x^k f(x) e^{itx} dx \ , \tag{9}$$

woraus durch Nullsetzen von t und nach Division durch i^k das k-te Moment der Verteilungsfunktion (6) folgt:

$$m_k = \frac{\varrho^{(k)}(0)}{i^k} \ . \tag{10}$$

Die charakteristische Funktion hat u. a. folgende und für die Lösung unseres Problemes wichtige Eigenschaft:

Die charakteristische Funktion einer Differenz von Zufallsvariablen $X_1 - X_2 - X_3 - \ldots$ erhält man aus der Beziehung

$$\varrho\,(t) = \varrho_1(t) \cdot \overline{\varrho_2(t)} \cdot \overline{\varrho_3(t)} \ \ldots \ , \tag{11}$$

wobei $\varrho_i(t)$ die charakteristischen Funktionen der Zufallsvariablen X_i und $\overline{\varrho_i(t)}$ die konjugiert komplexe Form von $\varrho_i(t)$ bedeuten.

Außerdem ist

$$\log \varrho\,(t) = \log \varrho_1(t) + \log \overline{\varrho_2(t)} + \ldots \tag{12}$$

Aus der logarithmischen Form der charakteristischen Funktion $\log \varrho\,(t)$ können analog zu den Momenten m_k Größen $\varkappa_k$ hergeleitet werden, die als Semiinvarianten oder Kumulanten in der Wahrscheinlichkeitsrechnung bekannt sind. Diese haben die günstige Eigenschaft, daß die k-te Semiinvariante der Verteilungsfunktion einer Summe oder Differenz von Zufallsvariablen als Summe oder Differenz der k-ten Semiinvarianten der Verteilungsfunktionen der einzelnen Zufallsvariablen gebildet werden kann, d. h.

$$\varkappa_k = \varkappa_{k1} + (-1)^k\ \varkappa_{k2} + \ldots\ . \tag{13}$$

Die Edgeworth-Reihe beruht auf dem Grundgedanken, die anzunähernde Verteilungsfunktion H(z) durch eine Summe von Gliedern darzustellen, die aus geeignet zu bestimmenden Vorzahlen sowie der Normalverteilung

$$\Phi(z) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du \tag{14}$$

und ihren Ableitungen $\Phi^{(\nu)}(z)$ bestehen. Die Reihenentwicklung lautet:

$$H(z) \approx V(z) = \sum_{\nu=0}^{\infty} a_\nu\ \Phi^{(\nu)}(z)\ . \tag{15}$$

Damit diese Reihe die anzunähernde Verteilungsfunktion möglichst gut approximiert, werden die Koeffizienten a_ν so bestimmt, daß das Integral über die Abstandsquadrate der wahren Verteilungsfunktion H(z) von der Näherung V(z) ein Minimum wird.

Um bei der Bestimmung der Bruchwahrscheinlichkeit die tabellierten Werte der Normalverteilung verwenden zu können, muß z normiert werden.

Wegen der Bruchbedingung $z = y - x < 0$ ist bis $z = 0$ zu integrieren – d. h. eigentlich V(0) zu berechnen. Die normierte Bruchbedingung lautet aber

$$r = \frac{0 - \mu}{\sigma}, \tag{16}$$

wobei $\mu = \mu_y - \mu_{x1} - \mu_{x2} - \dots$ und

$\sigma = \sqrt{\sigma_y^2 + \sigma_{x1}^2 + \sigma_{x2}^2 + \dots}$ ist. Außerdem müssen alle a_ν durch σ^ν dividiert werden. Gl. (15) erhält daher die endgültige Form

$$V(r) = \sum_{\nu=0}^{\infty} \frac{a_\nu}{\sigma^\nu} \Phi^{(\nu)}(r) = \Phi(r) - \frac{1}{3!}\frac{\varkappa_3}{\sigma^3}\Phi'''(r) \tag{17}$$

$$+ \frac{1}{4!}\frac{\varkappa_4}{\sigma^4}\Phi^{(IV)}(r) - \dots .$$

In der angegebenen Form enthält die Edgeworth-Reihe im ersten Glied die Faltung der beteiligten Verteilungsfunktionen als Normalverteilungen. Die weiteren Glieder sind Korrekturen, die auf Grund der höheren Semiinvarianten erfolgen. Diese Korrekturen werden wegen der Konvergenz der Edgeworth-Reihe von Glied zu Glied kleiner und es ist zu übersehen, wann die Genauigkeit der erreichten Annäherung ausreicht. Hier ist zu bemerken, daß wegen der besseren Konvergenz bei der praktischen Anwendung der Edgeworth-Reihe die Korrekturglieder umgeordnet wurden, also eine etwas andere Form, als hier angegeben verwendet wurde [4] .

Die durchgeführten Berechnungen ergaben, daß im allgemeinen bereits das erste Glied der Edgeworth-Reihe das Ergebnis in der richtigen Größenordnung angibt. Es sind also nur bei genaueren Untersuchungen die Korrekturglieder zu berücksichtigen. Da in Zuverlässigkeitsanalysen sehr geringe Bruchwahrscheinlichkeiten zu erwarten sind, wird in der Regel die Berechnung des ersten Gliedes der Edgeworth-Reihe genügen.

5. Schlußbemerkungen

Als Ergebnis durchgeführter Untersuchungen konnte folgendes festgestellt werden:

1. Die Schätzungen ergaben, daß im allgemeinen bereits das erste Glied der Edgeworth-Reihe das Ergebnis in der richtigen Größenordnung angibt. Es sind also nur bei genaueren Untersuchungen die Korrekturglieder zu berücksichtigen. Da in Zuverlässigkeitsanalysen sehr geringe Bruchwahrscheinlichkeiten zu erwarten sind, wird in der Regel die Berechnung des ersten Gliedes der Edgeworth-Reihe genügen,was einer einfach durchzuführenden Faltung von Normalverteilungen entspricht.
2. Die Bruchwahrscheinlichkeiten ausgeführter Tragwerke schwanken je nach getroffener Voraussetzungen in weiten Grenzen und liegen zwischen $P = 10^{-7}$ bis 10^{-2}; die richtige Verwendung der Extremwertverteilung ist für das Ergebnis von großer Bedeutung.

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ZUSAMMENFASSUNG

Die Schätzung der Bruchwahrscheinlichkeiten gelingt relativ einfach unter Anwendung der Edgeworth-Reihe. Dabei ist es erforderlich, bei mehrfach eingetragenen Belastungen die entsprechenden Extremwertverteilungen zu verwenden. Soll die Bruchwahrscheinlichkeit nur in der Größenordnung richtig geschätzt werden, so sind dazu nur die ersten beiden Momente der beteiligten Verteilungsfunktionen erforderlich. Damit reduziert sich das Problem auf eine Faltung von Normalverteilungen, die einfach durchzuführen ist.

SUMMARY

The estimation of the probability of fracture succeeds comparatively easy by the application of the Edgeworth-progression. Thereby it is necessary to use the corresponding extreme value distributions, if multiple stresses are inscribed. If the probability of **failure** is to estimate right only in the order of magnitude, there are necessary only the first two moments of the concerned distribution functions. With that the problem is decreased to the folding of normal distributions, which is easily to carry out.

RÉSUMÉ

Le calcul des probabilités de rupture réussit d'une façon relativement simple, si l'on emploie la progression d'aprés Edgeworth. Ce faisant, il est nécessaire d'utiliser, à une distribution multiple des charges, les répartitions de valeurs extrêmes correspondantes. Lorsque la probabilité de rupture ne doit être exactement évaluée qn'en ordre de grandeur, les deux premiers moments des fonctions de répartition engagées sont seulement nécessaires. Ainsi, le problème est réduit à une convolution des répartitions normales qui est facile à effectuer.

The Relation of Data to Calculated Failure Probabilities

Rapport entre les différents facteurs dans le calcul de la probabilité de rupture

Die Beziehung der Daten zur berechneten Bruchwahrscheinlichkeit

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By the methods of rational mechanics and the calculus of probability, we can now process the probability distributions for loads and material strengths relating to a proposed structure and calculate the 'probability of failure' to any desired number of decimal places, regardless of how scanty the data is or how poorly the curves fit the data. Clearly, the meaning of this calculated probability needs to be studied critically before it can be used with confidence in the design process. In particular, we must find ways to assess whether or not the data is really sufficient to warrant the probability statements used in the design.

The nature of probability has been studied extensively [1, 2]. In relation to the structural design problem the notion is fairly well defined; in most studies of the structural safety problem, 'probability' is usually taken in the sense of "probability-1" defined at length by CARNAP [2] (loosely called 'subjective probability'), or it is left as an undefined notion; "probability-2" ('objective probability') cannot properly be assigned any meaning in this context.

One way to employ probability(-1) in problems of structural safety is to adopt the viewpoint that it is merely a subjective measure of 'degree of belief,' or 'strength of belief'. The relation of data to the probability of failure is then very simple; data may rationally be assimilated into the input probabilities by Bayesian methods [3]. The question of what constitutes a sufficient amount of data to make a particular statement about the probability of failure, does not arise. Therefore, this paper is not relevant to 'Bayesian design.'

Alternatively, we may consider the probabilities associated with loads and strengths to be inherently unknown, auxiliary quantities. Objective statements about the probability of failure can then be made in the usual terms of statistical inference, and the subjective element in the justification of the design is greatly reduced. The viewpoint in the following, then, is that probability is not an absolute notion; rather, it has meaning only in relation to a specified body of evidence which, in this context, means: Actual results of load measurements, materials tests, model tests, prototype tests, etc., called the <u>data</u>. The advantage of this approach (when it is feasible) over the Bayesian approach is that it leads to propositions about the probability of failure that can be subjected to scientific inquiry.

Under normal conditions of practical design the data is, unfortunately, insufficient to make objective statements about the probability of failure of a proposed structure; for example, future loads must be guessed from measurements taken in the past. Nevertheless, it is instructive to study the rational inferences about the probability of failure that are possible under certain idealized conditions as models of reality, permitting us to estimate the amount of data required under less ideal conditions. In the following we will derive such a relationship (equation 12) between the necessary amount of data and various constants related to the design value of the probability of failure.

Consider a structure drawn at random from an infinite population of like structures and submitted to a single scalar load S drawn at random from an infinite population of loads. Let R denote the resistance of the structure, defined in such a way that <u>failure</u> is the event $R < S$. Resistance R and load S are assumed to be intrinsically positive, independent, continuous stochastic variables with unknown probability densities $p_R(R)$ and $p_S(S)$; information about these functions is assumed obtainable by random sampling. The data D is therefore a set of n_R resistance values and n_S load values:

$$D = \{R_i,\ i = 1,\dots,\ n_R;\ S_j,\ j = 1,\dots,\ n_S\}. \tag{1}$$

The probability of failure is

$$p_F = \int_{R<S} p_R(R)\ p_S(S)\,dS\,dR; \tag{2}$$

since p_R and p_S are unknown, p_F cannot be determined. The problem is instead to compute a suitable estimator C_F called the <u>calculated probability of failure</u>.

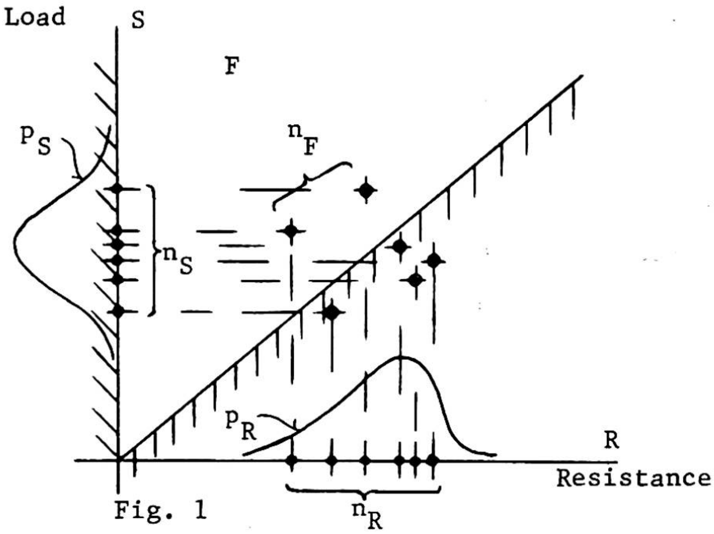

Fig. 1

To make inferences about the probability of failure p_F, it is necessary to derive suitable statistics of the stochastic variable C_F.

The simplest way to obtain such an estimator is to draw from the data D a sample W of n ($\leq n_R$, $\leq n_S$) pairs (R, S) of resistance and load values, at random and without replacement, see Fig. 1. Then, W is a random sample of the parent population $\{(R,S)\}$, and the elements of W are stochastically independent. Let n_F denote the number of outcomes of the failure event $R < S$ in the sample W. Evidently, n_F is the total number of "successes" in n independent Bernoulli trials with probability p_F of "success". Therefore, n_F is distributed according to the binomial distribution

$$b(1,n,p_F) = np_F(1-p_F)^{n-1} \tag{3}$$

with mean np_F and variance $np_F(1-p_F)$. It follows that the estimator $f_F=n_F/n$ is similarly distributed with mean $m = p_F$, variance $\sigma^2 = p_F(1-p_F)/n$, and coefficient of variation $v = \sigma/m = 1/\sqrt{np_F/(1-p_F)}$. The relative failure frequency f_F is therefore an unbiased estimator of p_F. It is discrete valued ($f_F \in \{0, 1/n, 2/n, \ldots, 1\}$), so that in order to get sufficient resolution it is required that n_F be large in comparison with unity. Assuming that n_F is greater than 9 and neglecting p_F in comparison with unity, it can be shown [4] that f_F is approximately normally distributed with mean p_F and coefficient of variation $1/\sqrt{np_F}$.

In this context, the most appropriate way to indicate the precision of an estimate of p_F is by means of confidence intervals [4]. First, a confidence coefficient α is selected. Taking the distribution to be normal with mean C_F and coefficient of variation $1/\sqrt{nC_F}$ gives the following approximate confidence limits for p_F computed from the calculated probability of failure:

$$L^- \approx C_F(1-N^{-1}(\alpha)/\sqrt{nC_F}), \quad L^+ \approx C_F(1+N^{-1}(\alpha)/\sqrt{nC_F}); \tag{4}$$

$N^{-1}(\ )$ denotes the inverse function of the normal probability integral. In a long sequence of repetitions the confidence interval between L^- and L^+ will contain the probability of failure p_F nearly a fraction α of the time.

To illustrate, assume that the data D consists of $n_S = 10^5$ and $n_R = 10^4$ random samples of load and resistance, respectively. The largest random sample W of independent elements that can be drawn contains $n = 10^4$ (R, S)-pairs. Assume that $n_F = nC_F = 16$ is the number of failure events in such a sample. If a confidence coefficient α = 90 per cent is considered suitable, we get from a table of the normal probability integral that $N^{-1}(0.9) = 1.645$. Equations (4) then give $L^- = (1-0.41)C_F$ and $L^+ = (1+0.41)C_F$. The following continued inequality may be written down:

$$(0.59)\left(\frac{16}{10^4}\right) < p_F < (1.41)\left(\frac{16}{10^4}\right); \tag{5}$$

it may be asserted that this inequality is satisfied with probability 0.9. In other words, chances are nine out of ten that the value of p_F lies between 0.00094 and 0.00226. Independent random pairing of load and resistance values is clearly a very inefficient way of processing the data, in the present case using only 10^4 out of a possible maximum of $n_R n_S = 10^9$ combinations of load and strength.

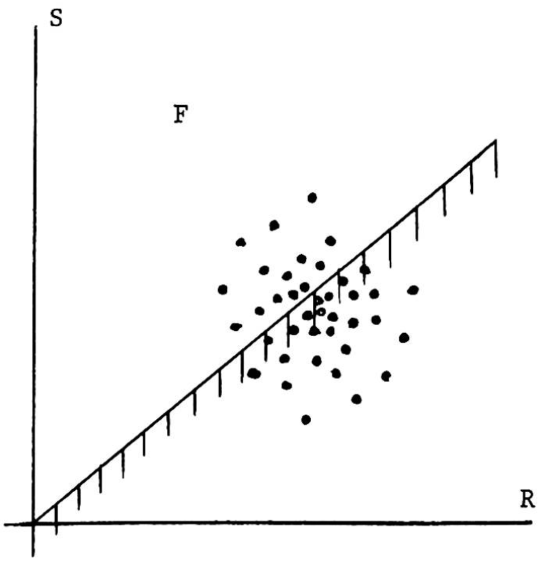

Fig. 2

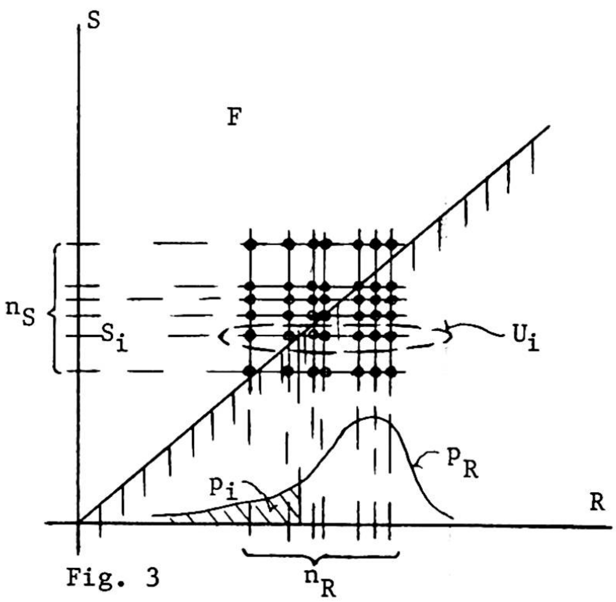

Fig. 3

Fig. 2 illustrates a sample consisting of a total of $n_R n_S$ pairs obtained by independent random sampling. Fig. 3 shows all the (R,S)- pairs that can be formed from the data. The ordering of the pairs in this figure suggests a stochastic dependence which, according to the sign of the correlation between sample elements, may either increase or decrease the variance of the estimated probability of failure in comparison with independent random sampling using the same sample size. Nevertheless, the relative failure frequency, C_F, in the sample is an unbiased estimator of the probability of failure,

$$m(C_F) = p_F, \tag{6}$$

since every sample element was obtained by random sampling. To compute the variance, consider a sub-sample U_i (Fig. 3) consisting of n_R pairs formed by one of the load values, S_i, paired with all the resistance values $R_1, \ldots, R_{n_R}$. A conditional probability of failure at this load level, p_i, may be associated with the sub-sample:

$$p_i = \int_0^{S_i} p_R(R)dR. \tag{7}$$

As before, the elements of the sub-sample constitute a sequence of n_R independent random Bernoulli trials. The number of failure events, n_i, at load level S_i is therefore binomially distributed with mean $n_R p_i$ and variance $n_R p_i(1-p_i)$. However, it is also observed that the n_S sub-samples constitute a sequence of independent random samples, for the n_R resistance values may be considered to be drawn a priori, thereby dividing the load range into $n_R + 1$ intervals establishing for each interval an associated probability that a load value will fall in the interval. As the loads are drawn independently and at random, the outcomes $n_i (i = 1, \ldots, n_S)$ are stochastically independent. Accordingly, the estimator

$$C_F = \sum_{i=1}^{n_S} \frac{1}{n_R n_S} n_i, \tag{8}$$

has the mean value

$$m(C_F) = \frac{1}{n_R n_S} \sum_{i=1}^{n_S} n_R p_i \tag{9}$$

and the variance

$$\sigma^2(C_F) = \frac{1}{(n_R n_S)^2} \sum_{i=1}^{n_S} n_R p_i (1-p_i). \tag{10}$$

Neglecting p_i in comparison with unity for all $i = 1, \ldots, n_S$, eliminating $m(C_F)$ from equations (7) and (9), and inserting the result into equation (10) gives for the estimator C_F the coefficient of variation

$$V(C_F) = \sigma(C_F)/m(C_F) \approx 1/\sqrt{n_R n_S P_F} \tag{11}$$

Thus, as a good approximation, the coefficient of variation of C_F has the same value as if all $n_R n_S$ sample pairs had been obtained by independent random sampling. We may therefore use equations (4) with $n = n_R n_S$ to determine the confidence limits for the probability of failure. To illustrate, let $n_S = n_R = 100$, yielding 10^4 (R,S)-pairs, and assume that 16 of these pairs represent failures. This data yields the same confidence interval as found above, equation (5). The calculated probability of failure, $C_F = n_F/n_R n_S$ according to Fig. 3, is believed to utilize the data in the most efficient way possible.

The amount of data required for a specified confidence coefficient α, a target "design" probability of failure P_F, and a specified maximum width βP_F of the confidence interval (symmetric about P_F) is easily computed from equation (4) to be

$$n_R n_S > [2N^{-1}(\alpha)/\beta]^2/P_F \tag{12}$$

For example, assume that we seek to design the structure so that the probability of failure "with 90 per cent confidence" ($\alpha = 0.9$) is a number between 10^{-3} and 10^{-4}. We select the target probability of failure $P_F = 5.5 \times 10^{-4}$ and choose $\beta = 9/5.5$ in order that the confidence limits $(1 \pm \beta)P_F$ coincide with the specified limits $p_F = 10^{-3}$ and $p_F = 10^{-4}$. Equation (12) gives the result that the product $n_R n_S$ must be greater than 7,500. For example, n_R must be greater than 150 is n_S equals 50. Alternatively, if we demand that the probability of failure equals $10^{-6} \pm 5\%$, with 95% confidence, the required amount of data is increased to $n_R n_S > 1.5 \times 10^9 = (50{,}000)(30{,}000)$.

While the specific case studied here is greatly idealized, it serves to give an idea of the amount of data required in probabilistic design, unless one is content with giving merely a subjective meaning to the term 'probability of failure'. The value of $n_S n_R$ according to equation 12 may be taken as a rough lower bound for the data required to make an objective statement about the probability of failure in the form of a confidence interval. The amount of data that, as a practical possibility, can be collected does not seem out of proportion to the amount required in probabilistic design, assuming that reasonable standards of precision are prescribed.

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Summary

Statistical considerations must be used to supplement purely probabilistic considerations in structural reliability studies if concepts such as the probability of failure are to have more than a mere subjective meaning. In this contribution, the amount of data required to make confidence interval statements about the probability of failure is estimated by the methods of mathematical statistics.

Résumé

Nous voulons ajouter des considérations statistiques aux considérations probabilistiques des études de sécurité dans le domaine de la construction, afin d'élever ces dernières au-dessus du niveau purement subjectif. Dans cette étude, nous proposons, à l'aide des méthodes de statistiques mathématiques, d'évaluer la quantité requise de données pour établir les intervalles de confiance autour de la probabilité de ruine.

Zusammenfassung

Ueber rein wahrscheinlichkeitstheoretische Ueberlegungen hinausgehende statistische Betrachtungen sind für die Studien der Sicherheitskriterien im Hochbau erforderlich, falls Begriffe wie "Bruchwahrscheinlichkeit" usw. mehr als mit bloss subjektiver Bedeutung belegt sein sollen. In der vorliegenden Arbeit wird aufgrund eines speziellen Modells eine Abschätzung für den Bedarf an Datenmaterial vorgenommen, um Konfidenzgrenzen für die berechnete Bruchwahrscheinlichkeit angeben zu können.

Safety Analysis of Suspension Bridges

Analyse de la sécurité de ponts suspendus

Sicherheitsbetrachtungen an Hängebrücken

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1. Introduction

Within the context of Theme I of the 8th congress, this paper establishes a method of structural safety analysis for the lateral vibration of aerodynamically stable suspension bridges under stormy winds.

The recent use of the so-called gust response factor in the dynamic analysis of structures subjected to gusty winds indicates an achievement of a higher level of sophistication in the structural safety analysis compared with the use of conventional safety factor, since the introduction of the gust response factor is based on the recognition that the wind velocity and hence the structural response have to be treated realistically as random processes.

The present paper demonstrates that a further effort will make it possible to estimate, in approximation, the probability of survival or failure of the suspension bridge (in the lateral mode of vibration) which is a more direct measure of safety in accordance with the probabilistic concept of structural safety[1] (*).

Since the type of failure considered in this paper is either buckling or yielding of a chord member of the stiffening truss due to its lateral bending under the wind pressure (this defines a critical bending moment at each cross-section), the linear equations of motion can be employed in the response analysis. Such failure modes are also assumed implicitly or explicitly in the previous papers[2,3] dealing with the same problem.

(*) Numerals indicate references at the end.

2. Structural Analysis

In the present paper, as in References 2 and 3, the wind velocity $U_z(t)$ at the height z above ground is the sum of the mean wind velocity $\bar{U}_z(t)$ and the fluctuating part $u_z(t,x)$.

The pressure due to the wind velocity $U_z(t)$ is, as usual, assumed to consist of two parts: the pressure due to the mean wind velocity

$$\bar{P}(t) = \frac{1}{2}\rho c A \bar{U}_z^2(t) \tag{1}$$

and the pressure due to the fluctuating part

$$p(t,x) = \rho c_d A \bar{U}_z(t) u_z(t,x) \tag{2}$$

where ρ is the density of air, c and c_d the static and the dynamic drag coefficient and A the exposed area of the structure considered.

It is usually observed from wind velocity records that $u_z(t,x)$ is nonstationary with a larger variance at a larger mean wind velocity. In the present study, however, it is assumed that $u_z(t,x)$ is stationary with a (constant) variance equal to that associated with the maximum mean wind velocity $\bar{U}_z$. Furthermore, $\bar{U}_z(t)$ in Eq.(2) is replaced by for simplicity. Hence, the following stationarized and conservative expression is used for $p(t,x)$.

$$p(t,x) = \rho c_d A \bar{U}_z u_z(t,x). \tag{3}$$

Since the variation of $\bar{P}(t)$ in time is much slow compared with the fundamental period of lateral vibration of the system of the cables and truss, the response $\bar{y}_T(t,x)$ and $\bar{y}_c(t,x)$ to $\bar{P}(t)$ is obtained performing a quasi-static analysis, while the mean square value of $y_T^*(t,x)$ and the bending moment $M^*(t,x)$ of the truss to $p(t,x)$ is evaluated on the basis of the standard equations of motion:

$$EI\,\bar{y}_T^{IV} + k(x)(\bar{y}_T - \bar{y}_c) = \bar{P}_T(t) \tag{4}$$

$$-H\,\bar{y}_c'' - k(x)(\bar{y}_T - \bar{y}_c) = \bar{P}_c(t) \tag{5}$$

$$m_T \ddot{y}_T^* + \mu_T \dot{y}_T^* + EI\,y_T^{*\,IV} + k(x)(y_T^* - y_c^*) = p_T(t,x) \tag{6}$$

$$m_c \ddot{y}_c^* + \mu_c \dot{y}_c^* - H\,y_c^{*\,''} - k(x)(y_T^* - y_c^*) = p_c(t,x) \tag{7}$$

with

$$k(x) = m_T g / h(x) \tag{8}$$

where the primes and the dots indicate differentiation with respect to x and t respectively, $h(x)$ is the hanger length, EI the bending rigidity of the truss in the horizontal direction, H the sum of the horizontal forces in the cables, m the mass per unit length, μ the linear viscous damping with subscripts T and C indicating that the quantities with T are associated with the truss and those with C are with the cables. The lateral bending moment of the truss can be obtained from its lateral displacement in the usual fashion.

The finite sine transform technique or the sine series expansion of $\bar{y}_T$ and $\bar{y}_C$ can be used to solve Eqs.(4) and (5) for $\bar{y}_T$ and $\bar{y}_C$. To evaluate the mean square response of M_T^* , the frequency response functions $H_{TT}(\omega, x, x_o)$ of $y_T^*(t,x)$ and $H_{TC}(\omega, x, x_o)$ of $y_C^*(t,x)$ due to an input $e^{i\omega t}\,\delta(x-x_o)$ applied at $x = x_o$ on the truss are first obtained by employing the finite sine transform technique. After some manipulation, one can show that the sine transforms $\widetilde{H}_{TT}(j) = \widetilde{H}_{TT}(\omega, j, x_o)$ and $\widetilde{H}_{TC}(j) = \widetilde{H}_{TC}(\omega, j, x_o)$ (with respect to x over $x = 0 \sim \ell$) of $H_{TT}(\omega, x, x_o)$ and $H_{TC}(\omega, x, x_o)$ satisfy the following equations.

$$\sum_{j=1}^{\infty} \widetilde{H}_{TT}(j)\, d_{nj} - \sum_{j=1}^{\infty} \widetilde{H}_{TC}(j)\, c_{nj} = \sin\frac{n\pi}{\ell}x, \qquad (n = 1, 2, \cdots) \tag{9}$$

$$-\sum_{j=1}^{\infty} \widetilde{H}_{TT}(j)\, c_{nj} + \sum_{j=1}^{\infty} \widetilde{H}_{TC}(j)\, e_{nj} = 0, \qquad (n = 1, 2, \cdots) \tag{10}$$

where

$$d_{nj} = (-\omega^2 m_T + i\omega\mu_T + EI\frac{n^4\pi^4}{\ell^4})\,\delta_{nj} + c_{nj} \tag{11}$$

$$c_{nj} = \frac{1}{\ell}\sum_{r=0}^{\infty} k_r \left\{ \delta_{(j+1),n} + \frac{|j-r|}{j-r}\,\delta_{|j-r|,n} \right\} \tag{12}$$

$$e_{nj} = (-\omega^2 m_C + i\omega\mu_C + H\frac{n^2\pi^2}{\ell^2})\,\delta_{nj} + c_{nj} \tag{13}$$

where ℓ is the span length, δ_{ij} the Kronecker delta, and k_r the coefficients of cosine series expansion of $k(x)$:

$$k(x) = \frac{2}{\ell}\sum_{r=0}^{\infty} k_r \cos\frac{r\pi}{\ell}x. \tag{14}$$

Eqs.(9) and (10) represent two sets of infinite number of equations for $\widetilde{H}_{TT}(n)$ and $\widetilde{H}_{TC}(n)$. By taking only first N terms each of $\widetilde{H}_{TT}(n)$ and $\widetilde{H}_{TC}(n)$ ($n, j = 1,2,\ldots, N$, and $r = 1,2,\ldots,2N$), one can obtain a set of $2N$ equations for $2N$ unknowns $\widetilde{H}_{TT}(n)$ and $\widetilde{H}_{TC}(n)$ ($n = 1, 2,\ldots,N$). Solving these and applying the inverse sine transformation, the frequency response function $H_{TT}(\omega, x, x_o)$ can be written as

$$H_{TT}(\omega, x, x_o) = \sum_{k=1}^{N} \alpha_k(\omega, x)\,\sin\frac{k\pi}{\ell}x_o \tag{15}$$

where

$$\alpha_k(\omega, x) = \sum_{j=1}^{N} a_{jk}^{-1}\,\sin\frac{j\pi}{\ell}x \tag{16}$$

In the Eq.(16) a_{jk}^{-1} is the $j - k$ member of the inverse matrix of a symmetric $2N$ x $2N$ matrix

$$A = \left[\begin{array}{c|c} D & -C \\ \hline -C & E \end{array}\right] \quad \text{with } D = [d_{ij}] \ , C = [c_{ij}] \quad \text{and } E = [e_{ij}] \ .$$

The frequency response functions $H_{CT}(\omega, x, x_0)$ of $y_T^*(t, x)$ and $H_{CC}(\omega, x, x_0)$ of $y_C^*(t, x)$ due to an input $e^{i\omega t}\,\delta(x - x_0)$ on the cable can also be obtained in a similar fashion.

$$H_{CT}(\omega, x, x_0) = \sum_{k=1}^{N} \beta_k(\omega, x) \sin\frac{k\pi}{\ell} x_0 \tag{17}$$

where

$$\beta_k(\omega, x) = \sum_{j=1}^{N} a_{j,N+k}^{-1} \sin\frac{j\pi}{\ell} x . \tag{18}$$

The functions $H_{CC}(\omega, x, x_0)$ and $H_{TC}(\omega, x, x_0)$ are not needed in the following analysis.

Making use of $\alpha''_k(\omega, x)$ and $\beta''_k(\omega, x)$, one can show that the mean square spectral density function of $M_T^*(t, x)$ is

$$\begin{aligned} S(\omega, x) = \sum_{r=1}^{N} \sum_{s=1}^{N} \Big[& \bar{\alpha}_r''(\omega, x)\, \alpha_s''(\omega, x)\, S_{rs}^{TT}(\omega) \\ & + 2\,Re \left\{ \bar{\alpha}_r''(\omega, x)\, \beta_s''(\omega)\, S_{rs}^{TC}(\omega) \right\} \\ & + \bar{\beta}_r''(\omega, x)\, \beta_s''(\omega, x)\, S_{rs}^{CC}(\omega) \Big] \end{aligned} \tag{19}$$

in which $Re\, z$ and $\bar{z}$ respectively indicate real part and complex conjugate of z ,

$$S_{rs}^{XY}(\omega) = \int_0^{\ell}\int_0^{\ell} S_{p_1 p_2}^{XY}(\omega) \sin\frac{r\pi}{\ell}x_1 \sin\frac{s\pi}{\ell}x_2 \, dx_1\, dx_2 \tag{20}$$

with X and Y standing either for T or C and $S_{p_1 p_2}^{XY}(\omega)$ being the cross-spectral density of $p_X(t, x_1)$ and $p_Y(t, x_2)$.

The variances σ_M^2 and $\sigma_{\dot{M}}^2$ of $M_T^*(t, x)$ and $\dot{M}_T^*(t, x)$ are then obtained as

$$\sigma_M^2 = \int_{-\infty}^{\infty} S(\omega, x)\, d\omega , \qquad \sigma_{\dot{M}}^2 = \int_{-\infty}^{\infty} \omega^2 S(\omega, x)\, d\omega \tag{21}$$

In the following discussion, however, the second and third terms within the square brackets of Eq.(19) are neglected because of their small contributions (as also done in Refs.2 and 3), and $S_{p_1 p_2}^{TT}$ is approximated by

$$S_{p_1 p_2}^{TT}(\omega) = \exp\left(-\frac{k\,\omega}{2\pi \bar{U}_z}\,|x_1 - x_2|\right)\Phi(\omega) \qquad (22)$$

where the exponential term is the square root of the coherence, $2\pi \bar{U}_z/(k\,\omega)$ the scale of turbulence at the wave length $2\pi \bar{U}_z/\omega$, and $\Phi(\omega)$ is the mean square spectral density of $p_T(t, x)$ and is given by[2]

$$\Phi(\omega) = 4\,(\rho\, C_{d_T} A_T \bar{U}_z)^2 K \frac{\bar{U}_{33}}{\omega} \frac{\left(\dfrac{\phi\,\omega}{\pi \bar{U}_{33}}\right)}{\left[1 + \left(\dfrac{\phi\,\omega}{\pi \bar{U}_{33}}\right)^2\right]^{4/3}} \qquad (23)$$

in which K is the surface drag coefficient, $\bar{U}_{33}$, the mean wind velocity at the reference height of 33 ft above ground, is related to $\bar{U}_z$ by

$$\bar{U}_1 = \bar{U}_z\left(\frac{33}{z}\right)^{\alpha} \qquad (24)$$

with α being a constant.

3. Safety Analysis

In previous papers[4, 5], one of the present authors developed a method of estimating upper and lower bounds of the probability that a Gaussian random process $z(t)$ will not be confined in a domain defined by $-a(t) \le z(t) \le a(t)$ in a specified time interval, where $a(t)$ (≥ 0) is a deterministic function of time.

Consider the standard design procedure for wind loads where the stiffening truss is designed so that it can withstand, with a safety factor n, the bending moment $M_d(x)$ produced by a specified (uniform) design wind pressure p_d. This implies that the critical bending moment at cross-section x is $nM_d(x)$. Suppose that the suspension bridge is subjected to a storm with mean wind velocity $\bar{U}(t)$ or mean wind pressure $\bar{p}(t)$ producing the bending moment $\bar{M}(t, x)$. Then, $a(t, x) = M^*(x) - M(t, x) = n M_d(x) - \bar{M}(t, x)$ is the maximum value of the bending moment $M^*(t, x)$ that the fluctuating part of wind pressure $p(t,x)$ can produce without failure. Since the variances of $M^*(t, x)$ and $\dot{M}^*(t, x)$ are evaluated in the preceding section, the method developed in References 4 and 5 can be applied to estimate upper and lower bounds of the probability of failure p_f or the probability that

$M^*(t,x)$ will not be confined in the domain defined by $-a(t,x) \le M^*(t,x) \le a(t,x)$.

Evidently, for a storm with a different mean value velocity function $\bar{U}_z(t)$, a different value of p_f is obtained. In fact, $\bar{U}_z(t)$ itself is usually a random function of time containing a number of random parameters, say $\bar{U}_z$ and T_0; $\bar{U}_z(t) = \bar{U}_z(t; \bar{U}_z, T_0)$. For example, the following forms of $\bar{U}_z(t, \bar{U}_z, T_0)$ are mathematically expedient and at the same time agree with observations reasonably well.

$$\bar{U}_z(t; \bar{U}_z, T_0) = \bar{U}_z \cdot e^{-(t/T_0)^2} \qquad -\infty < t < \infty \tag{25}$$

and

$$\begin{aligned} \bar{U}_z^2(t; \bar{U}_z, T_0) &= \bar{U}_z^2 (1 - |t|/T) && -T \le t \le T_0 \\ &= 0 && \text{otherwise} \end{aligned} \tag{26}$$

where T_0 is a measure of the duration of a storm in Eq. (1) while it is the duration in Eq.(2). Eq.(1) is used in Reference 3.

The probability of failure p_f is then computed for a storm with a particular set of $\bar{U}_z$ and T_0; $p_f = p_f(\bar{U}_z, T_0)$. Therefore, the probability of failure p_f^* due to a single application of a tatistical storm with $\bar{U}_z$ and T_0 being random is the expected value of $p(\bar{U}_z, T_0)$ with respect to $\bar{U}_z$ and T_0:

$$p_f^* = \iint p_f(\bar{U}_z, T_0)\, f(\bar{U}_z, T_0)\, d\bar{U}_z\, dT_0 \tag{27}$$

where $f(\bar{U}_z, T_0)$ is the joint density function of $\bar{U}_z$ and T_0. Hence, one can obtain the upper and lower bounds of p_f^* from those of $p_f(\bar{U}_z, T_0)$ using Eq.(27).

4. Numerical Example

As an example, a suspension bridge of the same dimension as the Forth Bridge is considered with $EI = 1.842 \times 10^{13}$ lb-ft^2, $h(x) = 309 - 1200(x/\ell)(1 - x/\ell)$ ft, $m_c g = 2.52 \times 10^3$ lb/ft, $m_T g = 8.38 \times 10^{13}$ lb/ft, $\ell = 3300$ ft, $H = 4.934 \times 10^7$ lb (Eqs.(4) - (8)), and such values of the linear viscous damping coefficients μ_T and μ_c (Eqs.(6) and (7)) that the logarithmic damping decrements of the first mode of independent lateral vibration of the truss and of the cables are both equal to 0.05. In Eqs.(15) - (19), $N = 5$ and in Eqs.(22) - (24), $k = 7$, $\phi = 2000$ ft, $K = 0.01$, $\alpha = 0.2$ and $z = 200$ ft (height of the truss above ground as in Refs.2 and 3).

With these parameter values, the variances of $M^*(t,x)$

and $\dot{M}^*(t,x)$ can now be evaluated numerically (IBM 7090 is used) following the method described in Section2. Because of the same assumption on the structure and the wind, the variance of $M^*(t,x)$ computed here is found to be close to those in Refs.2 and 3. Once these variances are computed, the bounding technique in Refs. 4 and 5 can be applied for the probability of failure $p_f(\bar{U}_z)$ with the time dependent barrier $a(t,x)$. Since in the present study, Eq.(26) is assumed for simplicity, $a(t,x)$ becomes

$$a(t,x) = \xi(x)\left(\frac{1}{2}\rho\, c\, A\right)\left\{ n\, U_d^2 - \bar{U}_z^2\left(1 - \frac{|t|}{T}\right)\right\} \tag{28}$$

where $\xi(x)$ is the bending moment of the truss at point x due to p_T = 1 lb/ft and p_c = 1/8.9 lb/ft (this value 8.9 is taken from Ref.3 and it is the ratio between the corresponding values of cA for the truss and the cables) and U_d is the design wind velocity which is taken as 110 mph in this study.

If the maximum mean wind velocity $\bar{U}_z$ is assumed to have the second asymptotic distribution of largest values[6] under a further assumption that $\bar{U}_z \geq$ 110 mph has a return period of 3450 years[3], then the density function $\bar{U}_z$ is given by

$$f(\bar{U}_z) = \frac{\gamma}{\bar{U}_c}\left(\frac{\bar{U}_z}{\bar{U}_c}\right)^{-\gamma-1} \exp\left[-\left(\frac{\bar{U}_z}{\bar{U}_c}\right)^{-\gamma}\right] \tag{29}$$

where γ is assumed to be 9.0 and $\bar{U}_c = 110\left[-\ln\left(1-\frac{1}{3450}\right)\right]^{1/\gamma}$ mph.

An additional assumption is made at this point that $\bar{U}_z^2$ and T_0 are proportional (or the intensity of storm and its duration are proportional) which appears to reflect the reality at least in approximation. In fact, a value $\bar{U}_z^2 / T_0 = 5\ \mathrm{ft/sec^3}$ observed from some Japanese records[3] is used here. Because of this assumption, Eq.(27) becomes a single integration hence considerably reducing the computational work:

$$p_f^* = \int_0^\infty p_f(U_z)\, f(\bar{U}_z)\, d\bar{U}_z \tag{30}$$

It is evident from Eq.(26) that $p_f(\bar{U}_z) = 1$ when $\bar{U}_z \geq \sqrt{n}\, U_d$. A further assumption $c_T = c_{dT}$ (see Eqs.(1) and (2)) is made here so that the following analysis becomes independent of the value of $\rho c_T A_T$.

The upper and lower bounds of p_f^* are computed as a function of the safety factor n (Fig.1). To be precise, the probability of failure $p_f(\bar{U}_z)$ and therefore p_f^* vary along x. However, the variation is negligible because the quantities $\xi(x) / \sigma_M(x)$ and $\xi(x) / \sigma_{\dot{M}}(x)$ on which the variation depends, are almost constant according to the

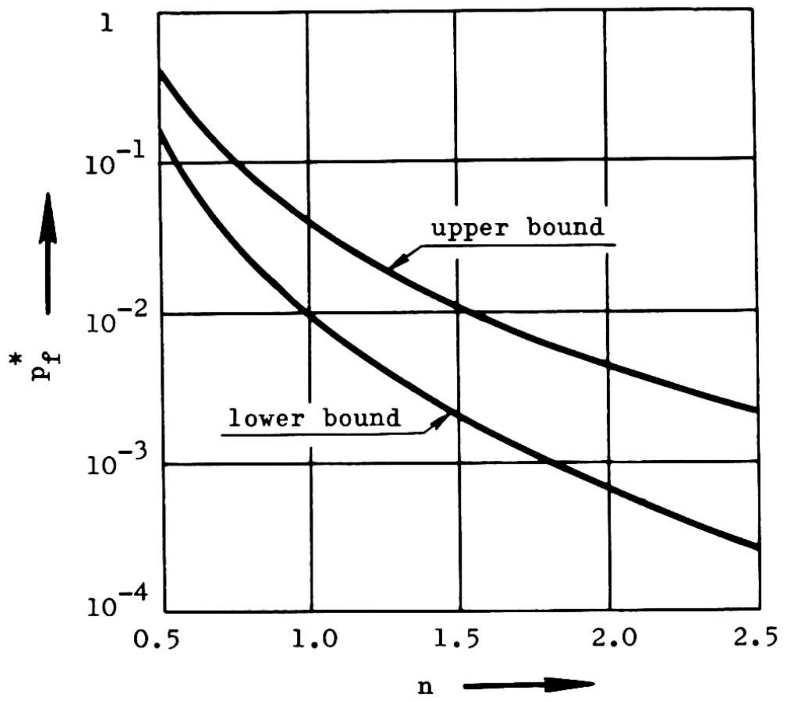

Fig. 1 Probability of Failure p_f^* as a function of safety factor n

numerical computation. In spite of the rather wide differences between the upper and lower bounds, the result shown in Fig.1 is quite useful in many respects. For example, using Fig.1 one can examine the effect of increasing the safety factor n. In fact, Fig. 1 indicates that the probability of failure decreases by one order of magnitude from the order of 10^{-2} to that of 10^{-3} by increasing n from 1.0 to 2.0. This implies the increase of the mean life by one order of magnitude from the order of 100 years to that of 1000 years, if it is assumed that significant storms occur on the average once a year. It is pointed out that from the view point of structural reliability analysis, the probability of failure estimated even only within the order of magnitude is a significant information.

5. Conclusion and Acknowledgement

A method of safety analysis by which the probability of failure of a suspension bridge due to lateral wind pressure caused by a (statistical) storm can be evaluated, is presented with a numerical example. The numerical example indicated that the probability can at least be estimated within the order of magnitude. This seems significant and satisfactory enough in view of the various assumptions one has to make as to structural response properties as well as statistical characteristics of the wind.

This study identified the information that is needed to make such an analysis more reliable. Other than those already identified elsewhere (for example, Refs. 2 and 3), the following quantities have to be known with reasonable accuracy; the cross-spectral density $S_{p_1 p_2}^{TC}$ (Eq.(20)) and more importantly, the mean wind velocity $\bar{U}_z(t)$ as a function of time t (Eqs.(25) and (26)) and its statistical nature, and the frequency of occurrence of significant storms.

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SUMMARY

This study presents a method of safety analysis of aerodynamically stable suspension bridges subjected to lateral wind pressure. The pressure is treated as a random process in space as well as in time. A numerical example is given under certain assumptions of statistical characteristics of the wind velocity. Importance of such a study lies not only in the development of a method of probabilistic safety analysis but also in the fact that it indicates what further information, statistical or otherwise, is needed to make the safety prediction more reliable.

RÉSUMÉ

Cette étude présente une méthode d'analyse de sécurité pour ponts suspendus aérodynamiquement stables soumis à une pression de vent latérale. La pression est supposée arbitraire dans l'espace et dans le temps. Un exemple numérique a été calculé à partir de certaines hypothèses des caractéristiques statistiques de la vitesse du vent. L'étude ne développe pas seulement une méthode d'analyse de sécurité probabiliste, elle indique avant tout quelles informations supplémentaires, statistiques ou autres, sont requises pour rendre les estimations de sécurité plus précises.

ZUSAMMENFASSUNG

Dieser Beitrag zeigt ein Verfahren für die Sicherheitsbetrachtung aerodynamisch stabiler Hängebrücken, die seitlichem Winddruck ausgesetzt sind. Der Druck wird als zufälliges Ereignis in Raum und Zeit behandelt. Ein numerisches Beispiel für bestimmte Annahmen der statistischen Charakteristiken der Windgeschwindigkeit wird angegeben. Die Wichtigkeit solcher Untersuchungen liegt nicht allein in der Entwicklung der wahrscheinlichen Sicherheit, sondern auch darin, daß erkannt wird, welche statistischen oder sonstigen Auskünfte künftig für die Sicherheitsvoraussage zuverlässig sein werden.

The Load Collapse for Elastic Plastic Trusses

La charge limite pour un treillis élasto-plastique

Traglast elasto-plastischer Fachwerke

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Abstract - The collapse load of a truss is investigated taking into consideration the way the bars actually behave, namely the effects of the strain hardening and the buckling respectively for the bars under tension and for those under compression.

During the buckling process the diagram which represents load versus axial deflection, on account of yelding of mid section,due to the bending, takes the form of a hiperbola branch (fig.1) [1] [2] [3]. At this stage, the bar, whose characteristic is a negative strain hardening - softening - becomes unstable. If, however, it is within a hyperstatic system, its buckling does not necessarily cause the collapse of the structure. Especially for multi-hyperstatic trusses, the collapse load may be found to be higher by far than the load generating the buckling condition of the first bar.

The problem has been put up with the restrictions as described in the following: The bars are pin hinged bars; the stress-strain relationship, as indipendent from the temperature and time,follows Prandtl's model [4] ; the deflections are assumed to be infinitesimal, that is finite but small, just that the geometry of the system and thereby the internal condition of the stresses are not affected at all; both localized and global bifurcation phenomena are ruled out. Of this structure are discussed the stability conditions in the classical meaning,that is for infinitesimal perturbances.

This problem has already been dealt with by other authors [5] , [6] [7] . From the stability postulate of Drucker's [8] [9] the sufficient conditions for stability and uniqueness of the solution have been deduced. In the discussion which follows only the first aspect of the question has been examined closely: By an original procedure, the necessary and sufficient stability conditions have been formulated.

The problem has been traced back to analysing the development to which is subjected the structural yield locus,which varies with the varying loads, under the action of incremental plastic deformations. Upon the external load reaching its critical value, to the increment of the plastic deformations corresponds a contraction into the yield locus which make it impossible to balance the original

load. From the discussion is possible to elaborate a graph which enables making a stability verification immediately, which can be made, hoverer, for pratical purposes, in the only case of two variables.

In the general case the problem is transferred into algebraic form: The parameter which indirectly furnishes the answer of the yield locus to an increase in the plastic deformations is determined by the energy irreversibly stored into the system: the elastic constrained energy and the energy dissipated through the plastic phenomena. If,to an increment whatever in the plastic deformation, the corresponding variation in the stored energy is still positive then the equilibrium is stable; if of no value or negative then at least in one case the equilibrium is neutral or unstable. The question is restricted to researching the sign of a quadratic form, associated with the matrix of rigidities, function of the plastic deformations and constrained thus by the signs of the latter.

These conditions can be brought to some other form as function of such parameters as are typical of the stability problems, that is the work done by the disturbing forces or the total energy of the system. It is demonstrable that if the variation occurring in the stored energy is either negative or zero the variation of the total energy of the resulting work done by the disturbing forces will likewise be either negative or zero. So we again come to a formulation which, though less praticable because of the further difficulty encontered in assessing the free elastic energy, connects directly to a principle which is as a rule normal within the elastic range or Drucker's postulate.

The problem is susceptible of generalizations.At this time the preference has been given to focussing the attention on the concepts rather than going deep into a more complex program.

The behaviour of the bars - The assumption is made that the bars,eithe in tension or compression, follow Prandtl's model [4] , indifferently.

In fig.1 is shown the curve relative to the relationship existing between axial force S, elongation or shrinkage δ for any bar in general The bar behaves elastically according to Hooke's law up to stress S_e; Past this point,plastic deformations take place, such that the linear trend of the line is changed. Upon relieving the load the representative point of the stress condition moves along the line parallel to O-A.

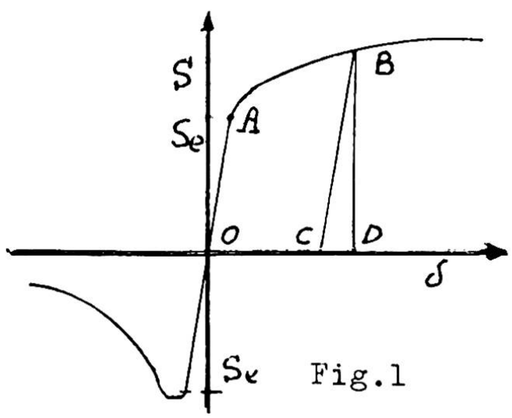


Fig.1

Segment O-C indicates the plastic deformation $\bar{\delta}$,at B,which at the time the load is relieved remains unaltered;segment C-D represents the elastic deformation δ_e. If the bar is isolated for $S=0,\delta=\bar{\delta}$;if it is within a hyperstatic system,for $S=0,\delta=\bar{\delta}+\bar{\delta}_e$,where $\bar{\delta}_e$ indicates the elastic deformation constrained whithin the system and recoverably only through cutting the bar.

Area OABD represents the total

work performed by the external forces which is necessary to achieve pattern B. In particular OABC is the graphical representation of as much amount of energy as is absorbed by the system and is dissipated through the plastic phenomena;the area CBD is the elastic energy which can be returned only if the bar it is isolated or part of an isostatic system.

Unlike the currently adopted convention on the signs for the axial forces S, a different one is being introduced here. The starting axial force S is assumed to be positive in all cases; increments are either positive or negative whether or not they are in accord with the starting force.

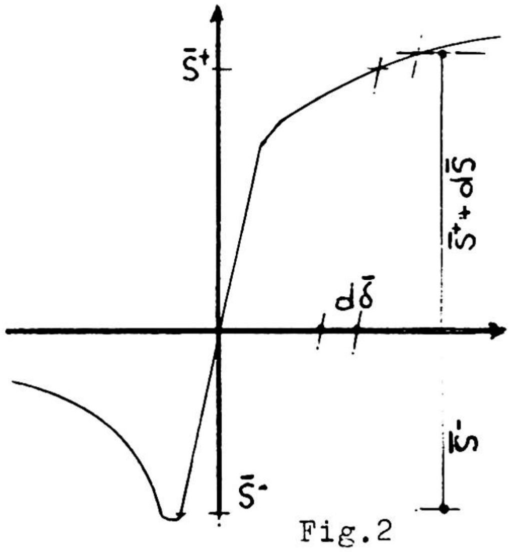


Fig.2

For assigned plastic deformation $\bar\delta$ (fig.2), $\bar S^+, \bar S^-$ are meant to be indicative of the limiting values within whose range the axial force can oscillate performing elastically.Therefore the yield locus shall be as established by the relation:

$$(1) \qquad S \le \bar S$$

where $\bar S$, generically,indicates the $\bar S^+, \bar S^-$ limiting values according to whether is correspondingly a traction or compression. If the verification yelds a disequality, the bar under test is in the elastic range, whereas the equality proves it is in the plastic range.

Where the bar is in the plastic range, i.e.if $S = \bar S$,the stress-strain relationship is linear, when the increments are infinitesimal: Curve $S(\delta)$ is substituted whith its tangential line at $\bar S$. Then by differentiating (1) in relation to δ or $\bar\delta$:

$$(2) \qquad dS \le \frac{dS}{d\delta}\, d\delta = W d\delta = \frac{d\bar S}{d\bar\delta}\, d\bar\delta = \bar W d\bar\delta = d\bar S$$

a limitation to the incremental relationship $\bar S - \delta$ is obtained. Owing to a $d\bar\delta$ increment in the plastic deformation the bar, initially stressed under $\bar S$, is now capable of taking a stress increment, at the limit, $d\bar S$: Therefore $d\bar S$ determines the dislocation of the yield locus (fig.2).

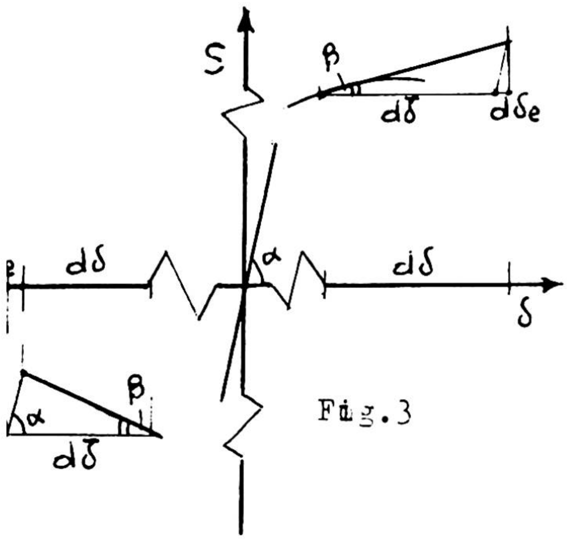


Fig.3

In the eq (2) W represents the differential rigidity, $\bar W$ the plastic differential rigidity (fig.3): the following is the correlation of the above rigidities to the elastic rigidity We:

$$\bar W = \frac{W \quad W_e}{W - W_e}$$

the result is that where $W \gtrsim 0$, $\bar W$ is likewise $\gtrsim 0$.The plastic deformation $d\bar\delta$ is restricted in sign by the relationship sign $d\bar\delta$ = sign $\bar S$,which, for the position of on the forces signs, is reduced to condition:

$$(3) \qquad d\bar\delta \gtrsim 0$$

The interval within which rigidity W is included is so defined:

$$-\infty < W < We$$

By combining eq. (2) with limitations:

$$d\bar{\delta} > 0 \qquad dS = dS = Wd\delta \quad (-\infty < W < We)$$

$$d\bar{\delta} = 0 \qquad dS = We d\delta \qquad (W = We)$$

the stress-strain incremental relationship is thus obtained. The eq. (2) covers the (4) and in a more general sense may be intended as relating to a cycle. At first, the incremental force dS verifies the equality with the bar being in the plastic range, subsequently is subjected to a reversal and thus verifies the disequality.

<u>The behaviour of the system</u> - As a reference, let it be taken a general type of reticular pin-hinged, made up by n bars, times r hyperstatic truss and let it be subjected to a loading pattern F: for an Fo load let Co be the corresponding in equilibrium and compatible pattern, typified by k number of bars ($K \gtrless r$) in plastic range, $\delta_1 \dots \delta_K$ being the corresponding elogations.

Let the displacements of the system be assumed as being infinitesimal, or finite, but such that they cannot affect the originary geometry of the system and, hence, indirectly, the stressed condition. This supposes that the strain condition which corresponds to Co can be regarded as borne by the plastic deformations $\bar{\delta}$, intended as distorsions, and by loads Fo, as applied to the elastic structure.

This as a reference Sei indicates the stress exercised by load Fo into bar "i"; Sij the stress transmitted to bar "i" through distorsion $\delta_j = 1$ at "j". Then the resulting stress in bar "i" is:

$$(5)\quad \bar{S}i = Sei + \sum_K Sij\,\bar{\delta}j \qquad (i = 1\dots n)$$

Eq.(5) is substituted in (1) by transferring to the right hand side the term relative to the distorsions:

$$(6)\quad Sei \leq \bar{S}i + \sum_K Sij\,\bar{\delta}j = \bar{\bar{S}}i$$

on the assumption that:

$$\bar{\bar{S}}i = \bar{S}i + \sum Sij\,\bar{\delta}j$$

The $\bar{\bar{S}}i$, different, whether tractive or compressive, are a generalization of the $\bar{S}i$ referred in (1) and define, within the space of the plastic deformations, the yield locus for pattern Co. If stresses Sei verify the inequality, the point representative of the stress condition falls inside yield locus. On the contrary, if for some of the bars the equality has been verified the representative point falls onto the edge of the yield locus and the structure is in the plastic range.

A variation is assigned to pattern Co by attributing to the bars in the plastic range a $d\bar{\delta}$ increment to the initial plastic deformations. on the assumption that the bars in the elastic range will stay such. The resulting C'o pattern is described as "perturbed" pattern. By differentiating (6) for the $d\bar{\delta}$ increments assigned and consistent with (3) we obtain the stress increments which C'o can absorb:

$$(7)\quad dSe_i \leq \bar{W}_i d\bar{\delta}i + \sum_K S_{ij} d\bar{\delta}_j = d\bar{\bar{S}}_i$$

Eq.(7) is a generalization of eq.(2). The dislocation of the initial yield locus $\bar{S}$, consequent to the assigned plastic deformations $d\delta j$ is just supplied by the $d\bar{\bar{S}}$. If the representative point of a new stress condition comes to fall inside of or into the edge of the yield locus, the equilibrium between the stresses and the strength of the bars is verified for pattern C_o' ; if outside, that is if for a certain number of bars: (8) $dSe_i > d\bar{\bar{S}}_i$
the equilibrium is impossible: the plastic deformations continue their pursuance to a new pattern C_o'' which may still verify eq.(7).

Stability of the system - A graphical method for the verification of the stability, in which the above indicated concepts ore expounded, is illustrated the problem being dealt with is limited to the case involving two placicized bars only. It will not be difficult but rather easy to extend, conceptually at least, the representation to the more generalized case.

As a reference let us consider a Cartesian system having as many axes as are the plasticized bars. Let us mark on the axes plastic deformations $d\delta$: The origin of the axes thus defines the pattern Co. As is conventional for the signs on plastic deformations (3), all C_o' patterns are comprehended within the quadrant of the positive $d\delta$. Chosing this as reference frame, we now draw as many straight lines $d\bar{\bar{S}}_i = 0$ as are the bars in the plastic range: the enveloping line defines the boundary of the plasticity field for that part which influences the stability of the system; on the perpendiculars are marked the stresses Se_i and the corresponding increments dSe_i. Therefore point Co sets also the initial stress condition in which $Se_i = \bar{\bar{S}}_i$.

Fixed the perturbed pattern C'o, the sides of the yield locus translate: according to $d\bar{\bar{S}}_i \gtrless 0$ it will correspondingly expand or contract: the new yield locus, so obtained, is defined "perturbed". The equilibrium in this stage is assuredly verified if the transposition to C_o' is considered as effected by forcing a set of supplemental restraints, non efficient in C_o. Point C_o' moreover establishes the elastic stresses dSe_i, relative to the reactions dF of the additional restraints constituting the, so called, perturbing forces.

The supplemental restraints are then removed and, hence, $dF \to 0$: Where $dSe_i \to 0$ the elastic stress condition C_o' has a tendency to resuming the initial position C_o. If C_o is found to fall inside the area of the perturbad yield locus, that is, if:

$$0 \leq d\bar{\bar{S}}_i$$

eq.(7) is verified: the pattern settles in C'o and the system behaves elastically again. If, on the contrary, for some of the bars eq. (8) is verified, that is

$$0 > d\bar{\bar{S}}_i$$

C_o comes to fall outside the perturbed field and there are no possibilities for an equilibrium. These bars keep being subjected to the plastic phenomenon with the field paralleley evolving in pursuance of a new pattern C_o'' which comprehend C_o. More forces are supposed to be interfering ot this stage such that a point-by-point equilibrium is assured.

For example, in the case illustrated in figura 4, what C'o might be, the resulting system is in any case that of equilibrium. Being tha at all times $d\bar{\bar{S}}_1 > 0$, $d\bar{\bar{S}}_2 > 0$, eq.(7) is verified, even where dSe$\rightarrow$0:The perturbed yield locus shall alwais comprehend the originating pattern C_o. In this case the equilibrium of pattern Co is stable.

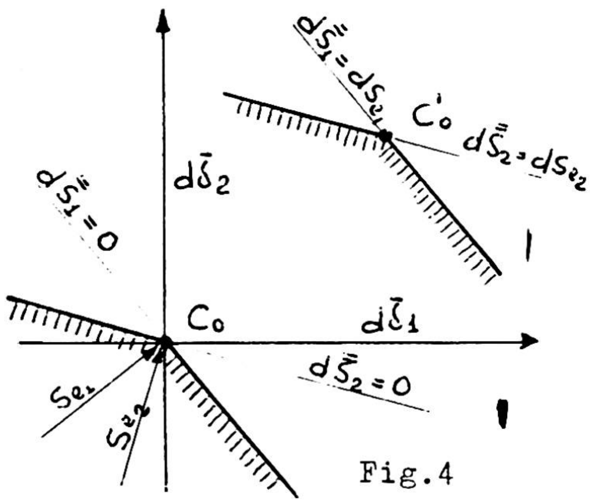


Fig.4

A diametrally opposed case, is that shown in figura 5. Whatever C'_o the resul is alwais $d\bar{\bar{S}}_i < 0$ $d\bar{\bar{S}}_2 < 0$. Hence by eliminating the perturbing forces eq.(8) is verified: within the two bars the plastic deformations increase. However,whatever the C''_o pattern which one can come to,during the unloading stage, the situation repeats itself again: the plastic deformations have a tendency to become infinitely great. Parallelely the edge of the yield locus, originally $\bar{\bar{S}}$, moves to $\bar{\bar{S}} - d\bar{\bar{S}}$: for $d\bar{\delta} \rightarrow \infty$, $\bar{\bar{S}} - d\bar{\bar{S}} \rightarrow 0$: the plasticity field for at least one its sides shrinks gradually up to becoming null. At C_o the equilibrium is therefore unstable.

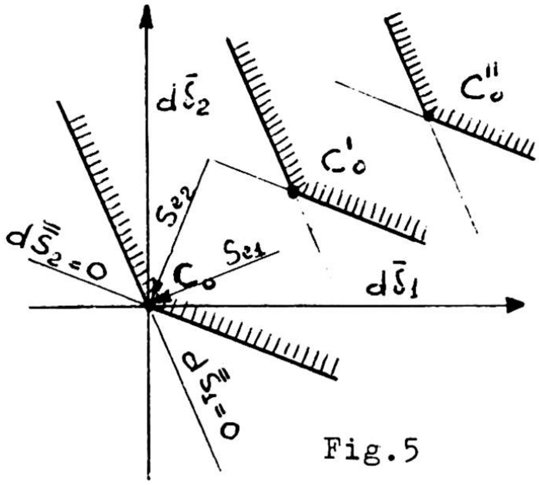


Fig.5

Figures (6) and (7) report some intermediate situations. The first shows a se of stable equilibrium, the second one a case of instability.

In fig. 8 is then illustrated a situation of neutral equilibrium. Whatever C'_o the system is apt to assuming an equilibrium pattern C''_o coincident or not whith the former. From this viewpoint the system is apparently stable. On the other hand, though, all patterns C'_o falling on straight line $d\bar{\bar{S}}_1 = d\bar{\bar{S}}_2 = 0$ are also corresponded by $dSe_1 = dSe_2 = 0$.All these patterns and, to the limit, the infinity one, are then attainable without the aid of a perturbing setup for forcing the system, ond hence without any energy dissipation. Along this directrix the system is seemingly worn out, unfit to counteract the modification of the original pattern C_o. The situation as illustrated in fig.7 is unstable although still presenting an indifference directrix.

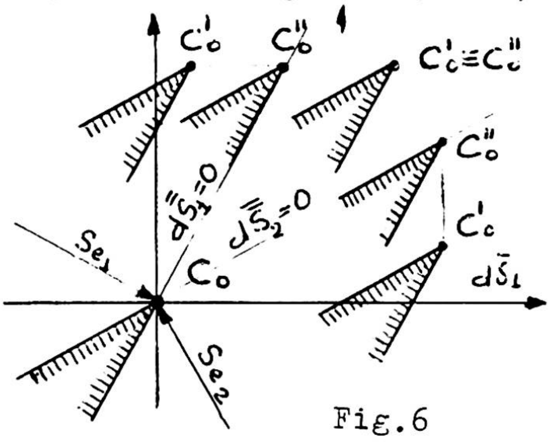


Fig.6

Even if hardly usable, owing to the unpratical possibility of extending it to an n dimension system, this graphical representation helps to clarify the problem and affords a comparison whith

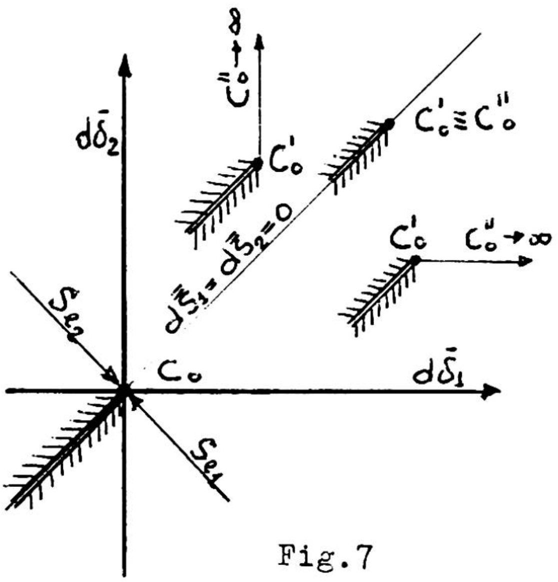


Fig.7

the analogus elastic problem.

In the elastic range, if the equilibrium is stable, $C'_0 \to C_0$ once eliminated the perturbation. In the elastic-plastic range we find that C'_0, apart from not returning in C_0 at all, may furtherly move away from it and reach C''_0, which alike C'_0, is very close to C_0. It follows that lacks the clear differentiation between a stable and a neutral equilibrium, as is found in the elastic range. The distinguishing point that differentiates the latter from the former lies only in the fact that, for translating the system from one pattern to another along the indifference directrix, there is no need of any external work.

<u>The system energy</u> - The stability conditions are algebraically expressed as functions of the energy. As an introduction some hint is therefore made about the energy stored in the system and its variations.

In an intermediate stage of the loading process 0-F_0, the work done by forces F in equilibrium with the internal stresses S, under the action of a d increment in the dispacements associated with an increment in the bar deformazion d is:

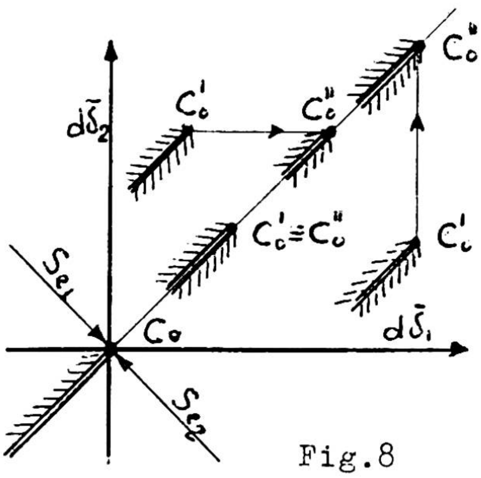


Fig.8

$$(9)\quad dL = \Sigma F d\eta = \sum_n S_i d\delta_i = \sum_n (Se_i - \sum_k S_{ij}\,\bar{\delta}_j)(d\delta e_i + d\bar{\delta} e_i + d\bar{\bar{\delta}}_i) = \sum_n Se_i\, d\delta e_i + \sum_k (\sum_k S_{ij}\,\bar{\delta}_j + \bar{\bar{S}}_i)\, d\bar{\delta}_i = \sum_n Se_i\, d\delta\, e_i + \sum_k Se_i\, d\bar{\delta}_i$$

the assumption having been made that in this stage too, K bars are plasticized.

The total work L, spent by the external forces for the development of pattern C_0 is:

$$(10)\quad L = \int F d\eta = \tfrac{1}{2} \sum_n Se_i\, \delta e_i + \sum_k (\sum_k S_{ij}\, \bar{\delta}_j)\, \bar{\delta}_i + \Sigma \int_0^{\delta_i} S_i\, d\bar{\delta}_i = E_e + E_v + E_p$$

The right hand side indicating the energy absorbed by the structure. In detail the first term, E_e, signifies the free elastic energy, in other words that quantity of energy which totally returns to the external forces at the unloading stage. The second term, E_v, the elastic ener_ gy constrained within the system by the plastic deformations which can be released to the outside only by making cuts in such a way that the structure becomes isostatic. The third term, Ep, the irreversible energy absorbed by the system, used to produce those alterations in the internal structure of the material which give origin to the plastic dislocations.

For translating the system from pattern C_0 to C_0' the work, d_2L, of the second order, done by the perturbing forces, taking into account the linearity of the stress-strain relationship, is

(11) $$d_2L = \frac{1}{2}\sum dF d\eta = \frac{1}{2}\sum_n dSe_i d\delta e_i + \frac{1}{2}\sum_\kappa(\sum_\kappa \bar{S}_{ij} d\bar{\delta}_j + \bar{W}_i d\bar{\delta}_i)\, d\bar{\delta}_i$$
$$= \sum_n dS_{ei}\, d\delta_{ei} + \sum_\kappa dSe_i\, d\bar{\delta}i = d_2E_1 + d_2Ev + d_2Ep = d_2E_1 + d_2\bar{E}$$

$d_2\bar{E}$ being the global constrained energy of the system both elastic and plastic.

The constrained energy d_2E is expressed by a homogeneous quadratic polynomial whose variables, however, are conditioned, in sign, by eq (3). For that part relative to the hiperquadrant 0 this polynomial coincides with the quadrantic form, associated to the matrix of the rigidities (7) and may result positive, null or negative: the last circumstance being possible in the sole case that, at least one bar be characterized by softening. The E_1 and E_v polinomials are instead always positive.

Generalizing the notion of the total energy of the system 10 by adding, in addition to the positional energy of the external agencies, and the free elastic energy, also the constrained energy, eq.(11),after transferring to the right hand side the external work, defines the variation prime, dE_t, of the total energy, stationary for the C_0 equilibrium pattern. Variation second d_2E_t is furnished instead by the right hand side of eq.(11).

<u>Stability conditions</u> - Let us suppose that the quadratic form $d_2\bar{E}$, devised for pattern C_0, is always positive for all the $d\bar{\delta}$ consistent with (3), but not simultaneonaly nought, that is:

(12) $$d_2\bar{E} = \sum d\bar{S}_i\, d\bar{\delta}_i = \sum(\sum \bar{S}i_j\, d\bar{\delta}_j + \bar{W}i\, d\bar{\delta}i)\, d\bar{\delta}_i > 0$$

In particular let for C_0' be:

$$dS_i = \left[\frac{d}{d\bar{\delta}_i}(d_2E)\right]_{C_0'} \geq 0$$

Eq (7) verified at the beginning in respect to the interference of the perturbing forces still rests verified for $dSe_i \to 0$: through the unloading stage the system behaves in an elastic way. In the space of the $d\bar{\delta}$ the pattern settles in C_0'.

Its supposed, instead, that for C_0' :

$$dS_i = \left[\frac{d}{d\bar{\delta}_i}(d_2E)\right]_{C_0'} \gtrless 0$$

In this case, although as a whole eq. (12) is verified, same of the addenda result as being negative. Whith the elimination of the perturbing forces for some of the bars eq.(8) is verified. For such bars the plastic phenomenon then progresses spontaneously and the system moves away passing from C'_o to C''_o. The second principle of the thermodynamics, as formulated by Lewis, [11] affirms that any spontaneaus phenomenon is corresponded by a decrease in the system energy which is transformed into the work of the balancing forces, that is dF, in the present case. Thus, if with $d_2\bar{E}_{c'_i}$ we designate the energy corresponding to travel C_o-C'_o, and $d_2\bar{E}_{c''_i}$ that relative to Co-C'o-C"o, the result will always yield:

$$(13) \qquad d_2E_{c'_i} > d_2\bar{E}_{c''_i}$$

But, for the supposition made in eq. (12), the verification of this relationship can only be ascertained where C"o within the space of the $\bar{\delta}$ - comes to falling around C'_o and, hence C_o. The pattern C''_o defines a relative extreme (minimum) of function $d_2\bar{E}$, conditioned by eq (3) and therefore:

$$d\bar{\bar{S}}_i = \left[\frac{d}{d\bar{\delta}_i} \quad (d_2E)\right]_{c''_i} \geq 0$$

Hence at C''_o, also for $dSe_i \rightarrow 0$, eq (7) is verified. So eq (12) represents a condition sufficient for Co being a pattern of stable equilibrium.

As a substitute of (12) let us assume:

$$(12') \qquad d_2\bar{E} \geqslant 0$$

In particular then let, for C'o, be $d_2\bar{E} = 0$: In the other case we come to fall again within the preceding situation.

Allowing for eq.(12') the risult will alvays yield:

$$d\bar{\bar{S}}_i = \left[\frac{d}{d\bar{\delta}_i} \quad (d_2E)\right]_{c'_i} \geq 0$$

Thus C'_o is a pattern of equilibrium with no interference of perturbing forces and as such are all those other patterns which fall into directrix Co-C'o which is justly typified by $d_2\bar{E}=0$. The system moves along this direction with no external work being done. Then the following is particularly to be verified:

$$d\,\bar{\bar{S}}_i > 0 \qquad \text{for} \qquad d\bar{\delta}_i = 0$$
$$d\,\bar{\bar{S}}_i = 0 \qquad \text{for} \qquad d\bar{\delta}_i > 0$$

Pattern C_o, which is corresponded by (12'), is then a pattern of neutral equilibrium.

For (12) let us assume as substitute:

$$(12'') \qquad d_2\,\bar{E} \gtrless 0$$

In particular is assumed as the assigned pattern C'_o that for which $d_2\bar{E} < 0$. In this case for some of the bars:

$$d\bar{\bar{S}}_i = \left[\frac{d}{d\bar{\delta}_i} \quad (d_2E)\right]_{c'_o} < 0$$

The perturbing forces eliminated, the plastic phenomenom then progress: the energy relative to a successive pattern C''_o is related to the energy at C'_o by eq.(13). In C''_o, and so for the successive patterns, is thus repeated the like situation as is found in C'_o. The plastic phenomenon keeps continuing indefinitely with the system never reaching a pattern of equilibrium with load Fo. Therefore if the pattern Co is associated to eq.(12") the equilibrium is unstable.

The considerations on the eq.(12'), (12") follows that eq.(12) represents also a condition necessary for the stability of the system.

Drucker's second stability postulate [8] [9] , as applied in the "small", fully confirms this result. In order that the system is stable the closed cycle work accomplished by the perturbing forces,applied at first and removed afterward, is to be positive. As this cycle terminates this work is found again under the form of stored energy: thus if $d_2\bar{E}>0$ the equilibrium is stable. On the contrary, if $d_2\bar{E}<0$ the result is that the cycle cannot be closed, that is the equilibrium is not verifiable without the introduction of an equilibrating system dF: then the equilibrium is unstable.

From the above it can be easy to deduce that, where the bars behav in an ideally plastic way (W=0), under the collapse load the equilibriu is neutral. True, in general $d_2\bar{E}\geq 0$ ($d_2Ep = 0$), particularly it nullifies for that $d\delta$ set which is corrisponded by the collapse mechanism. If the bars are instead strain hardened ($W>0$), $d_2\bar{E}>0$ as $d_2Ep>0$: In this case the equilibrium is stable.

The stability according to Drucher's postulate - The first postulate of Drucher's states that a system is stable, in the "small",if the work accomplished by whatever forces dF yields always a positive result. If these forces are supposed as acting in a proportional way,the work accomplished by forces dF is coincident with the energy stored by the system,(11),that is the total energy variation. In the following is the demonstration that this principle and the one expounded in the preceding paragraph match perfectly at least as far as concerns the specific case under consideration. It is demonstrated particulary that if $d_2\bar{E}>0$ or $d_2\bar{E}=0$, parallely, always does exist at least one perturbing pattern dF for which $d_2Et>0$ or $d_2Et = 0$.

Let us assume that $d_2\bar{E}>0$ and as dF a system of forces proportionate to load Fo acting in Co, characterized, thus, by a proporzionality factor $d\lambda$, infinitesimal. Since the system results being unstable for a given number of bars $d\bar{S}_i<0$. In order that C'_0 be an equilibrium pattern, eq.(7) must be verified and the result $dSe_i<0$ must thus be yielded. Since, for convention, stresses Se_i are positive, factor $d\lambda$ must be negative, or:

$$d\lambda\, Se_i = -\, dSe_i$$

The perturbing pattern dF must then result opposite to that Fo. In these conditions, at all times, eq.(7) is verified, even if plastic deformations are absent, in which case dSe_i 0. Among the C_0' solutions which verify eq.(7) there exists at least one, C_0' which verifies also eq.(4) in its generalized form, or:

$$(14)\quad \begin{aligned} -\, dSe_i &= d\bar{\bar{S}}_i & \qquad d\bar{\delta}_i &> 0 \\ -\, dSe_i &< d\bar{\bar{S}}_i & \qquad d\bar{\delta}_i &= 0 \end{aligned}$$

This solution defines one extreme of function $d_2\bar{E}$ [12] [13] [14] conditioned by eq. (7) and in particular for the assumption adopted on the sign,(12"), it defines a maximum. The work accomplished by forces dF, in moving the system from pattern Co to that C'_0, is then supplied by eq.(11) agrees with eq.(9) multiplied by the $\frac{1}{2}\,d\lambda$ negative factor.

Since is always: $dL > 0$

$$-\tfrac{1}{2}\, d\lambda \;\; dL = d_2L < 0$$

Obviously, if $d_2\, L < 0$, such is also the right hand side of eq. (11) that is the variation d_2E_t of the total energy. This implies that in C_o if (12") is verified, Ft definies a maximum and there exists, at least, one perturbed pattern C'_o for which $d_2L < 0$.

On the contrary if $d_2E \geqslant 0$, for the patterns C'_o falling on the indifference directrix:

$$dSe_i = \alpha_i dF = 0$$

then: $dF = 0$

$$d_2E_t = 0$$

If finally:

$$d_2F > 0$$

since $d_2F_1 > 0$, also $d_2E_t > 0$. In C_o the function E_t defines a minimum.

In the following a very simple example has been evolved. The stru_ cture is that as shown in fig.9. In figg.9, 9-a, 9-b, 9-c the graph shows plotted, in the upper part, the Ft force versus the slope δ, at C, for the beam, whose behaviour is supposed to be infinitely elastic; in the lower part of the same graph for the stanchion subjected to a buckling at A, assuming three different values for rigidity $\overline{W}a$. Star_ ting from pattern C_o, to which corresponds load $F_o = F_t + F_a$, an incre- ment $d\bar{\delta}$ is attributed to the plastic deformation and pattern C'_o is reached. Addenda $d_2\, E_1$, d_2E_V, d_2E_p, all coming within the energy ba- lance, hold as follows.

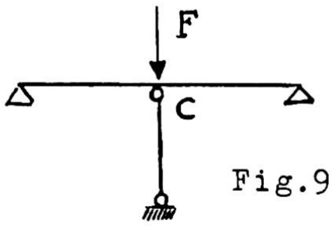


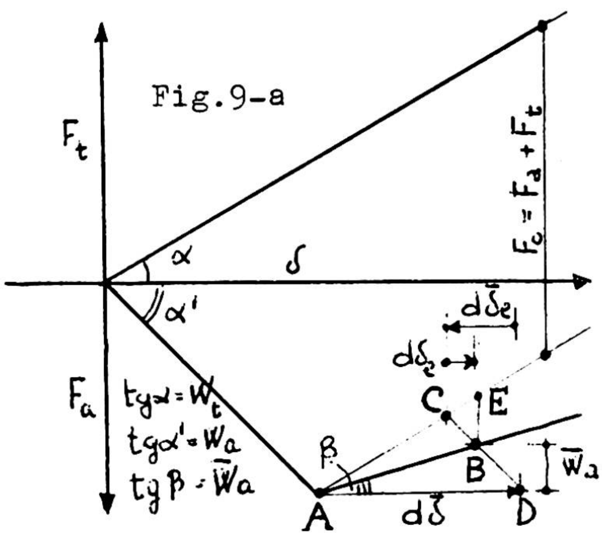
Fig.9

$$d_2E_e = \tfrac{1}{2}\;(W_a + W_t)d\,\delta\, e = \text{CBE area}$$

$$d_2E_V = \tfrac{1}{2}\;(W_a\; d\bar{\delta}e d\bar{\delta} = \text{ACD area}$$

$$d_2E_p = \tfrac{1}{2}\;\overline{W}_a\; d\,\bar{\delta}^{\,2} = \text{ABD area}$$

In particular, for.chart in fig.(9-a):

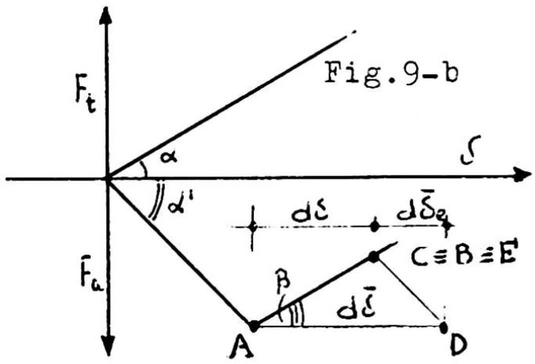


$$d_2\bar{F} = \text{ACD-ABD} = \text{ACB} > 0$$

$$d_2E_t = \text{ACB+CBE} = \text{ABE} > 0$$

The equilibrium is stable.

For chart in fig.(9-b):

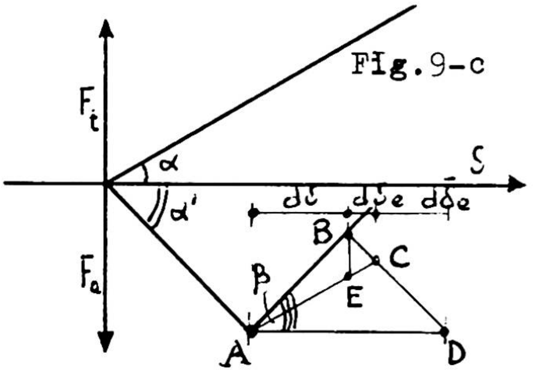


$d_2\bar{E}$ = ACD-ABD = 0

d_2E_t = 0

The equilibrium is neuter .

For chart in fig. (9-c)



$d_2\bar{E}$ = ACD-ABD = - ABC < 0

d_2E_t= ABC-CBE = - ABE < 0

The equilibrium is unstable.

Conclusions

The stability analyis of an olonomous system, whose components are stressed axially and are typified by positive and negative rigidities is led back to the study of function $d_2\ \bar{E}$, that is the quadratic form associated to the matrix of the differential rigidities within the hyperquadrant of the positive $d\bar{\delta}$. If, within this boundary, $d_2\bar{E} > 0$ then the equilibrium is stable: on the contrary it is neutral or unstable.

By the avail of the matrices theory [14] some conclusions can be drawn. If the quadratic form, associated to the matrix of the rigidities, is definite positive, such it will be also in the hyperquadrant $d\bar{\delta} > 0$: therefore the result is $d_2\ \bar{E} > 0$. Hence the equilibrium is stable. Instead if the quadratic form is definite negative, in like manner, $d_2\bar{E} < 0$: the equilibrium is then unstable. The same holds true if the quadratic form is semi-definite negative: the range of the matrix can never be less than one, and thus the indifference direction, at the limit, can only occupy a subspace of the positive hyperquadrant, the quadratic form in the complementary subspace remaining negative.

More complicated the question presents itself where the quadratic form is semidefinite positive or indefinite: In the first case $d_2\ \bar{E} > 0$ or $d_2\ \bar{E} = 0$, in the second case $d_2E \gtrless 0$ or the intermediate cases. The research of an algorism for the solution of this problem will be the subyect of a forthcoming information.

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SUMMARY

The stability analysis of an olonomous system, whose components are stressed axially and are typified by positive and negative rigidities is led back to the matrix of function $d_2 \bar{E}$, that is the quadratic form associated to the matrix of the differential rigidities within the hyperquadrant of the positive $d\sigma$. If, within this boundary, $d_2\bar{E} > 0$ then the equilibrium is stable: on the contrary it is neutral or unstable.

RÉSUMÉ

L'analyse de la stabilité d'un treillis, dont les barres ne subissent que des efforts axiaux, est déduite à l'étude de la fonction $d_2 \bar{E}$. Si $d_2 \bar{E} > 0$ le système est stable, sinon, il est neutre ou instable. Avec l'aide de la théorie des matrices [14] on peut tirer des conclusions sur la forme quadratique associée à la matrice. Le problème est plus ou moins simple, selon que cette forme quadratique est définie positive ou négative, ou semi-définie négative, ou alors si elle est semi-définie positive ou indéfinite. Ces derniers cas seront traités dans une information ultérieure.

ZUSAMMENFASSUNG

In diesem Beitrag wird die Stabilität unter Berücksichtigung der Traglast an einem Fachwerk, deren Stäbe achsialer Kräfte unterworfen sind, untersucht und mit Hilfe der Matrizenrechnung die Fälle des stabilen, labilen oder instabilen Gleichgewichts beschrieben.