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**I**  
**Sécurité**  
**Sicherheit**  
**Safety**

**Ia**

**Etude critique des critères de sécurité et de leurs  
fondements conceptuels.**

**Kritische Betrachtung der Sicherheitskriterien und  
ihrer grundsätzlichen Auffassungen.**

**Critical Appraisal of Safety Criteria and Their  
Basic Concepts.**



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# DISCUSSION PRÉPARÉE / VORBEREITETE DISKUSSION / PREPARED DISCUSSION

## Model Analysis for Structural Safety and Optimization

Analyse sur modèles de la sécurité et de l'optimisation des structures

Modelluntersuchung der Bausicherheit und -optimierung

GUIDO OBERTI

Prof.

Italy

### 1.- Foreword.

- a) In a short note presented at the Rio de Janeiro Congress of the IABSE in 1964 I stated that the possibility of analyzing on models, even to failure, of large structures, particularly plain or reinforced concrete structures, has long been proved by me in a great number of cases.

In fact, a model study under elastic conditions furnishes the values of the prototype stresses under working load, which is important for several reasons. Firstly, the results obtained, unaffected by the assumptions and limitations which impair the classical methods of calculation, can profitably be compared with those supplied by these methods. Secondly, it is not hard to solve on models unusual three-dimensional problems, contrary to what is the case with the conventional analytical procedures both because of extreme complexity (only partly reduced by the finite element method) and difficult mathematical schematization of accurate boundary conditions.

Extension beyond the elastic range is still always invaluable to the structural engineer as it may enable him to locate possible weak points in the design and thus assist in securing greater safety and optimization.

Models may be classified in elastic (tested within the elastic range only), structural (carried to failure) and geomechanical (when the foundation influences the structural performance).

Present trends, based on experience at ISMES, are:

- increasing emphasis on structural models;
- constant improvement of model materials to better suit the aims pursued;

- growing interest in thermal stress investigation, especially for concrete dams and reinforced (prestressed or not) concrete vessels of nuclear reactors;
- dynamic testing on large shake tables and marked concern for earthquake effects.

b) Theme I has been treated by the general reporters prof. A. M. Freudenthal and prof. J. Courbon.

In a first theoretical and critical paper regarding topic 1a, prof. Freudenthal deals with the evaluation of overall structural safety based on probabilistic criteria related to the operating loads, which seems fit for statically determinate structures only. In a second paper concerning topic 1b and also of a theoretical probabilistic nature, the same author discusses the possibility of predicting ultimate safety based on the physical properties of the materials and their influence at failure. None of the papers mentions structural model analysis.

Prof. Courbon's paper treats with topic 1c of Theme I. It concludes by mentioning, all too briefly, the great services rendered by model studies in the design of dams, thin shells and shields of nuclear reactors.

I, therefore, believe it of use to outline, the present-day possibilities of model analysis in evaluating the safety degree of large statically indeterminate structures.

c) Model investigation primarily concerns statically highly indeterminate structures and may be regarded as:

- I) a modern method of stress analysis;
- II) a tool for failure load evaluation.

In any case, it is possible to consider or predict the statistical dispersion of the operating loads and of the structural resistance of the prototype material.

In case II), when several models are tested, it is possible to evaluate the ultimate carrying capacity  $R$  of the structure for each type of load  $S$ , so that the model functions as a tool for determining the overall safety factor  $\gamma$ .

This factor may vary for each type of structure, depending on the probabilistic possibility assumed for the operating loads and the structural resistance, associated with a definite risk of failure.

Thus, for concrete dams, the loads are practically known (excepting those for earthquake-resistant design), and the uncertainties about the concrete resistance are quite small. The highly redundant type of these structures generally reduces the importance of the concrete strength dispersion. The safety factor in this case, therefore, serves rather as a coefficient of security against the insecurity of the analytical results, especially in relation to the real properties of the rock foundation.

## 2.- Actual possibilities of model analysis.

- a) Elastic models are based on linear elasticity (Hooke's law) and, hence, a superposition of effects is allowed. They also permit to proportionately modify the loadings so as to obtain the most suitable testing conditions. In particular, it is possible to operate at strains that are amplified with respect to those required by similitude (which demands that the strains in the prototype and in the model be the same).

Elastic models, widely used in "stress-analysis", may be divided in two groups.

The first group concerns plane elastic structures, and for them the deformer, photoelastic and Moiré methods are predominant.

Deformers are based on the well-known reciprocal theorems (Maxwell, Betti, Müller-Breslau). Photoelasticity is a first-rate research method, most used in structural engineering laboratories. The Moiré method is primarily used in flat slab investigations.

However, it should be observed that the importance of these methods has lately decreased due to the use of computers in solving problems relating to plane elastic structures.

The second group deals with three-dimensional models. In statical tests the loading equipment is usually made up of calibrated weights or hydraulic jacks, the pistons of which react against an external rigid frame; the loads are applied to the model through wooden cork-soled pads. Strain gages, ordinarily applied to the surface of the model, are used for measuring the direction and magnitude of the principal strains.

Young's modulus and Poisson's ratio of the model material are determined as usual, the former by tensile and flexural tests and the latter by torsional tests. The material may quite differ from that of the prototype, provided it obeys Hooke's law and its Poisson's ratio is similar to that of the structure. The model then functions as a "stress computer", and its results may be compared with the theoretical ones.

For elastic models, ISMES has recently succeeded in using epoxy resins mixed with various aggregates. They permit obtaining a wide range of elastic moduli in accordance with the requirements of each case, and stress-strain relationships that are similar even when the stresses are high.

- b) Structural models are best made of the same material as the prototype. This is generally possible for steel or prestressed concrete structures when suitable scale (1:4-1:20) models are used. But for very large structures, such as concrete dams, we are forced, also for economic

reasons, to adopt greatly reduced scales (1:30-1:100) and hence to use model materials whose mechanical characteristics are reduced compared with those of concrete in accordance with similitude requirements.

For structural models, I have long since used special materials simulating the mechanical properties of concrete, by introducing the technique of "wet models" (with a waterproof coating) practically free of internal stresses.

The tests are then divided into two successive stages. In the first stage, called "normal load tests", the deformations are investigated for values close to similitude conditions (that is,  $\epsilon = \epsilon'$ ) under loads corresponding to those of the structure in operation.

The second stage concerns ultimate load tests and the transition to them is gradual. The ratio of the highest actually supported load to that of the design load is generally assumed as overall "factor of safety".

This ratio can easily be referred to all the operating loads equally or differently increased following a probabilistic coefficient applied to each independent load. In the case of statically high indeterminate structures it differs from the classical ratio of ultimate to working stress, and its meaning is greater since it takes into account the bi- and triaxial strength of the material under stresses in different directions and the plastic adjustment.

One can by expedients increase on the model solely the loads which in the prototype may rise through extraordinary action. Such are wind load for skyscrapers and water pressure for dams. The horizontal loads alone may undergo increases of consequence for the stability of these structures. In setting up a model study it is, therefore, of basic importance that the factor of safety shall be evaluated as simultaneously affected by:

- loads having a fixed value (dead load);
- loads which may increase with respect to their ordinary value (wind effect);
- actions the occurrence of which is only probable (earthquake).

In practice, when the so-called "weight" of each of the above phenomena has been established, one can obtain

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(°) It is advisable to secure, through repeated loading cycles, non-elastic displacements (settlement of the foundation, adjustment and opening of joints, localized plasticity) which are likely to occur since the first loading in order to obtain an elastic and uniform model performance fit for repeated measurements and controls. This permits obtaining the stresses, displacements and structural behavior of the prototype under working conditions.

the factor of safety by experimentally increasing all the loads up to the failure of the model, considering the "weight" corresponding to each type of load

Therefore, not only one but a number of factors of safety can be secured, each of which corresponds to a given set of phenomena the influence of which is to be analyzed.

- c) Geomechanical models investigate structures on foundations whose equilibrium conditions may affect the safety of the structures, as is the case of dams, large bridges and power or highway tunnels.

The stability of block foundations has lately been simulated and studied on geomechanical models the characteristics of which had conveniently been schematized on the basis of geognostic tests.

It may also be pointed out that in-situ and laboratory investigations of the geomechanical features of the rock and soil mass are increasingly used and recommended as an aid to model studies.

The models, therefore, must faithfully simulate the rock and soil conditions and its mechanical properties. The tests are usually carried to failure.

These investigations are to be considered as basic when extensive discontinuities (faults, cavities) or a pronounced anisotropy (stratifications and diaclasses) are present in the rock mass, especially when sliding or least-resistance planes may develop or, more generally, when large low-strength block formations are involved.

In these models, cohesion and angle of friction must also be faithfully reproduced. The difficulty encountered in establishing the true values of the angles of friction makes it in the modeling conservative to assume reduced values which are still within the approximation allowed by field tests.

The prototype and model strains have to be the same and, therefore, the scale ratio must be reduced. The model materials then shall have high densities and low mechanical properties (i.e., very low moduli of elasticity, yield-point and ultimate loads) in order to comply with similitude.

### 3.- Assessment of (structural) safety at the design stage.

The adoption of model techniques is firstly of considerable importance at the design stage of structures, especially if these are statically highly indeterminate.

Structural safety can then be evaluated by modern probabilistic criteria as suggested in the Freudenthal report when:

- the expected statistical dispersion of the loads, i.e. of the external or operating forces  $S$ , is taken into consideration by determining the dimensions of the



prototype on the basis of a force  $\gamma_s \cdot S$  (where  $\gamma_s$ , the load safety factor, is  $>1$ ) and adopting equivalent working forces in the model;

- the statistical dispersion of the strength  $\sigma_R$  of the prototype material is taken into account by assuming an ultimate load, or a yield point, equal to  $\sigma_R / \gamma_R$  (with  $\gamma_R$ , the rupture safety factor,  $>1$ ) and comparing the highest internal stresses furnished by the model at that value.

The model then becomes a very efficient tool for a "structural analysis".

As typical examples I shall mention:

- the static and dynamic investigations carried out on an elastic model of the Polcevera viaduct, of the Maracaibo bridge type, designed by prof. R. Morandi (fig. 1);
- the far more elaborate analysis, made particularly on a structural model carried to failure, of the new San Francisco Cathedral designed by prof. P.L.Nervi (figs. 2, 3);
- the study of the safety degree of the Kurobe IV Dam and its foundation (figs. 4, 5, 6).

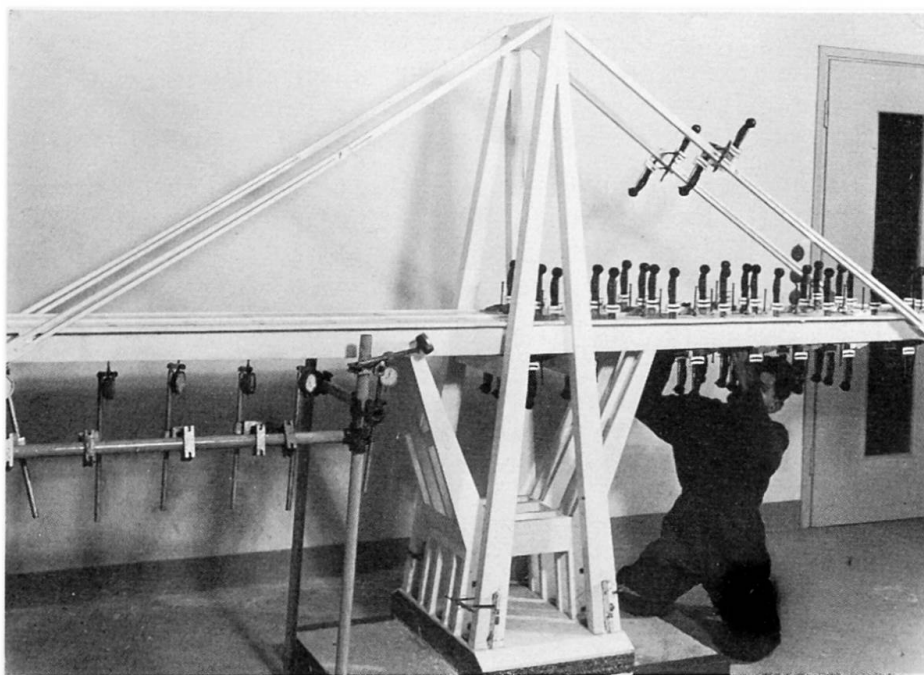


Fig. 1 Polcevera viaduct, Italy. Elastic model:  
Scale 1:50

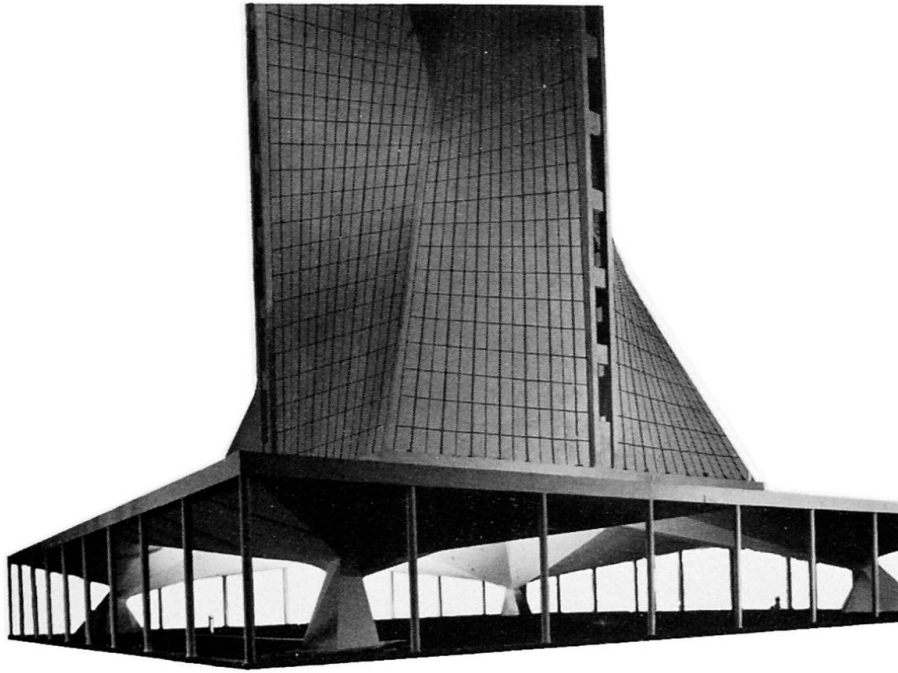


Fig. 2 San Francisco Cathedral, U.S.A.  
General view of model. Scale 1:15.

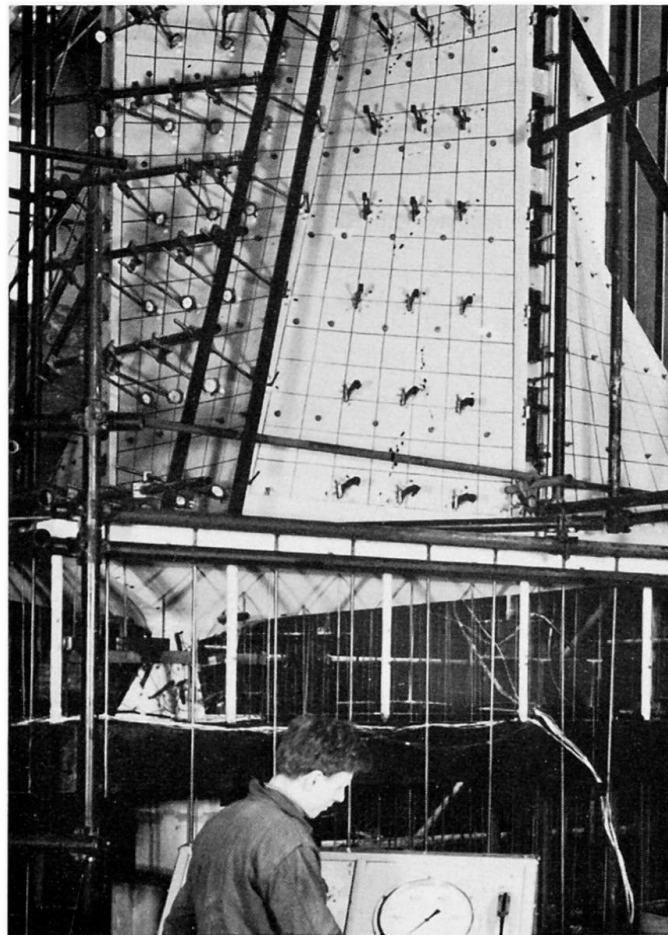


Fig. 3 San Francisco Cathedral.  
Structural model under failure tests.



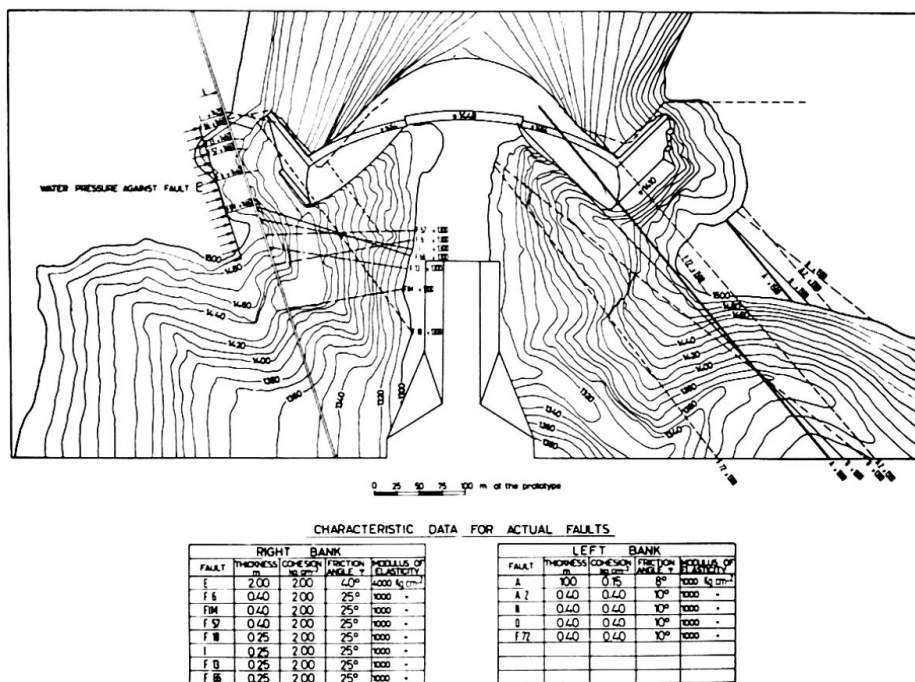


Fig.4 Kurobe IV Dam, Japan.  
General Layout of geomechanical model.

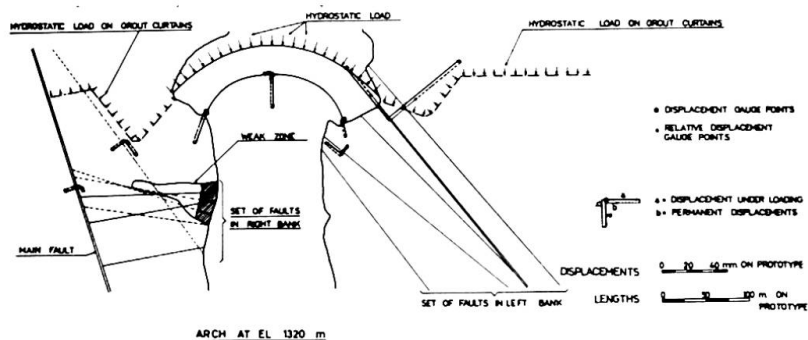


Fig.5 Kurobe IV Dam. Displacements recorded at a horizontal section (el. 1320 m).

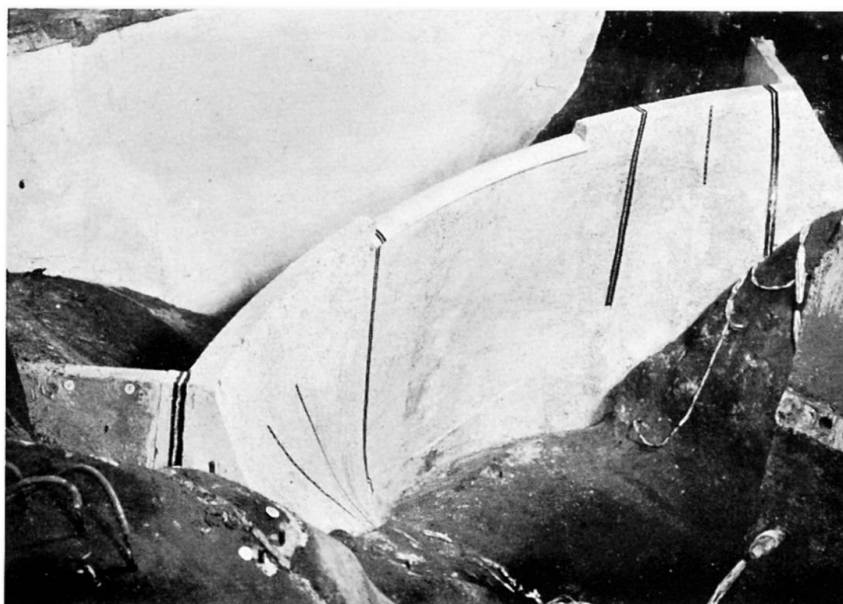


Fig.6 Kurobe IV Dam. Structural model at failure, showing the opening of joints.

#### 4.- Evaluation of the safety degree of an already existing structure.

Models may be of great assistance not only when structures are being designed but also when the stability and safety degree of erected works are being checked. This is particularly true when verifying large structures which have undergone statical conditions unpredicted or unpredictable at the design stage.

Here, too, I shall briefly illustrate some examples in which testing on models has yielded highly significant and conclusive results, with particular regard to the safety degree of the structure.

After a few years of operation, extensive subhorizontal microcracks were found at the upstream face of a large arch-gravity dam completed in 1958. The influence of these cracks on the structural performance and safety of the dam at full reservoir has been investigated on a large structural model in which the number and pattern of the microcracks had faithfully been reproduced (figs. 7, 8).

Interesting tests were also conducted on a 1:4 scale model to verify the compression safety degree of the main columns of the Cathedral in Milan (fig. 9). The two materials (Candoglio and Serizzo marbles), of highly different moduli of elasticity, and the geometry of the individual blocks were identical with those of the prototype (fig. 10). The pattern of the stresses in the masonry dome carrying the main spire of the Cathedral has then been analyzed on a large elastic model (fig. 11).

The effect of the horizontally stratified bedrock anisotropy on the stability of a recently constructed dam was investigated by means of geomechanical models. The various expedients devised to raise the safety degree of the dam-foundation unit were also examined (fig. 12).

Finally, the model tests carried out for the double-curvature arch Vajont Dam should be mentioned. As is known, this dam has brilliantly withstood the extraordinary sliding of Mount Toc into the partly filled reservoir and is now sustaining the enormous asymmetric mass of slide material (fig. 13). After the disaster, model studies were conducted to determine the safety degree of this imposing structure under the present exceptional live load (fig. 14).

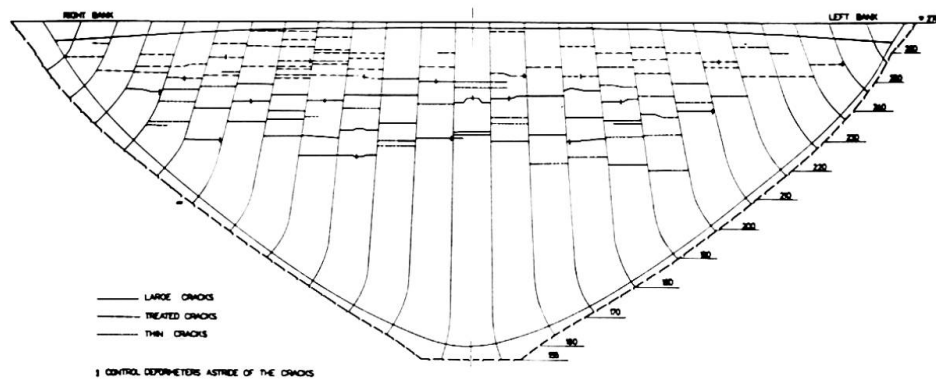


Fig. 7 Flumendosa Dam, Italy. Microcracks on upstream face of model.

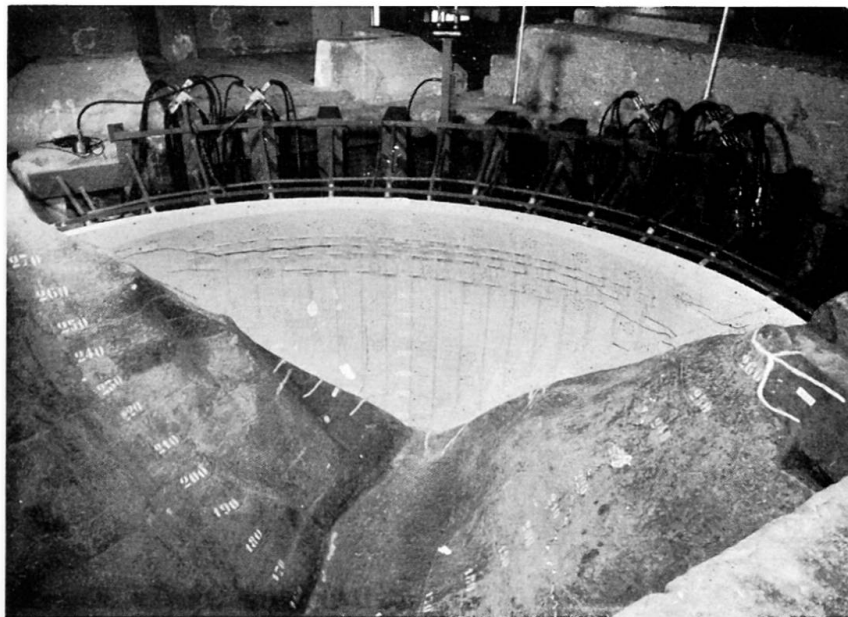


Fig. 8 Flumendosa Dam. Downstream view of model under test.

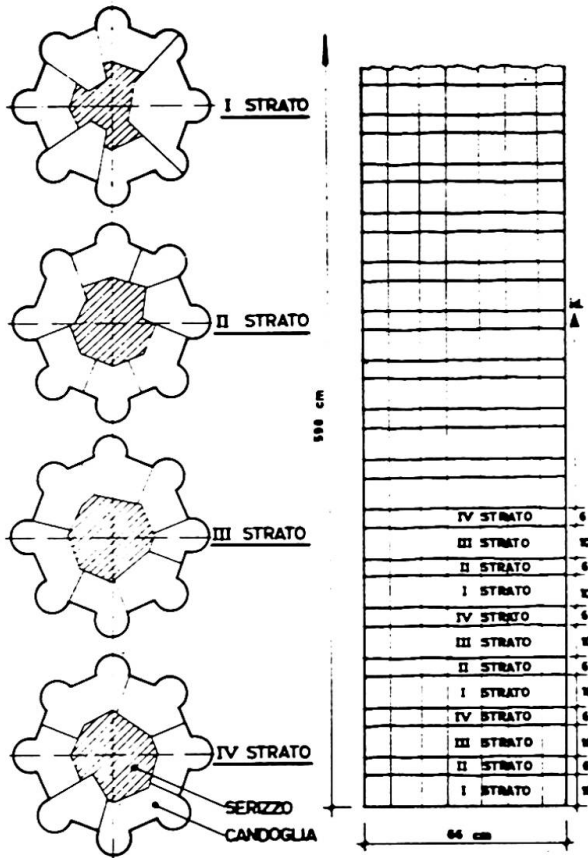


Fig. 10 Milan Cathedral. Cross-sections of columns.

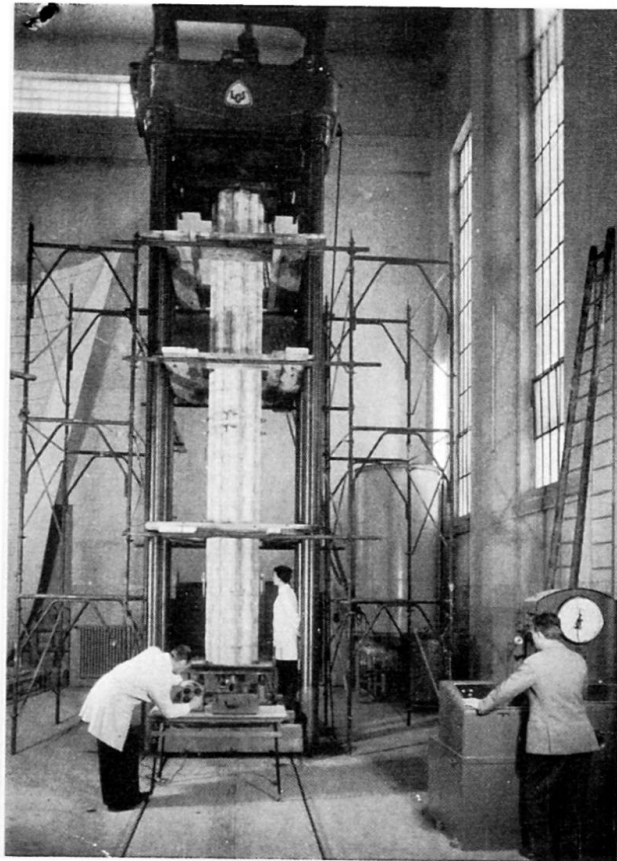


Fig. 9 Milan Cathedral. Model of column under test. Scale 1:4.

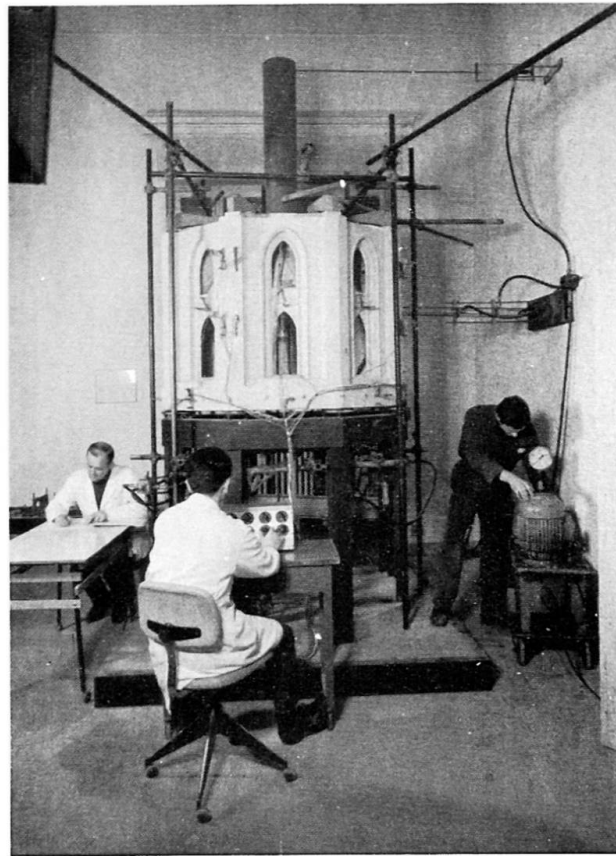


Fig. 11 Milan Cathedral. Elastic model of masonry dome.

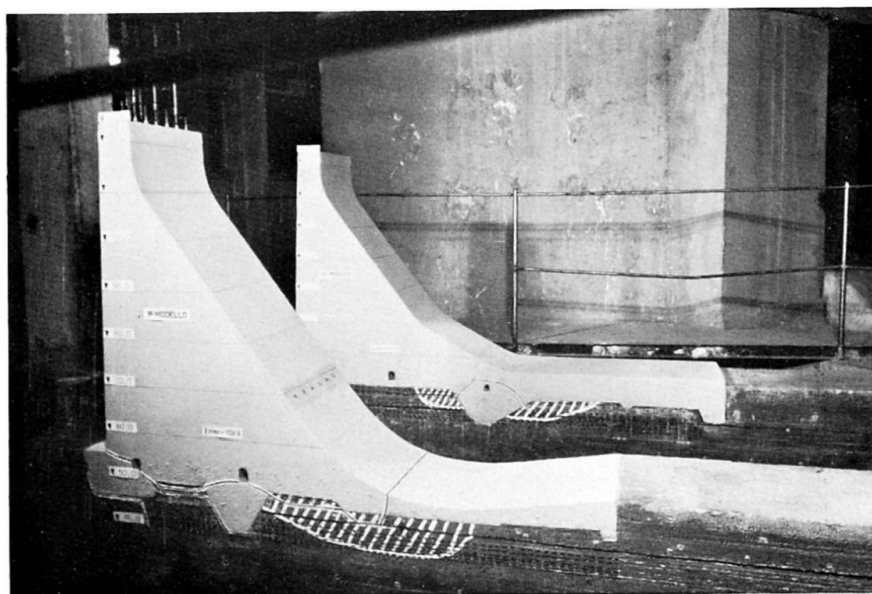


Fig. 12 Mequinenza Dam, Spain. Plane model on geomechanical foundation.

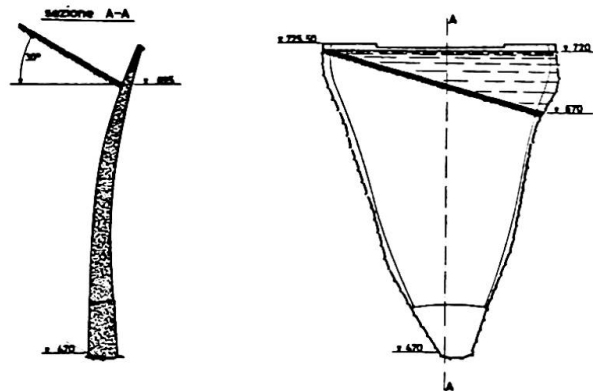


Fig. 13 Vajont Dam. Asymmetric slide material acting on upstream face.

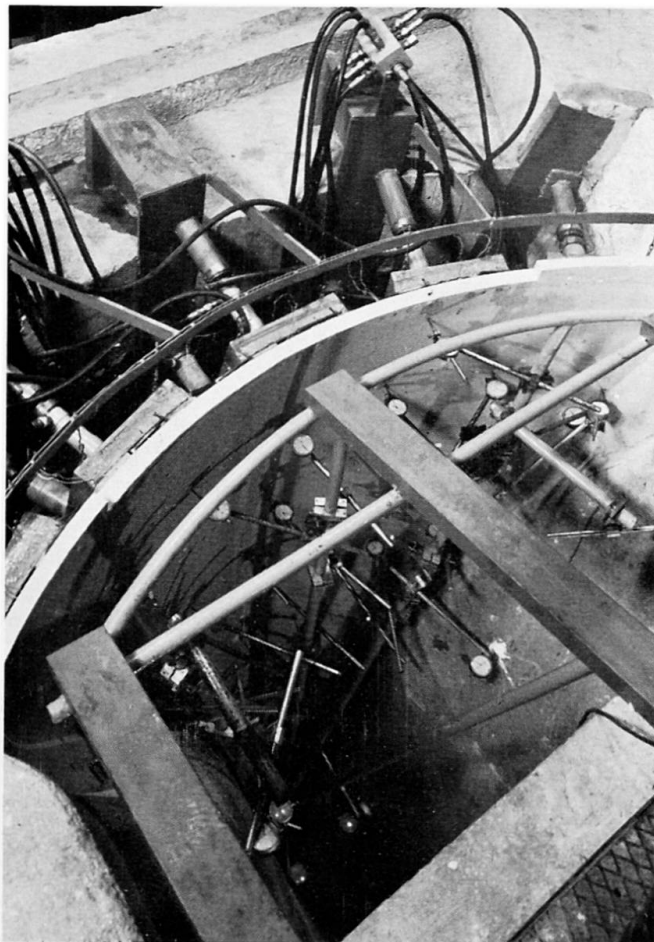


Fig. 14 Vajont Dam. Model under asymmetric load test.



## 5.- Conclusions.

It is believed that the above brief outline has clearly shown the contribution given, and which can be given, by testing on models when evaluating the safety degree of a structure at its design stage and after construction.

When it is assumed, in accordance with modern trends, that a rational determination of safety involves the adoption of an acceptable risk of failure, a design procedure for uniform safety, and hence optimization, can be based on structural model investigation.

Finally, the arduous problem of structural reliability of statically indeterminate structures, related to the failure mechanisms depending on the consecutive loads mentioned also by prof. Freudenthal at the end of his paper, can satisfactorily be solved through a judicious adoption of the present model test technique.

### SUMMARY

After a short introduction the paper outlines the actual possibilities of evaluating the safety degree of a structure by testing elastic, structural and geomechanical models.

The evaluation may concern: 1) structural safety at the design stage; 2) safety degree of an existent structure and of one operating under extraordinary conditions.

The importance of model investigation particularly for the optimization of statically highly indeterminate structures is then emphasized.

### RÉSUMÉ

Après quelques mots d'introduction le rapport souligne les possibilités actuelles données par les différents types de modèles (élastiques, structureaux, géomécaniques) pour l'analyse de la sécurité des grandes structures.

On considère après: 1) l'examen du coefficient de sécurité dans la phase du projet de l'ouvrage; 2) l'évaluation du degré de sécurité d'un ouvrage déjà achevé ou soumis à des actions exceptionnelles.

Le rapport termine en soulignant les possibilités des modèles surtout pour l'étude et l'optimization des structures hautement hyperstatiques.

### ZUSAMMENFASSUNG

Nach einer kurzen Einleitung werden die wirklichen Möglichkeiten einer Untersuchung des Sicherheitskoeffizienten eines Bauwerkes an elastischen, strukturellen und geomechanischen Modellen beschrieben.

Der untersuchte Sicherheitskoeffizient kann sich auf den Entwurf, ein bestehendes oder ein unter ausserordentlichen Verhältnissen befindliches Werk beziehen.

Die Wichtigkeit der Modelluntersuchungen für die Optimisierung statisch hochunbestimmter Werke wird nachdem besonders unterstrichen.



## **Critical Appraisal of Safety Criteria and their Basic Concepts**

Etude critique des critères de sécurité et de leurs fondements conceptuels

Kritische Betrachtung der Sicherheitskriterien und ihrer grundsätzlichen Auffassungen

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The subject of structural safety is primarily a matter of common sense and not of mathematics. This does not mean that mathematics should be excluded when safety standards are being established, but it means that its role must be subservient. The conclusions of a most erudite mathematical derivation are only as valid as the underlying assumptions. With this thought in view the writer intends to examine closely some of the propositions forming the basis of the author's mathematical development.

The author associates safety of structures with the concept of probability of failure and he outlines the method of derivation of the necessary relations based on this principle. He is careful however to point out that his formulae are not suitable for practical use for the reason of absence of the pertinent statistical data characterising the random variation of the relevant factors.

Furthermore he freely admits the presence of causes of failure unrelated to random factors and even holds mistakes in design of details as the usual cause of failure. In the light of these admissions one cannot see the virtue of the formulae associating failures solely with the random factors, seldom if ever responsible for the actual failure, and leaving out of consideration, of necessity, the really significant non-random causes.

The author's reference to the alleged use of the failure oriented probabilistic concept of factor of safety in the design of aeroplanes poses an interesting question as to the relevancy of this concept in the design of bridges and buildings. Once a person steps into an aeroplane the risk of failure and death, however remote, is tacitly accepted, and so it is not illogical to associate the design of the aeroplane structure with a probability of failure. The situation is however different in case of buildings and bridges. With his probabilistic approach the author in effect proposes an intentional reduction of safety, however small, compared to the one implied in the conventional design. Neither the society in general nor the engineering profession in particular would accept this idea. The present practice is, and hopefully will always remain, that the building should be designed as safe as humanly possible. This does not insure an absolute safety, because life is full of hazards. Factors responsible for these hazards are mostly of a non-random nature and unpredictable, although some of them, such as tornadoes

and earthquakes, excessively severe for a given region, are akin to the phenomena normally incorporated in design. It is no more rational to provide for these overviolent actions than for the acts of war, riots, collision with aircraft, gas and chemical explosions and other factors always left out of consideration.

The kind of reliability required for the design of structures seems to be provided adequately by the commonly used factor of safety covering the uncertainties and faults of all types, i.e. of design, construction, loads, materials and operation. This factor expresses the best collective judgment of engineering profession, and its value is subject to revision with improvement of all aspects of engineering practice.

The concept of failure as an integral part of the probabilistic theory, and several aspects of it, as used by the author, warrant close examination. A natural question is, how to analyze a particular structure for failure. The theory of ultimate or limit design gives in some cases an answer to this question. But this theory is highly controversial (50), and the acceptance of its answer means the endorsement of the theory. In other words, an expert on probability, and normally not an expert on structural theory, makes a decision for the designer, that of the two conflicting theories the elastic and the plastic, he must accept the latter.

Limit design procedure, right or wrong, is available only for low flexural frames. What should one do for the multitude of structures of other kinds? Wait until such solutions by ultimate theory become available, even if one has no confidence that they may be forthcoming?

No distinction is made in the author's theory between the actual physical failure and the functional failure, i.e. an excessively large deformation. This implies that in the author's view it does not matter whether people get killed in the collapse of a probabilistically designed structure or are merely inconvenienced by a large deformation,—a proposition, which is not likely to meet a ready acceptance.

A reader would find difficulty in following the author's argument that failure of a single member signifies failure of the whole structure irrespective of whether the latter is statically determinate or indeterminate.

A major impression which one gathers from the discussion of the probabilistic theory of failure is apparent lack of appreciation by its supporters of a bewildering multiplicity of causes affecting vitally the reliability of a structure. The writer wishes to illustrate this point by two examples.

Comparative stress analyses were made by the writer and his colleague (51) of a reinforced concrete barrel roof by two different methods: firstly, the theory of finite element, a new and highly effective tool of structural analysis, and secondly, by the equations of elasticity given in the Manual of Engineering Practice 31 of the American Society of Civil Engineers. Some significant stresses determined by the two methods differed greatly. How then should the choice between different discordant but still admissible methods be made by a probabilistic designer? By the way of explanation it

may be pointed out, that in the present design practice, once the disagreement of the existing methods is recognized, consensus is reached in a course of time leading to the acceptance of one method in preference to the other. In the meantime the factor of safety covers the uncertainty.

The situation in the example considered is however much more complicated than mere disagreement of the two sets of numerical results. Both methods of analysis were based on constant moment of inertia (i.e. an uncracked section), constant values of the modulus of elasticity and Poisson's ratio and the absence of creep and shrinkage. These assumptions are obviously not true. The designer would allow for these unknowable factors by judgment based on experience. Design is an art as well as a science, and is more than a mere substitution of numerical values into complicated probability formulae.

The other example is borrowed from the writer's discussion of a recent paper on probabilistic theory by the same author (52) .

"A collapse of an important bridge in the course of erection several years ago (accompanied by loss of life) was found to have been caused by the wrong design of a detail of the erection structure, accentuated by the contributing factors, including an unfortunate and destructive combination of the yielding of steel and crushing of plywood (a phenomenon neither described nor even recognized before), an inadequacy of prescribed allowable stress in the significant area, and two elementary blunders in calculation. Such nondescript errors would baffle any classification, yet they are real and not infrequent, although they are usually less drastic and seldom lead to failure",

In conclusion the writer recapitulates the reasons for his unqualified rejection of the probabilistic theory of safety of structures involving human occupancy.

1. The concept of the probability oriented factor of safety is unacceptable in principle.
2. The factors which usually cause failure are not of a random type.
3. The data for evaluation of parameters characterizing the random type factors are mostly unavailable.
4. The failure causing factors are so numerous and varied that they defy any classification and codification.
5. The value of the intensity of a given load pattern causing failure of a given structure is usually unknowable by a method of structural analysis and is questionable when such analysis is available.
6. Distinction between physical and functional failures and between determinate and redundant structures results in further difficulties for a probabilistic designer.

7. The usual concept of the factor of safety of the conventional elastic design is the best one available.

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- (50) A. Hrennikoff. Plastic and Elastic Designs Compared. Preliminary Publication. Seventh Congress, Rio de Janeiro, 1964. International Association for Bridge and Structural Engineering.
- (51) A. Hrennikoff and S. Tezcan. Analysis of Cylindrical Shells by the Finite Element Method. International Association on Shell Structures. Symposium. Leningrad, U.S.S.R. 1966.
- (52) A. Hrennikoff. Discussion. Analysis of Structural Safety by A. Freudenthal, J. Garrelts and M. Shinozuka. Journal of the Structural Division of A.S.C.E.

#### SUMMARY

The writer rejects the probabilistic method of design of structures involving human occupancy, because (1) it is unacceptable in principle, (2) leaves out of consideration the really significant non-random causes of failure, (3) is based only on a few random factors whose characteristic parameters incidentally are mostly unavailable and (4) for most structures, the condition of failure may not be identified by any existing method of analysis.

#### RÉSUMÉ

L'auteur rejette la méthode de projection de constructions qui se base sur la probabilité et tient compte de l'occupation humaine.

- 1 Le principe même de la méthode est inadmissible
- 2 Elle néglige les causes de ruine non-accidentelles vraiment importantes
- 3 Elle se base uniquement sur quelques facteurs aléatoires dont les paramètres caractéristiques sont le plus souvent inutiles
- 4 Pour la plupart des constructions, les conditions de ruine ne peuvent être déterminées par aucune méthode de calcul existante

## ZUSAMMENFASSUNG

Der Autor verwirft die wahrscheinlichkeitstheoretische Entwurfsmethode für Gebäude, die von Menschen bewohnt werden, weil sie

erstens im Prinzip unannehmbar ist,

zweitens die tatsächlich wichtigen, nicht zufälligen Bruchursachen auslöst,

drittens auf wenigen zufälligen Grössen gegründet ist, deren charakteristischen Parameter übrigens meist unbrauchbar sind,

und schliesslich viertens, weil für die meisten Bauwerke die Bruchlast mit keiner bestehenden analytischen Methode bestimmt werden kann.

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## **Critical Appraisal of Safety Criteria and their Basic Concepts**

Etude critique des critères de sécurité et de leurs fondements conceptuels

Kritische Betrachtung der Sicherheitskriterien und ihrer grundsätzlichen Auffassungen

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In his brilliant survey of the present status of structural safety problems Professor Freudenthal makes clear that engineers are not designing structures close enough to the "state of art" limit, that the rational approach to the problem of safety has to be a probabilistic one, and that absolute safety is no more than a convenient fiction.

The key to a rational approach to structural safety is in his own words the concept of an "acceptable risk of failure". But most engineers, because they believe they can or they have to design absolutely safe structures, are reluctant to accept such a concept.

The difference between the attitude of accepting or not accepting a risk of failure, be it a small one, is not an academic question, because structures will be designed quite differently depending on whether one does or does not recognize the impracticability of building absolutely safe structures. The consequences of these two opposite attitudes seem worthwhile emphasizing.

If the existence of risks is to be recognized, accepted and taken into consideration in the design of engineering structures, instead of trying to have uniform safety in all elements of a structure – which is an ideal recommended by several authors – one has to reduce the strength of the elements of which the failure will have less costly consequences, with a view to reinforcing those elements of which the failure would have costlier consequences. Such criteria will enable the design to be improved without increase in cost.

If the existence of risks is to be taken into consideration, one has to adjust the strength of the whole structure to the consequences of possible failures.



This implies building a dam, if located upstream of a town, stronger than one located downstream, even if both dams could otherwise be built perfectly alike.

If the existence of risks is recognized and accepted, the structure should be designed so as to reduce as much as possible the consequences of accidents. There will even be instances where it may be convenient to increase the probability of failure of a structure so as to reduce its cost, the savings being used to minimize the consequences of a possible failure. This will be the case with dykes against floods and sea invasions, where transverse dykes are built using the money that could otherwise be used to increase the height and reinforce the main dyke. The function of the transverse dykes is the reduction of the area flooded in case of failure of the main dyke rather than direct protection against sea invasion.

If the existence of risks is to be recognized and taken into consideration, structures will have to be designed so as to fail in the less inconvenient way. In some cases this will imply the use of devices similar in function to fuses, for instance when a lighter and lower dam is built on a secondary valley as a protection to a big earth or rock-fill dam on the main valley.

In spite of Professor Freudenthal's well presented arguments against redundant elements, the presence of such elements can, in some particular cases, contribute to increase the safety of the structure. Not only can the failure of redundant elements give warning to halt operation and avoid serious consequences of accidents, but in some other cases the presence of redundant elements will avoid complete collapse, that would, otherwise, have catastrophic consequences.

Some structures are intended to absorb energy rather than to hold forces. The amount of energy consumed in the destruction of redundant elements can, at least in some cases, be sufficient to save the structure. This is apparently the main reason why ships are always moored with a large number of redundant cables, instead of with a few strong cables.

## SUMMARY

The need to design structures accepting the existence of risks and taking into account the possible consequences of accidents may, or may not, be recognized. The practical implications of one or the other attitude are quite different.

If the existence of risks is recognized, the adoption of an uniform safety factor for all the elements of a structure, and the adoption of the same safety factor for structures submitted to the same loads but whose failure can have different consequences, should be discontinued.

It is also pointed out how redundant elements can contribute to increase the overall safety of a structure.

## RÉSUMÉ

Reconnaître ou ne pas reconnaître l'impossibilité de sécurité absolue quand on projette une structure peut avoir des conséquences pratiques très différentes.

Si cette impossibilité est reconnue, on doit choisir le coefficient de sécurité de chaque élément d'une structure, et de chaque structure en elle-même, d'après les conséquences des possibles accidents.

L'influence des éléments superabondants sur la sécurité d'une structure hyperstatique est aussi discutée.

## ZUSAMMENFASSUNG

Sehr verschiedene praktische Folgerungen beruhen auf dem Erkennen bzw. Nichterkennen der Möglichkeit, Tragwerke mit Ausschluss aller Risiken zu entwerfen.

Wenn das Bestehen von Risiken erkannt wird, muss der Sicherheitskoeffizient für jedes Element des Tragwerkes sowie das Tragwerk an sich nach dem Umfang der Folgen eventueller Unfälle gewählt werden, anstatt der üblichen Wahl von genormten Sicherheitskoeffizienten.

Der Einfluss der Verwendung zusätzlicher konstruktiver Elemente zur Erhöhung der Sicherheit von statisch unbestimmten Tragwerkssystemen wird ebenfalls beleuchtet.

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**Some Safety Problems**

Quelques questions de la sécurité

Einige Fragen zur Sicherheit

**E. MISTÉTH**

Budapest

**1./ Designations**

|                             |                                                                                     |
|-----------------------------|-------------------------------------------------------------------------------------|
| $R$                         | Internal breaking forces and moments                                                |
| $S$                         | Internal forces provoked by load                                                    |
| $Y = R - S$                 | The Basler reserve                                                                  |
| $\xi, \eta$                 | Probability variables                                                               |
| $\xi(t), \eta(t)$           | Stochastic processes                                                                |
| $a = E(\xi)$                | Expectation value                                                                   |
| $s = D(\xi)$                | Deviation                                                                           |
| $v = \frac{s}{a}$           | Relative deviation, variation coefficient                                           |
| $\mu_r = E[(\xi - a)^r]$    | Central moment of the $r^{\text{th}}$ order                                         |
| $f = \frac{\mu_3}{s^3}$     | Asymmetry                                                                           |
| $c = \frac{\mu_4}{s^4} - 3$ | Excess                                                                              |
| $h = \frac{\mu_5}{s^5}$     | Asymmetry of the fifth order                                                        |
| $\frac{1}{k}$               | Risk                                                                                |
| $t$                         | Time                                                                                |
| $T$                         | Lifetime of the construction                                                        |
| $q$                         | Interest factor                                                                     |
| $W$                         | Cross sectional quantity corresponding to the nature of internal forces and moments |
| $\sigma$                    | Stress corresponding to the nature of internal forces and moments                   |

$\eta = G(\xi_1, \xi_2, \dots, \xi_n)$  Functional relation of independent probability variables

|        |                                                                                                      |
|--------|------------------------------------------------------------------------------------------------------|
| $m$    | Independent variable of the standardised distribution function                                       |
| $C(k)$ | Cost of rebuilding /the bearing element/                                                             |
| $L(k)$ | Annual maintenance cost of the construction /the bearing element/                                    |
| $Q$    | Sum of the damage caused by the ruin of the construction /the bearing element/, profit lost included |

## 2./ Raising of the problem. Methods applied so far in calculating of dimensions

In dimensioning engineering structures for stability it is most essential to determine safety. The first question to be raised is whether an objective standard of safety can be found and what is the most economical magnitude thereof. Thus the general question of dimensioning is this: In what dimensions should be designed the bearing structure of an engineering construction at a time  $t = 0$ , if the construction is being designed for a lifetime  $t = T$ , with rebuilding cost of the bearing structure being  $C$ , and the annual maintenance cost of the bearing structure being  $L$ , sum of damages incurred by the ruin of the bearing structure, profit lost included, being  $Q$ .

The classical dimensioning specifications present safety in terms of the magnitude of allowable stress. Allowable stress is an empirical value: it is a quotient of breaking strength and safety. Present time specifications are threefold.

Into the first group come those specifications in which safety manifests itself in the measure of allowable stresses and the grouping of loadings. These specifications show, e.g. three groupings as to the combinations of loading forces: operational loading forces, extraordinary loads, catastrophal loads and influences. To each of the three groupings pertains a different allowable stress.

The second group comprises those specifications in which safety is divided in the grouping of loading forces, the dimensioning stresses and the cross section. These specifications proceed from the ruin of the construction and take every uncertainty, with a divided safety sector, at its proper place into consideration; to a greater relative deviation pertains a higher safety sector, to a smaller one a lower factor. The theoretical basis of these specifications was elucidated by Basler [3].

Specifications that come into the third group calculate safety on the basis of probability theory, with consideration given to loading forces and their deviation, rupture stress and its deviation. These specifications calculate with an undivided safety factor.

Safety factor is determined by a probability of rupture assumed in advance, and probability distribution. The assumption of the probability of rupture  $/10^{-3}, 10^{-4}, \text{etc.}/$  is a result of subjective evaluation, though it is much more perceptible than saying, that, e.g., a twofold safety is required. The function of the selected distribution is also based on individual judgement. The difference between, e.g., the Weibull and lognormal distribution in the rate of the safety factor can be 10-100 per cent [2].

By means of this procedure are calculated airplanes and solved the dimensioning problems connected with space travel. Theoretical considerations were set forth by Freudenthal [4] [5].

All of the three procedures, though at different places and in various ways, give the rate of safety on the basis of individual deliberation. This rate, the expression of safety, will be further on considered to include failures of an accidental character only.

Safety only provides an objective rate of measurement, as it will be demonstrated further below, together with economical considerations.

### 3./ A new procedure for dimensioning

The known basic relation for the calculation of dimensions, based on probability theory is, if the time parameter is also considered:

$$\lim_{t \rightarrow T} \min \{ [R(t) - S(t)] \geq 0 \} \geq 1 - \frac{1}{k} \quad \dots 1./$$

Expression 1./ says so much in words that, during the lifetime of the construction, the Basler reserve [3]  $Y/t/ = R/t/ - S/t/$  must be greater by a probability given in advance  $1 - \frac{1}{k}$  than zero.

The basic relation is not unequivocal without the time variable  $t$  /fig.1./

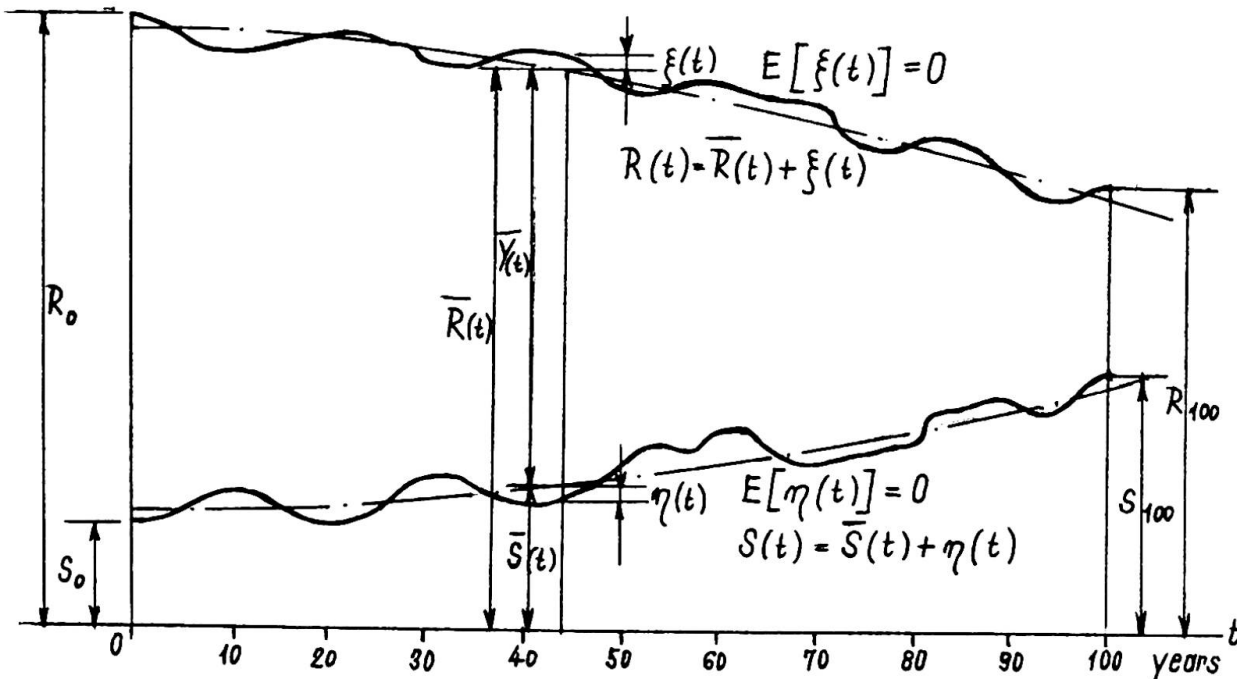


Fig. 1.

### 4./ Lifetime of the engineering structure

Safety of engineering structures can be related only to a certain lifetime. If  $T = \infty$ , the ruin of the structure is considered a certain event. The lifetime of engineering structures, therefore, has to be determined in advance. In respect of lifetime permanent and temporary structures can be dealt with.

For permanent structures lifetime has to be stated in  $T = 10$  years, if it is a vehicle, in  $T = 100$  years if it is an earthwork.

For temporary structures, when an earthwork is concerned,  $T = 10$  years, for structures of locomotion it may be that  $T = 1$ . Largely speaking, for permanent structures it is reasonable to count with  $T = 50$  years, for temporary ones with  $T = 5$ . From expression 1./ it is perceivable that if  $T$  is small, the difference,  $R/t/-S/t/$ , is greater than it is for a long  $T$  time. This holds particularly if  $R/t/$  and  $S/t/$  are stochastic processes with a notable trend /fig.1/.

### 5./ Loading forces, loading movements

Loading forces which are constant within time /dead load, earth pressure/, constitute a stochastic multitude, loading forces and movements which vary within time /useful load, water pressure, snow or wind pressure, variation of temperature, creeping, etv./ constitute a stochastic process. The periodical /e.g. annual/ maxima of these latter stochastic processes only form stochastic multitudes. With the processing of technical data it is reasonable to calculate four probability characteristics: the expectation value /a/, deviation /s/, asymmetry /f/ and excess /c/. The processing of the data must be performed on the basis of some textbook of mathematical statistics 1. In order to provide that the relative deviation of excess itself should not exceed 15 per cent, the number of elements of the multitude has to be selected  $\sim 500$  at the least. For the determination of the probability characteristics of the useful load it must be proceeded, with consideration given also to future development, from the loading spectre. Forces of a meteorological character must be processed from statistical data.

### 6./ Rupture stress, geometrical dimensions

Rupture stress within a  $t$  time, which constitutes a stochastic process with a trend, has to be determined in principle through precessing a stochastic multitude of rupture tests of the material taken at different times. In want of data it is supposed, in first approximation, that at the end of lifetime rupture stress can be taken as equal to the longtime stress limit and its probability characteristics are the following:

$$\left. \begin{aligned} \bar{\sigma}_R(T) &= \frac{\bar{\sigma}_{R0}}{\bar{\sigma}} \quad (e.g. \bar{\sigma} = 1,15) \\ s^{\sigma}(T) &= s^{\sigma}_0 \sqrt{\bar{\sigma}}; \quad f^{\sigma}(T) = f^{\sigma}_0; \quad c^{\sigma}(T) = c^{\sigma}_0 \end{aligned} \right\} \dots 2./$$

Geometrical dimensions always display a normal distribution. Because of the corrosion effect the geometrical dimensions have to be diminished at the end of the lifetime by a value  $\Delta$ , which may be, in absence of statistical data, 1 to 5 per cent of the dimension. Deviation of corrosion is taken equal to deviation of dimension:

$$\left. \begin{aligned} \bar{\mu}(T) &= \bar{\mu}_0 - \Delta \\ s^{\mu}(T) &= s^{\mu}_0 \sqrt{2}; \quad f^{\mu}(T) = f^{\mu}_0 = 0; \quad c^{\mu}(T) = c^{\mu}_0 = 0 \end{aligned} \right\} \dots 3./$$

### 7./ Probability characteristics of the function distribution

Probability characteristics of internal forces which cause rupture and internal forces which are the result of loading cannot be determined in a direct way, it is therefore necessary to determine from the probability characteristics of the components the probability characteristics of a quantity characterised by a functional relation. Exact formulation of the problem is this: the independent probability variables  $\xi_1, \xi_2, \dots, \xi_n$  are given, the probability characteristics of these,  $a_i, s_i, f_i, c_i$



$i=1, 2, \dots, n$  are also independent, and  $v_i < 0,5$ ; what are the amounts of the probability characteristics of the functional value  $\eta$ , characterised by the functional relation  $\eta = G(\xi_1, \xi_2, \dots, \xi_n)$ ?

If the function  $\eta$  is expanded in a Taylor series and also members of the third order are considered, the probability characteristics of a rational whole function can be determined according to the rule

$$\left. \begin{aligned} a_\eta &= E(\eta) = G(a_1, a_2, \dots, a_n) + \frac{1}{2} \sum_{i=1}^n G_{ii} s_i^2 + \dots \\ s_\eta^2 &= D(\eta) = \sum_{i=1}^n [G_i s_i]^2 + \sum_{i=1}^n G_i G_{ii} f_i s_i^3 + \dots \\ f_\eta &= \frac{1}{s_\eta^3} \left[ \sum_{i=1}^n G_i^3 f_i s_i^3 + \frac{3}{2} \sum_{i=1}^n G_i^2 G_{ii} (c_i + 2) s_i^4 + \sum_{j=1}^n G_i G_j G_{ij} s_i^2 s_j^2 + \dots \right] \\ c_\eta &= \frac{1}{s_\eta^4} \left[ \sum_{i=1}^n G_i^4 (c_i + 3) s_i^4 + 6 \sum_{j=1}^n (G_i G_j s_i s_j)^2 + \right. \\ &\quad \left. + 2 \sum_{i=1}^n G_i^3 G_{ii} (h_i - f_i) s_i^5 + 6 \sum_{j=1}^n (2 G_i G_{ij} + G_j G_{ii}) G_{ij} f_i s_i^3 s_j^3 + \dots \right] - 3 \end{aligned} \right\} \dots 4./$$

In expression 4./ e.g.

$$G_{ij} = \left[ \frac{\partial^2 G}{\partial \xi_i \partial \xi_j} \right]_{\substack{\xi_i = a_i \\ \xi_j = a_j}} \quad i, j = 1, 2, \dots, n \quad i \neq j$$

with the first two members. If  $0,15 < v_\eta \leq 0,35$ , all of the members written here must be counted with. Generally if  $v_\eta \leq 0,15$ , it suffices to calculate with the first, in the case of excess,  $v_\eta \leq 0,35$ , all of the members

If  $0,35 < v_\eta \leq 0,5$  it is necessary to calculate accordingly further members which are not written here. If the derivatives of function  $G$  are not limited derivatives but the function can be expanded to a Taylor series at the places  $a_1, a_2, \dots, a_n$ , also members of an order higher than the third may be required [6].

If  $|f_\eta| < 0,1$  and  $|c_\eta| \leq 0,2$ , the resultant distribution may be considered normal.

### 8./ Internal forces that cause rupture

They depend in general on rupture stress and cross section quantity and constitute a stochastical series

$$R(t) = \sigma_R(t) \cdot W(t) \quad \dots 5./$$

$$\text{or} \quad R(t) = H[\sigma_R^{(1)}(t), \sigma_R^{(2)}(t), \mu^{(3)}(t), \dots, \mu^{(n)}(t)] \quad \dots 6./$$

In expression 5./ internal forces causing rupture can generally be established as a product of rupture stress and cross section quantity. Expression 6./ refers to cases in which the bearing structure is not made of a homogenous material and the type of bearing is such as cannot be separated from the geometrical dimensions of the cross section /e.g. excentric internal forces within a r.-c. bearer/.

Probability characteristics of internal forces causing a rupture, if calculation has to be made on grounds of expression 5./, are

$$\begin{aligned}
E[R(t)] &= \bar{G}_R(t) \cdot \bar{W}(t) \\
[v^R(t)]^2 &= [v^G(t)]^2 + [v^W(t)]^2 + [v^G(t) \cdot v^W(t)]^2 \\
f^R(t) &= f^G(t) \left[ \frac{v^G(t)}{v^R(t)} \right]^3 \quad \dots 7./ \\
c^R(t) &= \frac{1}{[v^R(t)]^4} \left\{ [c^G(t) + 3][v^G(t)]^4 + 3[v^W(t)]^4 + 6[v^G(t) \cdot v^W(t)]^2 \right\} - 3
\end{aligned}$$

If the internal forces producing a rupture are to be calculated by expression 6./, probability characteristics must be determined with the use of formula 4./.

#### 9./ Determination of the cross section quantity

If probability characteristics of the Basler reserve  $Y/t/$  are to be determined from expression 1./, then, on the basis of expression 4./

$$\begin{aligned}
a^Y(t) &= \bar{Y}(t) = \bar{R}(t) - \bar{S}(t) \\
s^Y(t) &= \sqrt{[s^R(t)]^2 + [s^S(t)]^2} \\
f^Y(t) &= \frac{1}{[s^Y(t)]^3} \left\{ f^R(t)[s^R(t)]^3 - f^S(t)[s^S(t)]^3 \right\} \quad \dots 8./ \\
c^Y(t) &= \frac{1}{[s^Y(t)]^4} \left\{ [c^R(t) + 3][s^R(t)]^4 + [c^S(t) + 3][s^S(t)]^4 + 6[s^R(t) \cdot s^S(t)]^2 \right\} - 3
\end{aligned}$$

From expression 8./ it is to be seen that the dimensioning will be correct if

$$\begin{aligned}
R(t) &\geq S(t) + m s^Y(t) \\
\text{where } m &= m(f, c, k) \quad \dots 9./
\end{aligned}$$

The value of  $m$  depends on the selected distribution and the risk given in advance. Before proposing a type of distribution for the determination of the value of  $m$ , a simple relation can be given for the cross section quantity at a time  $t = 0$ , if internal forces that provoke rupture are such as according to expression 5./

$$W_0 = \Delta W_k + \frac{\bar{S}(T)}{\bar{G}_R(T)} \frac{1 + m \sqrt{[v^S(T)]^2 + \{1 - [m v^S(T)]^2\} \{[v^G(T)]^2 + [v^W(T)]^2\}}}{1 - m^2 \{[v^G(T)]^2 + [v^W(T)]^2\}} \quad \dots 10./$$

In expression 10./ the surplus cross section quantity, being a result of corrosion,  $\Delta W_k$ , depends on the value  $m = m(f, c, k)$ .  $S/T/$  is the expectation value of the sum of internal forces provoked by loadings, at the end of the service time,  $\bar{G}_R/T/$  is the expectation value of the rupture stress of the structural element in question,  $v/T/$  are the final values of the variation factors of the variable quantities.

If internal forces provoking rupture can be calculated by expression 6./, for the determination of the dimensions expression 5./ must be satisfied by way of the trial and error method.

### 10./ The selected type of distribution

The problem is what kind of a distribution function should be selected for the stochastical process  $Y/t/$  at a time  $t = T$  at the end of lifetime or any time  $t$ . From among the internal forces caused by loading meteorological forces and movements /internal forces caused by wind, snow, modifications of temperature/ can best be described theoretically with the use of the Weibull distribution [7]. A great part of useful loads and internal forces provoked by dead load do not follow the Weibull distribution pattern. The distribution of cross section dimensions is normal. The type of distribution drawn upon the rupture stress can be treated as though from among a homogenous multitude of bearers a discretionary one were selected and given to rupture. This problem is, in its essential conception, an urn-model to which one of the Pearson distributions will best apply. Since in the resultant distribution it is the rupture stress that generally has the greatest part and meteorological forces generally play but a slight part, for a resultant distribution the four parameter Pearson distribution, Pearson IV, can be recommended.

$$\gamma \int_{-\infty}^{\infty} \frac{e^{-\alpha \arctg \frac{x-d}{g}}}{[1 + (\frac{x-d}{g})^2]^{\beta}} dx = 1 - \frac{1}{k} \quad -\infty < x < \infty \quad \gamma > 0 \quad \beta > \frac{1}{2} \quad \dots 11./$$

The Pearson IV. distribution which is interpreted between  $-\infty < x < +\infty$  is not suitable because effective distribution is no clear urn model and, because distribution is dimensioned only for  $x > 0$ . If the four probability characteristics of distribution,  $a^Y/t/$ ,  $s^Y/t/$ ,  $f^Y/t/$ ,  $c^Y/t/$  are given,  $\beta$ ,  $\alpha$ ,  $d$  and  $g$  can be determined. The value of  $\gamma$  can be determined from the condition that the integral of expression 11./ between  $-\infty$  and  $+\infty$  is [1].

$$\left. \begin{aligned} \nu &= 6 \frac{c^Y + 2 - (f^Y)^2}{2c^Y - 3(f^Y)^2} \quad \text{auxiliary quantity} \\ \beta &= \frac{\nu}{2} + 1 \\ \alpha &= f^Y \cdot \nu(\nu-2) \sqrt{\frac{1}{16(\nu-1) - (f^Y)^2(\nu-2)^2}} \\ g &= \sqrt{\frac{\nu-1}{\alpha^2 + \nu^2}} \nu s^Y; \quad d = a^Y - \frac{g\alpha}{\nu} \\ \gamma &= \frac{1}{\int_{-\pi/2}^{+\pi/2} e^{-\alpha z} \cos^{\nu} z dz} \end{aligned} \right\} \dots 12./$$

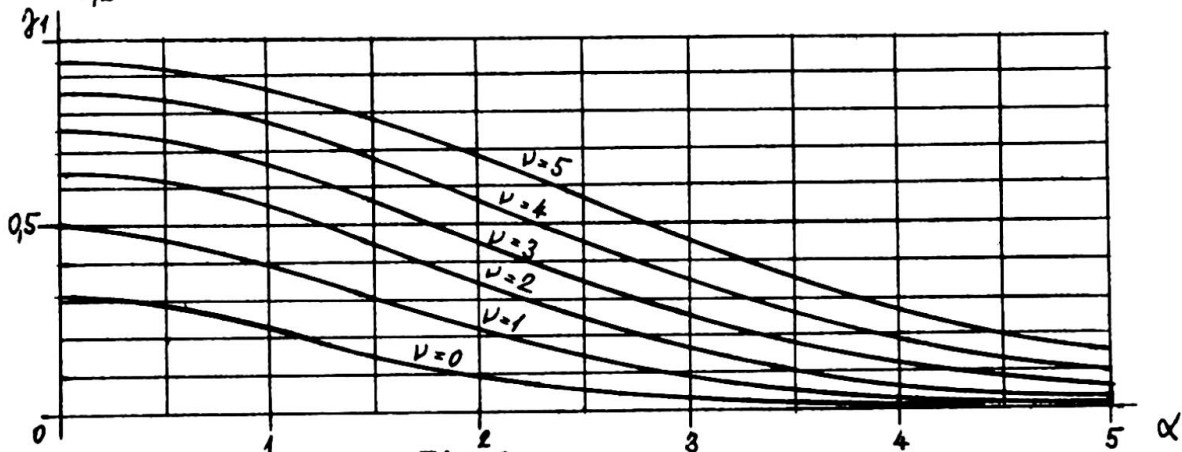


Fig.2.

From expression 12./ numerous conditions present themselves for the distribution characteristics which are not dealt with in this paper. The values of  $\gamma$  for whole number values of  $\alpha$  and  $\nu$  are shown in figur 2.

#### 11./ Optimum risk

The value of  $m = m/f, c, k/$  has furthermore to be determined in expression 10./ in order that the cross section quantity could be determined. The determination from expression 11./ is unequivocal if the value of  $k$  is given. For the determination of  $k$  two conditions can be offered.

The first condition is that the total cost of the installation should be a minimum. Supposing the interest factor to be  $q$ , the cost of rebuilding of the installation,  $C/k/$ , must be written off, during a service time  $T$ , and it is assumed that the installation will be ruined after a time  $t < T$  and the part of the construction not yet written off at that time will be capitalised to the date of the ruin. Then the total cost will be

$$K(k) = C(k) + \frac{1}{k} \left\{ C(k) \left[ 1 + \frac{q^T - q^t}{q^T - 1} \right] + Q \right\} \quad \dots 13./$$

A minimum cost can occur where the first derivative by  $k$  of expression 13./ is zero.

The costs of the bearing structure increase in a linear way with the cross section quantity,  $C/k/ = A + BW$ . This linearity will hold if the cross section quantity represents an area or if the ratio of cross section modulus to radius of inertia is constant. If expression 10./ is expanded to a series, it will hold, with a good approximation, that  $W_0 \approx W' + w_m$ .

If  $f$ ,  $c$  and  $s$  are constant, for the thinkable values of  $k$ ,  $m = \lambda_0 + \lambda_1 \log k$ . A deeper reason for linearity is to be found in the fact that distribution functions generally are exponential. For a normal distribution e.g. between  $2 < \log k < 6$ , it is true with a 3 per cent accuracy that  $m = 1,22 + 0,6 \log k/$ . Substituting the above expressions into each other,  $C/k/ = A + BW' + B \lambda_0 w/ + B \lambda_1 w \log k$  that is

$$C(k) = C_0 (1 + b_1 \log k) \quad \dots 14./$$

Expression 14./ reflects a stochastic relation which can be verified for a series of numerical examples with a difference less than 2 per cent, doing regressional calculations. The minimum that results from expressions 13./ and 14./ is

$$k(t) = \frac{1 + b_1 \log \frac{k}{e}}{b_1 \log e} \left[ \frac{Q}{C(\frac{k}{e})} \left( 1 + \frac{q^T - q^t}{q^T - 1} \right) \right] \quad \dots 15./$$

The maximum for expression 15./ produces itself at a time  $t=0$ . On an average, if the cost of the bearing structure of a building based on a 3 per cent risk, is  $C$ ,

$$k_{max} \approx \frac{2,6}{b_1} \left[ \frac{Q}{C} + 2 \right] \quad \dots 16./$$

It is apparent from expression 16./ that the more damage will be caused by the installation when it is ruined, the less will be the risk permitted to be taken. E.g. if  $T = 1$  and  $Q = 0$ ,  $k_{min} \sim 50$ . If  $T = 50$  and  $\frac{Q}{C} = 100$ ,  $k_{max} \approx 5000$ . Here the risk that is taken,

varying within a range of 2 per cent and 0,02 per cent, corresponds to the percentage wastes in non damaging industrial production, which is 1-2 per cent [8].

Another condition is that the annual quotient to write off for the installation is  $\frac{q^T/q-1}{q^T-1} C/k/$  and the total of annual maintenance,  $L/k/$  is the minimum of what is referred to as entire cost. The concept of entire cost was defined by a Congress held on "Perspective of the user and reliability of the system" in the United States in 1962. It is assumed that maintenance costs take the sum of

$$L(k) = L_0 \left( 1 + \frac{b_2}{\log k} \right) \quad \dots 17./$$

Expression 17./ is not proved, it merely appears to be logical upon the analogy of expression 14./ [9]. The minimum of all annual costs is secured by the expression

$$k = 10^{\sqrt{\frac{L_0 b_2 (q^T - 1)}{C_0 b_1 q^T (q - 1)}}} \quad \dots 18./$$

being satisfied. Expression 17./, if  $T \sim 50$  and  $q-1 = \frac{p}{100}$  where  $p$ , in percentage, is the interest rate, will be

$$k \approx 10^{9,5 \sqrt{\frac{L_0 b_2}{C_0 b_1 p}}} \quad \dots 19./$$

From the two expressions /15./ and 18./ that one must be satisfied which gives a larger value for  $k$ . From the comparison of expressions 14./ and 19./ results that when the condition

$$\frac{Q}{C} \geq \frac{b_1}{2,6} 10^{9,5 \sqrt{\frac{L_0 b_2}{C_0 b_1 p}}} - 2 \quad \dots 20./$$

is satisfied, that is, if the ratio of the damage incurred and the rebuilding costs is greater than the right side of expression 20./, the value of  $k$  must be determined on the basis of expression 15./ and /or 16./, if it is smaller, expressions 18./ and /or 19./ will give the value of  $k$ .

It should be noted that in the vicinity of the optimum value for extremely small differences will result in expression 13./, therefore the value of  $k$  must be determined with a rounding up and on a rather large scale.

## 12./ Other probabilities that may be considered

Beside what has been said above it is also essential, how many uniform structural elements are going to be built in. If  $n$  number of bearing elements are to be fabricated and a risk of  $\frac{1}{k}$  has to be taken for each of them, for one piece a risk of  $\frac{1}{k'}, \frac{1}{k'} < \frac{1}{k}$  must only be taken. The solution of the problem is

$$1 - \frac{1}{k} = \binom{n}{0} \left( \frac{1}{k'} \right)^0 \left( 1 - \frac{1}{k'} \right)^n \quad \dots 21./$$



Expression 21./ displays a binomial distribution.

$$\left. \begin{aligned} k' &= \frac{1}{1 - \sqrt[n]{1 - \frac{1}{k}}} \\ k &= \frac{1}{1 - (1 - \frac{1}{k'})^n} \end{aligned} \right\} \dots 22./$$

and/or If  $k > 50$ , then, with a good approximation

$$k' \cong nk \dots 23./$$

Thus, if it is required that no one of the  $n$  pieces should get ruined with a probability of  $\frac{1}{k}$ , every piece must be fabricated with a risk of  $\frac{1}{nk}$  being taken.

### 13./ Conclusions

1./ The following answer can be given to the problem raised: The correct dimensions are given in expressions 9./ or 10./, the value of  $m = m/f$ ,  $c$ ,  $k$  can be determined through the Pearson IV. distribution according to expression 11./, the most economical measure of risk taken against failure,  $1/k$ , is given by expressions 16/ or 19./. As to the measure of the assumed risk it must furthermore be considered, how many uniform elements in question are going to be fabricated.

The suggested method of calculation does not contain any subjective factors, all dimensions and safety can - on the basis of mathematical statistics - be determined solely upon economical considerations.

2./ Safety, consequently, is a mutual and unequivocal function of the probabilities for the installation that during its scheduled service time all circumstances provoking ruin can occur simultaneously in the most possibly unfavourable arrangement [3].

The lower this probability /the risk taken/ is, the greater is the safety. We suggest the acceptance, as a measuring value, of  $\log k$  the logarithm of the reciprocal value of the risk taken. This expression has proved suitable in information theory [10], as a quantity proportional to the measuring number of the information quantity. For a great safety a large amount of information is required about the given bearer. Investment expenditures increase with safety, maintenance costs decrease with it.

3./ For our bearing structures the principle of equal safety is in appropriate. The more damage is caused by the structure with its ruin, with so much more safety must it be designed. Secondary bearers, the ruin of which causes no damage, must be designed with a lesser safety. Installation with a short designed service time can be of smaller divisions, because within a shorter period the rupture strength of the load-bearing building material shows a lesser decrease and the probability for the occurrence of the loads, particularly meteorological ones, is lower within a shorter period. Consequently, if it is to proceed from the safety of the primary system of bearers of definite installations, the primary system of bearers of temporary installations can be fabricated with a lesser safety and so can the secondary bearers of the definite installations. Still lesser safety is required for secondary bearers of temporary installations.

4./ General rules for dimensioning are provided by deterministic interrelations in technical mechanics. By reason of a deviation of parameters in the functional interrelations the economical dimensions have to be determined with the aid of stochastic interrelations based on probability theory.

5./ A processing of statistical data is required. The described method, however, can be applied even without processing the data, in that case it has to be proceeded from loading given in the rules and from nominal geometrical and strength measures. For deviations there is to be taken, in the absence of data, one half of the tolerance. Tolerance, then, is based on statistical experience.

6./ Author considers the application of this procedure absolutely necessary: in setting up rules for dimensioning, stability calculations for elements of serial production and structures of high cost.

7./ Rules in operation at present are over-dimensioned even today. The degree of over-dimensionedness is, with various rules, in terms of costs 8-12 per cent for primary bearers at definite installations. Over-dimensionedness for secondary bearers and for temporary installations is 11-17 per cent. This can best be helped if calculations will be made, instead of the minimum values as specified in the rules, with their expectation values. If this proportion is considered, there will result economical dimensions /E.g. the expectable value of the yield point of St. 37 is, on the basis of statistical data, [11] ~ 2800 kp per square cm, whereas the rule specifies 2400 kp per square cm.

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#### Summary.

From all factors influencing the dimensions /loading, geometrical dimensions, crushing stress, etc./ the value and the probability variables of the load capacity reserve,  $Y/t$ , can be determined. By means of the Pearson IV. distribution the geometrical dimensions can be determined for a risk arbitrarily undertaken. The undertaken risk can be determined through only economical con- sider-



rations. The greater the damage caused, the lesser risk should be undertaken. If the damages are not significant, dimensions are influenced by the annual maintenance costs,  $/L/$ . For a safety rate the logarithm of the reciprocal value of risk  $/\log k/$  is suggested.

#### RÉSUMÉ

La valeur de la réserve de limite admissible de la charge,  $Y/t/$ , et ses variables de probabilité peuvent être déterminés à l'aide de tous les facteurs qui sont à même d'influencer les dimensions - tels la charge, les dimensions géométriques, tensions de rupture, etc. Par moyen de la distribution Pearson IV. les dimensions géométriques peuvent s'établir pour un risque arbitrairement entrepris. Or, le risque entrepris peut être déterminé moyennant des seules considérations économiques. Plus le dégât est grand, moins de risque doit être entrepris. Si le dégât n'est pas important, les dimensions seront influencées par les frais annuels d'entretien. Proposition est faite d'employer, pour mesure de sécurité, le logarithme de la valeur réciproque de la risque entreprise ( $\log k$ ).

#### ZUSAMMENFASSUNG

Aus sämtlichen, die Abmessungen beeinflussenden Faktoren - wie Belastung, geometrische Abmessungen, Bruchspannungen usw. - können der Wert  $Y/t/$  der Belastungsfähigkeitsreserve und seine Wahrscheinlichkeitsveränderlichen berechnet werden. Mit Hilfe der Verteilung Pearson IV. können die geometrischen Abmessungen zu einem beliebig unternommenen Risiko bestimmt werden. Das unternommene Risiko kann wieder auf Grund allein wirtschaftlicher Betrachtungen festgelegt werden. Je grösser der Schaden, ein um so niedrigeres Risiko darf unternommen werden. Bei nicht bedeutendem Schaden beeinflussen die jährlichen Wartungskosten,  $/L/$ , die Abmessungen. Als Sicherheitsmass wird der Logarithmus des Kehrwertes des unternommenen Risikos vorgeschlagen ( $\log k$ ).

### Safety of Structures as a Problem of Time

Sécurité des constructions en fonction du temps

Sicherheit der Bauten als ein Zeitproblem

C. EIMER  
Poland

First attempts of probabilistic approach to safety of structures were made as early as 1936 (W.Wierzbicki /1/, M.Prot /2/). The probabilistic philosophy has been discussed, for a long time, and gained its devotees and its skeptics, the problem being looked upon mainly from the point of view of a direct applicability to design and calculation. By the present writer's opinion, too little emphasis has ever been laid on the explanation of actual phenomena and interrelations that, in fact, has been dimmed by traditional methods, and this is the fundamental purpose of every theory. Once we realize we operate quantities affected with random scatter, we are induced, necessarily and at the same time, to the notion of safety and to probabilistic considerations, irrespective of whether we intend to establish a pure probabilistic theory of safety or to explain precisely the meaning of conventional coefficients of safety. Similar development can be noticed in those branches of technical activity where problems of reliability are of importance.

The present contribution aims at explaining the role of time in safety which, from the mathematical point of view, means making a step from random variables to random stochastic processes. So far, the basic end of the theory consisted in finding the probability of the strength criterium to be fulfilled, i.e. of the inequality  $P < R$ , where, loosely,  $P$  is the load and  $R$  the strength (carrying capacity). Since every structure is to be reliable during a limited period of time (called, in what follows, life time or period of exploitation),  $P$  denotes the maximum load that can occur in the course of this period and  $R$  is assumed to be independent of time and of previous history of loading. The former assumption presents serious difficulties as, in general, statistics containing long periods of time, within a more or less homogeneous population of structures,

are not available. The second assumption is only a crude approximation, e.g. it disregards phenomena connected with rheologic strength or fatigue. An attempt to avoid the first of the above assumptions is given by A.M.Freudenthal /3/. The author considers a sequence of load applications, the probability distribution of  $P$  in a single application being known. However, it is not always easy or even possible to say what is a single load application as the loading is a continuous process. Besides, in order to "locate" the process in time the intervals between those applications must be assumed. Thus, in general, the whole of the problem is to be discussed in the language of stochastic processes, the approximation with different discrete models being of course possible and valuable in view of effective calculations.

### 1. Measures of safety in time

A fundamental merit of the probabilistic approach to safety is the introduction of a unique and universal probabilistic measure of safety. We shall discuss here some basic notions following the very clear exposition of the subject in /3/. The generalization depends on passing from the discrete model to a continuous time process.

On the assumption that the carrying capacity  $R$  is independent of time (which will hold in this point) we define the probability of safety or the reliability,  $L$ , as the probability that the time to failure  $t_R$ , i.e. the effective life time of a structure exceeds the period of exploitation  $t = T$ ,  $T < t_R$ . This is equivalent to the condition  $P_{\max} < R$  if  $P_{\max}$  denotes the maximum load during  $T$ ; hence we have

$$L(t) = \Pr(t < t_R) = \Pr(P_{\max} < R), \quad t = T \quad (1.1)$$

The probability of failure within that period equals

$$F(t) = 1 - L(t) = \Pr(t > t_R). \quad (1.2)$$

The a priori probability density of failure at the instant  $t$  is

$$f(t) = \frac{dF(t)}{dt}. \quad (1.3)$$

The failure rate, accordingly to /3/, is the probability that a structure that has survived  $t$  will fail in a time unit at  $t$ ,

$$h(t) = \frac{f(t)}{L(t)} = - \frac{d}{dt} \ln L(t). \quad (1.4)$$

Obviously, the above formulae correspond to (2.1) ÷ (2.7) in /3/. Here,  $t_R$  is a random variable and denotes the time to first surpassing the value  $R$  by the load.

The load  $P$ , being a continuous time process which we denote by  $g(t)$ , the results of measurements can, depending on the type of measuring devices, be obtained in threefold form: (1) as a continuous graph (self-recording instruments), Fig.1, (2) as periodic readings at time intervals  $\delta t$  (points denoted by small circles), (3) as maximum values at fixed time intervals  $\Delta t$ , usually related to cyclicity of load occurrence, e.g. in 24 hours, a year, etc. (devices recording maximum values denoted by little crosses).

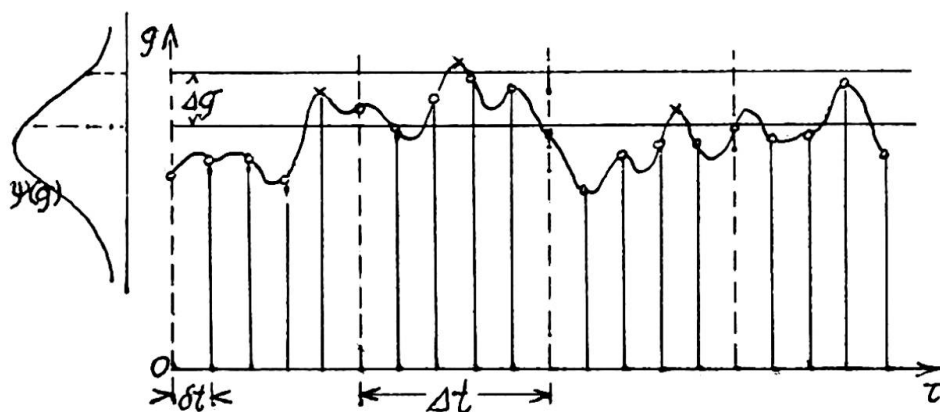


Fig.1

By taking the ratios of the number of points (marked with little circles) in the consecutive intervals  $\Delta g$  i.e. in consecutive horizontal bands to their total amount for a sufficiently long time interval  $t$  we obtain the frequency distribution and for  $t \rightarrow \infty$  the probability density of load at a given instant,  $\psi(g)$ , as shown on the left hand side of Fig.1. When the recording is continuous one can take arbitrary time intervals  $\delta t$ . On "matching" this probability density to that of  $R$  we arrive at the probability of failure at a single load application ( $p_F$  in /3/).

By considering the time interval  $\Delta t$  in which we are interested and a sufficiently long interval  $n \Delta t$ , and on establishing a constant value of  $g$ , we find the number  $m$  of intervals  $\Delta t$  in which

the latter has not been surpassed. The ratio  $m/n$  provides an approximate measure (exact for  $n \rightarrow \infty$ ) of the probability of not surpassing  $g$  in  $\Delta t$ . This probability is a function of the load,  $p = p(g)$  that can be found empirically by repeating the procedure for subsequent values of  $g$ . At the same time, it represents the distribution function of maximum load in  $\Delta t$ , since non-surpassing of  $g$  is equivalent to non-exceeding it by the maximum load. The probability density  $\pi(g)$  of maximum load in  $\Delta t$  can be obtained as the derivative  $dp(g)/dg$  or else directly from the graph, from the occurrences of "maximum" points in the consecutive intervals  $\Delta g$  (a procedure similar to the one already used for  $\psi(g)$ ).

If  $\Delta t$  were equal to the period of exploitation, the function  $p(g)$  would represent directly the distribution required. For practical purpose, however, it is important to arrive at some conclusions as to the distribution of maximum load in the period of exploitation  $T$  from the distribution of max  $g$  during an interval  $\Delta t$  that is, as a rule, shorter or, directly, from the density function  $\psi(g)$ , which implies two possible procedures, discussed in what follows.

In the first of them we find the probability of not exceeding  $g$  in  $n$  intervals  $\Delta t$  during the time  $T = n \Delta t$ ,

$$\Phi(g) = p^n(g), \quad (1.5)$$

valid under the restriction of independence of those events. Here,  $\Phi(g)$  represents the distribution of the maximum load. The probability density of this load is obtained by taking the derivative of

$$\varphi(g) = n p^{n-1}(g) \pi(g). \quad (1.6)$$

The above formula can also be obtained directly on taking into account that  $np^{n-1}$  represents the probability of not surpassing  $g$  in  $n - 1$  intervals  $\Delta t$ , whereas  $\pi(g) dg$  is the probability of the maximum load amounting to  $g$  in the remaining one interval  $\Delta t$ .

On establishing the load max  $g$  at a sufficiently high level so that higher values of  $g$  will occur but rarely, e.g. once in several months or even years, the interval  $\Delta t$  can be so far reduced that - without encroaching on the assumption of independent loads in consecutive intervals - very high values of  $p$  (near to 1)

are attained. The distribution (1.5) now tends to the Poisson distribution

$$\bar{\Phi}(g) = \exp [-\gamma(g) T], \quad (1.7)$$

with  $\gamma(g)$  denoting here the average number of events when  $g$  is surpassed per unit time ( $\gamma = 1/t_0$ , where  $t_0$  is the average time interval between such events), this number being dependent on the fixed value of  $g$ . The function  $\gamma(g)$  is found experimentally, e.g. by computing  $t_0$  for consecutive  $g$  (on a graph of the type of Fig.1).

The second procedure we mentioned above does not require determining of the function  $\gamma(g)$  or  $p(g)$  and is based on Fisher-Tippett asymptotic extremal distribution representing the distribution of the highest (or lowest) value in a test, where the number of particular test readings increases infinitely. Thus, it is a matter of finding the limiting distribution, for  $n \rightarrow \infty$  of the largest of  $n$  randomly chosen ordinates of points marked by little circles in Fig.1, at a fixed value of  $\delta t$  (so as to satisfy the requirement of independent loads). It is this form to which the distribution of  $\max g$  tends for  $t \rightarrow \infty$ , since a test of infinite number of test readings tends to become strictly representative. For finite  $n$  we obtain here Eqs. (1.5 and 1.6); albeit,  $p(g)$  and  $\mathcal{N}(g)$  have to be replaced by  $\bar{\Psi}(g)$  and  $\psi(g)$ , respectively (cf. Fig.1), i.e. by the probabilities of  $g$  at a given moment (in the experiment under consideration).

On introducing the new variables

$$z = n \left[ 1 - \bar{\Psi}(g) \right],$$

$$-u = \ln z = \ln n + \ln \int_0^\infty \psi(g) dg,$$

we obtain

$$\begin{aligned} \varphi(g) dg &= n \bar{\Psi}^{n-1} \psi dg = n \left( 1 - \frac{z}{n} \right)^{n-1} \psi dg \xrightarrow{n \rightarrow \infty} \\ &= n e^{-z} \psi \frac{1}{dz/dg} dz = -e^{-z} dz = e^{-e^{-u}} (-e^{-u}) du, \end{aligned}$$

whence the variable  $u$  is seen to possess the asymptotic density distribution

$$\omega(u) = \exp (-u - e^{-u}). \quad (1.8)$$

The variable  $u$  is seen to be related linearly to  $g$  if  $\psi(g)$  is exponential. Consequently,  $\varphi(g)$  can be obtained directly from Eq.



(1.8). It is proved in mathematical statistics that the asymptotic extremal distribution of Eq. (1.8) holds for normal distribution of  $\psi(g)$ , too. Obviously, this is but an approximate calculation, since the centre of the distribution is shifted proportionally to  $\ln n$ , at finite  $n = T/\delta t$  (to be calculated from the records), whereas the form of the distribution (1.8) itself is exact only for  $n \rightarrow \infty$  and does not depend on  $n$ .

Further calculations depend on the particular form of the distribution  $\psi(g)$  and, for different theoretical assumptions, are developed in the theory of extreme value distributions (cf. for instance [4]). Once we have found the extremal probability density  $\varphi(g)$  [from (1.6) or the derivative of (1.7) or else (1.8)] we insert it into the integral

$$L = \int \int_{P < R} \varphi(g) \psi(R) dg dR, \quad (1.9)$$

$\psi(R)$  being the probability density distribution for the strength  $R$ , where the integral is taken over the part of the plane  $(g, R)$  determined by the inequality  $g < R$ . Since  $\varphi(g)$  is a function of time, so is  $L = L(t)$  and our problem is solved.

## 2. Concept of damage and outline of a general theory

Precedent considerations were based on the assumption that  $R$  is constant which is but a crude approximation. We know that it depends, for instance, on the number of repeating load cycles in fatigue tests or on the time of loading if rheologic phenomena are involved. In order to describe this behavior the notion of "cumulative damage" is introduced in the theory of fatigue of materials and similar notions are also known from the general theory of reliability.

Let us generalize this notion and assume that the actual state of a structure (or a material) at a given instant (from the viewpoint of its carrying capacity) is defined by a unique positive number,  $\delta$ ,  $0 \leq \delta \leq 1$ , called damage, where zero damage ( $\delta = 0$ ) describes a perfect state and  $\delta = 1$  a complete failure. In general,  $\delta$  increases in the course of time, particularly, in the course of the loading process, which means that the ruin sets

in progressively and results in the "death" of the structure when  $\delta$  attains 1. For a simple example (not to be directly extended),  $\delta$  can represent the reduction of the cross-section of an axially loaded rod because of an expanding crack. Thus, our strength condition  $P < R$  is to be replaced by a more general one

$$\delta < 1. \quad (2.1)$$

Now, the problem consists in the prediction of the time  $t$  at which the damage becomes 1 or, in a probabilistic approach ( $\delta$  being a random variable), in the determination of the probability

$$L(t) = \Pr(\delta < 1) \quad (2.2)$$

for a given period of exploitation  $t = T$ .

For the classical case  $\delta$  remains 0 as long as  $P < R$ ,  $R$  being the carrying capacity. On surpassing  $R$  for the first time  $\delta$  suddenly increases to 1 and the structure fails (Fig.2). It is seen that  $\delta$  is defined to be a Heaviside function

$$\delta(t) = H(t - t_R),$$

$t_R$  denoting the time to first surpassing  $R$  by  $P = g(t)$ . The probability (2.2) reduces to (1.1) and exactly the theory in point 1 provides the solution.

In general, the hypothesis that the physical state of a structure can be determined by a unique parameter,  $\delta$ , is a considerable simplification of actual conditions, albeit it results in a far reaching gene-

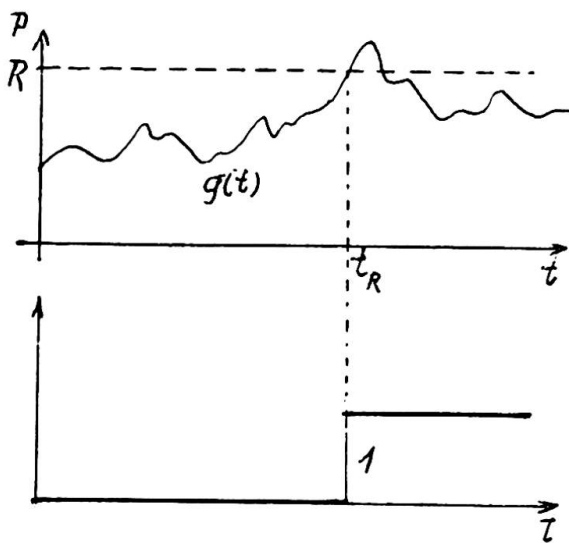


Fig.2

ralization of the former theories of safety. In fact,  $\delta$  can depend on the whole previous history of loading and, therefore, is a functional defined on the class of all possible functions  $P = g(t)$ . Depending on what phenomena are to be included (e.g. fatigue, rheology, etc.) and for the sake of effective calculation further restricting hypotheses have to be introduced. First of all,

we shall assume that  $\delta$  is cumulative so that we only examine the increments  $d\delta(t)$  which simply integrate in time. If, for the time being, we abstract from rheologic phenomena, we are able to make the assumption that  $d\delta$  depends only: (1) on the instantaneous internal state described by  $\delta$ , (2) on the external state described by  $P = g(t)$ , (3) on the change of the external state given by  $dP = dg(t)$ , (4) directly on time. Taking the increments in a time unit, i.e. replacing them by velocities (denoted with dots) we obtain

$$\dot{\delta} = f(\delta, g, \dot{g}, t). \quad (2.3)$$

The direct dependence on time reflects corrosion-like phenomena affecting  $\delta$  and will be neglected in further consideration. If we assume that damage is irreversible, the function  $f$  will be non-negative with respect to all arguments. If  $g$  approaches the limit strength  $R$  the velocity  $\dot{\delta}$  rapidly increases; if, furthermore,  $R$  depends on  $\delta$  and is independent of  $\dot{g}$ , we have  $f \rightarrow \infty$  for  $g \rightarrow R(\delta)$ . Further simplifying assumptions may state that  $\dot{\delta}$  does not depend on the sign of  $dg$  (internal friction-like phenomena at fatigue) - resulting in  $f$  symmetric respectively to  $\dot{g}$ , and that it is proportional to  $\dot{g}$  which gives the form

$$\dot{\delta} = f(\delta, g) |\dot{g}| \quad (2.4)$$

or, equivalently

$$d\delta = f(\delta, P) |dP|$$

Formulae of similar form, where instead of  $dP$  appears  $dn$  ( $n$  - number of cycles), can be found in the theory of fatigue (cf., for instance, /5/); however, those do not include any hypothesis as to the mechanism of failure and hold only within the above theory, for symmetric oscillations).

The simplest possible assumption for (2.4) is

$$f(\delta, g) = \beta = \text{const} \quad (2.5)$$

within the admissible region (Fig.3, shaded area) and  $f \rightarrow \infty$  for  $g \rightarrow R(\delta)$ , that is the actual smooth passage of the surface  $f(\delta, g)$  is replaced by a singularity. If, in particular,  $\beta = 0$ , we have the classical case, with the additional assumption that initial damage is possible and makes  $R$  lower (we are moving along vertical

lines in Fig.3). On integrating (2.4) for (2.5) we get  $\delta = \beta \sum |\Delta g|$

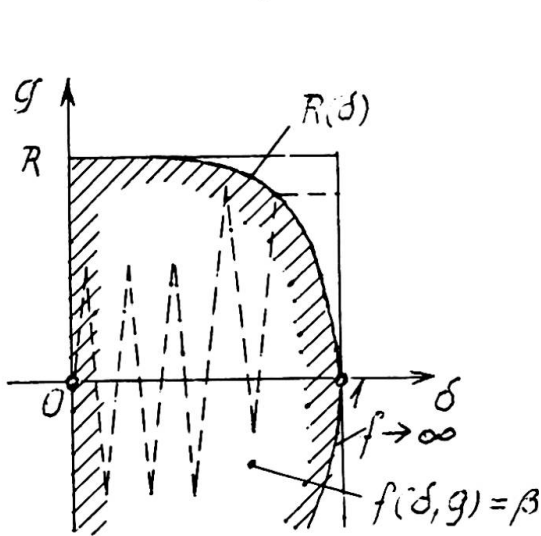


Fig.3

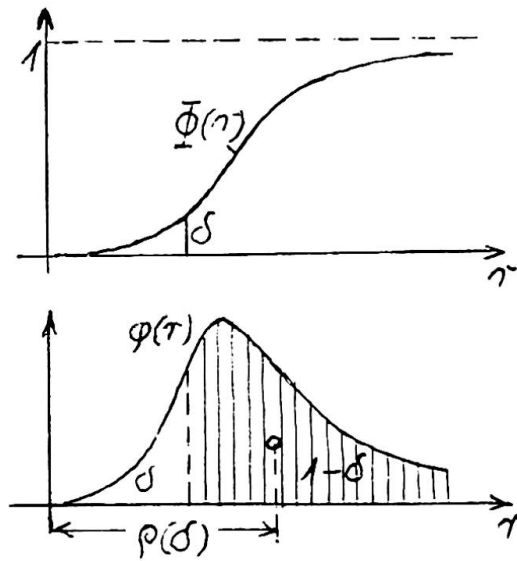


Fig.4

i.e.  $\delta$  is proportional to the sum of amplitudes of all load cycles, irrespective of the mean value (in Fig.5, below, the sum of segments  $\overline{01} + \overline{12} + \overline{23} + \dots$ ). In the case of simple (symmetric) oscillations it is proportional to the number of cycles,  $n$ ,

$$\delta = 4 n \beta g_m,$$

where  $g_m$  is the maximum load at one cycle. The path of loading is composed of straight segments with constant slope  $|d\delta/dg| = \beta$ , independent of the forms of "waves" in time, and on intersection of the curve  $R(\delta)$  it jumps horizontally till  $\delta = 1$  (Fig.3). This assumption is equivalent to the well-known Miner's hypothesis in the theory of fatigue of materials about a constant damage in a single oscillation with given amplitude. If, in particular, the curve  $R(\delta)$  coincides with the bounding straight segments,  $R = \text{const}$  and  $\delta = 1$  respectively, the equation of Wöhler's curve will result directly from the above formula for  $\delta = 1$ ,

$$g_m = \frac{1}{4\beta N}$$

which is the equation of a hyperbola. More generally, if the equation of Wöhler's curve  $g_m = W(N)$  is available, we obtain the curve  $R(\delta)$  solving for  $R$  the equation

$$R = W\left(\frac{\delta}{4\beta R}\right).$$

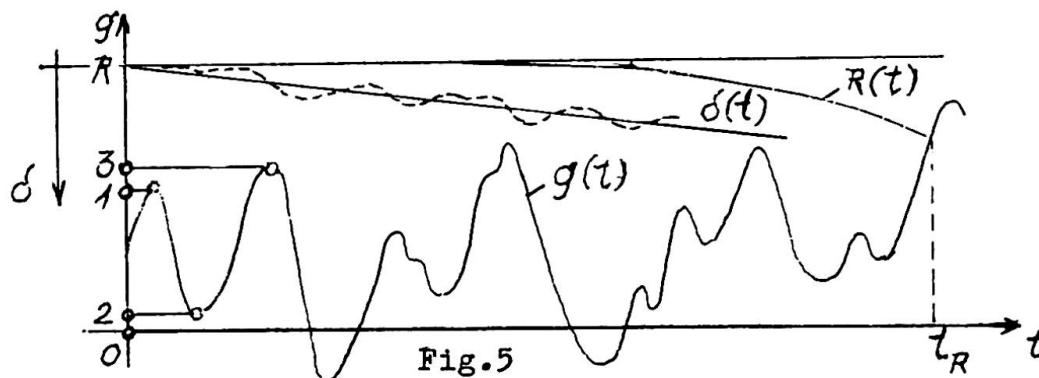
However, it must be pointed out that this curve might eventually not fit other non-zero values of the mean force, the assumption (2.5) being too simple.

The equation of the curve  $g = R(\delta)$  can also be argued theoretically, for instance, in the following way. Imagine a body composed of grains with variable strength properties and a process of damage that consists in consecutive failing of weaker grains. The volume proportion of elements at different levels of local strength,  $r$ , can be represented by an integral or differential probability distribution (Fig.4). Define the damage  $\delta$  as the part of the area (normalised to 1) under the curve  $\varphi(r)$  or else as an ordinate of the curve  $\bar{\Phi}(r)$ . The shaded part of the area,  $1-\delta$ , represents the actual carrying capacity (due to stronger grains). The equation sought for is

$$R(\delta) = (1 - \delta)\varphi(\delta)$$

where  $\varphi(\delta)$  is the abscissa of the centre of gravity of the shaded area.

The analysis of safety can be performed similarly to what has been said in point 1 (Fig.5). For a stationary process of loading



the damage  $\delta$  can be regarded, approximately, as proportional to time and assimilated to a straight line  $\delta = \delta_0 t$ , where  $\delta_0 = \beta \sum_{t=0}^1 |\Delta g|$  is the average damage during a time unit (obtained from the load curve by averaging over a sufficiently long time period). The strength curve is expressed in new units

$$g = R(\delta_0 t) \quad (2.6)$$

and failure appears at first intersection of this curve with  $P = g(t)$ , the problem being reduced to the one of a material with decreasing strength.

If strength properties (represented by (2.6)) are not affected with scatter, we can use, for instance, the same reasoning as for the formula (1.5). Assume, we have got records for a fixed period  $\Delta t$  and determined for this period the probability distribution  $p(g)$  (similarly as for (1.5)). Since the strength changes in time, we obtain

$$\bar{\Phi} = p(R_1) p(R_2) p(R_3) \dots \quad (2.7)$$

where, according to (2.6),  $R_k$  refers to the  $k$ -th sector  $\Delta t$  (cf. Fig.1). Taking logarithms of both sides

$$\ln \bar{\Phi} = \sum_k \ln p(R_k)$$

and taking for  $\ln p(R_k)$  its average value in the respective sector

$$\ln p(R_k) = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \ln p[R(t)] dt$$

we have

$$\ln \bar{\Phi} = \frac{1}{\Delta t} \int_0^{\tau} \ln p[R(t)] dt$$

and, finally,

$$\bar{\Phi} = \exp \frac{1}{\Delta t} \int_0^{\tau} \ln p[R(t)] dt. \quad (2.8)$$

This is an explicit function of parameters describing the function  $R(t)$ . Of course, this is but an approximate calculation, those parameters and, what more, the curve  $R(t)$  by itself being random (cf. Fig.5).

So far, the analysis was based on the assumptions (2.4) and (2.5) which is only a first step towards a theory including time - dependent phenomena. One of serious difficulties to be surmounted is connected with specifying the functions (2.3), (2.4). In general, if Wöhler - type curves for different non-zero mean stresses were available, we could come at a result on comparing them with respective solutions of the differential equation (2.4) for sinusoidal forms of load curves and for  $\delta = 1$ ,  $n = N$ ,  $\omega t = n$ ,  $a, b$  - constants,

$$\frac{d\delta}{dt} = f(\delta, b + a \sin \omega t) |a \omega \sin \omega t|.$$



Further generalizations could take into account rheologic phenomena and the formulae of the type (2.3) would be replaced by functionals, e.g. in an integral or an operational form. The simplified assumptions would, possibly, retain formulae of the type (2.3), introducing, however, some characteristic values of the load from the precedent history (e.g. the next local or the absolute maximum and minimum values of  $g$ ). The analysis, however, would be much more complex and is beyond the scope of this article.

In the present contribution we did not consider conventional measures of safety (e.g. coefficients of safety), as the methods of derivation of such measures have been discussed many times (cf. for instance, /3/, /5/) and a "pure" theory of safety can (and ought to) do without them.

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#### SUMMARY

The contribution is concerned with the problem of safety of structures on the basis of the theory of cumulative damage. The actual state of a structure (from the point of view of its carrying capacity) is described with a parameter, variable in time, depending on the previous course of loads. The latter is regarded as a stochastic process and a probabilistic measure of safety is derived.

#### RÉSUMÉ

L'auteur a examiné le problème de la sécurité de constructions du point de vue de la théorie du dommage cumulé. L'état actuel d'une construction est caractérisé par un paramètre unique (le dommage) variable avec le temps, dépendant des charges préalables. Celui-ci est considéré comme un processus stochastique et une mesure probabiliste de la sécurité est dérivée.

#### ZUSAMMENFASSUNG

Im vorliegenden Beitrag wird die Frage der Sicherheit einer Konstruktion auf Grund der Theorie der Anhäufung der Beschädigungen behandelt. Der Zustand der Konstruktion vom Standpunkt seiner Tragfähigkeit wird durch einen Parameter beschrieben, der die Beschädigung charakterisiert und von dem vorigen Verlauf der Belastung abhängig ist. Der obenerwähnte Verlauf wird als ein zufälliger Prozess aufgefasst und ein wahrscheinliches Mass der Sicherheit wird abgeleitet.

## **Zur Schätzung der Bruchwahrscheinlichkeiten der Tragwerke**

Estimation of the Probability of Failure of Structures

L'estimation de la probabilité de rupture des structures

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### **1. Einleitung**

Seit längerer Zeit laufen Bemühungen, die Sicherheitsuntersuchungen der Tragwerke aussagefähiger zu machen. Hierzu bieten sich die Methoden der Wahrscheinlichkeitsrechnung und mathematischen Statistik an [1]. Trotz zahlreicher Untersuchungen auf diesem Gebiet, die im Endergebnis auf eine Bestimmung der Bruch- bzw. Überlebenswahrscheinlichkeit der Tragwerke an Stelle der klassischen Sicherheitsberechnung hinzielen, haben diese modernen Methoden bisher noch keine oder nur sehr zögernde praktische Anwendung gefunden.

Der Grund dafür dürfte in den noch recht erheblichen Schwierigkeiten bei der Anwendung dieser Methoden liegen und in den dadurch bedingten bedeutenden Umstellungen für den Konstrukteur.

Solchen weitgreifenden Umstellungen werden erfahrungsgemäß berechtigte und unberechtigte Widerstände entgegengestellt, und es ergibt sich daraus die Aufgabe, Methoden und Möglichkeiten der Bestimmung von Bruch- bzw. Überlebenswahrscheinlichkeiten zu finden, die dem Konstrukteur für die Anwendung zumutbar, übersichtlich und in ihren Auswirkungen durchschaubar sind.

Diese Arbeit befaßt sich daher mit einer Möglichkeit, die Bruchwahrscheinlichkeit eines Tragwerkes mit möglichst einfachen Mitteln zu bestimmen. Die akzeptablen Bruchwahrscheinlichkeiten eines Tragwerkes sind relativ gering, so daß es genügt diese in der Größenordnung richtig zu schätzen. Eine größere Genauigkeit ist schon wegen der Schwierigkeit in der Präzisierung der Ausgangswerte kaum zu erhalten und für praktische Belange sicher

auch von weniger Interesse.

Demgegenüber bleiben die weiteren Probleme, die hiermit im Zusammenhang stehen und die z. B. FREUDENTHAL [1] ausführlich behandelt hat, unberührt.

## 2. Grundsätzlicher Lösungsweg und Schwierigkeiten

Bezeichnet  $y$  die Tragfähigkeit und  $x$  die Belastung, so ist die bisherige Sicherheitskonzeption durch das Verhältnis

$$v = \frac{Y}{X} > 1 \quad (1)$$

definiert. Ohne auf die Vor- oder Nachteile der einen oder anderen Darstellungsweise einzugehen, könnte auch

$$z = y - x > 0 \quad (2)$$

als ein solches Kriterium betrachtet werden.

Werden, was ihrem tatsächlichen Charakter besser entspricht, die Tragfähigkeit als Zufallsgröße  $Y$  und die Belastung als Zufallsgröße  $X$  betrachtet, so ist  $z$  selbst eine Zufallsgröße  $Z$ . Die Gl. (2) stellt hierbei für alle Zustände  $z = y - x \leq 0$  den Bruchbereich und für  $z = y - x > 0$  den Überlebensbereich dar. Gelingt es, die Wahrscheinlichkeitsverteilung für die Zufallsgröße  $Z$  zu formulieren und über  $-\infty < z \leq 0$  bzw.  $0 < z < +\infty$  zu integrieren, so ist die Bruch- bzw. Überlebenswahrscheinlichkeit bestimmt und das Problem gelöst.

Ist die Wahrscheinlichkeitsverteilung der Tragfähigkeit  $G(y)$  und der Belastung  $F(x)$  bekannt, so wird die Wahrscheinlichkeitsverteilung der neuen Zufallsvariablen  $Z$  durch Faltung [2] gefunden

$$H(z) = \iint_{x-y < z} df(z - y) dG(y). \quad (3)$$

Diesem formal einfachen Lösungsweg stellen sich praktische Schwierigkeiten entgegen, die vor allem folgende Gründe haben:

- a) Die mathematischen Modelle für die Wahrscheinlichkeitsverteilungen der Tragfähigkeit  $Y$  und der Belastung  $X$  sind häufig

nichtsymmetrische und oft ein- oder zweiseitig begrenzte Verteilungsfunktionen.

- b) Die Belastung auf ein Tragwerk besteht in der Regel aus einer Summe von Zufallsvariablen

$$X = \sum_{i=1}^k X_i \quad (4)$$

und die Bruchbedingung lautet daher

$$z = y - \sum_{i=1}^k x_i \leq 0 \quad (5)$$

- c) Die Belastung auf ein Tragwerk wird im allgemeinen wiederholt, d. h. häufig eingetragen, so daß nicht die Ausgangsverteilung der Belastung  $F(x)$ , sondern die Extremwertverteilungen  $F_n(x^{(n)})$  für  $n$  Belastungen maßgebend sind, wobei  $n$  auch durch die Zeit  $t$  ausgedrückt sein kann.
- d) Durch die wiederholte Belastung auf das Tragwerk wird oberhalb einer Anzahl  $n_0$  von Belastungen eine Minderung der Tragfähigkeit des Tragwerkes auftreten. Daher wird  $G(y)$  von der Zeit  $t$  abhängig und geht in den stochastischen Prozeß  $G(y, t)$  über.

Hierzu ist folgendes zu bemerken:

- zu a) Eine geschlossene Lösung des Faltungsintegrals (3) gelingt außer für Normalverteilungen bisher nur für Sonderfälle, die jedoch für das Bemessungsproblem wenig Bedeutung haben. Auch Potenzreihenentwicklungen führten bisher nicht zum Erfolg.
- zu b) Die Berücksichtigung der  $k$  Zufallsvariablen  $X_i$  der Belastung führt zu einer Mehrfachfaltung entsprechend Gl. (3), wodurch das mathematische Problem noch wesentlich komplizierter wird.
- zu c) Die Notwendigkeit, bei wiederholten Belastungen statt einer Ausgangsverteilung  $F(x)$  eine Extremwertverteilung  $F_n(x^{(n)})$  zu verwenden, ist leicht einzusehen, wenn die auftretenden Belastungen  $x_i$  nach Ranggrößen  $x^{(n)}$  geordnet werden, so daß

$$x^{(1)} < \dots < x^{(i)} < \dots < x_t^{(n)}$$

wobei die  $x^{(i)}$  der Ausgangsverteilung  $F(x)$  entsprechen und  $x_t^{(n)}$  die im Zeitraum  $t$  aufgetretene größte Belastung  $x^{(n)}$  bedeutet. Unter der Voraussetzung, daß die Tragfähigkeit zeitlich konstant ist, kann nur die größte Belastung zum Bruch führen und es ist die Extremwertverteilung des größten Wertes  $x_t^{(n)}$  maßgebend. Diese Extremwertverteilung ist abhängig von der Ausgangsverteilung und der Anzahl  $n$  der Belastungen [3, 4].

Wirken mehrere zufällige Belastungsgrößen  $X_i$ , so ist nur die maßgebende Extremwertverteilung mit den übrigen Ausgangsverteilungen zu falten, da sonst vorausgesetzt würde, daß mehrere Extremwerte gleichzeitig auftreten.

Mit der Annahme, daß der gesamte Belastungsablauf als diskreter stationärer Prozeß aufgefaßt werden kann, läßt sich die Richtigkeit und die Notwendigkeit der Verwendung von Extremwertverteilungen in der hier aufgeführten Art mit Hilfe der Übergangswahrscheinlichkeiten beweisen.

### 3. Die vier Fälle der Zuverlässigkeitsuntersuchung

Die Bestimmung einer Überlebenswahrscheinlichkeit unterscheidet sich von der Sicherheitsuntersuchung und wird als Zuverlässigkeitsuntersuchung bezeichnet.

Aus dem unter 2. Genannten ergeben sich vier Fälle der Zuverlässigkeitsuntersuchung:

#### 1. Statischer Fall

Die Tragfähigkeit  $G(y)$  ist zeitlich unbeeinflußt; für die einmalige Belastung ist die Ausgangsverteilung  $F(x)$  maßgebend.

#### 2. Quasi-statischer Fall

Die Tragfähigkeit  $G(y)$  ist zeitlich unbeeinflußt; für die  $n$ -malige Belastung ist die Extremwertverteilung  $F_n(x^{(n)})$  maßgebend.

#### 3. Betriebsfestigkeitsfall

Die Tragfähigkeit  $G(y)$  ist zeitlich beeinflusst und geht über in  $G(y,t)$ ; für die Tragwerksschädigung ist das ge-

samte Belastungskollektiv eines stochastischen Belastungsvorganges und für die Brucheinleitung die Extremwertverteilung  $F_n(x^{(n)})$  maßgebend.

#### 4. Zeit- oder Dauerfestigkeitsfall

Hierunter wird ein Belastungsvorgang mit konstanter Mittelspannung, Spannungsamplitude und Frequenz verstanden. Dieser Fall kann unter bestimmten Voraussetzungen auf den statischen Fall zurückgeführt werden.

In der angegebenen Form

$$H_1(u) = \int_{-\infty}^{+\infty} F(u-y) dG(y)$$

gilt Gl. (3) für den statischen Fall. Für den quasi-statischen Fall erhält sie die Form

$$H_2(u) = \int_{-\infty}^{+\infty} F_n(u-y) dG(y) \quad (3 \text{ a})$$

und für den Betriebsfestigkeitsfall

$$H_3(u) = \int_{-\infty}^{+\infty} F_n(u-y) dG(y,t) \quad (3 \text{ b})$$

Daraus folgt, daß der statische, der quasi-statische und unter gewissen Voraussetzungen auch der Zeit- bzw. Dauerfestigkeitsfall, ausgehend von Gl. (3) und Gl. (3 a) lösbar sind. Dagegen kann der Betriebsfestigkeitsfall wegen seines Charakters eines stochastischen Prozesses mit diesen Mitteln nicht gelöst werden, weshalb er hier zunächst nicht weiter behandelt wird.

#### 4. Lösung mit Hilfe der Edgeworth-Reihe

Zur näherungsweisen Bestimmung der Bruchwahrscheinlichkeit bietet sich eine Reihenentwicklung aus der Wahrscheinlichkeitsrechnung an, die unter dem Namen Edgeworth-Reihe bekannt ist. In der Form für die Wahrscheinlichkeitsdichte wird sie auch als Gram-Charlier-Reihe bezeichnet.



Wahrscheinlichkeitsverteilungen

$$F(x) = \int_{-\infty}^x f(t)dt \quad (6)$$

können auch durch die Gesamtheit ihrer Momente

$$m_k = \int_{-\infty}^{+\infty} x^k f(x)dx, \quad k = 1, 2, 3, \dots, \quad (7)$$

vollständig beschrieben werden.

Aus Gl. (7) lassen sich die Momente häufig nur schwer berechnen. Wendet man auf Gl. (6) eine Fourier-Stieltjes-Transformation an, so erhält man die sogenannte charakteristische Funktion der Verteilungsfunktion  $F(x)$ :

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} f(x)dx. \quad (8)$$

Die  $k$ -te Ableitung der charakteristischen Funktion nach  $t$  ergibt den Ausdruck

$$\varphi^{(k)}(t) = \int_{-\infty}^{+\infty} i^k x^k f(x) e^{itx} dx, \quad (9)$$

woraus durch Nullsetzen von  $t$  und nach Division durch  $i^k$  das  $k$ -te Moment der Verteilungsfunktion (6) folgt:

$$m_k = \frac{\varphi^{(k)}(0)}{i^k}. \quad (10)$$

Die charakteristische Funktion hat u. a. folgende und für die Lösung unseres Problems wichtige Eigenschaft:

Die charakteristische Funktion einer Differenz von Zufallsvariablen  $X_1 - X_2 - X_3 - \dots$  erhält man aus der Beziehung

$$\varphi(t) = \varphi_1(t) \cdot \overline{\varphi_2(t)} \cdot \overline{\varphi_3(t)} \dots, \quad (11)$$

wobei  $\varrho_i(t)$  die charakteristischen Funktionen der Zufallsvariablen  $X_i$  und  $\overline{\varrho_i(t)}$  die konjugiert komplexe Form von  $\varrho_i(t)$  bedeuten.

Außerdem ist

$$\log \varrho(t) = \log \varrho_1(t) + \log \overline{\varrho_2(t)} + \dots \quad (12)$$

Aus der logarithmischen Form der charakteristischen Funktion  $\log \varrho(t)$  können analog zu den Momenten  $m_k$  Größen  $\chi_k$  hergeleitet werden, die als Semiinvarianten oder Kumulanten in der Wahrscheinlichkeitsrechnung bekannt sind. Diese haben die günstige Eigenschaft, daß die  $k$ -te Semiinvariante der Verteilungsfunktion einer Summe oder Differenz von Zufallsvariablen als Summe oder Differenz der  $k$ -ten Semiinvarianten der Verteilungsfunktionen der einzelnen Zufallsvariablen gebildet werden kann, d. h.

$$\chi_k = \chi_{k1} + (-1)^k \chi_{k2} + \dots \quad (13)$$

Die Edgeworth-Reihe beruht auf dem Grundgedanken, die anzunähernde Verteilungsfunktion  $H(z)$  durch eine Summe von Gliedern darzustellen, die aus geeignet zu bestimmenden Vorzahlen sowie der Normalverteilung

$$\Phi(z) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du \quad (14)$$

und ihren Ableitungen  $\Phi^{(v)}(z)$  bestehen. Die Reihenentwicklung lautet:

$$H(z) \approx V(z) = \sum_{v=0}^{\infty} a_v \Phi^{(v)}(z) \quad (15)$$

Damit diese Reihe die anzunähernde Verteilungsfunktion möglichst gut approximiert, werden die Koeffizienten  $a_v$  so bestimmt, daß das Integral über die Abstandsquadrate der wahren Verteilungsfunktion  $H(z)$  von der Näherung  $V(z)$  ein Minimum wird.

Um bei der Bestimmung der Bruchwahrscheinlichkeit die ta-

bellierten Werte der Normalverteilung verwenden zu können, muß  $z$  normiert werden.

Wegen der Bruchbedingung  $z = y - x < 0$  ist bis  $z = 0$  zu integrieren - d. h. eigentlich  $V(0)$  zu berechnen. Die normierte Bruchbedingung lautet aber

$$r = \frac{0 - \mu}{\sigma} , \quad (16)$$

wobei  $\mu = \mu_y - \mu_{x1} - \mu_{x2} - \dots$  und

$$\sigma = \sqrt{\sigma_y^2 + \sigma_{x1}^2 + \sigma_{x2}^2 + \dots} \text{ ist. Außerdem müs-}$$

sen alle  $a_v$  durch  $\sigma^v$  dividiert werden. Gl. (15) erhält daher die endgültige Form

$$V(r) = \sum_{v=0}^{\infty} \frac{a_v}{\sigma^v} \Phi^{(v)}(r) = \Phi(r) - \frac{1}{3!} \frac{\kappa_3}{\sigma^3} \Phi'''(r) \quad (17)$$

$$+ \frac{1}{4!} \frac{\kappa_4}{\sigma^4} \Phi^{(IV)}(r) - \dots$$

In der angegebenen Form enthält die Edgeworth-Reihe im ersten Glied die Faltung der beteiligten Verteilungsfunktionen als Normalverteilungen. Die weiteren Glieder sind Korrekturen, die auf Grund der höheren Semiinvarianten erfolgen. Diese Korrekturen werden wegen der Konvergenz der Edgeworth-Reihe von Glied zu Glied kleiner und es ist zu übersehen, wann die Genauigkeit der erreichten Annäherung ausreicht. Hier ist zu bemerken, daß wegen der besseren Konvergenz bei der praktischen Anwendung der Edgeworth-Reihe die Korrekturglieder umgeordnet wurden, also eine etwas andere Form, als hier angegeben verwendet wurde [4] .

Die durchgeführten Berechnungen ergaben, daß im allgemeinen bereits das erste Glied der Edgeworth-Reihe das Ergebnis in der richtigen Größenordnung angibt. Es sind also nur bei genaueren Untersuchungen die Korrekturglieder zu berücksichtigen. Da in Zuverlässigkeitsanalysen sehr geringe Bruchwahrscheinlichkei-

ten zu erwarten sind, wird in der Regel die Berechnung des ersten Gliedes der Edgeworth-Reihe genügen.

## 5. Schlußbemerkungen

Als Ergebnis durchgeführter Untersuchungen konnte folgendes festgestellt werden:

1. Die Schätzungen ergaben, daß im allgemeinen bereits das erste Glied der Edgeworth-Reihe das Ergebnis in der richtigen Größenordnung angibt. Es sind also nur bei genaueren Untersuchungen die Korrekturglieder zu berücksichtigen. Da in Zuverlässigkeitsanalysen sehr geringe Bruchwahrscheinlichkeiten zu erwarten sind, wird in der Regel die Berechnung des ersten Gliedes der Edgeworth-Reihe genügen, was einer einfach durchzuführenden Faltung von Normalverteilungen entspricht.
2. Die Bruchwahrscheinlichkeiten ausgeführter Tragwerke schwanken je nach getroffener Voraussetzungen in weiten Grenzen und liegen zwischen  $P = 10^{-7}$  bis  $10^{-2}$ ; die richtige Verwendung der Extremwertverteilung ist für das Ergebnis von großer Bedeutung.

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## ZUSAMMENFASSUNG

Die Schätzung der Bruchwahrscheinlichkeiten gelingt relativ einfach unter Anwendung der Edgeworth-Reihe. Dabei ist es erforderlich, bei mehrfach eingetragenen Belastungen die entsprechenden Extremwertverteilungen zu verwenden. Soll die Bruchwahrscheinlichkeit nur in der Größenordnung richtig geschätzt werden, so sind dazu nur die ersten beiden Momente der beteiligten Verteilungsfunktionen erforderlich. Damit reduziert sich das Problem auf eine Faltung von Normalverteilungen, die einfach durchzuführen ist.

## SUMMARY

The estimation of the probability of fracture succeeds comparatively easy by the application of the Edgeworth-progression. Thereby it is necessary to use the corresponding extreme value distributions, if multiple stresses are inscribed. If the probability of failure is to estimate right only in the order of magnitude, there are necessary only the first two moments of the concerned distribution functions. With that the problem is decreased to the folding of normal distributions, which is easily to carry out.

## RÉSUMÉ

Le calcul des probabilités de rupture réussit d'une façon relativement simple, si l'on emploie la progression d'après Edgeworth. Ce faisant, il est nécessaire d'utiliser, à une distribution multiple des charges, les répartitions de valeurs extrêmes correspondantes. Lorsque la probabilité de rupture ne doit être exactement évaluée qu'en ordre de grandeur, les deux premiers moments des fonctions de répartition engagées sont seulement nécessaires. Ainsi, le problème est réduit à une convolution des répartitions normales qui est facile à effectuer.

## **The Relation of Data to Calculated Failure Probabilities**

Rapport entre les différents facteurs dans le calcul de la probabilité de rupture

Die Beziehung der Daten zur berechneten Bruchwahrscheinlichkeit

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By the methods of rational mechanics and the calculus of probability, we can now process the probability distributions for loads and material strengths relating to a proposed structure and calculate the 'probability of failure' to any desired number of decimal places, regardless of how scanty the data is or how poorly the curves fit the data. Clearly, the meaning of this calculated probability needs to be studied critically before it can be used with confidence in the design process. In particular, we must find ways to assess whether or not the data is really sufficient to warrant the probability statements used in the design.

The nature of probability has been studied extensively [1, 2]. In relation to the structural design problem the notion is fairly well defined; in most studies of the structural safety problem, 'probability' is usually taken in the sense of "probability-1" defined at length by CARNAP [2] (loosely called 'subjective probability'), or it is left as an undefined notion; "probability-2" ('objective probability') cannot properly be assigned any meaning in this context.

One way to employ probability(-1) in problems of structural safety is to adopt the viewpoint that it is merely a subjective measure of 'degree of belief,' or 'strength of belief'. The relation of data to the probability of failure is then very simple; data may rationally be assimilated into the input probabilities by Bayesian methods [3]. The question of what constitutes a sufficient amount of data to make a particular statement about the probability of failure, does not arise. Therefore, this paper is not relevant to 'Bayesian design.'



Alternatively, we may consider the probabilities associated with loads and strengths to be inherently unknown, auxiliary quantities. Objective statements about the probability of failure can then be made in the usual terms of statistical inference, and the subjective element in the justification of the design is greatly reduced. The viewpoint in the following, then, is that probability is not an absolute notion; rather, it has meaning only in relation to a specified body of evidence which, in this context, means: Actual results of load measurements, materials tests, model tests, prototype tests, etc., called the data. The advantage of this approach (when it is feasible) over the Bayesian approach is that it leads to propositions about the probability of failure that can be subjected to scientific inquiry.

Under normal conditions of practical design the data is, unfortunately, insufficient to make objective statements about the probability of failure of a proposed structure; for example, future loads must be guessed from measurements taken in the past. Nevertheless, it is instructive to study the rational inferences about the probability of failure that are possible under certain idealized conditions as models of reality, permitting us to estimate the amount of data required under less ideal conditions. In the following we will derive such a relationship (equation 12) between the necessary amount of data and various constants related to the design value of the probability of failure.

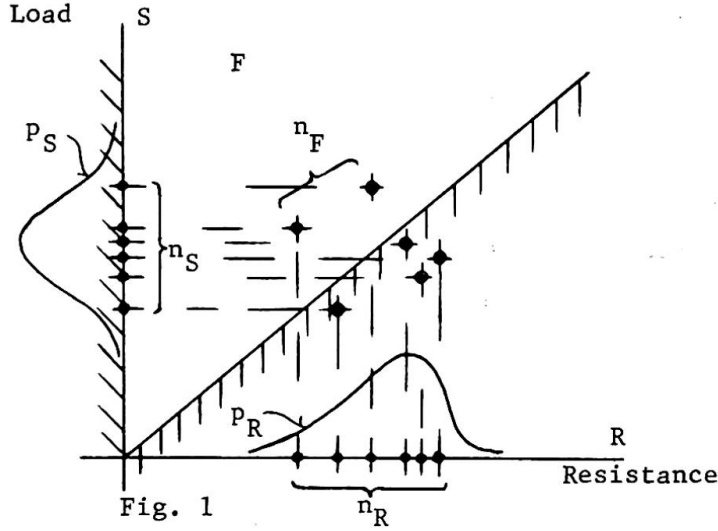
Consider a structure drawn at random from an infinite population of like structures and submitted to a single scalar load  $S$  drawn at random from an infinite population of loads. Let  $R$  denote the resistance of the structure, defined in such a way that failure is the event  $R < S$ . Resistance  $R$  and load  $S$  are assumed to be intrinsically positive, independent, continuous stochastic variables with unknown probability densities  $p_R(R)$  and  $p_S(S)$ ; information about these functions is assumed obtainable by random sampling. The data  $D$  is therefore a set of  $n_R$  resistance values and  $n_S$  load values:

$$D = \{R_i, i = 1, \dots, n_R; S_j, j = 1, \dots, n_S\}. \quad (1)$$

The probability of failure is

$$p_F = \int_{R < S} p_R(R) p_S(S) dS dR; \quad (2)$$

since  $p_R$  and  $p_S$  are unknown,  $p_F$  cannot be determined. The problem is instead to compute a suitable estimator  $C_F$  called the calculated probability of failure.



To make inferences about the probability of failure  $p_F$ , it is necessary to derive suitable statistics of the stochastic variable  $C_F$ .

The simplest way to obtain such an estimator is to draw from the data  $D$  a sample  $W$  of  $n$  ( $\leq n_R, \leq n_S$ ) pairs  $(R, S)$  of resistance and load values, at random and without replacement, see Fig. 1. Then,  $W$  is a random sample of the parent population  $\{(R, S)\}$ , and the elements of  $W$  are stochastically independent. Let  $n_F$  denote the number of outcomes of the failure event  $R < S$  in the sample  $W$ . Evidently,  $n_F$  is the total number of "successes" in  $n$  independent Bernoulli trials with probability  $p_F$  of "success". Therefore,  $n_F$  is distributed according to the binomial distribution

$$b(1, n, p_F) = np_F(1-p_F)^{n-1} \quad (3)$$

with mean  $np_F$  and variance  $np_F(1-p_F)$ . It follows that the estimator  $f_F = n_F/n$  is similarly distributed with mean  $m = p_F$ , variance  $\sigma^2 = p_F(1-p_F)/n$ , and coefficient of variation  $v = \sigma/m = 1/\sqrt{np_F(1-p_F)}$ . The relative failure frequency  $f_F$  is therefore an unbiased estimator of  $p_F$ . It is discrete valued ( $f_F \in \{0, 1/n, 2/n, \dots, 1\}$ ), so that in order to get sufficient resolution it is required that  $n_F$  be large in comparison with unity. Assuming that  $n_F$  is greater than 9 and neglecting  $p_F$  in comparison with unity, it can be shown [4] that  $f_F$  is approximately normally distributed with mean  $p_F$  and coefficient of variation  $1/\sqrt{np_F}$ .

In this context, the most appropriate way to indicate the precision of an estimate of  $p_F$  is by means of confidence intervals [4]. First, a confidence coefficient  $\alpha$  is selected. Taking the distribution to be normal with mean  $C_F$  and coefficient of variation  $1/\sqrt{nC_F}$  gives the following approximate confidence

limits for  $p_F$  computed from the calculated probability of failure:

$$L^- \approx C_F(1 - N^{-1}(\alpha)/\sqrt{nC_F}), \quad L^+ \approx C_F(1 + N^{-1}(\alpha)/\sqrt{nC_F}); \quad (4)$$

$N^{-1}(\cdot)$  denotes the inverse function of the normal probability integral. In a long sequence of repetitions the confidence interval between  $L^-$  and  $L^+$  will contain the probability of failure  $p_F$  nearly a fraction  $\alpha$  of the time.

To illustrate, assume that the data D consists of  $n_S = 10^5$  and  $n_R = 10^4$  random samples of load and resistance, respectively. The largest random sample W of independent elements that can be drawn contains  $n = 10^4$  (R,S)-pairs. Assume that  $n_F = nC_F = 16$  is the number of failure events in such a sample. If a confidence coefficient  $\alpha = 90$  per cent is considered suitable, we get from a table of the normal probability integral that  $N^{-1}(0.9) = 1.645$ . Equations (4) then give  $L^- = (1 - 0.41)C_F$  and  $L^+ = (1 + 0.41)C_F$ . The following continued inequality may be written down:

$$(0.59) \left( \frac{16}{10^4} \right) < p_F < (1.41) \left( \frac{16}{10^4} \right); \quad (5)$$

it may be asserted that this inequality is satisfied with probability 0.9. In other words, chances are nine out of ten that the value of  $p_F$  lies between 0.00094 and 0.00226. Independent random pairing of load and resistance values is clearly a very inefficient way of processing the data, in the present case using only  $10^4$  out of a possible maximum of  $n_R n_S = 10^9$  combinations of load and strength.

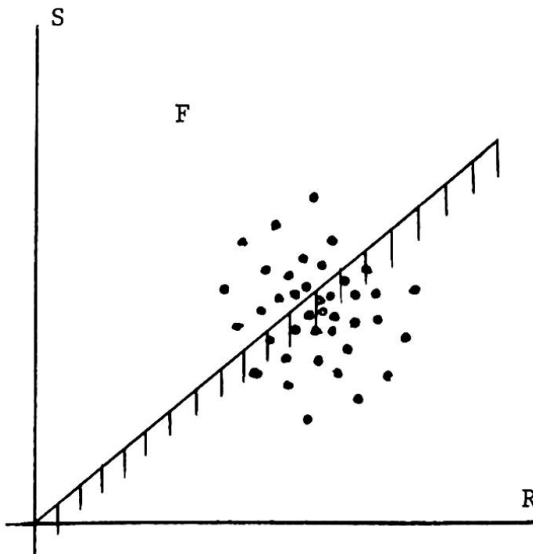


Fig. 2

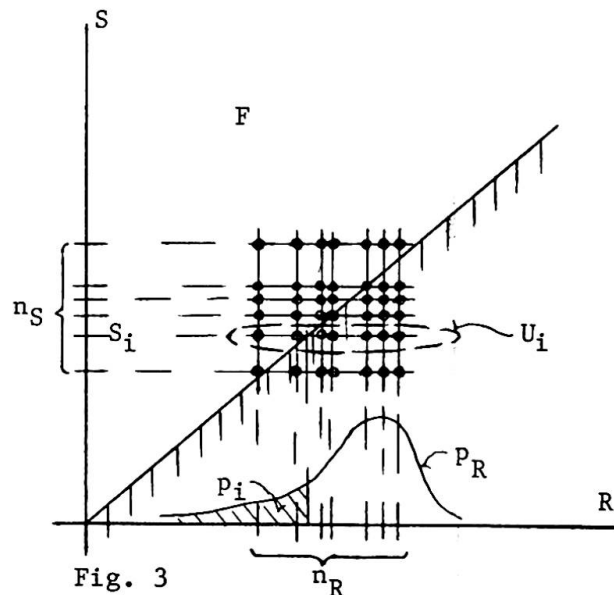


Fig. 3

Fig. 2 illustrates a sample consisting of a total of  $n_R n_S$  pairs obtained by independent random sampling. Fig. 3 shows all the (R,S)- pairs that can be formed from the data. The ordering of the pairs in this figure suggests a stochastic dependence which, according to the sign of the correlation between sample elements, may either increase or decrease the variance of the estimated probability of failure in comparison with independent random sampling using the same sample size. Nevertheless, the relative failure frequency,  $C_F$ , in the sample is an unbiased estimator of the probability of failure,

$$m(C_F) = p_F, \quad (6)$$

since every sample element was obtained by random sampling. To compute the variance, consider a sub-sample  $U_i$  (Fig. 3) consisting of  $n_R$  pairs formed by one of the load values,  $S_i$ , paired with all the resistance values  $R_1, \dots, R_{n_R}$ . A conditional probability of failure at this load level,  $p_i$ , may be associated with the sub-sample:

$$p_i = \int_0^{S_i} p_R(R) dR. \quad (7)$$

As before, the elements of the sub-sample constitute a sequence of  $n_R$  independent random Bernoulli trials. The number of failure events,  $n_i$ , at load level  $S_i$  is therefore binomially distributed with mean  $n_R p_i$  and variance  $n_R p_i (1-p_i)$ . However, it is also observed that the  $n_S$  sub-samples constitute a sequence of independent random samples, for the  $n_R$  resistance values may be considered to be drawn a priori, thereby dividing the load range into  $n_R + 1$  intervals establishing for each interval an associated probability that a load value will fall in the interval. As the loads are drawn independently and at random, the outcomes  $n_i$  ( $i = 1, \dots, n_S$ ) are stochastically independent. Accordingly, the estimator

$$C_F = \sum_{i=1}^{n_S} \frac{1}{n_R n_S} n_i, \quad (8)$$

has the mean value

$$m(C_F) = \frac{1}{n_R n_S} \sum_{i=1}^{n_S} n_R p_i \quad (9)$$

and the variance

$$\sigma^2(C_F) = \frac{1}{(n_R n_S)^2} \sum_{i=1}^{n_S} n_R p_i (1-p_i). \quad (10)$$

Neglecting  $p_i$  in comparison with unity for all  $i = 1, \dots, n_S$ , eliminating  $m(C_F)$  from equations (7) and (9), and inserting the result into equation (10) gives for the estimator  $C_F$  the coefficient of variation

$$V(C_F) = \sigma(C_F)/m(C_F) \approx 1/\sqrt{n_R n_S P_F} \quad (11)$$

Thus, as a good approximation, the coefficient of variation of  $C_F$  has the same value as if all  $n_R n_S$  sample pairs had been obtained by independent random sampling. We may therefore use equations (4) with  $n = n_R n_S$  to determine the confidence limits for the probability of failure. To illustrate, let  $n_S = n_R = 100$ , yielding  $10^4$  (R,S)-pairs, and assume that 16 of these pairs represent failures. This data yields the same confidence interval as found above, equation (5). The calculated probability of failure,  $C_F = n_F/n_R n_S$  according to Fig. 3, is believed to utilize the data in the most efficient way possible.

The amount of data required for a specified confidence coefficient  $\alpha$ , a target "design" probability of failure  $P_F$ , and a specified maximum width  $\beta P_F$  of the confidence interval (symmetric about  $P_F$ ) is easily computed from equation (4) to be

$$n_R n_S > [2N^{-1}(\alpha)/\beta]^2 / P_F \quad (12)$$

For example, assume that we seek to design the structure so that the probability of failure "with 90 per cent confidence" ( $\alpha = 0.9$ ) is a number between  $10^{-3}$  and  $10^{-4}$ . We select the target probability of failure  $P_F = 5.5 \times 10^{-4}$  and choose  $\beta = 9/5.5$  in order that the confidence limits  $(1 \pm \beta)P_F$  coincide with the specified limits  $p_F = 10^{-3}$  and  $p_F = 10^{-4}$ . Equation (12) gives the result that the product  $n_R n_S$  must be greater than 7,500. For example,  $n_R$  must be greater than 150 if  $n_S$  equals 50. Alternatively, if we demand that the probability of failure equals  $10^{-6} \pm 5\%$ , with 95% confidence, the required amount of data is increased to  $n_R n_S > 1.5 \times 10^9 = (50,000)(30,000)$ .

While the specific case studied here is greatly idealized, it serves to give an idea of the amount of data required in probabilistic design, unless one is content with giving merely a subjective meaning to the term 'probability of

failure'. The value of  $n_S n_R$  according to equation 12 may be taken as a rough lower bound for the data required to make an objective statement about the probability of failure in the form of a confidence interval. The amount of data that, as a practical possibility, can be collected does not seem out of proportion to the amount required in probabilistic design, assuming that reasonable standards of precision are prescribed.

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#### Summary

Statistical considerations must be used to supplement purely probabilistic considerations in structural reliability studies if concepts such as the probability of failure are to have more than a mere subjective meaning. In this contribution, the amount of data required to make confidence interval statements about the probability of failure is estimated by the methods of mathematical statistics.

#### Résumé

Nous voulons ajouter des considérations statistiques aux considérations probabilistiques des études de sécurité dans le domaine de la construction, afin d'élever ces dernières au-dessus du niveau purement subjectif. Dans cette étude, nous proposons, à l'aide des méthodes de statistiques mathématiques, d'évaluer la quantité requise de données pour établir les intervalles de confiance autour de la probabilité de ruine.

Zusammenfassung

Ueber rein wahrscheinlichkeitstheoretische Ueberlegungen hinausgehende statistische Betrachtungen sind für die Studien der Sicherheitskriterien im Hochbau erforderlich, falls Begriffe wie "Bruchwahrscheinlichkeit" usw. mehr als mit bloss subjektiver Bedeutung belegt sein sollen. In der vorliegenden Arbeit wird aufgrund eines speziellen Modells eine Abschätzung für den Bedarf an Datenmaterial vorgenommen, um Konfidenzgrenzen für die berechnete Bruchwahrscheinlichkeit angeben zu können.



## **Safety Analysis of Suspension Bridges**

Analyse de la sécurité de ponts suspendus

Sicherheitsbetrachtungen an Hängebrücken

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### 1. Introduction

Within the context of Theme I of the 8th congress, this paper establishes a method of structural safety analysis for the lateral vibration of aerodynamically stable suspension bridges under stormy winds.

The recent use of the so-called gust response factor in the dynamic analysis of structures subjected to gusty winds indicates an achievement of a higher level of sophistication in the structural safety analysis compared with the use of conventional safety factor, since the introduction of the gust response factor is based on the recognition that the wind velocity and hence the structural response have to be treated realistically as random processes.

The present paper demonstrates that a further effort will make it possible to estimate, in approximation, the probability of survival or failure of the suspension bridge (in the lateral mode of vibration) which is a more direct measure of safety in accordance with the probabilistic concept of structural safety<sup>1 (\*)</sup>.

Since the type of failure considered in this paper is either buckling or yielding of a chord member of the stiffening truss due to its lateral bending under the wind pressure (this defines a critical bending moment at each cross-section), the linear equations of motion can be employed in the response analysis. Such failure modes are also assumed implicitly or explicitly in the previous papers<sup>2,3</sup> dealing with the same problem.

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(\*) Numerals indicate references at the end.

## 2. Structural Analysis

In the present paper, as in References 2 and 3, the wind velocity  $U_z(t)$  at the height  $z$  above ground is the sum of the mean wind velocity  $\bar{U}_z(t)$  and the fluctuating part  $u_z(t, x)$ .

The pressure due to the wind velocity  $U_z(t)$  is, as usual, assumed to consist of two parts: the pressure due to the mean wind velocity

$$\bar{P}(t) = \frac{1}{2} \rho c A \bar{U}_z^2(t) \quad (1)$$

and the pressure due to the fluctuating part

$$p(t, x) = \rho c_d A \bar{U}_z(t) u_z(t, x) \quad (2)$$

where  $\rho$  is the density of air,  $c$  and  $c_d$  the static and the dynamic drag coefficient and  $A$  the exposed area of the structure considered.

It is usually observed from wind velocity records that  $u_z(t, x)$  is nonstationary with a larger variance at a larger mean wind velocity. In the present study, however, it is assumed that  $u_z(t, x)$  is stationary with a (constant) variance equal to that associated with the maximum mean wind velocity  $\bar{U}_z$ . Furthermore,  $\bar{U}_z(t)$  in Eq. (2) is replaced by  $\bar{U}_z$  for simplicity. Hence, the following stationarized and conservative expression is used for  $p(t, x)$ .

$$p(t, x) = \rho c_d A \bar{U}_z u_z(t, x). \quad (3)$$

Since the variation of  $\bar{P}(t)$  in time is much slow compared with the fundamental period of lateral vibration of the system of the cables and truss, the response  $\bar{y}_T(t, x)$  and  $\bar{y}_C(t, x)$  to  $\bar{P}(t)$  is obtained performing a quasi-static analysis, while the mean square value of  $y_T^*(t, x)$  and the bending moment  $M^*(t, x)$  of the truss to  $p(t, x)$  is evaluated on the basis of the standard equations of motion:

$$EI \bar{y}_T^{IV} + k(x)(\bar{y}_T - \bar{y}_C) = \bar{P}_T(t) \quad (4)$$

$$-H \bar{y}_C'' - k(x)(\bar{y}_T - \bar{y}_C) = \bar{P}_C(t) \quad (5)$$

$$m_T \ddot{y}_T^* + \mu_T \dot{y}_T^* + EI y_T^{*IV} + k(x)(y_T^* - y_C^*) = p_T(t, x) \quad (6)$$

$$m_C \ddot{y}_C^* + \mu_C \dot{y}_C^* - H y_C^{*''} - k(x)(y_T^* - y_C^*) = p_C(t, x) \quad (7)$$

with

$$k(x) = m_T g / h(x) \quad (8)$$

where the primes and the dots indicate differentiation with respect to  $x$  and  $t$  respectively,  $h(x)$  is the hanger length,  $EI$  the bending rigidity of the truss in the horizontal direction,  $H$  the sum of the horizontal forces in the cables,  $m$  the mass per unit length,  $\mu$  the linear viscous damping with subscripts  $T$  and  $C$  indicating that the quantities with  $T$  are associated with the truss and those with  $C$  are with the cables. The lateral bending moment of the truss can be obtained from its lateral displacement in the usual fashion.

The finite sine transform technique or the sine series expansion of  $\bar{y}_T$  and  $\bar{y}_C$  can be used to solve Eqs.(4) and (5) for  $\bar{y}_T$  and  $\bar{y}_C$ . To evaluate the mean square response of  $M_T^*$ , the frequency response functions  $H_{TT}(\omega, x, x_0)$  of  $y_T^*(t, x)$  and  $H_{TC}(\omega, x, x_0)$  of  $y_C^*(t, x)$  due to an input

$e^{i\omega t} \delta(x-x_0)$  applied at  $x = x_0$  on the truss are first obtained by employing the finite sine transform technique. After some manipulation, one can show that the sine transforms  $\tilde{H}_{TT}(j) = \tilde{H}_{TT}(\omega, j, x_0)$  and  $\tilde{H}_{TC}(j) = \tilde{H}_{TC}(\omega, j, x_0)$  (with respect to  $x$  over  $x = 0 \sim \ell$ ) of  $H_{TT}(\omega, x, x_0)$  and  $H_{TC}(\omega, x, x_0)$  satisfy the following equations.

$$\sum_{j=1}^{\infty} \tilde{H}_{TT}(j) d_{nj} - \sum_{j=1}^{\infty} \tilde{H}_{TC}(j) c_{nj} = \sin \frac{n\pi}{\ell} x, \quad (9)$$

( $n = 1, 2, \dots$ )

$$-\sum_{j=1}^{\infty} \tilde{H}_{TT}(j) c_{nj} + \sum_{j=1}^{\infty} \tilde{H}_{TC}(j) e_{nj} = 0, \quad (10)$$

( $n = 1, 2, \dots$ )

where

$$d_{nj} = (-\omega^2 m_T + i\omega \mu_T + EI \frac{n^4 \pi^4}{\ell^4}) \delta_{nj} + c_{nj} \quad (11)$$

$$c_{nj} = \frac{1}{\ell} \sum_{r=0}^{\infty} k_r \left\{ \delta_{(j+r), n} + \frac{|j-r|}{j-r} \delta_{|j-r|, n} \right\} \quad (12)$$

$$e_{nj} = (-\omega^2 m_C + i\omega \mu_C + H \frac{n^2 \pi^2}{\ell^2}) \delta_{nj} + c_{nj} \quad (13)$$

where  $\ell$  is the span length,  $\delta_{ij}$  the Kronecker delta, and  $k_r$  the coefficients of cosine series expansion of  $k(x)$  :

$$k(x) = \frac{2}{\ell} \sum_{r=0}^{\infty} k_r \cos \frac{r\pi}{\ell} x. \quad (14)$$

Eqs.(9) and (10) represent two sets of infinite number of equations for  $\tilde{H}_{TT}(n)$  and  $\tilde{H}_{TC}(n)$ . By taking only first  $N$  terms each of  $\tilde{H}_{TT}(n)$  and  $\tilde{H}_{TC}(n)$  ( $n, j = 1, 2, \dots, N$ , and  $r = 1, 2, \dots, 2N$ ), one can obtain a set of  $2N$  equations for  $2N$  unknowns  $\tilde{H}_{TT}(n)$  and  $\tilde{H}_{TC}(n)$  ( $n = 1, 2, \dots, N$ ). Solving these and applying the inverse sine transformation, the frequency response function  $H_{TT}(\omega, x, x_0)$  can be written as

$$H_{TT}(\omega, x, x_0) = \sum_{k=1}^N \alpha_k(\omega, x) \sin \frac{k\pi}{\ell} x_0 \quad (15)$$

where

$$\alpha_k(\omega, x) = \sum_{j=1}^N a_{jk}^{-1} \sin \frac{j\pi}{\ell} x \quad (16)$$

In the Eq.(16)  $a_{jk}^{-1}$  is the  $j - k$  member of the inverse matrix of a symmetric  $2N \times 2N$  matrix

$$A = \begin{bmatrix} D & -C \\ -C & E \end{bmatrix} \quad \text{with } D = [d_{ij}] \text{ , } C = [c_{ij}] \text{ and } E = [e_{ij}] .$$

The frequency response functions  $H_{CT}(\omega, x, x_0)$  of  $y_T^*(t, x)$  and  $H_{CC}(\omega, x, x_0)$  of  $y_C^*(t, x)$  due to an input  $e^{i\omega t} \delta(x-x_0)$  on the cable can also be obtained in a similar fashion.

$$H_{CT}(\omega, x, x_0) = \sum_{k=1}^N \beta_k(\omega, x) \sin \frac{k\pi}{\ell} x_0 \quad (17)$$

where

$$\beta_k(\omega, x) = \sum_{j=1}^N a_{j, N+k}^{-1} \sin \frac{j\pi}{\ell} x. \quad (18)$$

The functions  $H_{CC}(\omega, x, x_0)$  and  $H_{TC}(\omega, x, x_0)$  are not needed in the following analysis.

Making use of  $\alpha_k''(\omega, x)$  and  $\beta_k''(\omega, x)$ , one can show that the mean square spectral density function of  $M_T^*(t, x)$  is

$$\begin{aligned} S(\omega, x) = & \sum_{r=1}^N \sum_{s=1}^N \left[ \bar{\alpha}_r''(\omega, x) \alpha_s''(\omega, x) S_{rs}^{TT}(\omega) \right. \\ & + 2 \operatorname{Re} \left\{ \bar{\alpha}_r''(\omega, x) \beta_s''(\omega) S_{rs}^{TC}(\omega) \right\} \\ & \left. + \bar{\beta}_r''(\omega, x) \beta_s''(\omega, x) S_{rs}^{CC}(\omega) \right] \quad (19) \end{aligned}$$

in which  $\operatorname{Re} z$  and  $\bar{z}$  respectively indicate real part and complex conjugate of  $z$ ,

$$S_{rs}^{XY}(\omega) = \int_0^\ell \int_0^\ell S_{p_1 p_2}^{XY}(\omega) \sin \frac{r\pi}{\ell} x_1 \sin \frac{s\pi}{\ell} x_2 dx_1 dx_2 \quad (20)$$

with  $X$  and  $Y$  standing either for  $T$  or  $C$  and

$S_{p_1 p_2}^{XY}(\omega)$  being the cross-spectral density of  $p_x(t, x_1)$  and  $p_y(t, x_2)$ .

The variances  $\sigma_M^2$  and  $\sigma_{\dot{M}}^2$  of  $M_T^*(t, x)$  and  $\dot{M}_T^*(t, x)$  are then obtained as

$$\sigma_M^2 = \int_{-\infty}^{\infty} S(\omega, x) d\omega, \quad \sigma_{\dot{M}}^2 = \int_{-\infty}^{\infty} \omega^2 S(\omega, x) d\omega \quad (21)$$

In the following discussion, however, the second and third terms within the square brackets of Eq.(19) are neglected because of their small contributions (as also done in Refs.2 and 3), and  $S_{p_1 p_2}^{TT}$  is approximated by

$$S_{p_1 p_2}^{TT}(\omega) = \exp\left(-\frac{k\omega}{2\pi\bar{U}_z} |x_1 - x_2|\right) \Phi(\omega) \quad (22)$$

where the exponential term is the square root of the coherence,  $2\pi\bar{U}_z/(k\omega)$  the scale of turbulence at the wave length  $2\pi\bar{U}_z/\omega$ , and  $\Phi(\omega)$  is the mean square spectral density of  $p_T(t, x)$  and is given by<sup>2</sup>

$$\Phi(\omega) = 4 (\rho c_{d\tau} A_\tau \bar{U}_z)^2 K \frac{\bar{U}_{33}}{\omega} \frac{\left(\frac{\phi\omega}{\pi\bar{U}_{33}}\right)}{\left[1 + \left(\frac{\phi\omega}{\pi\bar{U}_{33}}\right)^2\right]^{4/3}} \quad (23)$$

in which  $K$  is the surface drag coefficient,  $\bar{U}_{33}$ , the mean wind velocity at the reference height of 33 ft above ground, is related to  $\bar{U}_z$  by

$$\bar{U}_1 = \bar{U}_z \left(\frac{33}{z}\right)^\alpha \quad (24)$$

with  $\alpha$  being a constant.

### 3. Safety Analysis

In previous papers<sup>4, 5</sup>, one of the present authors developed a method of estimating upper and lower bounds of the probability that a Gaussian random process  $z(t)$  will not be confined in a domain defined by  $-a(t) \leq z(t) \leq a(t)$  in a specified time interval, where  $a(t)$  ( $\geq 0$ ) is a deterministic function of time.

Consider the standard design procedure for wind loads where the stiffening truss is designed so that it can withstand, with a safety factor  $n$ , the bending moment  $M_d(x)$  produced by a specified (uniform) design wind pressure  $p_d$ . This implies that the critical bending moment at cross-section  $x$  is  $nM_d(x)$ . Suppose that the suspension bridge is subjected to a storm with mean wind velocity  $\bar{U}(t)$  or mean wind pressure  $\bar{p}(t)$  producing the bending moment  $\bar{M}(t, x)$ . Then,  $a(t, x) = M^*(x) - \bar{M}(t, x) = nM_d(x) - \bar{M}(t, x)$  is the maximum value of the bending moment  $M^*(t, x)$  that the fluctuating part of wind pressure  $p(t, x)$  can produce without failure. Since the variances of  $M^*(t, x)$  and  $\dot{M}^*(t, x)$  are evaluated in the preceding section, the method developed in References 4 and 5 can be applied to estimate upper and lower bounds of the probability of failure  $p_f$  or the probability that

$M^*(t, x)$  will not be confined in the domain defined by  
 $-a(t, x) \leq M^*(t, x) \leq a(t, x)$ .

Evidently, for a storm with a different mean value velocity function  $\bar{U}_z(t)$ , a different value of  $p_f$  is obtained. In fact,  $\bar{U}_z(t)$  itself is usually a random function of time containing a number of random parameters, say  $\bar{U}_z$  and  $T_0$ ;  $\bar{U}_z(t) = \bar{U}_z(t; \bar{U}_z, T_0)$ . For example, the following forms of  $\bar{U}_z(t, \bar{U}_z, T_0)$  are mathematically expedient and at the same time agree with observations reasonably well.

$$\bar{U}_z(t; \bar{U}_z, T_0) = \bar{U}_z \cdot e^{-(t/T_0)^2} \quad -\infty < t < \infty \quad (25)$$

and

$$\begin{aligned} \bar{U}_z^2(t; \bar{U}_z, T_0) &= \bar{U}_z^2 (1 - |t|/T) \quad -T \leq t \leq T_0 \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (26)$$

where  $T_0$  is a measure of the duration of a storm in Eq. (1) while it is the duration in Eq. (2). Eq. (1) is used in Reference 3.

The probability of failure  $p_f$  is then computed for a storm with a particular set of  $\bar{U}_z$  and  $T_0$ ;  $p_f = p_f(\bar{U}_z, T_0)$ . Therefore, the probability of failure  $p_f^*$  due to a single application of a statistical storm with  $\bar{U}_z$  and  $T_0$  being random is the expected value of  $p_f(\bar{U}_z, T_0)$  with respect to  $\bar{U}_z$  and  $T_0$ :

$$p_f^* = \iint p_f(\bar{U}_z, T_0) f(\bar{U}_z, T_0) d\bar{U}_z dT_0 \quad (27)$$

where  $f(\bar{U}_z, T_0)$  is the joint density function of  $\bar{U}_z$  and  $T_0$ . Hence, one can obtain the upper and lower bounds of  $p_f^*$  from those of  $p_f(\bar{U}_z, T_0)$  using Eq. (27).

#### 4. Numerical Example

As an example, a suspension bridge of the same dimension as the Forth Bridge is considered with  $EI = 1.842 \times 10^{13}$  lb-ft<sup>2</sup>,  $h(x) = 309 - 1200(x/\ell)(1 - x/\ell)$  ft,  $m_c g = 2.52 \times 10^3$  lb/ft,  $m_r g = 8.38 \times 10^{13}$  lb/ft,  $\ell = 3300$  ft,  $H = 4.934 \times 10^7$  lb (Eqs. (4) - (8)), and such values of the linear viscous damping coefficients  $\mu_r$  and  $\mu_c$  (Eqs. (6) and (7)) that the logarithmic damping decrements of the first mode of independent lateral vibration of the truss and of the cables are both equal to 0.05. In Eqs. (15) - (19),  $N = 5$  and in Eqs. (22) - (24),  $k = 7$ ,  $\phi = 2000$  ft,  $K = 0.01$ ,  $\alpha = 0.2$  and  $z = 200$  ft (height of the truss above ground as in Refs. 2 and 3).

With these parameter values, the variances of  $M^*(t, x)$

and  $\dot{M}^*(t, x)$  can now be evaluated numerically (IBM 7090 is used) following the method described in Section 2. Because of the same assumption on the structure and the wind, the variance of  $M^*(t, x)$  computed here is found to be close to those in Refs. 2 and 3. Once these variances are computed, the bounding technique in Refs. 4 and 5 can be applied for the probability of failure  $p_f(\bar{U}_z)$  with the time dependent barrier  $a(t, x)$ . Since in the present study, Eq. (26) is assumed for simplicity,  $a(t, x)$  becomes

$$a(t, x) = \xi(x) \left( \frac{1}{2} \rho c A \right) \left\{ n U_d^2 - \bar{U}_z^2 \left( 1 - \frac{|t|}{T} \right) \right\} \quad (28)$$

where  $\xi(x)$  is the bending moment of the truss at point  $x$  due to  $\bar{p}_r = 1$  lb/ft and  $\bar{p}_c = 1/8.9$  lb/ft (this value 8.9 is taken from Ref. 3 and it is the ratio between the corresponding values of  $cA$  for the truss and the cables) and  $U_d$  is the design wind velocity which is taken as 110 mph in this study.

If the maximum mean wind velocity  $\bar{U}_z$  is assumed to have the second asymptotic distribution of largest values<sup>6</sup> under a further assumption that  $\bar{U}_z \geq 110$  mph has a return period of 3450 years<sup>3</sup>, then the density function  $\bar{U}_z$  is given by

$$f(\bar{U}_z) = \frac{\gamma}{\bar{U}_c} \left( \frac{\bar{U}_z}{\bar{U}_c} \right)^{-\gamma-1} \exp \left[ - \left( \frac{\bar{U}_z}{\bar{U}_c} \right)^{-\gamma} \right] \quad (29)$$

where  $\gamma$  is assumed to be 9.0 and  $\bar{U}_c = 110 \left[ - \ln \left( 1 - \frac{1}{3450} \right) \right]^{1/\gamma}$  mph.

An additional assumption is made at this point that  $\bar{U}_z^2$  and  $T_0$  are proportional (or the intensity of storm and its duration are proportional) which appears to reflect the reality at least in approximation. In fact, a value  $\bar{U}_z^2 / T_0 = 5$  ft/sec<sup>3</sup> observed from some Japanese records<sup>3</sup> is used here. Because of this assumption, Eq. (27) becomes a single integration hence considerably reducing the computational work:

$$p_f^* = \int_0^\infty p_f(U_z) f(\bar{U}_z) d\bar{U}_z \quad (30)$$

It is evident from Eq. (26) that  $p_f(\bar{U}_z) = 1$  when  $\bar{U}_z \geq \sqrt{n} U_d$ . A further assumption  $c_r = c_{dr}$  (see Eqs. (1) and (2)) is made here so that the following analysis becomes independent of the value of  $\rho c_r A_r$ .

The upper and lower bounds of  $p_f^*$  are computed as a function of the safety factor  $n$  (Fig. 1). To be precise, the probability of failure  $p_f(\bar{U}_z)$  and therefore  $p_f^*$  vary along  $x$ . However, the variation is negligible because the quantities  $\xi(x) / \sigma_M(x)$  and  $\xi(x) / \sigma_{M_r}(x)$  on which the variation depends, are almost constant according to the



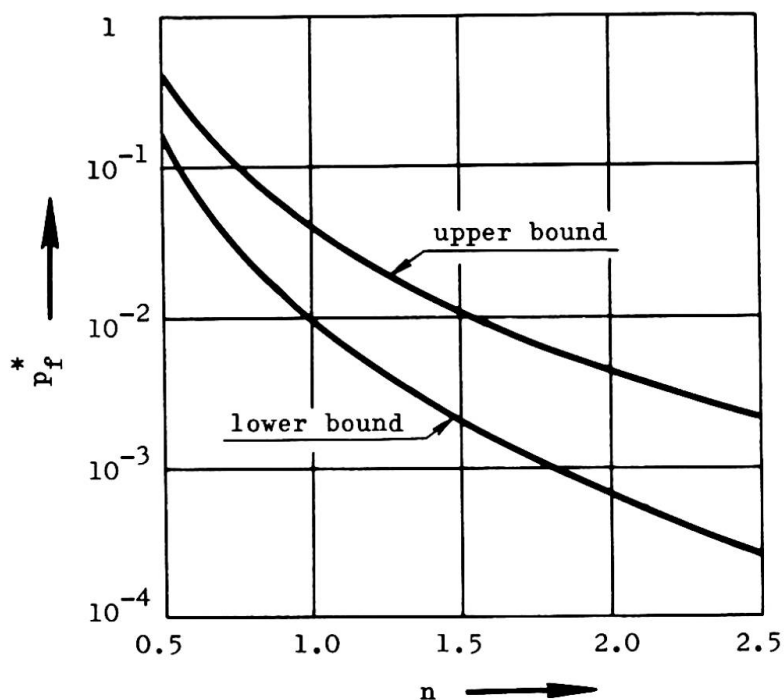


Fig. 1 Probability of Failure  $p_f^*$  as a function of safety factor  $n$

numerical computation.

In spite of the rather wide differences between the upper and lower bounds, the result shown in Fig. 1 is quite useful in many respects.

For example, using Fig. 1 one can examine the effect of increasing the safety factor

$n$ . In fact, Fig. 1 indicates that the probability of failure decreases by one order of magnitude from the order of  $10^{-2}$  to that of  $10^{-3}$  by increasing  $n$  from 1.0 to 2.0. This implies the increase

of the mean life by one order of magnitude from the order of 100 years to that of 1000 years, if it is assumed that significant storms occur on the average once a year. It is pointed out that from the view point of structural reliability analysis, the probability of failure estimated even only within the order of magnitude is a significant information.

## 5. Conclusion and Acknowledgement

A method of safety analysis by which the probability of failure of a suspension bridge due to lateral wind pressure caused by a (statistical) storm can be evaluated, is presented with a numerical example. The numerical example indicated that the probability can at least be estimated within the order of magnitude. This seems significant and satisfactory enough in view of the various

assumptions one has to make as to structural response properties as well as statistical characteristics of the wind.

This study identified the information that is needed to make such an analysis more reliable. Other than those already identified elsewhere (for example, Refs. 2 and 3), the following quantities have to be known with reasonable accuracy; the cross-spectral density  $S_{p_i p_j}^{\tau c}$  (Eq.(20)) and more importantly, the mean wind velocity  $\bar{U}_z(t)$  as a function of time  $t$  (Eqs.(25) and (26)) and its statistical nature, and the frequency of occurrence of significant storms.

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## SUMMARY

This study presents a method of safety analysis of aerodynamically stable suspension bridges subjected to lateral wind pressure. The pressure is treated as a random process in space as well as in time. A numerical example is given under certain assumptions of statistical characteristics of the wind velocity. Importance of such a study lies not only in the development of a method of probabilistic safety analysis but also in the fact that it indicates what further information, statistical or otherwise, is needed to make the safety prediction more reliable.

## RÉSUMÉ

Cette étude présente une méthode d'analyse de sécurité pour ponts suspendus aérodynamiquement stables soumis à une pression de vent latérale. La pression est supposée arbitraire dans l'espace et dans le temps. Un exemple numérique a été calculé à partir de certaines hypothèses des caractéristiques statistiques de la vitesse du vent. L'étude ne développe pas seulement une méthode d'analyse de sécurité probabiliste, elle indique avant tout quelles informations supplémentaires, statistiques ou autres, sont requises pour rendre les estimations de sécurité plus précises.

## ZUSAMMENFASSUNG

Dieser Beitrag zeigt ein Verfahren für die Sicherheitsbetrachtung aerodynamisch stabiler Hängebrücken, die seitlichem Winddruck ausgesetzt sind. Der Druck wird als zufälliges Ereignis in Raum und Zeit behandelt. Ein numerisches Beispiel für bestimmte Annahmen der statistischen Charakteristiken der Windgeschwindigkeit wird angegeben. Die Wichtigkeit solcher Untersuchungen liegt nicht allein in der Entwicklung der wahrscheinlichen Sicherheit, sondern auch darin, daß erkannt wird, welche statistischen oder sonstigen Auskünfte künftig für die Sicherheitsvoraussage zuverlässig sein werden.

## The Load Collapse for Elastic Plastic Trusses

La charge limite pour un treillis élasto-plastique

Traglast elasto-plastischer Fachwerke

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**Abstract** - The collapse load of a truss is investigated taking into consideration the way the bars actually behave, namely the effects of the strain hardening and the buckling respectively for the bars under tension and for those under compression.

During the buckling process the diagram which represents load versus axial deflection, on account of yielding of mid section, due to the bending, takes the form of a hiperbola branch (fig.1) [1] [2] [3]. At this stage, the bar, whose characteristic is a negative strain hardening - softening - becomes unstable. If, however, it is within a hyperstatic system, its buckling does not necessarily cause the collapse of the structure. Especially for multi-hyperstatic trusses, the collapse load may be found to be higher by far than the load generating the buckling condition of the first bar.

The problem has been put up with the restrictions as described in the following: The bars are pin hinged bars; the stress-strain relationship, as independent from the temperature and time, follows Prandtl's model [4]; the deflections are assumed to be infinitesimal, that is finite but small, just that the geometry of the system and thereby the internal condition of the stresses are not affected at all; both localized and global bifurcation phenomena are ruled out. Of this structure are discussed the stability conditions in the classical meaning, that is for infinitesimal perturbances.

This problem has already been dealt with by other authors [5], [6] [7]. From the stability postulate of Drucker's [8] [9] the sufficient conditions for stability and uniqueness of the solution have been deduced. In the discussion which follows only the first aspect of the question has been examined closely: By an original procedure, the necessary and sufficient stability conditions have been formulated.

The problem has been traced back to analysing the development to which is subjected the structural yield locus, which varies with the varying loads, under the action of incremental plastic deformations. Upon the external load reaching its critical value, to the increment of the plastic deformations corresponds a contraction in the yield locus which make it impossible to balance the original

load. From the discussion is possible to elaborate a graph which enables making a stability verification immediately, which can be made, however, for practical purposes, in the only case of two variables.

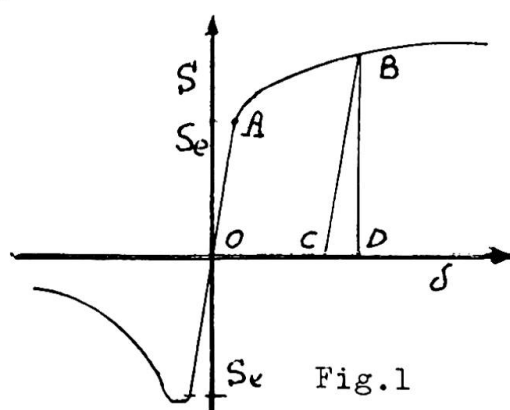
In the general case the problem is transferred into algebraic form: The parameter which indirectly furnishes the answer of the yield locus to an increase in the plastic deformations is determined by the energy irreversibly stored into the system: the elastic constrained energy and the energy dissipated through the plastic phenomena. If, to an increment whatever in the plastic deformation, the corresponding variation in the stored energy is still positive then the equilibrium is stable; if of no value or negative then at least in one case the equilibrium is neutral or unstable. The question is restricted to researching the sign of a quadratic form, associated with the matrix of rigidities, function of the plastic deformations and constrained thus by the signs of the latter.

These conditions can be brought to some other form as function of such parameters as are typical of the stability problems, that is the work done by the disturbing forces or the total energy of the system. It is demonstrable that if the variation occurring in the stored energy is either negative or zero the variation of the total energy of the resulting work done by the disturbing forces will likewise be either negative or zero. So we again come to a formulation which, though less practicable because of the further difficulty encountered in assessing the free elastic energy, connects directly to a principle which is as a rule normal within the elastic range or Drucker's postulate.

The problem is susceptible of generalizations. At this time the preference has been given to focussing the attention on the concepts rather than going deep into a more complex program.

The behaviour of the bars - The assumption is made that the bars, either in tension or compression, follow Prandtl's model [4], indifferently.

In fig.1 is shown the curve relative to the relationship existing between axial force  $S$ , elongation or shrinkage  $\delta$  for any bar in general. The bar behaves elastically according to Hooke's law up to stress  $S_e$ ; Past this point, plastic deformations take place, such that the linear trend of the line is changed. Upon relieving the load the representative point of the stress condition moves along the line parallel to  $O-A$ .



Segment  $O-C$  indicates the plastic deformation  $\bar{\delta}$ , at  $B$ , which at the time the load is relieved remains unaltered; segment  $C-D$  represents the elastic deformation  $\delta_e$ . If the bar is isolated for  $S=0$ ,  $\delta=\bar{\delta}$ ; if it is within a hyperstatic system, for  $S=0$ ,  $\delta=\bar{\delta}+\bar{\delta}_e$ , where  $\bar{\delta}_e$  indicates the elastic deformation constrained within the system and recoverably only through cutting the bar.

Area  $OABD$  represents the total

work performed by the external forces which is necessary to achieve pattern B. In particular OABC is the graphical representation of as much amount of energy as is absorbed by the system and is dissipated through the plastic phenomena; the area CBD is the elastic energy which can be returned only if the bar it is isolated or part of an isostatic system.

Unlike the currently adopted convention on the signs for the axial forces  $S$ , a different one is being introduced here. The starting axial force  $S$  is assumed to be positive in all cases; increments are either positive or negative whether or not they are in accord with the starting force.

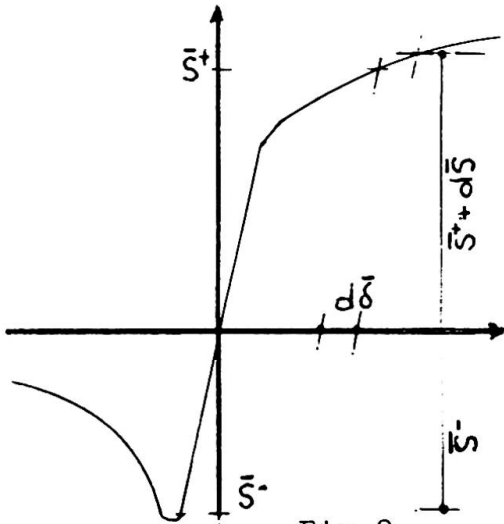


Fig. 2

For assigned plastic deformation  $\bar{\delta}$  (fig. 2),  $\bar{S}^+, \bar{S}^-$  are meant to be indicative of the limiting values within whose range the axial force can oscillate performing elastically. Therefore the yield locus shall be as established by the relation:

$$(1) \quad S \leq \bar{S}$$

where  $\bar{S}$ , generically, indicates the  $\bar{S}^+, \bar{S}^-$  limiting values according to whether is correspondingly a traction or compression. If the verification yields a disequality, the bar under test is in the elastic range, whereas the equality proves it is in the plastic range.

Where the bar is in the plastic range, i.e. if  $S = \bar{S}$ , the stress-strain relationship is linear, when the increments are infinitesimal: Curve  $S(\delta)$  is substituted with its tangential line at  $\bar{S}$ . Then by differentiating (1) in relation to  $\bar{\delta}$  or  $\delta$ :

$$(2) \quad dS \leq \frac{dS}{d\delta} d\delta = W d\delta = \frac{dS}{d\bar{\delta}} d\bar{\delta} = \bar{W} d\bar{\delta} = d\bar{S}$$

a limitation to the incremental relationship  $\bar{S}-\delta$  is obtained. Owing to a  $d\bar{\delta}$  increment in the plastic deformation the bar, initially stressed under  $\bar{S}$ , is now capable of taking a stress increment, at the limit,  $d\bar{S}$ : Therefore  $d\bar{S}$  determines the dislocation of the yield locus (fig. 2).

In the eq (2)  $\bar{W}$  represents the differential rigidity,  $\bar{W}$  the plastic differential rigidity (fig. 3): the following is the correlation of the above rigidities to the elastic rigidity  $W_e$ :

$$\bar{W} = \frac{W}{W - W_e}$$

the result is that where  $W \geq 0$ ,  $\bar{W}$  is likewise  $\geq 0$ . The plastic deformation  $d\bar{\delta}$  is restricted in sign by the relationship  $\text{sign } d\bar{\delta} = \text{sign } \bar{S}$ , which, for the position of on the forces signs, is reduced to condition:

$$(3) \quad d\bar{\delta} \geq 0$$

The interval within which rigidi-

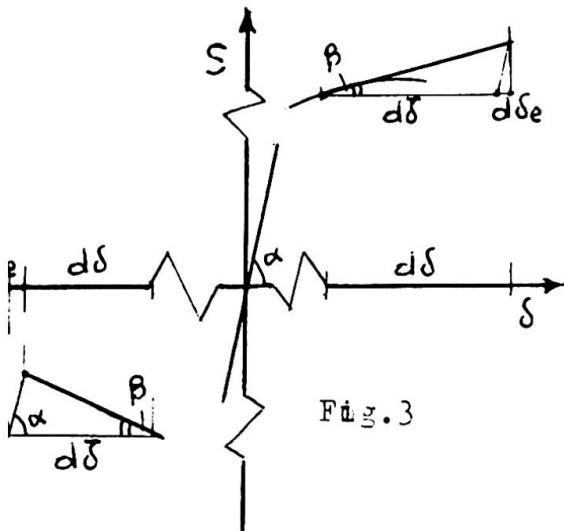


Fig. 3



ty  $W$  is included is so defined:

$$-\infty < W < W_e$$

By combining eq. (2) with limitations:

$$d\bar{\delta} > 0 \quad dS = dS = Wd\bar{\delta} \quad (-\infty < W < W_e)$$

$$d\bar{\delta} = 0 \quad dS = W_e d\bar{\delta} \quad (W = W_e)$$

the stress-strain incremental relationship is thus obtained. The eq. (2) covers the (4) and in a more general sense may be intended as relating to a cycle. At first, the incremental force  $dS$  verifies the equality with the bar being in the plastic range, subsequently is subjected to a reversal and thus verifies the disequality.

The behaviour of the system - As a reference, let it be taken a general type of reticular pin-hinged, made up by  $n$  bars, times  $r$  hyperstatic truss and let it be subjected to a loading pattern  $F$ : for an  $F_0$  load let  $C_0$  be the corresponding in equilibrium and compatible pattern, typified by  $k$  number of bars ( $k \geq r$ ) in plastic range,  $\bar{\delta}_1 \dots \bar{\delta}_k$  being the corresponding elongations.

Let the displacements of the system be assumed as being infinitesimal, or finite, but such that they cannot affect the ordinary geometry of the system and, hence, indirectly, the stressed condition. This supposes that the strain condition which corresponds to  $C_0$  can be regarded as borne by the plastic deformations  $\bar{\delta}$ , intended as distortions, and by loads  $F_0$ , as applied to the elastic structure.

This as a reference  $Se_i$  indicates the stress exercised by load  $F_0$  into bar "i";  $S_{ij}$  the stress transmitted to bar "i" through distortion  $\bar{\delta}_j = 1$  at "j". Then the resulting stress in bar "i" is:

$$(5) \bar{S}_i = Se_i + \sum_k S_{ik} \bar{\delta}_k \quad (i = 1 \dots n)$$

Eq.(5) is substituted in (1) by transferring to the right hand side the term relative to the distortions:

$$(6) Se_i \leq \bar{S}_i + \sum_k S_{ik} \bar{\delta}_k = \bar{S}_i$$

on the assumption that:

$$\bar{S}_i = Se_i + \sum_k S_{ik} \bar{\delta}_k$$

The  $\bar{S}_i$ , different, whether tractive or compressive, are a generalization of the  $Se_i$  referred in (1) and define, within the space of the plastic deformations, the yield locus for pattern  $C_0$ . If stresses  $Se_i$  verify the inequality, the point representative of the stress condition falls inside yield locus. On the contrary, if for some of the bars the equality has been verified the representative point falls onto the edge of the yield locus and the structure is in the plastic range.

A variation is assigned to pattern  $C_0$  by attributing to the bars in the plastic range a  $d\bar{\delta}$  increment to the initial plastic deformations. on the assumption that the bars in the elastic range will stay such. The resulting  $C'_0$  pattern is described as "perturbed" pattern. By differentiating (6) for the  $d\bar{\delta}$  increments assigned and consistent with (3) we obtain the stress increments which  $C'_0$  can absorb:

$$(7) dSe_i \leq \bar{W}_i d\bar{\delta}_i + \sum_k S_{ik} d\bar{\delta}_k = d\bar{S}_i$$



Eq.(7) is a generalization of eq.(2). The dislocation of the initial yield locus  $\bar{S}$ , consequent to the assigned plastic deformations  $d\bar{\sigma}_j$  is just supplied by the  $d\bar{S}$ . If the representative point of a new stress condition comes to fall inside of or into the edge of the yield locus, the equilibrium between the stresses and the strength of the bars is verified for pattern  $C'_0$ ; if outside, that is if for a certain number of bars: (8)  $dSe_i > d\bar{S}_i$

the equilibrium is impossible: the plastic deformations continue their pursuance to a new pattern  $C''_0$  which may still verify eq.(7).

Stability of the system - A graphical method for the verification of the stability, in which the above indicated concepts are expounded, is illustrated the problem being dealt with is limited to the case involving two plasticized bars only. It will not be difficult but rather easy to extend, conceptually at least, the representation to the more generalized case.

As a reference let us consider a Cartesian system having as many axes as are the plasticized bars. Let us mark on the axes plastic deformations  $d\bar{\sigma}$ : The origin of the axes thus defines the pattern  $C_0$ . As is conventional for the signs on plastic deformations (3), all  $C'_0$  patterns are comprehended within the quadrant of the positive  $d\bar{\sigma}$ . Choosing this as reference frame, we now draw as many straight lines  $d\bar{S}_i = 0$  as are the bars in the plastic range: the enveloping line defines the boundary of the plasticity field for that part which influences the stability of the system; on the perpendiculars are marked the stresses  $Se_i$  and the corresponding increments  $dSe_i$ . Therefore point  $C_0$  sets also the initial stress condition in which  $Se_i = \bar{S}_i$ .

Fixed the perturbed pattern  $C'_0$ , the sides of the yield locus translate: according to  $d\bar{S}_i \geq 0$  it will correspondingly expand or contract: the new yield locus, so obtained, is defined "perturbed". The equilibrium in this stage is assuredly verified if the transposition to  $C'_0$  is considered as effected by forcing a set of supplemental restraints, non efficient in  $C_0$ . Point  $C'_0$  moreover establishes the elastic stresses  $dSe_i$ , relative to the reactions  $dF$  of the additional restraints constituting the, so called, perturbing forces.

The supplemental restraints are then removed and, hence,  $dF \rightarrow 0$ : Where  $dSe_i \rightarrow 0$  the elastic stress condition  $C'_0$  has a tendency to resuming the initial position  $C_0$ . If  $C_0$  is found to fall inside the area of the perturbed yield locus, that is, if:

$$0 \leq d\bar{S}_i$$

eq.(7) is verified: the pattern settles in  $C'_0$  and the system behaves elastically again. If, on the contrary, for some of the bars eq. (8) is verified, that is

$$0 > d\bar{S}_i$$

$C_0$  comes to fall outside the perturbed field and there are no possibilities for an equilibrium. These bars keep being subjected to the plastic phenomenon with the field parallelly evolving in pursuance of a new pattern  $C''_0$  which comprehend  $C_0$ . More forces are supposed to be interfering at this stage such that a point-by-point equilibrium is as-

sured.

For example, in the case illustrated in figura 4, what  $C'_0$  might be, the resulting system is in any case that of equilibrium. Being that at all times  $d\bar{S}_1 > 0$ ,  $d\bar{S}_2 > 0$ , eq.(7) is verified, even where  $dS_e \rightarrow 0$ : The

perturbed yield locus shall always comprehend the originating pattern  $C_0$ . In this case the equilibrium of pattern  $C_0$  is stable.

A diametrically opposed case, is that shown in figura 5. Whatever  $C'_0$  the result is always  $d\bar{S}_1 < 0$ ,  $d\bar{S}_2 < 0$ . Hence by eliminating the perturbing forces eq.(8) is verified: within the two bars the plastic deformations increase. However, whatever the  $C''_0$  pattern which one can come to, during the unloading stage, the situation repeats itself again: the plastic deformations have a tendency to become infinitely great. Parallely the edge of the yield locus, originally  $\bar{S}$ , moves to  $\bar{S} - d\bar{S}$ : for  $d\bar{S} \rightarrow \infty$ ,  $\bar{S} - d\bar{S} \rightarrow 0$ : the plasticity field for at least one of its sides shrinks gradually up to becoming null. At  $C_0$  the equilibrium is therefore unstable.

Figures (6) and (7) report some intermediate situations. The first shows a case of stable equilibrium, the second one a case of instability.

In fig. 8 is then illustrated a situation of neutral equilibrium. Whatever  $C'_0$  the system is apt to assuming an equilibrium pattern  $C''_0$  coincident or not with the former. From this viewpoint the system is apparently stable. On the other hand, though, all patterns  $C'_0$  falling on straight line  $d\bar{S}_1 = d\bar{S}_2 = 0$  are also corresponded by  $dS_{e1} = dS_{e2} = 0$ . All these patterns and, to the limit, the infinity one, are then attainable without the aid of a perturbing set-up for forcing the system, and hence without any energy dissipation. Along this directrix the system is seemingly worn out, unfit to counteract the modification of the original pattern  $C_0$ . The situation as illustrated in fig.7 is unstable although still presenting an indifference directrix.

Even if hardly usable, owing to the unpractical possibility of extending it to an n dimension system, this graphical representation helps to clarify the problem and affords a comparison with

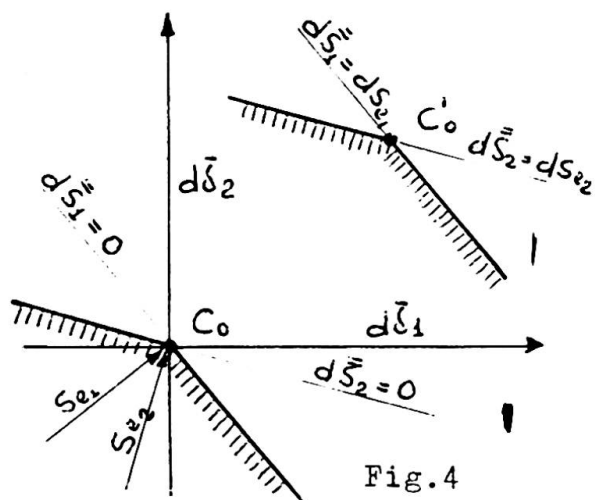


Fig.4

to  $\bar{S} - d\bar{S}$ : for  $d\bar{S} \rightarrow \infty$ ,  $\bar{S} - d\bar{S} \rightarrow 0$ : the plasticity field for at least one of its sides shrinks gradually up to becoming null. At  $C_0$  the equilibrium

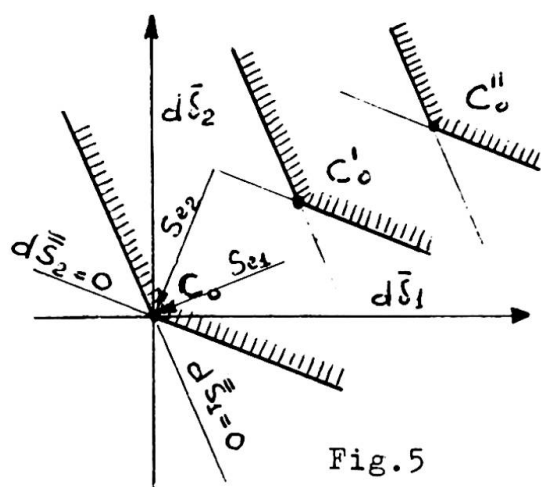


Fig.5

finity one, are then attainable without the aid of a perturbing set-up for forcing the system, and hence without any energy dissipation.

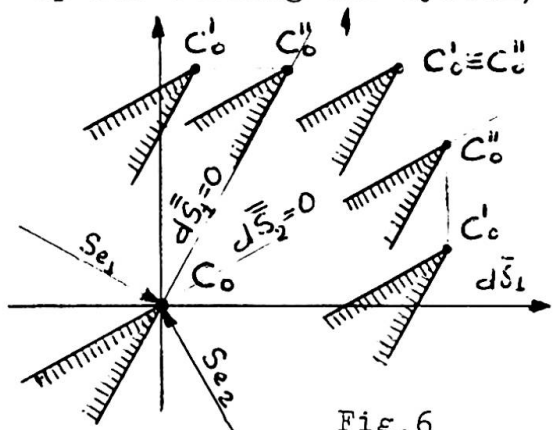


Fig.6



gy constrained within the system by the plastic deformations which can be released to the outside only by making cuts in such a way that the structure becomes isostatic. The third term,  $E_p$ , the irreversible energy absorbed by the system, used to produce those alterations in the internal structure of the material which give origin to the plastic dislocations.

For translating the system from pattern  $C_0$  to  $C_0'$  the work,  $d_2L$ , of the second order, done by the perturbing forces, taking into account the linearity of the stress-strain relationship, is

$$(11) \quad d_2L = \frac{1}{2} \sum dF d\eta = \frac{1}{2} \sum_n dS_{ei} d\delta_{ei} + \frac{1}{2} \sum_k \left( \sum_j S_{kj} d\bar{\delta}_j + \bar{W}_i d\bar{\delta}_i \right) d\bar{\delta}_i \\ = \sum_n dS_{ei} d\delta_{ei} + \sum_k dS_{ei} d\bar{\delta}_i = d_2E_1 + d_2E_v + d_2E_p = d_2E_1 + d_2\bar{E}$$

$d_2\bar{E}$  being the global constrained energy of the system both elastic and plastic.

The constrained energy  $d_2E$  is expressed by a homogeneous quadratic polynomial whose variables, however, are conditioned, in sign, by eq (3). For that part relative to the hiperquadrant 0 this polynomial coincides with the quadrantic form, associated to the matrix of the rigidities (7) and may result positive, null or negative: the last circumstance being possible in the sole case that, at least one bar be characterized by softening. The  $E_1$  and  $E_v$  polinomials are instead always positive.

Generalizing the notion of the total energy of the system 10 by adding, in addition to the positional energy of the external agencies, and the free elastic energy, also the constrained energy, eq.(11), after transferring to the right hand side the external work, defines the variation prime,  $dE_t$ , of the total energy, stationary for the  $C_0$  equilibrium pattern. Variation second  $d_2E_t$  is furnished instead by the right hand side of eq.(11).

Stability conditions - Let us suppose that the quadratic form  $d_2\bar{E}$ , devised for pattern  $C_0$ , is always positive for all the  $d\bar{\delta}$  consistent with (3), but not simultaneonally nought, that is:

$$(12) \quad d_2\bar{E} = \sum d\bar{S}_i d\bar{\delta}_i = \sum \left( \sum_j S_{ij} d\bar{\delta}_j + \bar{W}_i d\bar{\delta}_i \right) d\bar{\delta}_i > 0$$

In particular let for  $C_0'$  be:

$$dS_i = \left[ \frac{d}{d\bar{\delta}_i} (d_2E) \right]_{C_0'} \geq 0$$

Eq (7) verified at the beginning in respect to the interference of the perturbing forces still rests verified for  $dS_{ei} \rightarrow 0$ : through the unloading stage the system behaves in an elastic way. In the space of the  $d\bar{\delta}$  the pattern settles in  $C_0'$ .

Its supposed, instead, that for  $C_0'$  :

$$dS_i = \left[ \frac{d}{d\bar{\delta}_i} (d_2E) \right]_{C_0'} \geq 0$$

In this case, although as a whole eq. (12) is verified, same of the addenda result as being negative. Whith the elimination of the perturbing forces for some of the bars eq.(8) is verified. For such bars the plastic phenomenon then progresses spontaneously and the

system moves away passing from  $C'_0$  to  $C''_0$ . The second principle of the thermodynamics, as formulated by Lewis, [11] affirms that any spontaneous phenomenon is corresponded by a decrease in the system energy which is transformed into the work of the balancing forces, that is  $dF$ , in the present case. Thus, if with  $d_2 \bar{E}_{C'_0}$  we designate the energy corresponding to travel  $C_0-C'_0$ , and  $d_2 \bar{E}_{C''_0}$  that relative to  $C_0-C'_0-C''_0$ , the result will always yield:

$$(13) \quad d_2 E_{C'_0} > d_2 \bar{E}_{C''_0}$$

But, for the supposition made in eq. (12), the verification of this relationship can only be ascertained where  $C''_0$  within the space of the  $\bar{S}$  - comes to falling around  $C'_0$  and, hence  $C_0$ . The pattern  $C''_0$  defines a relative extreme (minimum) of function  $d_2 \bar{E}$ , conditioned by eq (3) and therefore:

$$d\bar{S}_i = \left[ \frac{d}{d\bar{S}_i} (d_2 E) \right]_{C''_0} \geq 0$$

Hence at  $C''_0$ , also for  $dS_{e1} \rightarrow 0$ , eq (7) is verified. So eq (12) represents a condition sufficient for  $C_0$  being a pattern of stable equilibrium.

As a substitute of (12) let us assume:

$$(12') \quad d_2 \bar{E} \geq 0$$

In particular then let, for  $C'_0$ , be  $d_2 \bar{E} = 0$ : In the other case we come to fall again within the preceding situation.

Allowing for eq.(12') the result will always yield:

$$d\bar{S}_i = \left[ \frac{d}{d\bar{S}_i} (d_2 E) \right]_{C'_0} \geq 0$$

Thus  $C'_0$  is a pattern of equilibrium with no interference of perturbing forces and as such are all those other patterns which fall into directrix  $C_0-C'_0$  which is justly typified by  $d_2 \bar{E}=0$ . The system moves along this direction with no external work being done. Then the following is particularly to be verified:

$$\begin{aligned} d\bar{S}_i &> 0 & \text{for } d\bar{S}_i &= 0 \\ d\bar{S}_i &= 0 & \text{for } d\bar{S}_i &> 0 \end{aligned}$$

Pattern  $C_0$ , which is corresponded by (12'), is then a pattern of neutral equilibrium.

For (12) let us assume as substitute:

$$(12'') \quad d_2 \bar{E} \leq 0$$

In particular is assumed as the assigned pattern  $C'_0$  that for which  $d_2 \bar{E} < 0$ . In this case for some of the bars:

$$d\bar{S}_i = \left[ \frac{d}{d\bar{S}_i} (d_2 E) \right]_{C'_0} < 0$$

The perturbing forces eliminated, the plastic phenomenon then progress: the energy relative to a successive pattern  $C''_0$  is related to the energy at  $C'_0$  by eq.(13). In  $C''_0$ , and so for the successive patterns, is thus repeated the like situation as is found in  $C'_0$ . The plastic phenomenon keeps continuing indefinitely with the system never reaching a pattern of equilibrium with load  $F_0$ . Therefore if the pattern  $C_0$  is associated to eq.(12'') the equilibrium is unstable.

The considerations on the eq.(12''), (12'') follows that eq.(12) represents also a condition necessary for the stability of the system.



Drucker's second stability postulate [8] [9], as applied in the "small", fully confirms this result. In order that the system is stable the closed cycle work accomplished by the perturbing forces, applied at first and removed afterward, is to be positive. As this cycle terminates this work is found again under the form of stored energy: thus if  $d^2\bar{E} > 0$  the equilibrium is stable. On the contrary, if  $d^2\bar{E} < 0$  the result is that the cycle cannot be closed, that is the equilibrium is not verifiable without the introduction of an equilibrating system  $dF$ : then the equilibrium is unstable.

From the above it can be easy to deduce that, where the bars behave in an ideally plastic way ( $W=0$ ), under the collapse load the equilibrium is neutral. True, in general  $d^2\bar{E} \geq 0$  ( $d^2E_p = 0$ ), particularly it nullifies for that  $d\delta$  set which is corresponded by the collapse mechanism. If the bars are instead strain hardened ( $W > 0$ ),  $d^2\bar{E} > 0$  as  $d^2E_p > 0$ : In this case the equilibrium is stable.

The stability according to Drucker's postulate - The first postulate of Drucker's states that a system is stable, in the "small", if the work accomplished by whatever forces  $dF$  yields always a positive result. If these forces are supposed as acting in a proportional way, the work accomplished by forces  $dF$  is coincident with the energy stored by the system, (11), that is the total energy variation. In the following is the demonstration that this principle and the one expounded in the preceding paragraph match perfectly at least as far as concerns the specific case under consideration. It is demonstrated particularly that if  $d^2\bar{E} > 0$  or  $d^2\bar{E} = 0$ , parallelly, always does exist at least one perturbing pattern  $dF$  for which  $d^2Et > 0$  or  $d^2Et = 0$ .

Let us assume that  $d^2\bar{E} > 0$  and as  $dF$  a system of forces proportionate to load  $F_0$  acting in  $C_0$ , characterized, thus, by a proportionality factor  $d\lambda$ , infinitesimal. Since the system results being unstable for a given number of bars  $d\bar{S}_i < 0$ . In order that  $C'_0$  be an equilibrium pattern, eq.(7) must be verified and the result  $dSe_i < 0$  must thus be yielded. Since, for convention, stresses  $Se_i$  are positive, factor  $d\lambda$  must be negative, or:

$$d\lambda Se_i = - dSe_i$$

The perturbing pattern  $dF$  must then result opposite to that  $F_0$ . In these conditions, at all times, eq.(7) is verified, even if plastic deformations are absent, in which case  $dSe_i = 0$ . Among the  $C'_0$  solutions which verify eq.(7) there exists at least one,  $C'_0$  which verifies also eq.(4) in its generalized form, or:

$$(14) - \begin{aligned} dSe_i &= d\bar{S}_i & d\bar{S}_i &> 0 \\ - dSe_i &< d\bar{S}_i & d\bar{S}_i &= 0 \end{aligned}$$

This solution defines one extreme of function  $d^2\bar{E}$  [12] [13] [14] conditioned by eq. (7) and in particular for the assumption adopted on the sign, (12"), it defines a maximum. The work accomplished by forces  $dF$ , in moving the system from pattern  $C_0$  to that  $C'_0$ , is then supplied by eq.(11) agrees with eq.(9) multiplied by the  $\frac{1}{2} d\lambda$  negative factor.

Since is always:  $dL > 0$

$$- \frac{1}{2} d\lambda \quad dL = d_2L < 0$$

Obviously, if  $d_2L < 0$ , such is also the right hand side of eq. (11) that is the variation  $d_2E_t$  of the total energy. This implies that in  $C_0$  if (12") is verified,  $F_t$  defines a maximum and there exists, at least, one perturbed pattern  $C'_0$  for which  $d_2L < 0$ .

On the contrary if  $d_2E \geq 0$ , for the patterns  $C'_0$  falling on the indifference directrix:

$$dSe_1 = \alpha_1 dF = 0$$

then:  $dF = 0$

$$d_2E_t = 0$$

If finally:

$$d_2E > 0$$

since  $d_2E_1 > 0$ , also  $d_2E_t > 0$ . In  $C_0$  the function  $E_t$  defines a minimum.

In the following a very simple example has been evolved. The structure is that as shown in fig.9. In figg.9, 9-a, 9-b, 9-c the graph shows plotted, in the upper part, the  $F_t$  force versus the slope  $\delta$ , at C, for the beam, whose behaviour is supposed to be infinitely elastic; in the lower part of the same graph for the stanchion subjected to a buckling at A, assuming three different values for rigidity  $\bar{W}_a$ . Starting from pattern  $C_0$ , to which corresponds load  $F_0 = F_t + F_a$ , an increment  $d\delta$  is attributed to the plastic deformation and pattern  $C'_0$  is reached. Addenda  $d_2E_1$ ,  $d_2E_v$ ,  $d_2E_p$ , all coming within the energy balance, hold as follows.

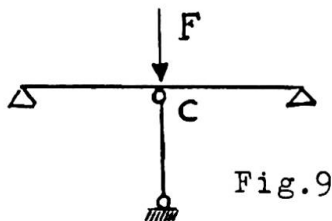


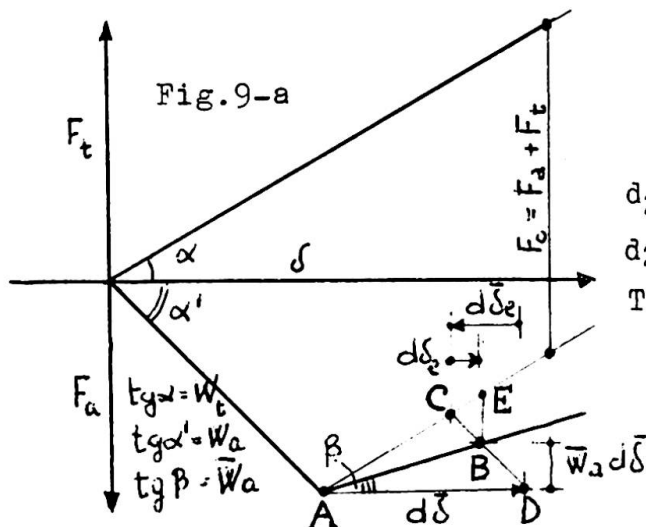
Fig.9

$$d_2E_e = \frac{1}{2} (W_a + W_t) d\delta_e = \text{CBE area}$$

$$d_2E_v = \frac{1}{2} (W_a d\bar{\delta} ed\bar{\delta}) = \text{ACD area}$$

$$d_2E_p = \frac{1}{2} \bar{W}_a d\bar{\delta}^2 = \text{ABD area}$$

In particular, for chart in fig.(9-a):



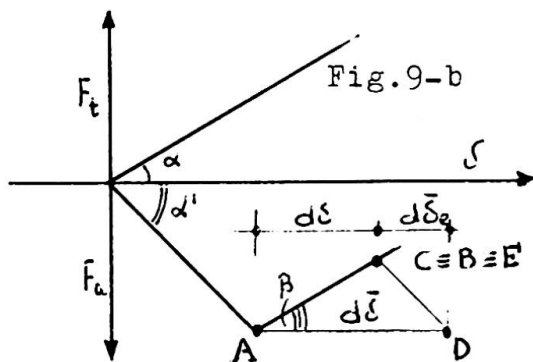
$$d_2\bar{F} = \text{ACD} - \text{ABD} = \text{ACB} > 0$$

$$d_2E_t = \text{ACB} + \text{CBE} = \text{ABE} > 0$$

The equilibrium is stable.



For chart in fig.(9-b):

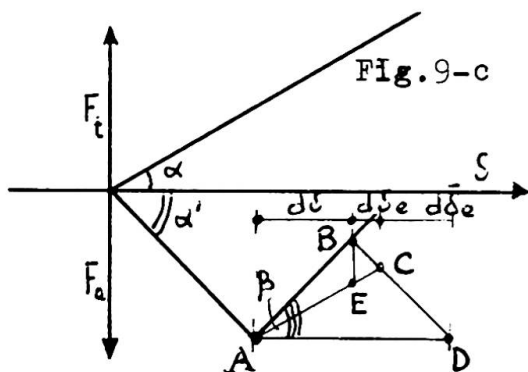


$$d_2 \bar{E} = ACD - ABD = 0$$

$$d_2 E_t = 0$$

The equilibrium is neuter.

For chart in fig. (9-c)



$$d_2 \bar{E} = ACD - ABD = -ABC < 0$$

$$d_2 E_t = ABC - CBE = -ABE < 0$$

The equilibrium is unstable.

### Conclusions

The stability analysis of an onomous system, whose components are stressed axially and are typified by positive and negative rigidities is led back to the study of function  $d_2 \bar{E}$ , that is the quadratic form associated to the matrix of the differential rigidities within the hyperquadrant of the positive  $d\bar{\delta}$ . If, within this boundary,  $d_2 \bar{E} > 0$  then the equilibrium is stable: on the contrary it is neutral or unstable.

By the avail of the matrices theory [14] some conclusions can be drawn. If the quadratic form, associated to the matrix of the rigidities, is definite positive, such it will be also in the hyperquadrant  $d\bar{\delta} > 0$ : therefore the result is  $d_2 \bar{E} > 0$ . Hence the equilibrium is stable. Instead if the quadratic form is definite negative, in like manner,  $d_2 \bar{E} < 0$ : the equilibrium is then unstable. The same holds true if the quadratic form is semi-definite negative: the range of the matrix can never be less than one, and thus the indifference direction, at the limit, can only occupy a subspace of the positive hyperquadrant, the quadratic form in the complementary subspace remaining negative.

More complicated the question presents itself where the quadratic form is semidefinite positive or indefinite: In the first case  $d_2 \bar{E} > 0$  or  $d_2 \bar{E} = 0$ , in the second case  $d_2 \bar{E} \geq 0$  or the intermediate cases. The research of an algorithm for the solution of this problem will be the subject of a forthcoming information.

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## SUMMARY

The stability analysis of an olnomous system, whose components are stressed axially and are typified by positive and negative rigidities is led back to the matrix of function  $d_2 \bar{E}$ , that is the quadratic form associated to the matrix of the differential rigidities within the hyperquadrant of the positive  $d\bar{E}$ . If, within this boundary,  $d_2 \bar{E} > 0$  then the equilibrium is stable: on the contrary it is neutral or unstable.

## RÉSUMÉ

L'analyse de la stabilité d'un treillis, dont les barres ne subissent que des efforts axiaux, est déduite à l'étude de la fonction  $d_2 \bar{E}$ . Si  $d_2 \bar{E} > 0$  le système est stable, sinon, il est neutre ou instable. Avec l'aide de la théorie des matrices [14] on peut tirer des conclusions sur la forme quadratique associée à la matrice. Le problème est plus ou moins simple, selon que cette forme quadratique est définie positive ou négative, ou semi-définie négative, ou alors si elle est semi-définie positive ou indéfinie. Ces derniers cas seront traités dans une information ultérieure.

## ZUSAMMENFASSUNG

In diesem Beitrag wird die Stabilität unter Berücksichtigung der Traglast an einem Fachwerk, deren Stäbe achsialer Kräfte unterworfen sind, untersucht und mit Hilfe der Matrizenrechnung die Fälle des stabilen, labilen oder instabilen Gleichgewichts beschrieben.

## DISCUSSION LIBRE / FREIE DISKUSSION / FREE DISCUSSION

**Essai sur modèle d'un pont courbe précontraint**

Modellversuche einer vorgespannten, gekrümmten Brücke

Trials on a curved and prestressed bridge model

R. BAUS — J.B. SCHLEICH — J. D'HUART — G. CLAUDE

**1. Caractéristique de l'ouvrage**

L'Administration des Ponts et Chaussées de Belgique vient d'entamer la Construction d'un complexe routier à Thuin, comprenant notamment un important viaduc franchissant la vallée de la Sambre.

L'ouvrage comporte 6 travées d'approche de 9,50 m, 12 m ou 21,5 m et trois travées centrales respectivement de 34 m, 60 m et 34 m. La longueur totale de l'ouvrage est de 235,5 m (fig. 1).

Les appuis sont constitués de 2 culées d'extrémité et de 8 piles intermédiaires ; la pile No 4 est encastrée à la base et au sommet, tandis que les sept autres piles sont encastrées au sommet et reposent à la base sur des appuis constitués de plaques de téflon (fig. 2).

Le pont a une section transversale en caisson, à inertie variable dans sa partie centrale, et à inertie constante dans les travées d'approche ; il est précontraint par câbles.

L'ouvrage se caractérise également par une pente longitudinale, une pente transversale variable, et un tracé courbe avec changement de courbure.

L'étude de l'ouvrage sur la base d'un avant-projet établi par l'Administration des Ponts et Chaussées a été confiée :

- au Bureau PRECO (Ing. Chaikès) de Bruxelles, pour les travées à inertie variable ;
- au Bureau RONDAS de Bruxelles pour les travées d'approche ; et
- au Bureau JEUNEHOMME et JOASSIN de Liège, pour l'infrastructure.

Les travaux de construction sont réalisés par une firme liègeoise : "La Compagnie Générale des Conduites d'Eau".

## 2. Caractéristiques du modèle

La complexité de l'ouvrage et l'incertitude sur la validité des hypothèses simplificatrices admises pour les calculs, ont conduit le maître de l'oeuvre à imposer une étude sur modèle.

Cette étude est actuellement en cours au Laboratoire de l'Institut du Génie Civil de l'Université de Liège.

Le modèle est réalisé à l'échelle 1/35e (fig. 3, 4, 5). Le matériau choisi est une résine époxy (araldite) chargée de sable. Les principales caractéristiques de ce matériau sont les suivantes :

- Résistance à la compression : 500 à 600 kg/cm<sup>2</sup>
- Résistance à la traction : 70 à 75 kg/cm<sup>2</sup>
- Résistance à la flexion : 100 kg/cm<sup>2</sup>
- module d'élasticité : 100 000 kg/cm<sup>2</sup>
- coefficient de Poisson : 0,27

La simulation du poids mort et des surcharges est réalisée à l'aide d'une série de poids suspendus au modèle (fig. 6). Un plancher mobile reposant sur des vérins hydrauliques permet d'effectuer à volonté la mise en charge uniforme et progressive de l'ensemble du modèle (fig. 7).

La mise en précontrainte de ce modèle est réalisée au moyen de 6 fils de 2 mm de diamètre, munis de dispositifs dynamométriques de mise en tension permettant un contrôle précis des efforts appliqués (fig. 8).

Les fils sont logés dans les gânes de laiton de manière à annuler les pertes par frottement (fig. 9).

### 3. Programme des essais

Des dispositifs dynamométriques disposés sous les piles permettent de déterminer les composantes des réactions d'appuis (fig. 10).

La détermination des flèches conduit, pour la travée centrale de 60 m, à un rapport  $f/l$  égal à  $1/1200$  sous poids mort et à  $-1/4000$  sous poids mort + précontrainte (flèche négative).

La détermination des moments de flexion et des moments de torsion a été prévue dans 23 sections du modèle. Chaque section comporte 18 jauges de contrainte "rosettes" permettant de déterminer les contraintes principales et les contraintes de cisaillement (fig. 9).

Dans les sections où les effets de courbure sont très faibles, les valeurs des moments de flexion et de torsion sous poids mort obtenus dans les essais sur modèle sont en excellent accord avec les résultats du calcul des moments flexions effectué sans tenir compte de la courbure ; l'écart maximum n'est en effet que de 6%.

Par contre, dans les zones où les effets de courbure sont importants, on observe sur le modèle des moments de flexion inférieurs de 15 à 20% à ceux calculés sans tenir compte des effets de courbure, ni de l'encastrement de la pile No 4.

La seconde phase des essais qui permettra de vérifier l'ouvrage sous poids mort + précontrainte est actuellement en voie d'achèvement.

Une troisième phase d'essais est envisagée pour permettre le tracé des surfaces d'influence des réactions, moments de flexion et moments de torsion.



# PONT COURBE PRECONTRAIT DE THUIN

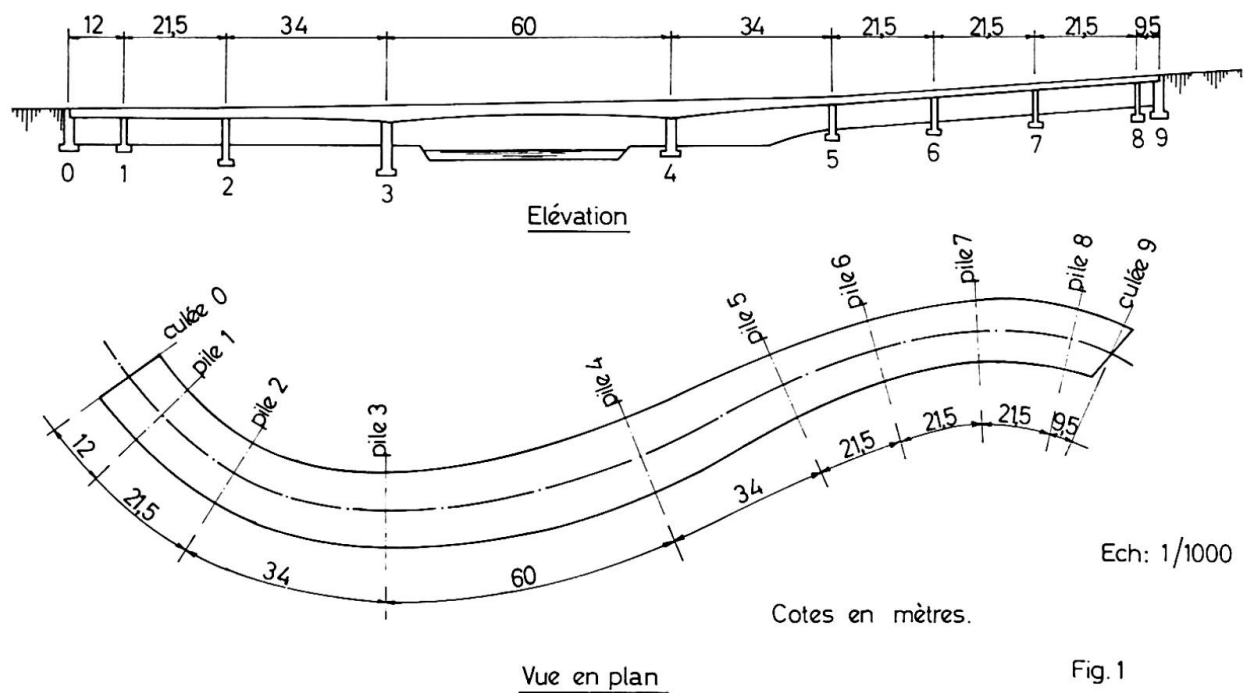


Fig. 1

# PONT COURBE PRECONTRAIT DE THUIN

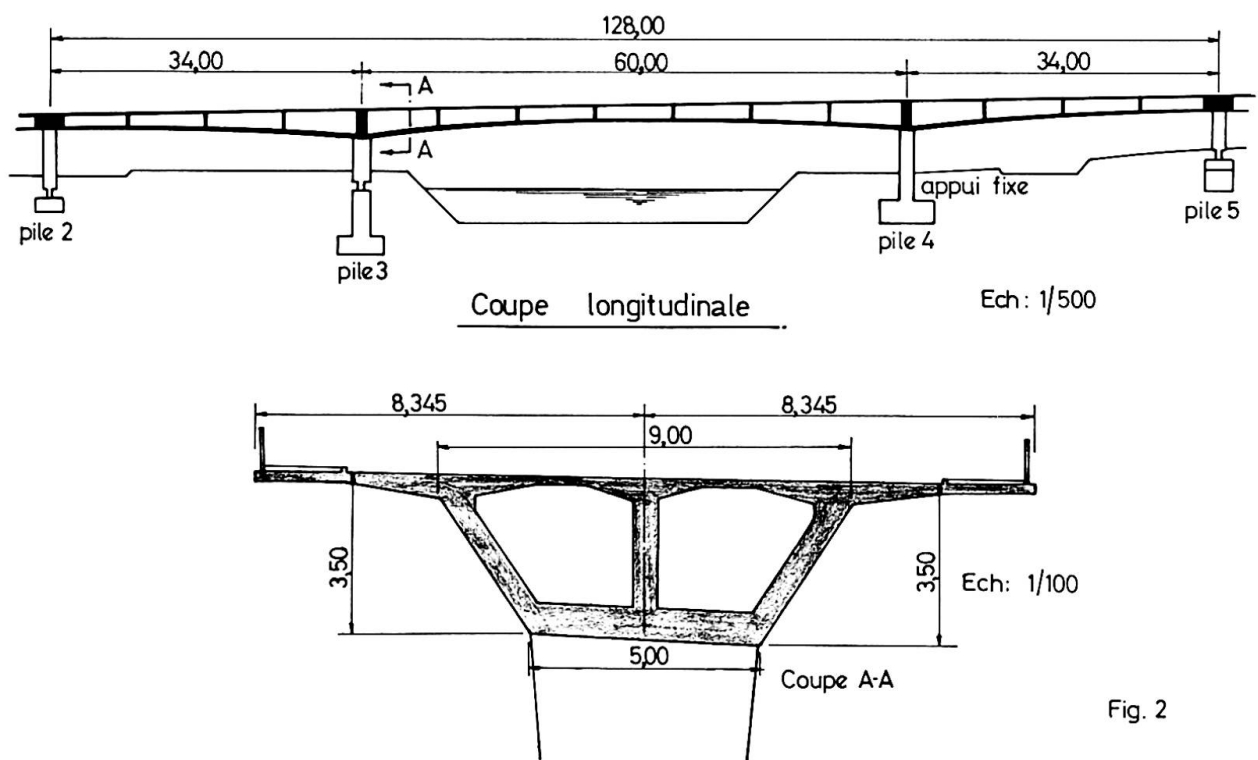


Fig. 2



Fig. 3.

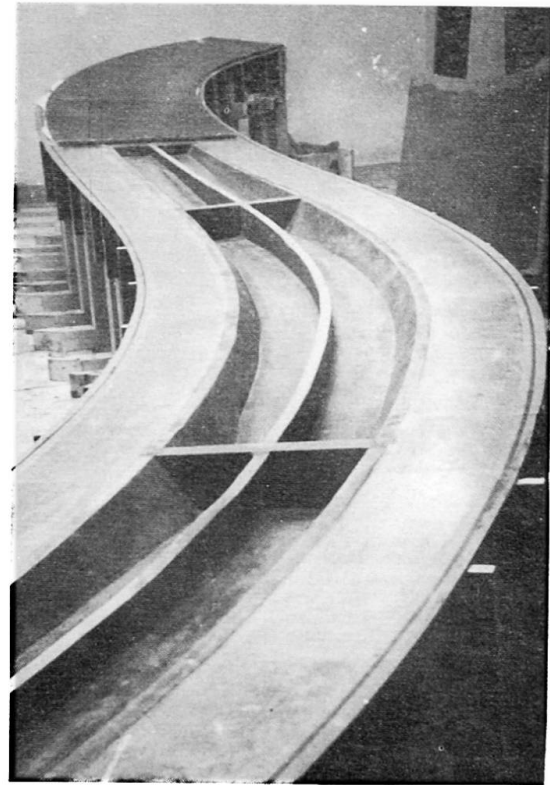


Fig. 4.

Vues partielles du modèle dans son moule.

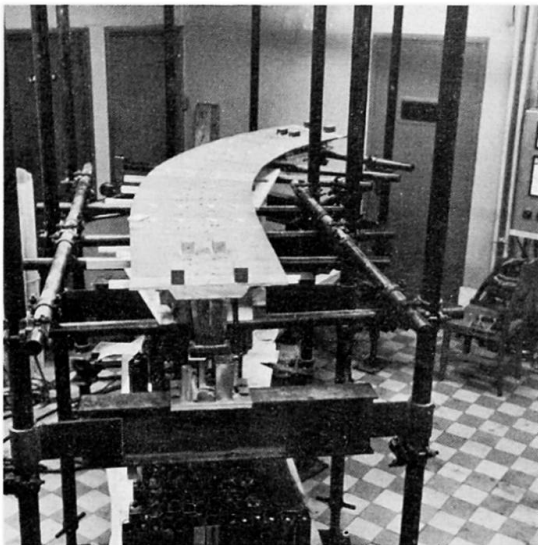


Fig. 5.

Partie centrale du modèle dans le montage d'essai et détail d'une pile.

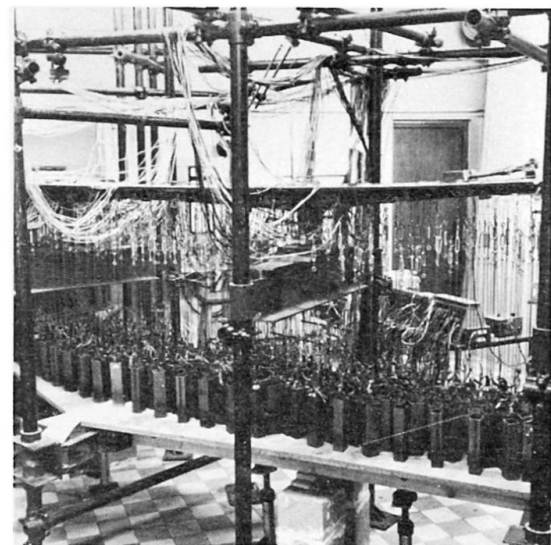


Fig. 6.

Simulation du poids mort et des surcharges au moyen de poids suspendus au modèle.

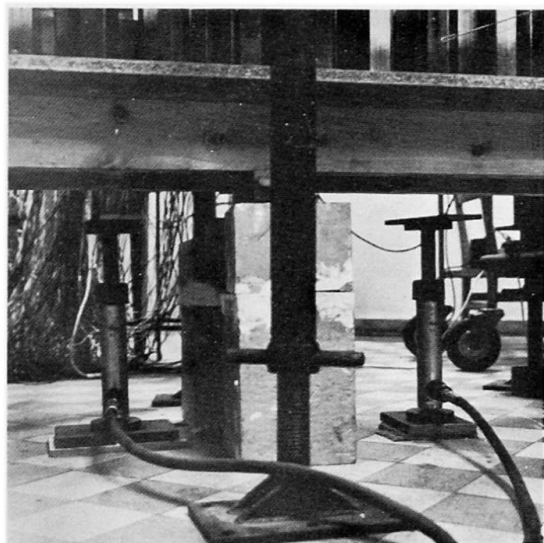


Fig. 7. Plancher mobile permettant la mise en charge uniforme et progressive du modèle.

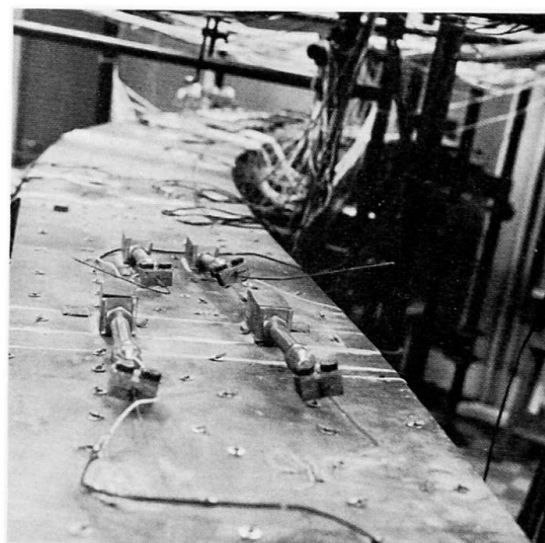


Fig. 8. Dispositif d'ancrage des fils de précontrainte avec boulons dynamométriques.

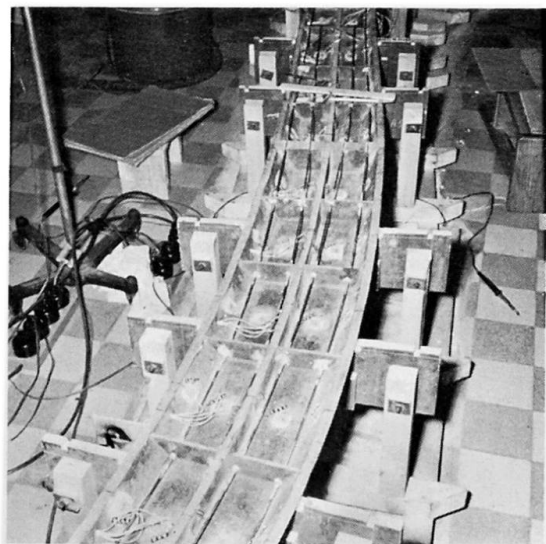


Fig. 9. Disposition des gâines de précontrainte et des jauges de contrainte à l'intérieur du caisson.

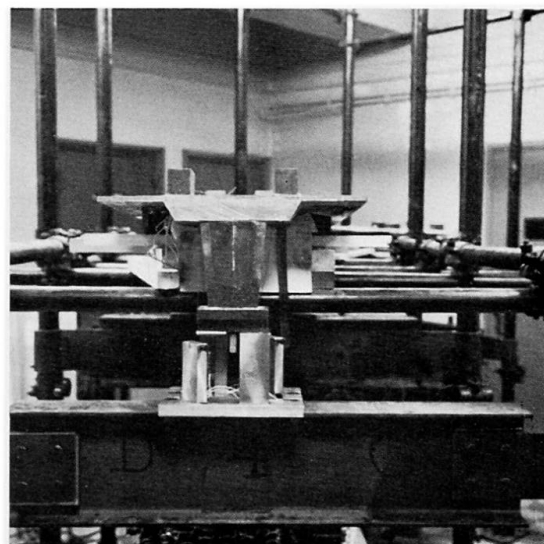


Fig. 10. Dispositif de mesure des composantes des réactions d'appuis.

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**MARIA EMILÍA CAMPOS E MATOS**  
Mozambique

À propos de la communication du prof. Oberti nous voulons informer que la sécurité et l'optimisation de la majorité des grandes structures tels que ponts, barrages, jetées, bâtiments, etc. projetées et construites dans les différents territoires Portugais depuis 20 ans, ont été déterminées expérimentalement sur des modèles réduits.

D'ailleurs le comportement de ces structures est toujours vérifié pendant et après construction.

Cela confirme les conclusions qu'on peut déduire de la communication du prof. Oberti.

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Freie Diskussion

Free Discussion

D. SFINTESCO

Par son intervention virulente, le Professeur Hrennikoff vient d'apporter la "contestation" dans la discussion sur le calcul probabiliste des constructions.

Parmi les arguments qu'il a développés, il y en a bon nombre qui paraissent incontestables et auxquels tout praticien sensé et réaliste ne peut que souscrire. Et cependant, il n'en va pas de même des conclusions qu'il en tire pour condamner le calcul probabiliste, et je suis heureux de constater que c'est également la position du Professeur Freudenthal.

Bien sûr, nous devons reconnaître que l'application pratique du concept probabiliste ne peut être que très imparfaite, tant que nous manquons encore de données statistiques suffisantes, notamment quant aux surcharges de caractère aléatoire, - le vent, par exemple. Toutefois, cela ne doit pas être une raison pour "jouer perdants", en nous figeant dans les errements d'un mode de calcul que l'on sait maintenant inapte à révéler le degré de sécurité réel des ouvrages : il vaut mieux oeuvrer vigoureusement pour obtenir ces données, au fur et à mesure que cela se pourra.

Certes, aucun coefficient de sécurité et aucun calcul ne sauraient prémunir contre les fautes graves de conception ou de réalisation - et cela revient à dire que tout calcul, quel qu'il soit, doit nécessairement se placer dans les conditions de respect des règles de l'art.

Un exemple très significatif est fourni par le danger de rupture fragile, source bien connue d'accidents. Or, celui-ci ne peut être évité que par une conception adéquate des formes constructives, mais jamais par l'introduction d'un coefficient supplémentaire dans le calcul, dont l'effet serait d'ailleurs exactement contraire au but recherché, car il ne ferait qu'augmenter la raideur, là où il faut la souplesse. Voilà donc un exemple qui illustre les arguments du Professeur Hrennikoff.

Cependant, les méthodes très exactes et très élaborées de calcul auxquelles on fait appel de plus en plus seraient dépourvues de sens si l'on n'adoptait pas un concept de base permettant de déterminer honnêtement le degré de sécurité, en se référant à un critère de ruine, défini dans le sens probabiliste et reflétant le comportement physique de l'élément ou de l'ouvrage considéré.

Ce serait une erreur de considérer que cela impliquerait obligatoirement le recours au calcul à l'état limite de ruine, soit le "plastic design", puisque le concept probabiliste s'applique également au calcul en élasticité.

Il suffit pour cela d'introduire des coefficients de pondération, appliqués aux charges et de se référer à un critère de ruine défini avec la même probabilité pour chaque mode de sollicitation. En procédant ainsi, on peut alors envisager un degré de sécurité homogène, indépendant du mode de sollicitation et pouvant être assez correctement chiffré. D'ailleurs, même la méthode actuelle, dite "déterministe", est assortie de probabilisme lorsqu'elle prévoit des contraintes admissibles différentes, suivant les cas de charges.

Mais il est temps de se libérer de ce système hybride, peu scientifique et parfois dangereux. En effet, le calcul pratiqué en appliquant aux contraintes un coefficient minorateur global, - donc le calcul par les contraintes admissibles, - n'est valable que dans les cas particuliers où il y a proportionnalité des charges et des contraintes, par exemple en traction pure, mais dans tous les cas impliquant des phénomènes d'instabilité, on est contraint de recourir à l'artifice des coefficients variables, arbitraires et dépourvus de signification.



Lorsqu'il y a superposition de plusieurs charges donnant lieu à des contraintes de signes contraires, c'est encore pire, et cette méthode est alors carrément dangereuse. En effet, une légère variation de l'une de ces charges peut donner lieu à une majoration démesurée de la contrainte, ou même à un renversement d'effort, que le calcul classique en contraintes admissibles n'annonce pas.

Le fait que d'innombrables ouvrages calculés ainsi se comportent parfaitement, n'est pas un argument. Dans les siècles passés, on en a bien construit d'autres sans les calculer du tout. Fallait-il alors en rester là ?

Une application du concept probabiliste de la sécurité au calcul en élasticité est préconisée dans les Recommandations de la Convention Européenne de la Construction Métallique. Les règles françaises CM 1966 en constituent un exemple pratique. Elles en ont adopté le principe et en ont fait l'application, dans les limites possibles aujourd'hui.

Il convient encore de signaler qu'un examen en commun des principes de sécurité est en cours, par la Convention Européenne de la Construction Métallique et le Comité Européen du Béton. Cet examen a déjà révélé :

- 1) que les principes probabilistes adoptés par les deux organismes dans leurs Recommandations respectives sont identiques, au point que l'on envisage d'en faire un document unique pour tous les matériaux,
- 2) que les modalités d'application de ces principes sont nécessairement différentes, compte tenu des caractéristiques particulières de chaque matériau et des conditions de réalisation des ouvrages respectifs.

En effet, les dispersions relatives aux divers paramètres ne sont pas les mêmes. De plus, certains paramètres importants pour un matériau n'entrent pas en ligne de compte pour l'autre. Il en résulte que les coefficients de pondération doivent prendre des formes et des valeurs différentes si l'on veut obtenir un même degré de sécurité des ouvrages.

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## **The Importance of Supervision of Workmanship in Building Construction**

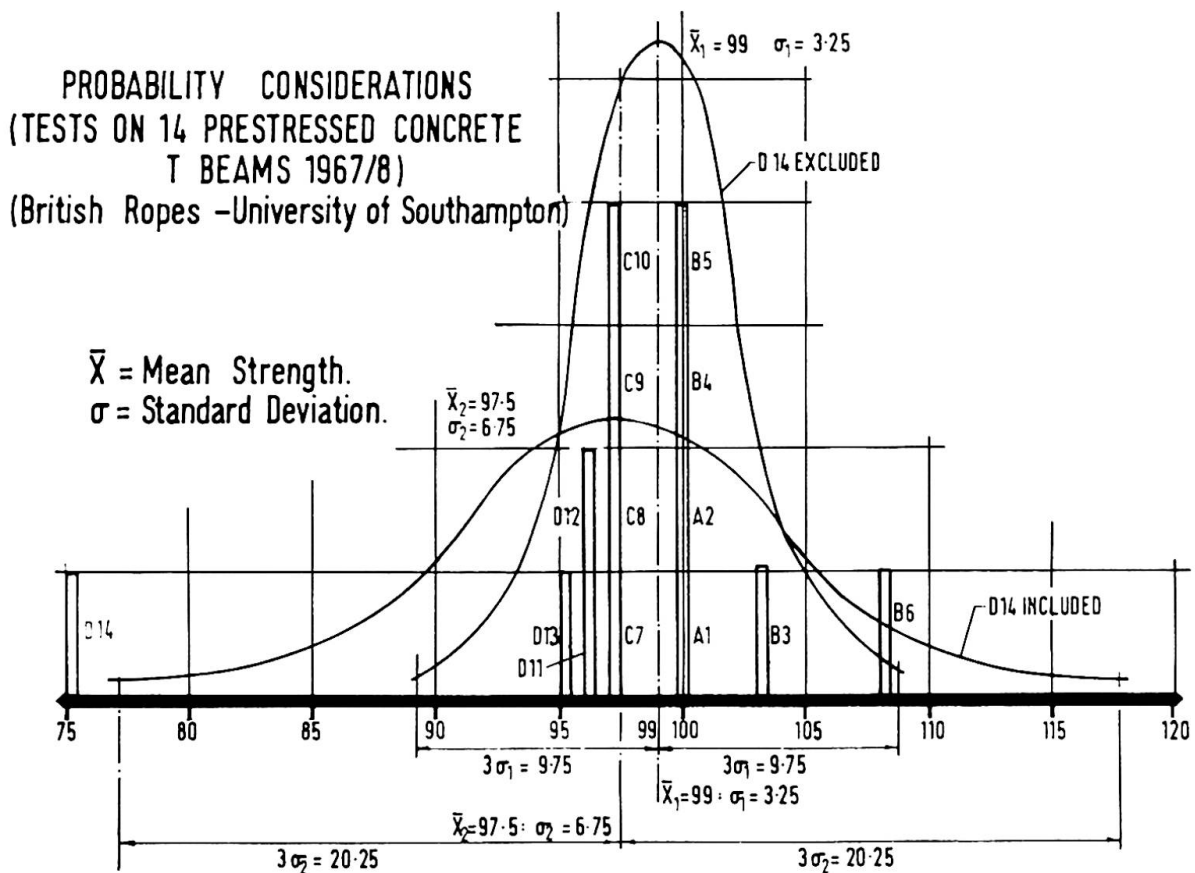
L'importance de surveiller la main d'oeuvre dans la construction

Über die Notwendigkeit der Überwachung der Arbeiten im Bauwesen

P. W. ABELES

Prof. Freudenthal has been one of the pioneers of the introduction of new safety criteria based on a new philosophy of design and his present paper represents a critical appraisal.

With structural members the workmanship is of utmost importance and in this respect the degree of supervision is of great influence which is, however, most difficult to assess at the time when the design is made. In IVb5 (Fig. 12), an example is shown, taken from the contribution in which thirteen of fourteen prestressed concrete beams agreed very well with the expected failure loads, whereas one beam failed at 75% of this load. This relates to partially prestressed T-shaped beams of 24' 6" span, the types A to D having different reinforcements comprising tensioned and non-tensioned strands. All beams are under-reinforced and were designed to fail by yielding. Casting as well as prestressing were specially supervised by one of the authors of paper IVb5. However, at the last beam D14 the supervising engineer had to leave shortly before the flange of 2' widths and 2" depth was concreted. At the test it turned out that a portion in the compression flange of this beam was not properly compacted and consequently early compression failure took place at 75% of the expected failure load.



This Figure shows the distribution curves if in one case all fourteen results are duly taken into account comparing calculated failure moments taken as 100 with actual failure moments. In this case the mean value is 97.5 instead of 100 with a standard deviation of 6.75, whereas these values are greatly improved to a mean of 99 with a standard deviation of 3.25 if beam D.14 is excluded. The corresponding Gauss distributions clearly indicate that it would be most disastrous if such faults in manufacture were expected to occur in practice due to lack of supervision. This example illustrates the difficulty of assessing the magnitude of standard deviation at the design and finding a satisfactory distribution.

This contribution is not intended to query the approach based on probabilistic methods, but rather to give an example of the difficulties which are encountered in ordinary building constructions.

**Ib**

**Combinaison des théories de l'élasticité, de la plasticité et de la viscosité dans l'étude de la sécurité des structures.**

**Untersuchung der Tragwerkssicherheit mittels der Elastizitäts-, Plastizitäts- und Viskositätstheorie.**

**Combination of the Theories of Elasticity, Plasticity and Viscosity in Studying the Safety of Structures.**

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## DISCUSSION PRÉPARÉE / VORBEREITETE DISKUSSION / PREPARED DISCUSSION

**Creep Failure of Nonlinear Rotational Shells**

Rupture par fluage de voiles minces axisymétriques non-linéaires

Kriechbruch nichtlinearer Rotationsschalen

W. OLSZAK  
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Poland

Z. BYCHAWSKI  
Assoc.Prof.Dr.**1. Introduction**

The authors have established [1] a criterion of the attainment of critical states in linear viscoelastic bodies. The idea of the criterion can also be extended to the range of nonlinear viscoelastic behavior, if the phenomenon of failure is considered as a critical state.

The criterion is founded on an energetical basis and for a certain group of nonlinear viscoelastic materials it states that such a critical state as, for example, creep rupture depends in general on a function of the accumulated energy and the dissipated power accompanying the deformation process. Thus, if  $W_E$  stands for the accumulated energy and  $\dot{W}_D$  is the dissipated power, the condition of creep rupture is stated as follows:

$$f(W_E, \dot{W}_D) = \text{const.} \quad (1.1)$$

In some cases, however, the accumulated part of energy may be neglectfully small. Moreover, there are materials which are not able to accumulate energy at all as, for example, the pure creeping ones. In these cases it seems reasonable to represent the criterion (1.1) in the following different form:

$$f(W_D) = \text{const.} \quad (1.2)$$

where  $W_D$  is the deviatoric dissipated energy per unit volume of the body. It follows from the condition (1.2) that the dissipated energy is considered as a certain measure of the attainment of the critical state. The correct form of this condition should be founded on experimental results.

The problem of attainment of a critical state as, for example, creep rupture may turn out to be essential when analysing the conditions occurring for thin-walled metallic structures un-



der high pressure, especially, high temperature containers, pneumatic structures, etc.

For such problems, we apply the criterion (1.2) to geometrically nonlinear rotational shells in the membrane state under internal pressure in order to evaluate the critical time of failure as a consequence of the creep process. Accordingly, we assume that the material of shells exhibits pure creep only.

## 2. Physical and geometrical equations

In general, we assume that an isotropic incompressible material of shells creeps according to the integral law [2]

$$e_{ij} = N s_{ij} , \quad (2.1)$$

where  $e_{ij}$  stands for the creep strain tensor,  $s_{ij}$  is the stress deviator and  $N$  denotes a nonlinear integral operator of the form

$$N s_{ij} = - \int_{t_0}^t s_{ij}(\tau) \partial_{\tau} H[t, \tau, s(\tau)] d\tau . \quad (2.2)$$

Here,  $H$  is the generalized creep function depending on the effective stress

$$s(t) = \frac{3}{2} [s_{ij}(t) s_{ij}(t)] , \quad (2.3)$$

$t$  standing for time,  $t_0$  being the initial instant and  $\partial_{\tau} = \partial/\partial\tau$ . As shown in [2] the generalized creep function  $H$  covers both the linear and nonlinear range of creep. However, in the present paper we use only its nonlinear representation.

In particular, the creep function may be assumed in such a form as to satisfy the following condition:

$$\partial_{\tau} H[t, \tau, s(t)] = F[s(\tau)] \partial_{\tau} C(t-\tau) , \quad (2.4)$$

where  $F$  is a nonlinear magnifying factor and  $C$  the creep factor. The last representation of the creep function is very useful when considering the non-steady states of creep in which the deformation stabilizes after an infinite period of time.

For the state of creep of metallic materials, the derivative of the creep factor becomes a constant, i.e.,  $C$  is a linear function of time

$$C(t-\tau) = c(t-\tau) , \quad (2.5)$$

where  $c$  is a constant. In the last case, Eq.(2.2) takes the form

$$N_0 s_{ij} = \int_{t_0}^t s_{ij}(\tau) F_0[s(\tau)] d\tau \quad (2.6)$$

As it is seen from Eq.(2.6), we assume that the state of stress is variable with time. We shall show later that in the case of creep

of nonlinear shells in membrane state, the stress state is always a non-steady one in the presence of a constant internal pressure. The stresses are found to drop to zero in an infinite period of time.

According to Eqs. (2.2) and (2.6), the initial conditions at  $t$  equal  $t_0$  are assumed to be of zero value, i.e., there are no deformations at the initial instant. However, these conditions may not be zero, if we consider a creep process at  $t > t_0$ . In this case there is an initial deformation state expressed instantaneously by the values of integrals within the limits  $t_0, t$ .

The equations (2.1) and (2.6) can be written in terms of strain rates as follows:

$$\dot{\epsilon}_{ij} = \dot{N} s_{ij} \quad , \quad \dot{\epsilon} = \dot{N}_0 s_{ij} \quad , \quad (2.7)$$

where the dots over the operators are symbolic. For example, in the case of the second relation (2.7) we find

$$\dot{N}_0 s_{ij} = s_{ij}(t) F_0[s(t)] \quad . \quad (2.8)$$

We apply the physical equations (2.1) or (2.6) in order to investigate the critical states of rotational membranes of small and constant thickness which deform under constant internal pressure. In deriving the geometrical relations for such membranes we assume that in the time-interval considered, the strain tensor and strain rate tensor are small quantities, the rotation angles being also small; the normal component of displacement is supposed to have a finite value. Further, we assume that the undeformed surface is generated by the revolution of a plane curve which does not imply any singularities. In order to simplify the equations we restrict ourselves to shallow rotational membranes. For such membranes we obtain a set of two equations of equilibrium [2]

$$d_\varphi(\varphi \sigma_1) = \sigma_2 \quad , \quad (2.9)$$

$$\sigma_1(\bar{k}_1 - d_\varphi^2 \bar{w}) + \sigma_2(\bar{k}_2 - \frac{1}{\varphi} d_\varphi \bar{w}) = \frac{p}{h} \quad , \quad (2.10)$$

where  $\sigma_{ij}$  and  $\sigma_2$  are the stresses in the directions of the main curvatures  $k_1$  and  $k_2$ , respectively,  $\bar{w}$  is the displacement normal to the surface,  $p$  the constant internal pressure,  $h$  the thickness of the membrane and  $\varphi$  denotes the surface coordinate. The symbol  $d_\varphi$  represents the derivative  $d/d\varphi$ .

To the set of Eqs. (2.9) and (2.10) we now join the equation of compatibility of deformations [2]

$$\varphi d_\varphi e_2 + e_2 - e_1 = -\frac{1}{2}(d_\varphi \bar{w})^2 + d_\varphi(\varphi \bar{k}_2 \bar{w}) - \bar{k}_1 \bar{w} \quad , \quad (2.11)$$

where  $e_1 = e_{11}$ ,  $e_2 = e_{22}$  are the main strains.

Further, we introduce the following substitutions:

$$r = \left(\frac{\varphi}{R}\right)^2, \quad z = \frac{r}{D} \sigma_1, \quad D = \frac{p}{g}, \quad g = \frac{h}{R} \quad , \quad (2.12)$$

$$w = \frac{\bar{w}}{h}, \quad k_1 = R \bar{k}_1, \quad k_2 = R \bar{k}_2 \quad ,$$

where  $R$  is the value of the larger radius of main curvatures.

On the basis of Eq. (2.12), the stress deviator components expressed by stress components become

$$s_1 = s_{11} = \frac{1}{3} (3\sigma_1 - 2\sigma_2) = \frac{1}{3} D \left( 3\frac{z}{r} - 2d_r z \right), \quad (2.13)$$

$$s_2 = s_{22} = \frac{1}{3} (3\sigma_2 - 2\sigma_1) = \frac{1}{3} D (4d_r z - 3\frac{z}{r}). \quad (2.14)$$

By introducing the quantities of Eq. (2.12), we satisfy Eq. (2.9) and Eq. (2.10) takes the form

$$\frac{z}{r} [k_1 - 4g\sqrt{r} d_r (\sqrt{r} d_r w)] + (2d_r z - \frac{z}{r}) (k_2 - 2gd_r w) = 1. \quad (2.15)$$

If the use of the physical relations (2.1) is made, then the main strains become

$$e_1 = Ns_1 = N \left[ \frac{1}{3} D \left( 3\frac{z}{r} - 2d_r z \right) \right], \quad e_2 = Ns_2 = N \left[ \frac{1}{3} D (4d_r z - 3\frac{z}{r}) \right], \quad (2.16)$$

and the condition of compatibility may be written as follows [see Eq. (2.11)]:

$$2\sqrt{r} d_r (\sqrt{r} Ns_2) - Ns_1 = -2g^2 r (d_r w)^2 + 2g\sqrt{r} d_r (\sqrt{r} k_2 w) - gk_1 w. \quad (2.17)$$

The set of equations (2.15) and (2.17) is a system determining two unknown functions: the non-dimensional stress function  $z$  and the non-dimensional deflection  $w$ . Thus, the solution of the above system of equations gives the solution of the problem of creeping nonlinear membranes.

In the particular case of physical relations (2.6), we put in Eq. (2.17)  $N_0$  instead of  $N$ . In this very case the condition of compatibility may be presented in terms of strain rates and the relations (2.8) applied.

### 3. Concept of analogy

It has been found [2] that in the case of purely creeping nonlinear rotational membranes it is possible to obtain the creep solution by separating the variables  $r$  and  $t$  in the fundamental set of Eqs. (2.15) and (2.17). Then the time-independent set of equations is analogous to the corresponding system of the instantaneous problem, if only the nonlinear functions of deviatoric stress intensity are of similar forms in both cases. The time-dependent set of equations can be solved in a closed form. It follows from the last solution that the creep process of nonlinear membranes is always unsteady. If the form of the generalized creep function is assumed according to Eq. (2.4), where the creep factor is expressed by exponential functions, then the solution describes a stabilizing process of creep. On the other hand, if the particular case of Eq. (2.6) is considered and creep is unlimited, a complete relaxation of stresses occurs after an infinite period of time and strains become infinite. From the physical point of view such a state of the membrane cannot occur and at a certain finite time-instant the creep rupture takes place. We assume as a measure of reaching this point the value of dissipated energy during the creep process. Thus,

if a certain critical value of energy is dissipated through creep resistance, the membrane is considered as collapsed. From an appropriate condition of the form (1.2) it is then possible to find the critical time of creep failure.

#### 4. Shallow spherical membrane

We shall consider in detail a shallow spherical membrane with the radius of curvature  $R$ . For such a membrane we obtain the following system of equations

$$\sigma_1(\bar{k} - d_\varphi^2 \bar{w}) + \sigma_2(\bar{k} - \frac{1}{g} d_\varphi \bar{w}) = \frac{P}{h}, \quad \bar{k} = 1/\bar{R}, \quad (4.1)$$

$$g d_\varphi e_2 + e_2 - e_1 = -\frac{1}{2} (d_\varphi \bar{w})^2 + d_\varphi (g \bar{k} \bar{w}) - \bar{k} \bar{w}, \quad (4.2)$$

and Eq. (2.9).

The equation (4.1) may be at once integrated by using Eq. (2.9), the constant of integration being equal zero. Thus, instead of Eq. (4.1), we obtain

$$d_\varphi \bar{w} = g(\bar{k} - \frac{P}{2h\sigma_1}). \quad (4.3)$$

Further, we introduce the following notations:

$$\tau = \left(\frac{g}{1}\right)^2, \quad z = \frac{\tau}{D} \sigma_1, \quad D = \rho \left(\frac{1}{2h}\right)^2, \quad w = \frac{\bar{w}}{h}, \quad (4.4)$$

$$k = 1/R, \quad R = \bar{R}/h, \quad g = h/1$$

where 1 is the maximum value of the variable  $\varphi$  (for  $\varphi = 1$ ,  $\tau = 1$ ).

With the above notations, the stress deviator components have the form of Eqs. (2.13) and (2.14), and the strains are given by the formula (2.16).

Considering the radius of curvature  $R$  as time dependent, we differentiate Eq. (4.2) with respect to time and thus obtain

$$g d_\varphi \dot{e}_2 + \dot{e}_2 - \dot{e}_1 = -d_\varphi \dot{\bar{w}} d_\varphi \bar{w} + d_\varphi (g \dot{\bar{k}} \bar{w}) + d_\varphi (g \bar{k} \dot{\bar{w}}) - \dot{\bar{k}} \bar{w} - \bar{k} \dot{\bar{w}} \quad (4.5)$$

Finally, by using the second of Eqs. (2.7) and introducing the notations (4.4), we represent Eqs. (4.3) and (4.5) in the following form:

$$d_r w = \frac{1}{2} \frac{k}{g^2} - \frac{\tau}{z}, \quad (4.6)$$

$$2r d_r \dot{N}_0 \left[ \frac{1}{3} D (4d_r z - 3 \frac{z}{\tau}) \right] + \dot{N}_0 \left[ \frac{1}{3} D (4d_r z - 3 \frac{z}{\tau}) \right] - N_0 \left[ \frac{1}{3} D (3 \frac{z}{\tau} - 2d_r z) \right] + 2\tau \left[ d_r \dot{w} (2g^2 - d_r w - k) + \frac{k}{\dot{k}} d_r w \right] = 0. \quad (4.7)$$

Here the operator  $\dot{N}_0$  is given by Eq. (2.8).

It is seen from the set of Eqs. (4.6) and (4.7) that the displacement  $w$  can be easily eliminated from the second equation by means of the first one.

Now, let us assume that the nonlinear function of effective stress appearing in Eq. (2.8) is a power function of the form

$$F_0[s(t)] = \frac{3}{2} B s^{n-1}(t) , \quad (4.8)$$

where B and n are physical constants, the last being an odd natural number.

According to Eq. (2.3) and Eqs. (2.13), (2.14), the effective stress is expressed as follows

$$s^2(r, t) = D^2 \Omega(r, t) , \quad (4.9)$$

where

$$\Omega(r, t) = \Omega(z) = 4(d_r z)^2 - 6d_r z \frac{z}{r} + 3\left(\frac{z}{r}\right)^2 . \quad (4.10)$$

Introducing Eq. (4.8) together with Eqs. (4.9) and (4.10) and eliminating the displacement in Eq. (4.7), we finally obtain

$$\Omega^{\frac{1}{2}(n-3)} [8\Omega d_r^2 z + (n-1)(4d_r z - 3\frac{z}{r})d_r \Omega] = 2\gamma \left[ \frac{z}{r} \left( \frac{r}{z} \right)^2 - \left( \frac{2Dk}{p} \right)^2 \right] \frac{k}{k} \quad (4.11)$$

where

$$\gamma = \frac{p}{BD^{n+1}} . \quad (4.12)$$

The method of solution of the problem for a creeping membrane is founded on the basis of an analogy as stated above. We assume the solution of Eq. (4.11) in the form

$$z(r, t) = z^0(r) \varphi(t) , \quad (4.13)$$

and put

$$k(t) = k^0 / \varphi(t) . \quad (4.14)$$

If we introduce the solution (4.13) into Eq. (4.11), then after separating the variables we find

$$\begin{aligned} \frac{1}{\gamma} \left[ \left( \frac{2Dk}{p} \right)^2 - \left( \frac{r}{z^0} \right)^2 \right]^{(-1)} \Omega_0^{\frac{1}{2}(n-1)} [8\Omega_0 d_r^2 z^0 + (n-1)(4d_r z^0 - 3\frac{z^0}{r})d_r \Omega_0] &= (4.15) \\ &= -2 \frac{\dot{\varphi}(t)}{[\varphi(t)]^{n+3}} = \lambda , \end{aligned}$$

where

$$\Omega_0(r) = \Omega_0(z^0) = 4(d_r z^0)^2 - 6d_r z^0 \frac{z^0}{r} + 3\left(\frac{z^0}{r}\right)^2 , \quad (4.16)$$

$\lambda$  being a constant.

The time-independent part of Eq. (4.15) is analogous to the equation for an instantaneous problem, if only the physical equation is of a form analogous to Eq. (4.8). Thus, if the solution of the instantaneous problem is known, we are able to obtain the creep solution in a formal way. The time-independent solution is obtained by representing the stress function in the form of power series.

On the other hand, the time-dependent part of Eq.(4.15) may be written as follows:

$$\dot{\varphi}(t) + \frac{1}{2} \lambda [\varphi(t)]^{n+3} = 0. \quad (4.17)$$

The variables in Eq.(4.17) are separable and the solution is given by the formula

$$\varphi(t) = \varphi_0 \left[ 1 + \frac{1}{2} \lambda \varphi_0^{n+1} (n+2)(t-\bar{t}) \right]^{-\frac{1}{n+2}}, \quad (4.18)$$

where the constant of integration  $\varphi_0 = \varphi(\bar{t})$ . According to the solution (4.18) we consider as initial instant of the observed creep process a certain intermediate time-point at which the past creep effects are taken into account instantaneously.

In order to obtain the appropriate solution for the displacement  $w$ , we assume the last in the form

$$w(r, t) = w^0(r) \psi(t), \quad (4.19)$$

and by putting it into Eq.(4.6) we obtain

$$\left[ \frac{1}{2} \frac{k}{g^2} - \frac{r}{z_0} \right]^{(-1)} d_r w^0 = \frac{1}{\varphi(t) \psi(t)} = 1. \quad (4.20)$$

From the last result we obtain the relation between the two time functions  $\psi$  and  $\varphi$

$$\psi(t) = [\varphi(t)]^{(-1)}. \quad (4.21)$$

As may be seen from Eq.(4.18), the function  $\varphi$  tends to zero with time tending to infinity. This means that the stresses [see Eq.(4.13)] drop to zero and their relaxation is complete after an infinite period of time. On the other hand, the function  $\psi$  increases infinitely with time and thus the displacement  $w$  [see Eq.(4.19)] becomes infinite.

## 5. Critical time of failure

In order to find the critical time of failure we use the criterion for the critical creep state in the form (1.2) where the function  $f$  is assumed as a linear one. Thus, we obtain the condition

$$W_D = \text{const.} = K^2, \quad (5.1)$$

where  $K^2$  is the critical value of dissipated energy through the creep resistance.

The power of dissipation is given by the relation containing the stress components and strain rate components

$$\dot{W}_D = \sigma_{ij}(r, t) \dot{\epsilon}_{ij}(r, t), \quad (5.2)$$

and the condition takes the form



$$W_D = \int_{\bar{t}}^{t^*} \sigma_{ij}(\tau) \dot{\epsilon}_{ij}(\tau) d\tau = K^2 \quad (5.3)$$

Here, by  $t^*$  we denote the time-instant at which the creep rupture takes place. Evidently, the initial instant  $\bar{t}$  should also be considered as a certain critical time-point as, for example, the instant of reaching the stage at which the elastic effects can be neglected. In this case, the value of dissipated energy  $W_D(\bar{t}) = W$  characterizes the process up to this stage.

In the particular case of a spherical membrane, the condition (5.3) takes the form

$$W_D = \int_{\bar{t}}^{t^*} [\sigma_1(r, \tau) \dot{\epsilon}_1(r, \tau) + \sigma_2(r, \tau) \dot{\epsilon}_2(r, \tau)] d\tau = K^2, \quad (5.4)$$

where

$$\sigma_1(r, t) = \frac{D}{r} z^0(r) \varphi(t), \quad \sigma_2(r, t) = D \left[ 2d_r z^0(r) - \frac{1}{r} z^0(r) \right] \varphi$$

$$\dot{\epsilon}_1(r, t) = \varphi^n(t) L^0 \left[ \frac{1}{3} D \left( 3 \frac{z^0}{r} - 2d_r z^0 \right) \right], \quad \dot{\epsilon}_2(r, t) = \varphi(t) L^0 \left[ \frac{1}{3} D \left( 4d_r z^0 - 3 \frac{z^0}{r} \right) \right]. \quad (5.5)$$

In the Eq.(5.5)  $L^0$  is related to the operator  $\dot{L}$  as follows

$$\dot{L} = \varphi^{n-1}(t) L^0 \quad (5.6)$$

On the basis of Eqs.(5.4) and (5.5), the condition (5.4) may now be written

$$W_D = W^0(r) \int_{\bar{t}}^{t^*} [\varphi(\tau)]^{n+1} d\tau = K^2, \quad (5.7)$$

where  $W^0$  stands for a time-independent energetical coefficient the value of which can easily be evaluated on the basis of Eqs. (5.4) and (5.5).

In order to obtain the critical time of creep rupture, we calculate the value of the integral

$$\int_{\bar{t}}^{t^*} [\varphi(\tau)]^{n+1} d\tau = \frac{K^2}{W^0}, \quad (5.8)$$

by substituting the function  $\varphi$  according to the solution (4.18).

Denoting by

$$x = 1 + \frac{1}{2} \lambda \varphi_0^{n+2} (n+2)(\tau - \bar{t}), \quad A = \frac{1}{2} \lambda \varphi_0^{n+2} (n+2), \quad (5.9)$$

we obtain, instead of Eq.(5.8),



$$\frac{\phi_0^{n+1}}{A} \int_1^{x^*} x^{-\frac{n+1}{n+2}} dx = \frac{K^2}{W_0}, \quad (5.10)$$

where  $x^* = x(t^*)$ .

Carrying out the integration in Eq.(5.10), we finally obtain

$$t^* = \bar{t} + \frac{1}{A} \left\{ \left[ 1 + \frac{1}{2} \lambda \varphi_0 \frac{K^2}{W} \right]^{n+2} - 1 \right\} \quad (5.11)$$

It is seen from the result obtained that, if the criterion (5.3) is applied, we are able to predict the critical time of creep rupture and thus bound the unlimited creep process predicted by the creep solution. Since only dissipation is involved during the process, it seems reasonable to found the prediction of creep rupture on the basis of the amount of dissipated energy which thus constitutes a certain measure of reaching this critical state.

- [1] Z.Bychawski, W.Olszak, Energetic interpretation of critical states in viscoelastic bodies (in Polish), IBTP Reports, No.2, Warsaw, 1967.
- [2] Z.Bychawski, W.Olszak, Rheological states of geometrically nonlinear rotational membranes, The Second IUTAM Symposium on the Theory of Thin Shells, Copenhagen, 1967.

#### SUMMARY

On the basis of the authors criterion of attainment of critical states in viscoelastic bodies, the problem of creep failure of nonlinear rotational shells is investigated. For a spherical membrane the critical time of failure is found by introducing the dissipated energy through creep resistance as a measure of attainment of this state.

#### RÉSUMÉ

En se basant sur la condition des états critiques proposée par les auteurs, on considère le problème de la rupture par fluage pour les voiles minces nonlinéaires. Pour une membrane, le temps critique de rupture est calculé en introduisant l'énergie dissipée par la résistance de fluage comme une mesure pour atteindre cet état.

## ZUSAMMENFASSUNG

Die Verfasser haben ein Kriterium für das Erreichen des kritischen Zustandes infolge Kriecherscheinungen formuliert und dasselbe zur Analyse des Problems des Kriechbruches von nicht-linearen Schalen im Membranzustand angewandt. Als Resultat findet man die kritische Zeit, in welcher Kriechbrucherscheinungen in einer sphärischen Membran eintreten. Als entsprechendes Mass wird dabei voraussetzungsgemäss die durch den Kriechwiderstand zerstreute (dissipierte) Energie eingeführt.

## Der Einfluß des zeitabhängigen Verhaltens bei Hänge- und Schrägseilbrückensystemen

Time-Dependent Response of Suspension and Cable-Stayed Bridges

L'influence du temps dans le comportement de ponts suspendus ou haubannés

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Es ist allgemein bekannt, daß bei fast allen heute gebräuchlichen Baustoffen - insbesondere unter Dauerbelastung - Formänderungen auftreten. Der vorliegende Beitrag befaßt sich mit dem zeitabhängigen Verhalten bei Hänge- und Schrägseilbrückensystemen, wobei speziell der Einfluß des Kabelkriechens (auch Nachlängung der Tragkabel genannt) betrachtet werden wird.

### Diskussion über erfolgte Messungen

Zwei die zeitliche Deformation charakteristisch beschreibende Darstellungen sind bei rheologischen Untersuchungen üblich geworden: der Verlauf der Dehnung bei konstanter Last und der Verlauf der Spannung bei konstanter Gesamtdehnung, beides in Abhängigkeit von der Zeit bei konstant gehaltener Temperatur; man spricht bei dem einen Verlauf vom "Kriechen" und bei der zweiten Funktion von der "Relaxation". Wir werden uns hier mit dem ersten Phänomen befassen.

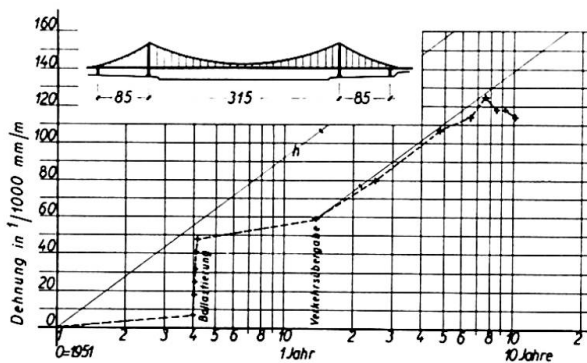


Fig.1

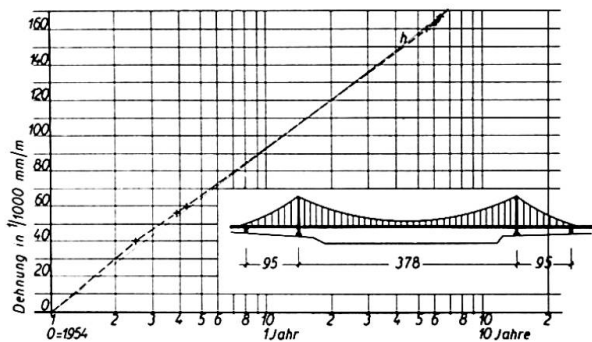


Fig.2

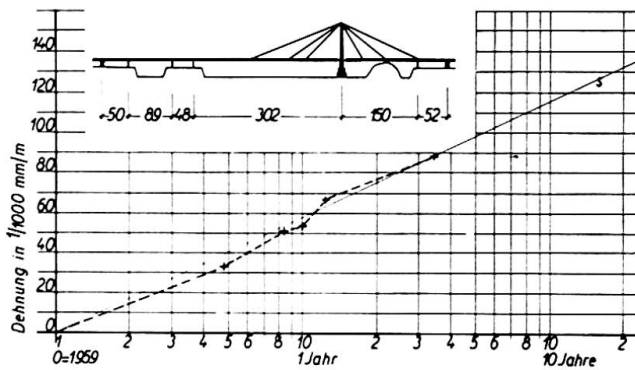


Fig.3

In dankenswerter Weise hat ein angesehenes deutsches Draht- und Kabelwerk<sup>1)</sup> an einigen bekannten Rheinbrücken Messungen vorgenommen, womit es möglich geworden ist, nunmehr auch an kompletten Hänge- und Schrägseilbrückentragwerken zeitlich ablaufende Deformationen zu studieren. Die Fig.1 und Fig.2 zeigen die aufgenommenen Kriechkurven für die Tragkabel bei Hängebrücken; die Fig.3 das gleiche bei einer Schrägseilbrücke. Es handelt sich im einzelnen um die nachfolgend aufgezählten Rhein-

brücken im Raume von Köln in der Bundesrepublik Deutschland.

Hängebrücke Köln-Mülheim: In der Fig.1 maßgebend ist erst der Kurvenzug ab etwa 1,4 Jahren, nach erfolgten Ballastierungen, Probebelastungen und endgültiger Fertigstellung. Spannung aus ständiger Last im Tragkabel i.M. 33 kp/mm<sup>2</sup>; aus Verkehrslast pro Tag i.M. 5 bis 8 kp/mm<sup>2</sup>.

Hängebrücke Köln-Rodenkirchen: Meßpunkte in Fig.2 ab Verkehrsübergabe 1954 aufgetragen. Spannungen aus ständiger Last i.M. 41 kp/mm<sup>2</sup>; aus Verkehrslast pro Tag i.M. 3-5 kp/mm<sup>2</sup>.

Schrägseilbrücke Köln-Deutz (Severinsbrücke): Meßpunkte in Fig.3 ab Verkehrsübergabe im November 1959 aufgetragen. Spannungen aus ständiger Last i.M. 38 kp/mm<sup>2</sup>; aus Verkehrslast pro Tag bis 9 kp/mm<sup>2</sup>.

Ein Vergleich des dargestellten Meßverlaufes für das Kabelkriechen der jeweiligen Brücken zeigt eine offensichtliche Übereinstimmung in der Tendenz bei den beiden Hängebrücken und ebenso deutlich eine Abweichung bei der einzigen hier betrachteten Schrägseilbrücke; beide Systeme zeigen in der semilogarithmischen Darstellung im wesentlichen ein lineares Kriechen, wie wir es prinzipiell schon von den kaltgezogenen Einzeldrähten her kennen. In Fig.1 und Fig.2 wurde jeweils eine für Hängebrücken als typisch angenommene mittlere Gerade h mit gleicher Steigung (120 mm/m in 2 Jahren) eingetragen. Die entsprechende Gerade s bei der betrachteten Schrägseilbrücke verläuft mit 75 mm/m in 2 Jahren etwas flacher (vergl.Fig.3); dieser Brückentypus ist sichtlich kriechsteifer als eine Hängebrücke. In der folgenden Abhandlung soll der Einfluß des aufgezeigten Kriechverhaltens der Tragkabel auf das Gesamttragwerk gezeigt und wenn möglich mit anderen üblichen Lastfällen verglichen werden.

### Hängebrücke

Die statische Berechnung einer erdverankerten Hängebrücke wird üblicherweise an einem Ersatzsystem durchgeführt<sup>[1]</sup>; Fig.4 zeigt ein solches für den Fall, daß der Versteigungsträger über 3 Felder durchläuft. In den meisten Fällen kann  $y_1'' = y_2'' = y'' = 8f/l^2 = \text{const}$  angenommen werden.

<sup>1)</sup> Felten & Guilleaume Carlswerk Eisen und Stahl Aktiengesellschaft Köln.

Hängebrücke mit durchlaufendem Versteifungsträger über drei Öffnungen

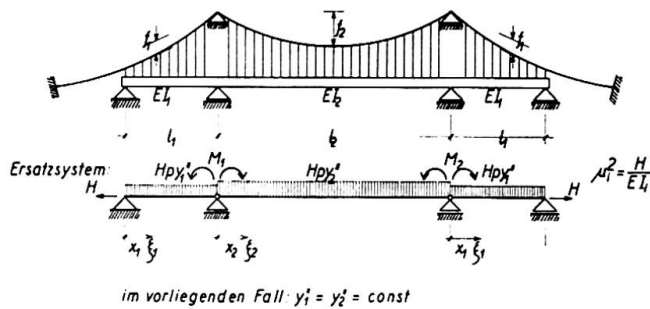


Fig. 4

sigt werden dürfen. Unter diesen Voraussetzungen erhält man auch im vorliegenden Fall die I. Grundgleichung in der Form

$$EJ v^{IV} - (H_g + H_p) v'' - (p - H_p \frac{8f}{l^2}) = 0 \quad (1)$$

welche gleichzeitig die Differentialgleichung des Ersatzsystems (Träger) darstellt. Dabei wurde angenommen, daß die Trägerachse identisch mit der x-Achse und die Ordinaten des Tragwerkes mit y, die entsprechenden Verformungen mit u bzw. v bezeichnet sind.

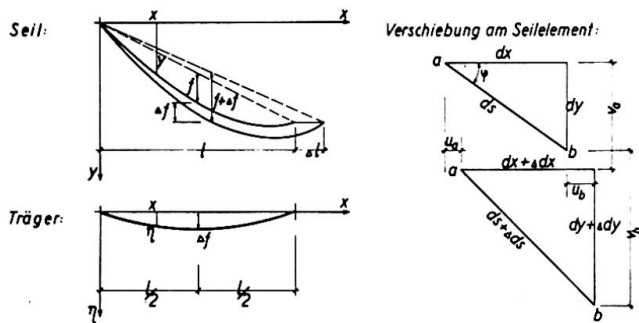


Fig. 5

Die Brücke sei in üblicher Weise so montiert, daß der Anteil  $g = \text{const}$  der ständigen Brückenbelastung vom Tragkabel allein getragen wird. Ferner wird vorausgesetzt, daß die Hänger vertikal und ungedehnt bleiben, daß die Hängerkräfte stetig über den Versteifungsträger verteilt sind, daß die Tragkabel keine Biegesteifigkeit besitzen, daß die Sattellager waagrecht verschiebbar sind und daß Pylonenverkürzungen und Schubverzerrung des Versteifungsträgers vernachlässigt werden dürfen.

Der Verfasser setzt als bekannt voraus, daß die II. Grundgleichung mit dem Ansatz

$$\int_0^L \Delta dx = 0 \quad (2)$$

gefunden wird. In Fig. 5 sind alle erforderlichen Zusammenhänge zwischen der Verformung des Seiles und des Trägers, sowie am Seilelement selbst dargestellt.

Es gilt dann

$$\Delta ds = \frac{H_p ds}{E_k F_k \cos \varphi} + \chi \log t ds \pm \alpha_T \Delta T ds \quad (3)$$

worin  $H_p$  den Horizontalzugzuwachs gegenüber dem Normalzustand  $H_g$  (ständige Last),  $L$  die gesamte Länge des Versteifungsträgers,  $\chi$  die Steigung der Geraden  $h$  in Fig. 1 und Fig. 2 in den dortigen semilogarithmischen Darstellungen, mit  $t$  als Zeit in Jahren, und  $\Delta T$  eine Temperaturänderung darstellt. Nach Erfüllung von (2) erhält man schließlich mit (3) eine erweiterte II. Grundgleichung

$$H_p \frac{L}{E_k F_k} + (\chi \log t \pm \alpha_T T) L_T + \int_0^L y'' v dx = 0, \quad (4)$$

die den funktionellen Zusammenhang zwischen Tragkabel und Versteifungsträger herstellt. Der Lastfall "Kabelkriechen" erscheint im 2. Term dieser Gleichung als ein ebenfalls erweiterter Temperaturlastfall;  $L_T$  bedeutet dabei  $\int_0^L dx / \cos^2 \varphi$  (später wird auch noch

$L_k = \int_0^L dx / \cos^3 \varphi$  gebraucht). Die praktische Auswertung erfolgt

zweckmäßig nach K. H o e n i n g [2], wo z.B. speziell aus Kabelkriechen

$$\Delta H = - \frac{\chi \log t L_T \frac{l^2}{8(f+\Delta f)}}{\frac{8f}{l^2} F + \frac{l^2}{8(f+\Delta f)} \cdot \frac{L_k}{E_k F_k}}, \quad (5)$$

mit  $F$  als die gesamte Biegefläche aller Öffnungen unter der Einheitsgleichlast, bestimmt werden kann etc.

Unter diesen einfachen und anschaulichen Voraussetzungen kann bereits brauchbar etwas über die Auswirkung des Kabelkriechens ausgesagt werden.

#### Beispiel:

Hängebrücke Köln-Mülheim [3]: Wenn man die Gerade  $h$  in Fig.1 zugrundelegt und  $E_k$  konstant annimmt<sup>2)</sup> erhält man z.B. für 3, 10 und 100 Jahre ein  $\Delta H$  von rund -6Mp, -12 Mp und -24 Mp bei einem  $H$  von 3180 Mp! Der entsprechende, äquivalente Temperaturlastfall<sup>g</sup> entspräche einer Temperaturdifferenz von ca 2,1°C, 4,5°C und 9°C, d.h. der letzte höchste Wert entspricht etwa 1/4 des in der ursprünglichen Berechnung berücksichtigten Temperaturanteils.

#### Schrägseilbrücke

Auch für dieses Brückensystem (Fig.3) kann, wie F.W. W a l t k i n g in einer unveröffentlicht gebliebenen Arbeit [4] gezeigt hat, ein Ersatzsystem gefunden werden, das eine geschlossene Lösung und eine sehr anschauliche baustatische Deutung des elastischen Problems ermöglicht. Im Anhang zur vorliegenden Arbeit wird im einzelnen gezeigt, daß die Lösung des als kontinuierliche Wand von Schrägseilen (Zügelseilen) aufgefaßten Ersatzsystems (vergl. nebenstehende Fig.6) auf einen "Träger auf elastischer Unterlage" führt.

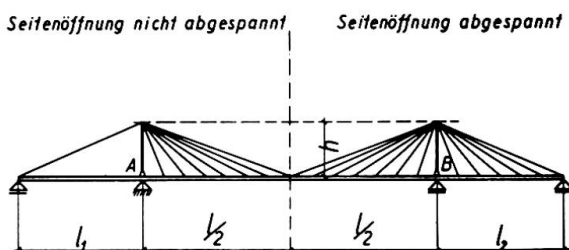


Fig.6

Der Verfasser sieht bei Benutzung eines solchen Ersatzsystems grundsätzlich die Möglichkeit, zeit- und auch temperaturabhängige Vorgänge an Schrägseilbrücken aber auch an ähnlich seilabgespannten Konstruktionen ohne größeren Aufwand zu untersuchen. Es genügen wohl nicht mehr einfache geometrische Überlegungen wie vorhin

<sup>2)</sup> Der E-Modul des Tragkabels ist streng genommen ebenfalls eine Funktion der Zeit, was bei der Durchrechnung des Beispiels vernachlässigt werden konnte. – Bei der Behandlung der Schrägseilbrücke wird gezeigt werden, welche Form der Grenzwert  $E_{k\infty}$  annimmt.



bei der Behandlung der Hängebrücke aufgezeigt, aber auch die hier anwendbaren Prinzipien der linearen Viskoelastizitätstheorie bieten noch die Möglichkeit einer anschaulichen baustatischen Behandlung des Problems.

Zur ausführlichen Theorie selbst sei auf die entsprechende Literatur verwiesen ([5] u.a.). Hier soll nur das Wesentlichste zum besseren Verständnis der Abhandlung herausgehoben und den nachfolgenden Überlegungen vorangestellt werden:  
Das betreffende Bauglied (hier die Schrägseilkabel) oder ganz allgemein das Tragwerk als ganzes (wenn z.B. neben dem Kabelkriechen auch das Kriechen des Versteifungsträgers berücksichtigt werden sollte) wird als aus einem ideal viskoelastischen Baustoff hergestellt angesehen, wo zwei Medien vereint gedacht sind, nämlich der ideal elastische Körper und die zähe Flüssigkeit. Die elastischen Effekte sollen dabei dem Hookeschen, die zähen Effekte dem Newtonschen Gesetz unterliegen. Es gibt mehrere Modelle, je nachdem wie diese Effekte nebeneinander und/oder hintereinander geschaltet gedacht sind. Die Auswahl erfolgt im Idealfall nach aufgenommenen Kriech- und Relaxationskurven am tatsächlichen Bauglied bzw. Tragwerk.- Grundsätzlich existieren Analogien zwischen einer Lösung des viskoelastischen und einer entsprechenden des sogenannten "begleitenden elastischen" Systems, die in Form von Korrespondenzprinzipien in der Theorie erarbeitet worden sind.

Gelingt es also für ein gewisses Problem die Lösung des "begleitenden elastischen" Systems zu finden, so besteht grundsätzlich auch die Möglichkeit, die Lösung des viskoelastischen Problems mithilfe eines der Korrespondenzprinzipien zu erhalten.

Im vorliegenden Fall ist es durch das Ersatzsystem von Waltring möglich, eine elastische Lösung zu finden; sie lautet nach (A 13)

$$\left. \begin{aligned} y &= \eta + \frac{p}{4k^4 EJ} + \frac{\mathfrak{F}_A}{h} x \\ \eta &= A \cos kx \cos kx + B \sin kx \cos kx + C \cos kx \sin kx + D \sin kx \sin kx \end{aligned} \right\} (6)$$

Die Lösung entspricht, wie im Anhang näher ausgeführt, der eines Balkens auf elastischer Unterlage, wobei hier speziell noch die Unterlage viskoelastisch sein soll. D i s t e f a n o hat in einer seiner Arbeiten gezeigt [6], daß der Grenzwert der Lösung des viskoelastischen Problems in diesem Fall mit der Lösung des reinen elastischen Problems dann übereinstimmt, wenn der Koeffizient der viskoelastischen Unterlage durch

$$k^* = \frac{k}{1 + k \mathfrak{F}} \quad (7)$$

ersetzt wird, wobei  $\mathfrak{F} = \int_0^\infty f(t) dt$  und  $f(t)$  die Kriechfunktion darstellt.

Im Falle der Schrägseilbrücke hängt nach (A 8) dieser Koeffizient lediglich vom E-Modul  $E_k$  ab, wofür wir nach D i s c h i n g e r auch den Wert <sup>3)</sup>

$$E_k^* = \frac{E_k}{1 + \varphi} \quad (8)$$

anschreiben können; dabei wird in



$$\varphi = \frac{\epsilon_{el}}{\gamma_o}$$

für  $\epsilon_{el}$  die elastische Dehnung und für  $\gamma_o$  die Endkriechdehnung gesetzt.  $\varphi$  kann z.B. aus Fig.3 ermittelt werden, wenn z.B. für  $\gamma_o$  der Dehnwert für 1000 Jahre eingesetzt wird. Mit diesem angenommenen Grenzwert kann nach (A 11) der Koeffizient  $k$  und damit unter Beachtung der Randbedingungen des vorliegenden Brückensystems  $y$  und somit auch alle Schnittkräfte, wie beispielsweise das Biegemoment

$$M = - EJy'' = 2EJk^2 \left( A \sin kx \sin kx + B \sin kx \sin kx - C \sin kx \cos kx - D \sin kx \cos kx \right) \quad (10)$$

gefunden werden.

Beispiel:

Schrägseilbrücke nach Fig.6, linker Teil: Mit  $E = 21 \cdot 10^6$  Mp/m<sup>2</sup>

$l_1 = 108$  m       $F_1 = 0,06489$  m<sup>2</sup>       $E_k(o) = 16 \cdot 10^6$  "

$l = 260$  "       $f_k = 0,4626 \cdot 10^{-4}$  m       $J = 0,6$  m<sup>4</sup>

$h = 40$  "       $p = 4,5$  Mp/m       $J_1 = 1,1$  "

sowie mit  $\epsilon_{el} = 2,0625 \cdot 10^{-3}$  erhält man extrapoliert aus Fig.3

$$\gamma_o = 0,464 \text{ mm/m} = 0,464 \cdot 10^{-3}$$

womit man folgende Durchbiegungen in der Mittelöffnung erhält  
aus der elastischen Lösung       $y_m = 53,2$  cm

aus der viscoelastischen Lösung       $y_m^v = 70,7$  cm.

#### Zur Tragwerkssicherheit

In den vorangegangenen Abschnitten der Abhandlung hat der Verfasser zeigen können, welchen Verlauf das Kablekriechen bei den beiden hier untersuchten Brückensystemen jeweils hat, wie das Phänomen praktisch rechnerisch erfaßt werden kann und welchen Einfluß es im allgemeinen hat. Es konnte zumindest qualitativ festgestellt werden, daß Hängebrücken ein stärkeres Kabelkriechen aufweisen als Schrägseilbrücken. Schließlich kann ganz allgemein noch erwähnt werden, daß trotz aller Komplexität davon ausgegangen werden kann, daß im Falle der hier vorerst in Betracht gezogenen, stählernen Brücken ein weitgehend lineares Verformungsverhalten vorliegt.

Ergänzend zu diesen Erkenntnissen wollen wir nun festhalten, welche "konventionellen" Sicherheitsfaktoren

$$\bar{V} = R_{\min}/S_{\max}$$

nach Gleichung (2.17) auf Seite 19 des Vorberichtes zum laufenden 8. Kongreß – bei der jeweiligen Bemessung der Tragkabel bestimmend waren (vergl. [3] und [7]), wobei wir stellvertretend je ein ausgeführtes Bauwerk für jedes Brückensystem herausgreifen:

- 3) Nach einem modifizierten Voigt'schen Modell lautet die entsprechende Kriechfunktion  $f(t) = \frac{\varphi}{E_k} (1 - e^{-\delta t})$

|                   |                             |                                    |
|-------------------|-----------------------------|------------------------------------|
| Hängebrücke       | : Köln-Mülheim .....        | $\bar{V} = 2,61$                   |
| Schrägseilbrücke: | Köln-Deutz (Severinsbrücke) | $\bar{V} = 2,55 \text{ bis } 3,06$ |

Dazu muß bemerkt werden, daß eine Hängebrücke eine weit geringere Anzahl statisch Überzähliger aufweist, als eine Schrägseilbrücke, was bei der Bestimmung einer realistischen Bruchwahrscheinlichkeit u.a. aber eine Rolle spielen wird. Auch auf sonstige unterschiedliche Verhaltenserscheinungen der beiden Brückensysteme, wie etwa die aerodynamische Stabilität etc., kann an dieser Stelle nicht näher eingegangen werden.

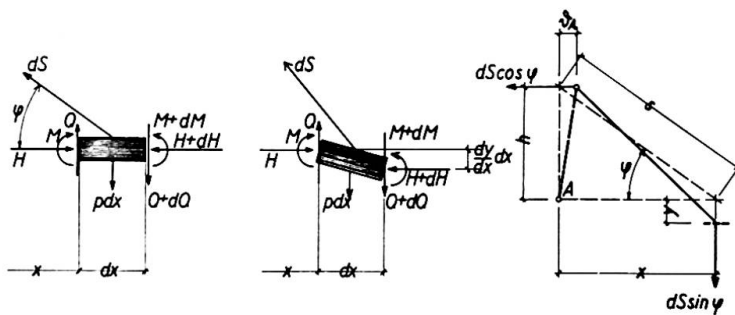
Beim Abwägen der oben zahlenmäßig angeführten "konventionellen" Sicherheitsfaktoren wird man aber sicher schon bei Berücksichtigung der vorangegangenen Betrachtungen zugeben müssen, daß offensichtlich ein Widerspruch zwischen den bisher angenommenen Sicherheitszahlen mehr oder weniger konstanter Größe und dem tatsächlichen Verhalten der verglichenen Brückensysteme vorliegt.- Es wäre nun an der Zeit, solche und ähnliche historisch entstandene und heute einfach üblich gewordene Zahlen kritisch zu durchleuchten und wenn, wie hier gezeigt, notwendig, zu korrigieren.

Es scheint bei dem hier aufgezeigten Sachverhalt wiederum eine Bestätigung dafür vorzuliegen, daß wir uns im Bauingenieurwesen mehr als bisher mit der Wahrscheinlichkeit des Zusammenbruches bzw. des im vorliegenden Fall maßgebend sein werdenden Unbrauchbarwerdens eines Tragwerkes befassen müssen. Dem Ziele, ein "gleichmäßiges" Sicherheitsmaß für alle Tragglieder eines Bauwerkes, aber auch für alle Tragwerke gleicher oder ähnlicher Beanspruchung zu erreichen, können wir nur durch konsequente Verfolgung stochastischer Prinzipien näherkommen. Dazu benötigen wir mehr als bisher Beobachtungen und Messungen an ausgeführten Bauwerken. Bei Vorliegen einer genügend großen Dichte solcher Beobachtungen und Messungen ist es aber z.B. im vorliegenden Fall - wo ein weitgehend lineares Verformungsverhalten vorliegt - ohne weiteres möglich, die vom Verfasser vorgeschlagenen Differentialgleichungen der jeweiligen Tragwerksmechanik in Gleichungen mit stochastischen Koeffizienten umzuformen. Ähnlich wie hier von der elastischen zur viskoelastischen Lösung kann dabei auch von der entsprechenden Lösung der klassischen Gleichungen als erste Näherung auf den zu "erwartenden" Wert nach der Wahrscheinlichkeitsdichtefunktion geschlossen werden.

Anhang  
Vorschlag für ein Ersatzsystem für Schrägseil-  
brücken von der Art von Zügelseilbrücken nach  
F.W. Waltking 4)

Einzelne Schrägseile sollen hier durch eine kontinuierliche Wand von Zügelseilen (Fig.6) ersetzt werden. Alle nachfolgenden Betrachtungen beschränken sich auf symmetrische Systeme mit gelenkigen Pylonenfüßen.

Die auf ein Längenelement  $dx$  des Versteifungsträgers entfallende Schrägseil-Querschnittsfläche sei  $f dx$ , wobei  $f$  eine Funktion von  $x$  sei.  $H(x)$  werde die mit  $x$  ebenfalls veränderliche Längskraft im Versteifungsträger benannt; dabei soll Druck positiv gekennzeichnet werden (entspricht den zugehörigen Seil- Zugkräften  $S$ !).



Nach Fig.7 ergibt sich bei Gleichgewicht:

$$dS = - \frac{dH}{\cos \varphi} ; \quad (A 1)$$

$$Q = \frac{dM}{dx} ; \quad (A 2)$$

$$\frac{dQ}{dx} = -p - \frac{dH}{dx} \operatorname{tg} \varphi \quad (A 3)$$

Fig. 7

$$\text{bzw. } \frac{d^2}{dx^2} (EJ \frac{d^2 y}{dx^2}) = p(x) + \frac{dH}{dx} \operatorname{tg} \varphi \quad (A 4)$$

Die elastische Arbeit bei der Verformung der Brücke infolge Dehnung des betrachteten Seiles:

$$dS \Delta s = \frac{(dS)^2 s}{E_k f dx} \quad \text{oder in Termen}$$

$$dS \sin \varphi \cdot y - dS \cos \varphi \vartheta_A = \frac{(dS)^2 s}{E_k f dx}$$

$$\text{Mit } s = h/\sin \varphi \text{ und } \operatorname{tg} \varphi = h/x \quad (A 5)$$

wird

$$\frac{dH}{dx} = \frac{E_k}{h} f (y \sin^2 \varphi \cos \varphi - \vartheta_A \sin \varphi \cos^2 \varphi) \quad (A 6)$$

und daraus gemeinsam mit (A 4)

$$\frac{d^2}{dx^2} (EJ \frac{d^2 y}{dx^2}) + \frac{E_k}{h} f(x) \sin^3 \varphi \cdot y = p(x) + \vartheta_A \frac{E_k}{h} f(x) \cdot \sin^2 \varphi \cos \varphi \quad (A 7)$$

4) Auszug aus einer unveröffentlichten Arbeit von F.W. Waltking [4]

die Differentialgleichung der Zügelseilbrücke gefunden. Eine Erweiterung der gefundenen Differentialgleichung nach den Prinzipien der Verformungstheorie (Theorie II. Ordnung genannt) wäre ohne Schwierigkeiten möglich. Eine von W a l t k i n g vorgenommene Untersuchung zeigt aber, daß bei Schrägseilbrücken von der Bauart einer Zügelseilbrücke bei idealen Montagezielen - der Versteifungsträger weise unter ständiger Last weder Durchbiegungen noch Biegemomente auf - im abgespannten Teil der Einfluß aus der Verformungstheorie maximal eine 1,5 bis 2 - prozentige Erhöhung ergibt. Bei einer üblichen Überhöhung der Brückenlängsachse wird auch dieser geringe Einfluß zum größten Teil aufgehoben.

Zur Lösung der Differentialgleichung des Ersatzsystems werden die Hilfsfunktionen

$$\left. \begin{aligned} c(x) &= \frac{E_k}{h} f(x) \sin^3 \varphi \\ \text{und} \quad \bar{p}(x) &= \frac{E_k}{h} \vartheta_A f(x) \sin^2 \varphi \cos \varphi \end{aligned} \right\} \quad (\text{A } 8)$$

eingeführt. Man erkennt sofort, daß die Differentialgleichung des Systems (A 7) der Differentialgleichung eines Trägers auf elastischer Bettung entspricht. Zweckmäßig wird hier eine gleichförmige Nachgiebigkeit, d.h.  $c(x) = \text{const}$ , angestrebt. Setzt man demnach

$$f(x) \cdot \sin^3 \varphi = f_k \quad (\text{A } 9)$$

so erhält man zusammen mit der weiteren praktisch möglichen Vereinfachung,  $J = \text{const}$ , die vereinfachte Differentialgleichung des Ersatzsystems

$$EJ y^{IV} + \frac{E_k f_k}{h} y = p(x) + \vartheta_A \frac{E_k f_k}{h^2} x \quad (\text{A } 10)$$

Die Lösung des Problems erhält man schließlich mit

$$k^4 = \frac{E_k f_k}{4 h EJ} \quad (\text{A } 11)$$

womit man in der an sich bekannten Art die Lösung der homogenen Differentialgleichung in der Form

$$\eta = A \cos kx + B \sin kx + C \cosh kx + D \sinh kx \quad (\text{A } 12)$$

und schließlich für einen Trägerabschnitt innerhalb dessen  $p = \text{const}$  ist, die Lösung der vollständigen Differentialgleichung

$$y = \eta + \frac{p}{4k^4 EJ} + \frac{\vartheta_A}{h} x \quad (\text{A } 13)$$

findet.

Zur Bestimmung der Konstanten A, B, C und D, sowie der Verschiebung  $\vartheta_A$  an der Pylonenspitze werden die Randbedingungen

$$\begin{aligned} y(0) &= 0 ; & y'(0) &= y_1' ; & -EJ y''(0) &= M_A \\ y'(1/2) &= 0 & \text{und} & y'''(1/2) &= 0 \end{aligned} \quad (\text{A } 14)$$

sowie die Gleichgewichtsbedingung

$$H_0 = \frac{pl^2}{8h} + \frac{E_k f_k}{h^2} \int_0^{l/2} x \eta dx = H_1 \quad (A 15)$$

herangezogen, wobei je nach Abspannart der Seitenspanne

$$H_1 = \frac{E_k F_1}{h} \vartheta_1 \cos^2 \varphi_1 \sin \varphi_1 \quad (A16)$$

oder 
$$H_1 = \frac{pl_1^2}{2h} + \frac{E_k f_k}{h^2} \int_0^{l_1} x \eta dx \quad (A 17)$$

verwendet wird.

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## ZUSAMMENFASSUNG

Die Abhandlung bringt einen Beitrag zur Untersuchung der Tragwerks-sicherheit mittels der Viskoelastizitätstheorie, wobei Ergebnisse aus Lösungen "begleitender" Probleme der Elastizitätstheorie herangezogen wurden. Im speziellen ist das Phänomen des Kriechens der Tragkabel untersucht worden. Für Schrägseilbrücken wurde ein weniger bekanntes Ersatzsystem nach Walting vorgeschlagen und verwendet. Es konnten baupraktische Schlüsse gezogen werden.

## SUMMARY

This publication gives a contribution about the researches of the safety of structures with the theory of viscosity, whereat results with solutions of "attendant" problems of the theory of elasticity were drawn near. The creeping phenomenon of the main cables has been investigated specially. For diagonal cable-bridges there was proposed and used a less known substitution system of Walting. Practically conclusions were able to make.

## RÉSUMÉ

Les memoires apportent une contribution pour le calcul de la sécurité des structures au moyen des théories de la viscosité où on a attiré des résultats des solutions des problèmes "accompagnés" de la théorie de l'élasticité. Spécialment on a examiné le phénomène du fluage des cables porteurs. Pour les ponts de fil incliné on a proposé et employé un système équivalent selon Walting. On a pu prendre des conclusions pratiques pour la construction.

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**L'influence du fluage linéaire sur l'équilibre des systèmes hyperstatiques en béton précontraint**

Einfluß des geradlinigen Kriechens auf das Gleichgewicht der statisch unbestimmten Spannbetonsysteme

Influence of Linear Creep on the Equilibrium of Prestressed Indeterminate Systems

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PREMIERE PARTIE - LES LOIS de FLUAGE et de RELAXATION du BETON

I - LOI de DEFORMATION du BETON

Imposons à un prisme de béton une contrainte de compression constante  $\sigma$  à partir de l'âge  $t_0$ . Si la contrainte  $\sigma$  n'est pas trop grande, inférieure par exemple au tiers de la contrainte de rupture, l'expérience montre que le raccourcissement unitaire du béton  $\varepsilon(t)$  à l'âge  $t > t_0$  est proportionnel à la contrainte  $\sigma$ ; donc :

$$(1) \quad \varepsilon(t) = \frac{\sigma}{E(t_0, t)}$$

Le module de déformation du béton est donc une fonction des deux variables  $t_0$  et  $t$ . Pour  $t = t_0$ , nous obtenons le module instantané  $E(t_0)$ , et pour  $t = \infty$ , le module différé  $K(t_0)$  :

$$(2) \quad E(t_0) = E(t_0, t_0), \quad K(t_0) = E(t_0, \infty)$$

La relation (1) peut également s'écrire :

$$(3) \quad \varepsilon(t) = \frac{\sigma}{E(t_0)} + \sigma F(t_0, t) \quad \text{avec :} \quad F(t_0, t) = \frac{1}{E(t_0, t)} - \frac{1}{E(t_0, t_0)}$$

La déformation apparaît ainsi comme la somme de la déformation élastique instantanée et de la déformation différée  $\sigma F(t_0, t)$ , proportionnelle à la contrainte et croissant avec le temps, appelée fluage linéaire. On notera que  $F(t_0, t_0)$  est nul.

Seule l'expérience permet de connaître la fonction  $E(t_0, t)$ . Diverses expressions analytiques ont été proposées pour la représenter. La plus simple est

$$(4) \quad \frac{1}{E(t_0, t)} = \frac{1}{E(t_0)} + \left[ \frac{1}{K(t_0)} - \frac{1}{E(t_0)} \right] \cdot \left[ 1 - e^{-\beta(t-t_0)} \right]$$

$\beta$  caractérise la vitesse de fluage.

Lorsque le béton n'est chargé qu'à un âge assez grand, on peut admettre que  $E(t_0)$  et  $K(t_0)$  sont des constantes  $E$  et  $K$ , de sorte que :

$$(5) \quad \frac{1}{E(t_0, t_0)} = \frac{1}{E} + \left( \frac{1}{K} - \frac{1}{E} \right) \left[ 1 - e^{-\beta(t-t_0)} \right]$$

$E(t_0, t)$  ne dépend alors que de la variable  $t - t_0$ .

Lorsque la contrainte  $\sigma(t)$  appliquée au béton dans l'intervalle de temps  $(t_0, \infty)$  est variable, on trouve, en appliquant la loi (1) que le raccourcissement  $\varepsilon(t)$  a pour expression :

$$(6) \quad \varepsilon(t) = \frac{\sigma(t_0)}{E(t_0, t)} + \int_{t_0}^t \frac{\sigma'(\theta) d\theta}{E(\theta, t)}$$

## II - LOI de RELAXATION du BETON

L'expérience montre que si l'on impose à un prisme de béton un raccourcissement unitaire  $\varepsilon$  à partir de l'âge  $t_0$ , la contrainte de compression initialement égale à  $\sigma(t_0) = \varepsilon E(t_0, t_0)$  décroît dans le temps et reste proportionnelle à  $\varepsilon$  ; donc :

$$(7) \quad \sigma(t) = \varepsilon R(t_0, t) \quad \text{avec : } R(t_0, t_0) = E(t_0, t_0)$$

La formule (7) peut également se mettre sous la forme :

$$(8) \quad \sigma(t) = \sigma(t_0) - \varepsilon G(t_0, t) \quad \text{avec : } G(t_0, t) = R(t_0, t_0) - R(t_0, t)$$

qui met en évidence la diminution de contrainte ou relaxation

Lorsque le raccourcissement  $\varepsilon(t)$  imposé au béton dans l'intervalle de temps  $(t_0, \infty)$  est variable, on trouve, en appliquant la loi (7) que la contrainte  $\sigma(t)$  a pour expression :

$$(9) \quad \sigma(t) = \varepsilon(t_0) R(t_0, t) + \int_{t_0}^t \varepsilon'(\theta) R(\theta, t) d\theta$$

## III - LOI de RELAXATION DEDUITE de la LOI de FLUAGE

Si l'on se donne  $\varepsilon(t)$ , la contrainte  $\sigma(t)$ , solution de l'équation intégrale (6) est donnée par l'expression (9). Cette solution est donc connue si l'on sait déterminer la fonction  $R(t_0, t)$  connaissant la fonction  $E(t_0, t)$

Nous désignerons par  $\Phi(t_0, t)$  l'unique solution de l'équation intégrale

$$(10) \quad \int_{t_0}^t \frac{\Phi'(\theta) d\theta}{E(\theta, t)} = F(t_0, t)$$

qui s'annule pour  $t = t_0$ . L'équation (10) peut se ramener à une équation de VOLTERRA.

Dans le cas général, le calcul des valeurs de la fonction  $\Phi(t_0, t)$  s'effectue sans difficulté par intégration numérique.

L'intégration formelle est possible dans le cas de la loi de déformation (5) où l'on trouve :

$$(11) \quad \Phi(t_0, t) = \left(1 - \frac{\kappa}{E}\right) \left[1 - e^{-\gamma(t-t_0)}\right], \text{ avec : } \gamma = \beta \frac{E}{\kappa}$$

et dans le cas de la loi de déformation (4) où l'on trouve :

$$(12) \quad \Phi(t_0, t) = \beta \left[ \frac{1}{\kappa(t_0)} - \frac{1}{E(t_0)} \right] \int_{t_0}^t E(u) e^{-\gamma(u)} du, \text{ avec : } \gamma(t) = \beta \int_{t_0}^t \frac{E(u)}{\kappa(u)} du$$

Ceci posé, pour déterminer la fonction  $R(t_0, t)$  donnons à  $\varepsilon(t)$  la valeur constante  $\varepsilon$  dans la relation (6) ; nous obtenons ainsi :

$$\varepsilon = \frac{\sigma(t_0)}{E(t_0, t_0)} = \frac{\sigma(t_0)}{E(t_0, t)} + \int_{t_0}^t \frac{\sigma'(\theta) d\theta}{E(\theta, t)}$$

de sorte que la diminution de contrainte ou relaxation  $\varphi(t) = \sigma(t_0) - \sigma(t)$  est la solution de l'équation intégrale :

$$\int_{t_0}^t \frac{\varphi'(\theta) d\theta}{E(\theta, t)} = \sigma(t_0) F(t_0, t)$$

qui s'annule pour  $t = t_0$ . Donc :

$$\varphi(t) = \sigma(t_0) \Phi(t_0, t)$$

et par suite :

$$\sigma(t) = \sigma(t_0) [1 - \Phi(t_0, t)] = \varepsilon E(t_0, t_0) [1 - \Phi(t_0, t)]$$

En comparant avec la formule (7), nous voyons donc que :

$$(13) \quad R(t_0, t) = E(t_0, t_0) [1 - \Phi(t_0, t)]$$

Ainsi, dans le cas de la loi de déformation (5), nous obtenons :

$$(14) \quad R(t_0, t) = E - (E - \kappa) [1 - e^{-\gamma(t-t_0)}] \quad \left(\gamma = \beta \frac{E}{\kappa}\right)$$

Le coefficient  $\gamma$  caractérise la vitesse de relaxation ; ce coefficient est environ trois fois plus grand que le coefficient  $\beta$  qui caractérise la vitesse de fluage. La relaxation est donc plus rapide que le fluage.

Il est également possible de déduire la fonction  $E(t_0, t)$  de la fonction  $R(t_0, t)$  par une méthode analogue à celle qui vient d'être exposée.

## DEUXIEME PARTIE - APPLICATION aux SYSTEMES HYPERSTATIQUES

### I - EQUILIBRE sous l'ACTION d'un SYSTEME de FORCES DONNÉES

Considérons un système  $n$  fois hyperstatique  $(\Sigma)$  et désignons par  $X_i$  les  $m$  composantes des forces appliquées aux points  $A_i$  et par  $R_j$  les  $n$  composantes des réactions hyperstatiques appliquées aux points  $B_j$ . Associons au système  $(\Sigma)$  le système isostatique  $(\Sigma')$  obtenu en supprimant les liaisons surabondantes correspondant aux réactions  $R_j$ . Si, au temps  $t_0$  on applique au système  $(\Sigma')$  un ensemble de forces constantes  $X_i$  et  $R_j$  aux points  $A_i$  et  $B_j$ , le déplacement  $v_k$  du point  $B_k$  dans la direction de la force  $R_k$  aura pour valeur au temps  $t > t_0$ , dans l'hypothèse du fluage linéaire :

$$(15) \quad v_k(t) = \frac{1}{E(t_0, t)} \left[ \sum_{i=1}^{i=m} a_k^i X_i + \sum_{j=1}^{j=n} b_k^j R_j \right]$$

$a_k^i$  et  $b_k^j$  étant des constantes caractéristiques du système étudié

Il en résulte que les réactions hyperstatiques  $R_j$  sont données par le système :

$$(16) \quad \sum_i a_k^i X_i + \sum_j b_k^j R_j = 0$$

dans lequel le déterminant  $|b_k^j|$  n'est pas nul (configuration non critique).

Donc, dans l'hypothèse du fluage linéaire, l'équilibre d'un système hyperstatique, soumis à des forces extérieures données constantes dans le temps, est indépendant du temps et identique à l'équilibre élastique déterminé avec un module de déformation constant.

Si les forces appliquées  $X_i(t)$  et  $R_j(t)$  au système isostatique associé  $(\Sigma')$  dépendent du temps, le déplacement  $v_k(t)$  au temps  $t$  postérieur au temps  $t_0$  début de l'application des forces a pour expression :

$$(17) \quad v_k(t) = \frac{1}{E(t_0, t)} \left[ \sum_i a_k^i X_i(t_0) + \sum_j b_k^j R_j(t_0) \right] + \int_{t_0}^t \frac{1}{E(\theta, t)} d \left[ \sum_i a_k^i X_i(\theta) + \sum_j b_k^j R_j(\theta) \right]$$

ou, en intégrant par parties :

$$(18) \quad v_k(t) = \frac{1}{E(t, t)} \left[ \sum_i a_k^i X_i(t) + \sum_j b_k^j R_j(t) \right] + \int_{t_0}^t \left[ \sum_i a_k^i X_i(\theta) + \sum_j b_k^j R_j(\theta) \right] \frac{\partial}{\partial \theta} \left[ \frac{1}{E(\theta, t)} \right] d\theta$$

Il en résulte que les réactions hyperstatiques  $R_j$  sont données par le système :

$$(19) \quad \sum_{i=1}^{l=m} a_{\bar{k}}^i X_i(t) + \sum_{j=1}^{j=m} b_{\bar{k}}^j R_j(t) = 0$$

Donc, dans l'hypothèse du fluage linéaire, l'équilibre d'un système hyperstatique, soumis à des forces variables dans le temps, se confond à tout moment avec l'équilibre élastique déterminé avec un module de déformation constant.

En particulier, les résultats précédents s'appliquent aux calculs des réactions hyperstatiques dues à la précontrainte, puisque la précontrainte est équivalente à l'application d'un système de forces données : forces concentrées aux ancrages des armatures et forces réparties provoquées par la courbure des armatures.

## II - EQUILIBRE sous l'EFFET de DEFORMATIONS IMPOSEES

### A - Compensation des systèmes hyperstatiques

Pour diminuer les contraintes d'un système hyperstatique, on impose souvent à ce système des déformations maintenues par les liaisons surabondantes (par exemple : dénivellation des appuis d'une poutre continue). On introduit ainsi dans le système des efforts dits de compensation qui s'opposent aux efforts provoqués par les forces données. Le problème qui se pose est de savoir comment les efforts de compensation évoluent dans le temps.

En imposant à l'âge  $\tau_0$  des déplacements  $v_{\bar{k}}^0$  aux points d'application  $B_{\bar{k}}$  dans la direction des réactions  $R_{\bar{k}}$ , on introduit dans le système des réactions compensatrices  $R_{\bar{k}}(t)$  qui sont des fonctions du temps.

Au temps  $\tau_0$ , les valeurs  $R_j(t_0)$  sont données par le système :

$$v_{\bar{k}}^0 = \frac{1}{E(t_0, t_0)} \sum_j b_{\bar{k}}^j R_j(t_0)$$

Au temps  $\tau$ , les valeurs  $R_j(t)$  vérifient les équations intégrales :

$$v_{\bar{k}}^0 = \frac{1}{E(t_0, t)} \sum_j b_{\bar{k}}^j R_j(t_0) + \sum_j b_{\bar{k}}^j \int_{t_0}^t \frac{dR_j(\theta)}{E(\theta, t)} d\theta$$

La comparaison des deux équations précédentes donne :

$$\sum_j b_{\bar{k}}^j \int_{t_0}^t \frac{R_j'(\theta) d\theta}{E(\theta, t)} = - F(t_0, t) \sum_j b_{\bar{k}}^j R_j(t_0)$$

ou, puisque le déterminant  $|b_{\bar{k}}^j|$  est différent de zéro :

$$\int_{t_0}^t \frac{R_j'(\theta) d\theta}{E(\theta, t)} = -R_j(t_0) F(t_0, t)$$

Il en résulte que les diminutions  $S_j(t) = R_j(t_0) - R_j(t)$  des réactions compensatrices sont les solutions des équations intégrales :

$$\int_{t_0}^t \frac{S_j'(\theta) d\theta}{E(\theta, t)} = R_j(t_0) F(t_0, t)$$

Donc :  $S_j(t) = R_j(t_0) \Phi(t_0, t)$  et par suite :

$$(20) \quad R_j(t) = R_j(t_0) [1 - \Phi(t_0, t)]$$

Les réactions hyperstatiques introduites par la compensation à l'âge  $t_0$  diminuent donc dans le temps. A l'âge  $t > t_0$  les efforts de compensation (moments fléchissants, contraintes, etc..) sont égaux aux efforts de compensation à l'âge  $t_0$  multipliés par le coefficient de réduction  $1 - \Phi(t_0, t)$ . Ce coefficient de réduction ne dépend que des propriétés du béton et non des caractéristiques du système hyperstatique. Il peut être calculé une fois pour toutes pour un béton donné.

Dans le cas particulier de la loi de déformation ( 5 ), la formule ( 20 ) devient :

$$(21) \quad R_j(t) = R_j(t_0) \left[ \frac{K}{E} + \left( 1 - \frac{K}{E} \right) e^{-\gamma(t-t_0)} \right]$$

Faisons tendre  $t$  vers l'infini, nous obtenons :

$$R_j(\infty) = \frac{K}{E} R_j(t_0)$$

Dans ce cas, les efforts de compensation initiaux sont réduits à la longue dans le rapport du module de déformation différée au module de déformation instantanée.

#### B - Effet de déformations imposées dans le cas général

Imposons aux  $m$  points  $A_i$  du système  $n$  fois hyperstatique  $(\Sigma)$  des déplacements donnés dans la direction des forces  $X_i$ . Ceci revient à introduire dans le système  $(\Sigma)$   $m$  liaisons supplémentaires, donc à le transformer en un système  $(\Sigma_1)$   $m+n$  fois hyperstatique. Il est même possible que le système  $(\Sigma)$  soit isostatique ; dans ce cas le système  $(\Sigma_1)$  est  $m$  fois hyperstatique.

En appliquant les résultats obtenus ci-dessus pour la compensation au système hyperstatique  $(\Sigma_i)$  que nous supposons ne pas être à configuration critique, nous voyons que les efforts, provoqués dans un système isostatique ou hyperstatique par des déformations imposées à l'âge  $t_0$ , diminuent dans le temps. Au temps  $t > t_0$ , les efforts sont égaux aux efforts initiaux multipliés par un coefficient de réduction égal à  $1 - \Phi(t_0, t)$

### III - INFLUENCE du MODE de CONSTRUCTION - DEFORMATIONS DIFFEREEES

Supposons que pour construire le système hyperstatique  $(\Sigma)$  on exécute d'abord un système isostatique associé  $(\Sigma')$  et qu'on réalise ensuite à l'âge  $t_0$  les liaisons surabondantes au moyen d'armatures de précontrainte. Ce cas se rencontre en particulier dans la construction en encorbellement, et également lorsqu'on réalise une poutre continue à partir de poutres préfabriquées posées d'abord sur appuis simples.

#### A - Calcul des réactions hyperstatiques dues à la réalisation des liaisons

Nous désignons par  $X_i$  les forces appliquées en permanence (y compris éventuellement les forces dues à la précontrainte isostatique) à partir de l'instant  $t_0$ . Nous pouvons faire abstraction des forces appliquées postérieurement à  $t_0$ , car nous savons que les réactions hyperstatiques correspondantes sont celles que l'on calcule en supposant le module de déformation constant. Si l'on avait construit d'emblée le système hyperstatique, par exemple en l'exécutant sur cintre, les réactions hyperstatiques auraient eu les valeurs  $R_j^*$  données par les équations

$$(22) \quad \sum_i a_k^i X_i + \sum_j b_k^j R_j^* = 0$$

Supposons d'abord que les liaisons surabondantes sont réalisées par précontrainte concordante, donc que les réactions hyperstatiques  $R_j(t)$  sont nulles pour  $t = t_0$ .

Avant la réalisation des liaisons surabondantes, les déplacements des points  $B_k$  du système isostatique associé  $(\Sigma')$  ont pour valeurs à l'instant  $t_0$ , en supposant, pour simplifier l'exposé que les forces sont appliquées à partir de l'instant  $t_0$  :

$$v_k = \frac{1}{E(t_0, t_0)} \sum_i a_k^i X_i$$

Ces déplacements ne varient plus lorsque les liaisons surabondantes sont réalisées ; nous avons donc à l'instant  $t$  :



$$v_k = \frac{1}{E(t_0, t)} \sum_i a_k^i X_i + \sum_j b_k^j \int_{t_0}^t \frac{R_j'(\theta) d\theta}{E(\theta, t)}$$

Eliminons  $v_k$  entre les deux équations précédentes ; nous obtenons, en tant compte de la relaxation ( 22 ) :

$$\sum_j b_k^j \int_{t_0}^t \frac{R_j'(\theta) d\theta}{E(\theta, t)} = F(t_0, t) \sum_j b_k^j R_j^*$$

soit, puisque le déterminant  $|b_k^j|$  est différent de zéro :

$$(23) \quad \int_{t_0}^t \frac{R_j'(\theta) d\theta}{E(\theta, t)} = R_j^* F(t_0, t)$$

Les équations ( 23 ) sont des équations intégrales du type ( 10 ) ; donc :

$$(24) \quad R_j(t) = R_j^* F(t_0, t)$$

Dans le cas particulier de la loi de déformation (5), nous avons

$$(25) \quad R_j(t) = R_j^* \left(1 - \frac{\kappa}{E}\right) [1 - e^{-r(t-t_0)}]$$

Nous voyons donc que les réactions hyperstatiques varient constamment dans le même sens depuis les valeurs initiales  $R_j(t_0) = 0$  jusqu'aux valeurs limites :

$$R_j(\infty) = R_j^* \left(1 - \frac{\kappa}{E}\right)$$

Ainsi les valeurs limites des réactions hyperstatiques peuvent atteindre les deux tiers des valeurs correspondant à la réalisation directe du système hyperstatique.

Le cas où la précontrainte de liaison n'est pas concordante se ramène immédiatement au cas précédent, puisque la précontrainte peut être considérée comme un système de forces extérieures appliquées à l'instant  $t_0$ . Nous aurons donc pour valeurs des réactions hyperstatiques dues à la réalisation des liaisons surabondantes :

$$(26) \quad \begin{aligned} R_j(t_0) &= S_j && \text{à l'instant } t_0 \\ R_j(t) &= S_j + R_j(t) && \text{à l'instant } t > t_0 \end{aligned}$$

les valeurs  $R_j(t)$  étant données par ( 24 )

#### B - Etude des déformations différées

Le déplacement d'un point  $M^*$  du système hyperstatique  $(\Sigma)$  sous l'effet des forces  $X_i$  et de la précontrainte (supposées, pour simplifier l'exposé, appliquées à partir de l'instant  $t_0$ ) est égal au déplacement du point  $M$  du

système isostatique associé  $(\Sigma')$  sous l'effet des forces  $X_i$ , de la précontrainte et des réactions hyperstatiques  $\mathcal{R}_j(t)$  dues à la réalisation des liaisons surabondantes. A l'instant  $t > t_0$ , ce déplacement aura donc une expression de la forme :

$$(27) \quad v_M(t) = \frac{1}{E(t_0, t)} \left[ V_M + \sum_i \alpha_M^i X_i + \sum_j \beta_M^j \mathcal{R}_j(t_0) \right] + \sum_j \beta_M^j \int_{t_0}^t \frac{\mathcal{R}_j'(\theta) d\theta}{E(\theta, t)}$$

$\alpha_M^i$  et  $\beta_M^j$  étant des constantes et  $\frac{V_M}{E(t_0, t)}$  le déplacement provoqué par la précontrainte.

Le déplacement différé au temps  $t$  est :

$$(28) \quad \delta_M(t) = v_M(t) - v_M(t_0)$$

Dans le cas d'un système isostatique, les réactions  $\mathcal{R}_j(t)$  sont identiquement nulles, et l'on déduit de (27) et (28) :

$$\delta_M(t) = \left[ \frac{E(t_0, t_0)}{E(t_0, t)} - 1 \right] v_M(t_0), \quad \delta_M(\infty) = \left[ \frac{E(t_0)}{K(t_0)} - 1 \right] v_M(t_0)$$

A la limite, la déformation différée peut donc être le double de la déformation instantanée.

Dans le cas d'un système hyperstatique  $(\Sigma)$  obtenu par réalisation des liaisons surabondantes dans le système isostatique associé  $(\Sigma')$  par précontrainte concordante, la formule (27) devient :

$$v_M(t) = \frac{1}{E(t_0, t)} \left[ V_M + \sum_i \alpha_M^i X_i \right] + \sum_j \beta_M^j \int_{t_0}^t \frac{\mathcal{R}_j'(\theta) d\theta}{E(\theta, t)}$$

soit, compte tenu de la relation (23) :

$$(29) \quad v_M(t) = \frac{1}{E(t_0, t)} \left[ V_M + \sum_i \alpha_M^i X_i \right] + F(t_0, t) \sum_j \beta_M^j \mathcal{R}_j^*$$

Nous obtenons donc pour valeur du déplacement différé :

$$(30) \quad \delta_M(t) = F(t_0, t) \left[ V_M + \sum_i \alpha_M^i X_i + \sum_j \beta_M^j \mathcal{R}_j^* \right]$$

expression que l'on peut également mettre sous la forme :

$$(31) \quad \delta_M(t) = \left[ \frac{E(t_0, t_0)}{E(t_0, t)} - 1 \right] v_M(t_0) + F(t_0, t) \sum_j \beta_M^j \mathcal{R}_j^*$$

Dans le cas où la précontrainte de liaison n'est pas concordante, les réactions  $\mathcal{R}_j(t)$  sont données par (26), et l'on trouve sans difficulté que la

formule (30) doit être remplacée par la formule :

$$(32) \quad \delta_M(t) = F(t_0, t) \left[ V_M + \sum_i \alpha_M^i X_i + \sum_j \beta_M^j (R_j^* + S_j) \right]$$

la formule (31) demeurant valable.

En général, les termes entre crochets des formules (30) et (32) sont très petits, de sorte que les déformations différées des systèmes hyperstatiques sont faibles. Dans des cas pratiques, nous avons trouvé, et observé sur les ouvrages, des déformations différées vingt fois moindres que celles de l'ouvrage isostatique associé.

#### IV - EFFET du RETRAIT, de la TEMPERATURE et des TASSEMENTS d'APPUI

Dans le système isostatique associé  $(\Sigma')$ , le retrait, la température ou des tassements d'appui, agissant à partir de l'instant  $t_0$ , donnent des déplacements des points  $B_k$  égaux à  $v_k(t)$

Considérons alors le système hyperstatique  $(\Sigma)$ , et supposons d'abord le module de déformation constant et égal du module instantané ; dans cette hypothèse, on obtiendrait des réactions hyperstatiques  $R_j^*(t)$  données par les équations :

$$(33) \quad v_k(t) + \frac{1}{E(t_0, t)} \sum_j b_k^j R_j^*(t) = 0$$

$R_j^*(t)$

Nous connaissons donc les fonctions

En réalité, le module de déformation n'est pas constant, et les réactions hyperstatiques  $R_j(t)$  sont données par les équations intégrales :

$$v_k(t) + \frac{1}{E(t_0, t)} \sum_j b_k^j R_j(t_0) + \sum_j b_k^j \int_{t_0}^t \frac{R_j'(\theta) d\theta}{E(\theta, t)}$$

L'élimination de  $v_k(t)$  entre les deux équations précédentes donne les équations :

$$\sum_j b_k^j \int_{t_0}^t \frac{R_j'(\theta) d\theta}{E(\theta, t)} + \frac{1}{E(t_0, t)} \sum_j b_k^j R_j(t_0) = \frac{1}{E(t_0, t_0)} \sum_j b_k^j R_j^*(t)$$

équivalentes, puis le déterminant  $|b_k^j|$  est différent de zéro, aux équations :

$$(34) \quad \int_{t_0}^t \frac{R_j'(\theta) d\theta}{E(\theta, t)} + \frac{R_j(t_0)}{E(t_0, t)} = \frac{R_j^*(t)}{E(t_0, t_0)}$$

Les équations (34) sont des équations intégrales du type de l'équation (6); leurs solutions sont donc données par les formules :

$$(35) \quad R_j(t) = \frac{1}{E(t_0, t_0)} \left[ R_j^*(t_0) R(t_0, t) + \int_{t_0}^t R_j'^*(\theta) R(\theta, t) d\theta \right]$$

qu'une intégration par parties permet également d'écrire :

$$(36) \quad R_j(t) = \frac{1}{E(t_0, t_0)} \left[ R_j^*(t) R(t, t) - \int_{t_0}^t R_j^*(\theta) \frac{\partial R(\theta, t)}{\partial \theta} d\theta \right]$$

Dans le cas particulier de la loi de déformation ( 5 ), il est possible de faire de nombreuses applications des résultats précédents.

## RÉSUMÉ

Lorsque le raccourcissement du béton sous contrainte constante est proportionnel à la contrainte mais dépend de l'âge du béton et de la durée du chargement, l'équilibre d'un système hyperstatique sous l'action de forces données est l'équilibre élastique. Par contre, l'équilibre, sous l'effet de déformations imposées dues au retrait, à la température ou au mode de construction évolue dans le temps et tend vers un équilibre limite.

## SUMMARY

When the strain of concrete under constant stress is proportional to the stress, but depends on the age of the concrete and the time the load is applied the equilibrium resulting from the action of given forces is the elastic equilibrium. But, under imposed deformations resulting from shrinkage, temperature or mode of construction, the equilibrium evolves in time and tends toward a limit equilibrium.

## ZUSAMMENFASSUNG

Wenn die Verkürzung des Betons unter ständiger Spannung proportional zur Spannung bleibt, aber vom Alter des Betons und von der Dauer der Belastung abhängt, ist das Gleichgewicht eines statisch unbestimmten Systems bei gegebenen Kräften das elastische Gleichgewicht. Unter den aufgezwungenen Verformungen des Schwindens, der Temperatur oder der Bauart entwickelt sich hingegen das Gleichgewicht mit der Zeit zu einer Grenzlage.

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Comments by the author of the introductory report  
Remarques de l'auteur du rapport introductif  
Bemerkungen des Verfassers des Einführungsberichtes

ALFRED M. FREUDENTHAL  
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The discussion contributed to Theme Ia ranges from a rejection of the approximate probabilistic approach to structural safety based on the introduction of load and of carrying capacity as random variables as mathematically not rigorous enough (Misteh, Eimer, Konishi) to its rejection as being "mathematics" instead of being "common sense" (Hrennikoff), whatever this may mean. It is encouraging to those who, over the years, have attempted to promote a rational probabilistic approach to the concept of structural safety, that among the 8 contributions to Theme Ia only a single one (Hrennikoff) repeats the familiar argument of the "practical engineer" that problems of safety should be left to the "collective judgment of the profession" which will protect society from "erudite mathematical derivations" which can obviously not estimate the chances of incompetence in analysis, design and construction. Since Prof. Costa, in his discussion, has refuted this point of view by summarizing the principal arguments for the probabilistic approach in a most effective manner I shall comment only on the other extreme, namely the proposition to base the approach to structural safety on the theory of stochastic processes.

While, in principle, there can be no objection to this approach, a closer consideration of its practical applicability reveals that even an approximate solution of the problem of the estimation of the time to failure ("first exceedance" or first passage time) presupposes the introduction of such drastic simplifying assumptions concerning the character of the random process, the response of the structure and the nature of the failure process that the physical significance of the solution becomes dubious, to say

the least. Even with these simplifying assumptions not even an approximate solution can be obtained if the resistance of the structure is a statistical variable subject to time-or load history effects. It appears that under these conditions the engineering relevance of the stochastic approach to structural safety is open to serious doubts.

It is therefore more expedient to develop the approximate probabilistic approach reviewed in the Introductory Report and dealt with in the contributions by Prof. Lind and Dr. Koch. However, I should like to express some apprehension concerning the use of the Gram-Charlier expansions in fitting distribution functions. These expansions produce negative ordinates at not too large distances from the mean and are therefore unsuitable in the low probability range characteristic of safety analysis. Also selection of distribution functions on the basis of curve-fitting near the center of the distribution is an irrelevant procedure. Distribution functions that can be extrapolated towards the tails may be rationally selected only on the basis of physical argument by which a certain probability model can be justified.

In the case of structures the loads of which are of a clearly stochastic nature, such as towers subject to wind, maritime structures subject to waves and swell and flexible structures subject to earthquake accelerations, a synthesis must be attempted between the approximate probabilistic and the rigorous stochastic approach to safety analysis on the basis of which rational design criteria for such structures are developed. An illustration of such a procedure for maritime structures is presented by the author at the 22nd International Navigation Congress in Paris in 1969.

Considering the elaborate analytical methods of safety analysis in the inelastic range, as illustrated by the various contributions to Theme Ib, and the dubious physical assumptions concerning



the material response underlying such analysis (linear visco-elastic, ideal elastic-plastic, steady state creep, etc.), it would seem that structural model analysis represents, so far, the only really reliable method for the establishment of the critical failure mechanism of any but the simplest structural forms on which a rational safety analysis can be based. The fact that it has not been specifically referred to in the Introductory Reports, on which Prof. Oberti comments, is simply a tacit expression of the conviction that it is so well-established a tool that it is unfailingly used whenever the results of a theoretical analysis are either physically suspect or unobtainable.

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**Ic**

**Optimisation des structures  
Optimierung von Tragwerken  
Optimisation of Structures**

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## DISCUSSION PRÉPARÉE / VORBEREITETE DISKUSSION / PREPARED DISCUSSION

**Beitrag zur Frage der Optimierung von Tragwerken**

Contribution to the Question of Optimisation of Structures

Contribution à la question de l'optimisation des structures

**F. FALTUS**  
 Prof.Ing.Dr., Dr. Sc.  
 Tschechoslowakei

Herr Courbon hat in seinem Einführungsartikel eine beachtenswerte Zusammenstellung allgemeiner Gesichtspunkte zur Frage der Optimierung von Tragwerken gegeben, welche zeigt, wie vielseitig die Aufgabe ist, auch wenn wir uns nur auf Tragwerke bzw. Bauten beschränken und nicht das ganze Bauvorhaben mit seinen mannigfaltigen ökonomischen Zusammenhängen ins Auge fassen. Ich möchte zu diesen Ausführungen einige Bemerkungen hinzufügen.

1) Die einfachste Optimierungsaufgabe ergibt sich, wenn als Kriterium lediglich das Minimum des Aufwandes eines Baustoffes gestellt wird. Da bei Stahlkonstruktionen der Materialpreis die Gesamtkosten wesentlich beeinflusst und die Anarbeitungskosten bei nicht zu grosser Änderung der Technologie dem Gewicht etwa proportional sind, genügt oft diese vereinfachte Aufgabenstellung. Sie ist sogar zu gewissen Zeiten mit Stahlmangel ausschlaggebend. Courbon weist richtig darauf hin, dass das Gewichtsminimum gewöhnlich sehr flach verläuft, wenn nur geometrische Abmessungen variiert werden. Es werden dann auch ziemlich grosse Abweichungen von der optimalen Form nur geringfügige Änderungen des Gewichtes bringen. Es sollte daher auch immer dieser Zusammenhang untersucht werden, da dann scheinbar untergeordnete, und daher vernachlässigte Einflüsse die Lage des Minimums stark verschieben können. Es soll noch hervor-  
gehoben werden, dass sich eine Tragkonstruktion minimalen Gewichtes i.A. nicht aus Bauteilen zusammensetzt, die an sich minimale Gewichte haben.

Grössere Stahleinsparungen bringen nur prinzipielle Änderungen.

der Tragkonstruktion. Als Beispiel sei eine Überlegung erwähnt, die beim Entwurf der Bogenbrücke über den Stausee der Moldau südlich Prag gemacht wurde. Es handelt sich um eine Strassenbrücke von 380 m lichter Weite, deren Fahrbahnkonstruktion auf vollwandigen Zweigelenkbogen von 330 m Stützweite aufgeständert ist (Abb. 1 und 2). Es wurde die Frage aufgeworfen, ob bei dieser grossen Stützweite die Fahrbahnplatte als leichte orthotrope Platte, oder wie bei Brücken kleinerer Spannweiten besser als Stahlbetonplatte in Verbund mit den Fahrbahnhauptträgern von 23 m Spannweite ausgeführt werden soll. Eine Vergleichsrechnung zeigte, dass der Stahlbedarf und dieser war zur Zeit der Projektverfassung ausschlaggebend für eine orthotrope Platte grösser ist, als die Vermehrung des Stahlgewichtes der Bogenträger zufolge der Vergrösserung der ständigen Last. Einen grossen Einfluss auf dieses Ergebnis hatte auch der Umstand, dass mit Grenzlaster gerechnet wurde, d.h. dass in die Berechnung nach der Theorie zweiter Ordnung das Eigengewicht mit einem kleineren Überlastfaktor eingeführt wurde als die Nutzlast.

2) Wesentlich schwieriger ist die Optimierung mit Hinblick auf die Gestehungskosten. Die in Betracht gezogene Technologie der Fertigung hat einen entscheidenden Einfluss auf die optimale Form des Tragwerkes. Denken wir z.B. an die Verschiebung der ökonomischen Grenzen zwischen Fachwerk und Vollwandträger zu Gunsten des letzteren durch die Einführung der Schweisstechnik und Rückkehr zu Fachwerkträgern im Leichtbau. Die Erzeugungskosten werden wesentlich von der Fabrikationseinrichtung und diese von den herzustellenden Stückzahlen beeinflusst. Bei Massenerzeugung sind auch komplizierte Formen mit grossen Anforderungen an die Genauigkeit zulässig, wie die Fachwerkplatten und Kuppeln beweisen. Ganz wesentlich wirkt sich auch das Verhältniss zwischen Materialpreis und Lohnkosten aus, sodass die Ergebnisse nicht einfach auf andere Wirtschaftsgebiete übertragen werden können.

Ein typisches und nur scheinbar einfaches Beispiel einer solchen Aufgabe ist die optimale Aussteifung eines Stehbleches. Theorie und Versuch zeigen, dass die tragfähigkeitserhöhende Wirkung einer Aussteifung nicht linear mit ihrem Querschnitt zunimmt und dass sich nach einer gewissen Grösse der Steife die Tragfähigkeit des Stehblechs nur langsam seinem Maximum nähert. Es besteht also sicherlich ein optimaler Steifenquerschnitt und mit Hinblick auf die Herstellungskosten auch eine optimale Steifenzahl. Diese Aufgabe wurde

eigentlich noch nicht gelöst, insbesondere nicht für die überkritische Beulfestigkeit. Form und Querschnitt der Gurtungen eines Vollwandträgers beeinflussen sowohl die Tragfähigkeit des Stehbleches, als auch die Kippgefahr des Trägers. In der Praxis werden sehr verschiedene Lösungen verwendet, wie Abb. 3 an einigen Beispielen zeigt. Entscheidend ist hier wieder der Gesamterfolg.

Dies möge noch ein Beispiel von dem Bau der oben erwähnten Bogenbrücke zeigen. Die Brücke ist mit einer Breite von 13,0 m zwischen den Geländern schmal im Verhältnis zur Spannweite (Abb. 3). Um eine genügende Seitenstabilität zu sichern, war ursprünglich vorgesehen, die Bogen schräg zu stellen und die damit verbundenen Erzeugungs und Montagekosten in Kauf zu nehmen. Nach entsprechenden Studien wurde jedoch folgende, wesentlich einfachere Lösung zur Ausführung bestimmt. Die beiden Bogen wurden in einem Abstand von 13 m angeordnet und mit Windverbänden bei beiden Gurtungen als torsionssteifes Gebilde ausgeführt. Die Säulen, welche die in einem Abstand von 12 m liegenden Fahrbahnhauptträger tragen, sind über den inneren Stehblechen aufgeständert. Zur Seitensteifigkeit und Stabilität der Brücke trägt wesentlich die Fahrbahn bei, welche als durchlaufender, wagrechter Balken auf fünf Stützen die Windkräfte auf den Fahrbahnträger und die Fahrzeuge nur zu einem kleinen Teil im Bogenscheitel abgibt. In Abb. 4 sind die horizontalen Reaktionen des Fahrbahnträgers bei verschiedenen Anordnungen gezeigt. Im Bogenscheitel würde bei der üblichen Ausführung (Abb. 4d) eine Reaktion von 132 Mp zu übertragen sein, bei der Ausführung als Durchlaufträger über 5 Stützen (Abb. 4b) beträgt die Reaktion nur 32 Mp, da sich der Bogen unter seiner auf ihn entfallenden Windlast auch horizontal durchbiegt. Durch diese Ausnützung des räumlichen Zusammenwirkens konnte viel an Anarbeitungskosten gespart werden. Es sei nur nebenbei bemerkt, dass der horizontale Träger mit einer Länge von 542 m wohl der längste Vollwandträger ist, der je gebaut wurde. Den Querschnitt bildet die 12 m breite, ohne Dilatationsfugen durchlaufende Stahlbetontafel als "Stehblech" in Verbund mit den 1,8 m hohen Fahrbahnhauptträgern als Gurtungen. Der provisorische Windverband zwischen den Fahrbahnhauptträgern wurde nach Erhärten der Betonplatte entfernt.

3) Eine besondere Art der Optimierungsaufgaben bildet die Festlegung von Typenreihen. Hier kommt als neue Veränderliche noch die Grösse des Typenschrittes hinzu, welche die wirtschaftlichen Ergeb-



nisse stark beeinflussen kann. Die optimale Grösse des Typenschrittes hängt wesentlich von den zu erwartenden Ausführungszahlen ab. Grosse Schritte erhöhen den Materialverbrauch, bringen jedoch durch die Vergrösserung der Stückzahlen gleicher Ausführung Einsparungen an Fertigungskosten.

4) Wesentlich komplizierter wird die Auffindung der optimalen Lösung, wenn verschiedene Baumateriale, also zB. Stahl, Beton, Holz usw. in Wettbewerb treten. Die Vergleichsbasis sollte nicht, wie es oft geschieht, die Gestehungskosten, sondern der Gebrauchswert der einzelnen Ausführungen sein. Dass in solchen Fällen die optimalen Lösungen für die einzelnen Bauarten verglichen werden sollten, ist wohl selbstverständlich. Unter Verbrauchswert können wir die Zusammenfassung aller während der Exploitationszeit anfallenden Kosten verstehen, also die Gestehungskosten mit entsprechender Berücksichtigung des günstigen Einflusses einer Verkürzung der Bauzeit, die Erhaltungskosten, die Möglichkeit von Einsparungen durch bessere Adaptibilität der Konstruktion bei etwaiger Änderung der Anforderungen an den Bau, und schliesslich noch die Abbruchkosten und der Altmaterialwert. Der letztgenannte Posten hat in erzarmen Ländern einen nicht ausser Acht zu lassenden Einfluss.

5) Noch schwieriger ist die Optimierungsaufgabe, welche in Ländern mit strenger Planwirtschaft gestellt wird : die geplanten Bauvorhaben sind optimal auf die vorhandenen, bzw. noch zu schaffenden Erzeugungskapazitäten der einzelnen, auf verschiedene Baumaterialien eingestellten Betriebe zu verteilen. Für gewisse Baugruppen ist es von allem Anfang an klar, dass sie bei Erfüllung der technischen Erfordernisse nur in bestimmten Materialien ausgeführt werden können. Es bleibt jedoch ein, mit dem technischen Fortschritt sich vergrösserndes Gebiet übrig, auf dem die verschiedenen Baustoffe in Wettbewerb treten können. Da keine Reserven vorhanden sind und die einzelnen Materialien daher nicht optimal eingesetzt werden können, besteht die Aufgabe darin, die durch die weniger vorteilhafte Verwendung der Baumateriale entstehenden Verluste möglichst klein zu halten. Es kann sich auch ergeben, dass auch tragende Konstruktionen aus Leichtmetall so vorteilhaft ihren Platz im Bauplan zugewiesen bekommen. Es ist einzusehen, dass die Lösung einer so umfangreichen Aufgabe mit den zur Verfügung stehenden Mitteln und Kenntnissen nicht immer restlos gelingt.

6) Als höchste und schwierigste Optimierungsaufgabe, die weit

über das hinausgeht, was hier behandelt wird und die nur der Vollständigkeit halber angeführt wird, ist die Abschätzung der optimalen Zusammensetzung der Baukapazitäten, welche ein Land braucht, um den anfallenden Baubedarf in günstigster Weise zu befriedigen. Hierbei sind auch die notwendigen Nebenbetriebe usw. ins Kalkül zu ziehen.

Es ist sehr zu begrüßen, dass unter dem Stichwort "Optimierung" auf diesem Kongress auch ökonomische Fragen zur Sprache kommen. Der Theoretiker betrachtet oft als vordringlichste Aufgabe die möglichst exakte Berechnung der Spannungen in einer gegebenen Konstruktion, für eine gegebene Belastung. In Wirklichkeit ist die Aufgabe immer umgekehrt gestellt: für einen gegebenen Zweck ist das nach Form, Technologie und Gebrauchswert optimale Bauwerk zu entwerfen. Eine allgemeine Lösung ist ausser für einfachste Fälle noch unbekannt. Es fehlt eine tragfähige Theorie des Konstruierens und des Entwerfens.

Bei dem Fortschritt und der Verfeinerung der Berechnungsmethoden und der Möglichkeit des Einsatzes von Rechenautomaten sollte die Fragestellung mit Einschluss der Wirtschaftlichkeit immer mehr an Bedeutung gewinnen. Das Ziel einer Berechnung sollte nicht der Nachweis des Spannungszustandes oder der Sicherheit einer gegebenen Konstruktion sein, sondern direkt das Auffinden der optimalen Gestalt derselben. Es sollte sich also der Übergang von der Beurteilung eines Querschnittes zu dessen Bemessung und weiter zu seiner optimalen Formgebung vollziehen. Wir sind erst bei wenigen einfachen Aufgaben bei dieser dritten Stufe angelangt. Leider wird auch bei den Statikern in der Praxis oft mehr Wert auf eine (oft nur scheinbar) genaue Berechnung gelegt, als auf eine kritische Beurteilung der Ergebnisse und hauptsächlich auf die aus ihnen zu ziehenden Lehren für eine wirtschaftlichere Gestaltung der Konstruktion. Trotz elektronischer Rechenautomaten bleibt das Schaffensfeld des schöpferischen Gestalters der Konstruktion noch vollkommen uneingeschränkt. Dies auch deswegen, weil der Begriff "optimale Konstruktion" mathematisch schwer formulierbar bleibt.

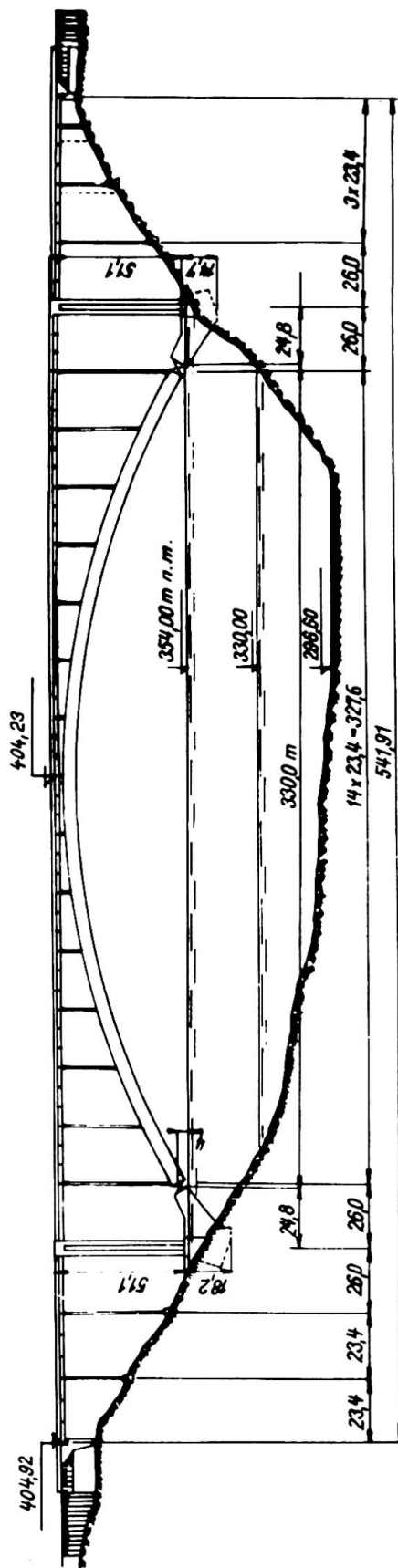


Abb. 1 Ansicht der Zweigelenkbogenbrücke

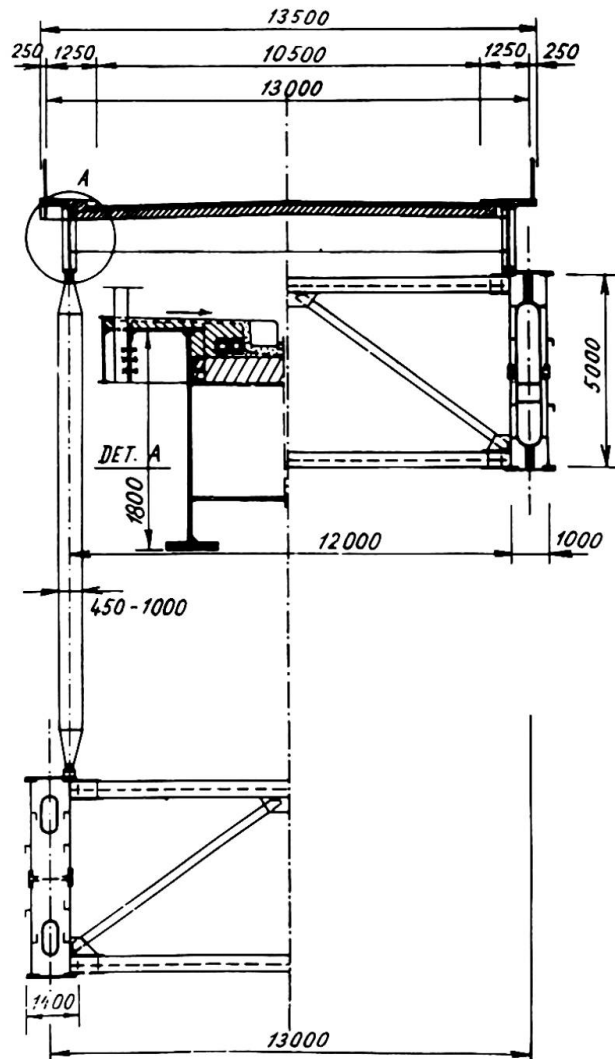


Abb.2 Querschnitt im Bogenviertel  
und im Bogenscheitel

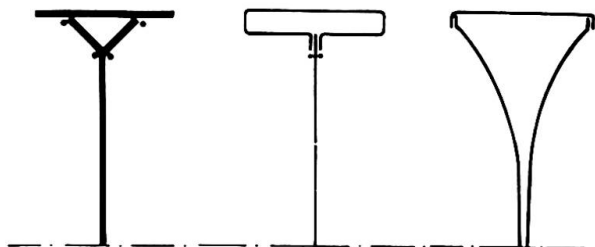


Abb.3 Querschnitte von Trägern  
mit vergrößerter Kippsteifigkeit

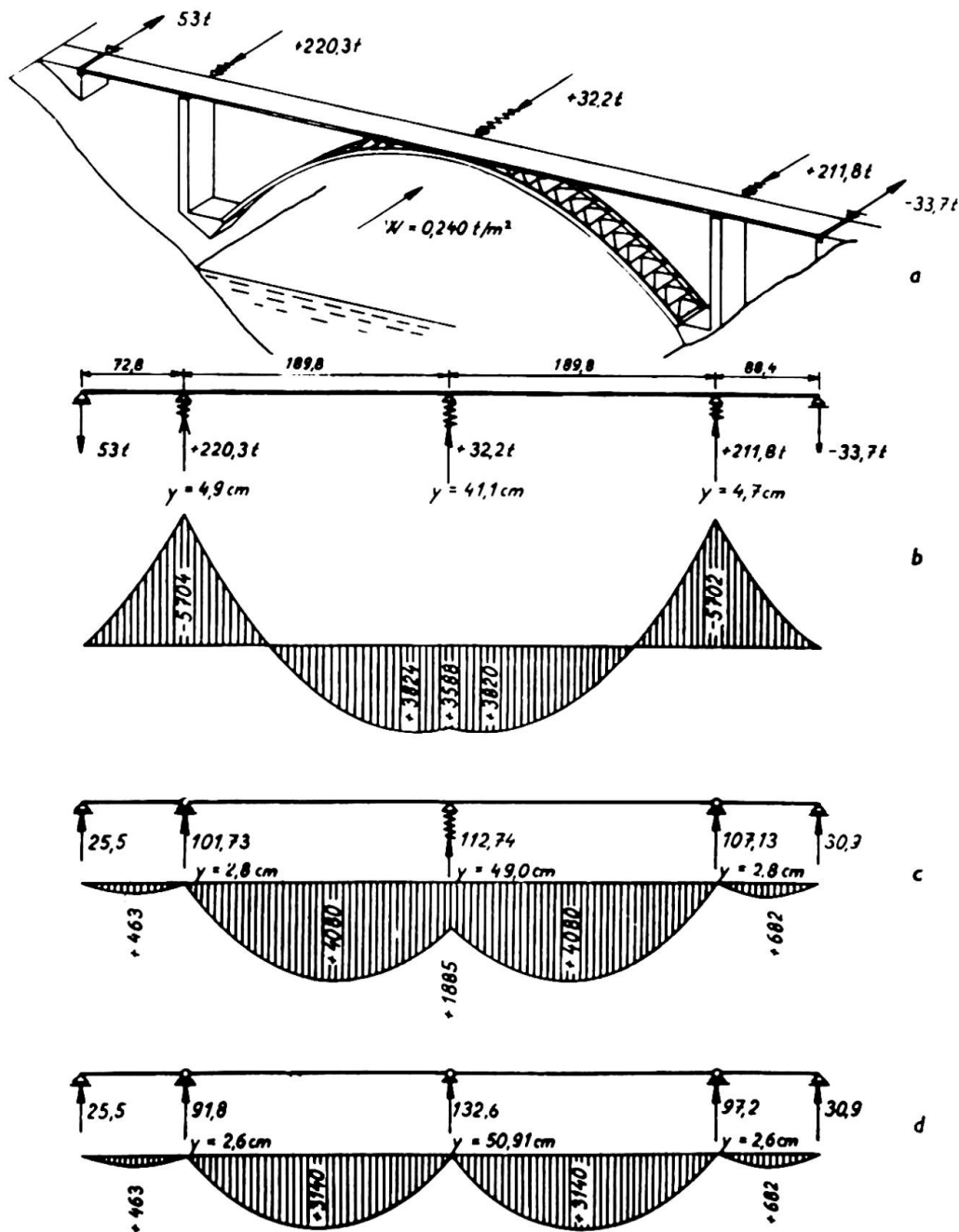


Abb. 4. Die Brücke als Raumkonstruktion bei Windbelastung  
 a) Ansicht der Brücke. Horizontale Reaktion der Fahrbahnplatte und Biegemoment bei Ausbildung  
 b) als Durchlaufträger über die ganze Brückenlänge  
 c) als Durchlaufträger nur im Mittelfeld  
 d) als 4 freiaufliegende Träger

Legend of figures:

- Fig. 1 Elevation of the two-hinged arch bridge near Prague  
 Fig. 2 Cross-section of the road bridge in the quarter and in the middle of the span  
 Fig. 3 Different sections with enlarged stiffness against lateral buckling  
 Fig. 4 The bridge as a space structure with wind load  
     a) perspective view  
         Horizontal reactions and bending moments in the horizontal girder constructed  
     b) as a continuous girder over the whole length of the bridge  
     c) as a continuous girder only over the length of the arch  
     d) as four simply supported girders

Légende de figures:

- Fig. 1 Elévation du pont en arc à deux articulations près de Prague  
 Fig. 2 Coupe transversale au quart et au sommet de l'arc  
 Fig. 3 Différentes sections à résistance agrandie contre déversement latéral  
 Fig. 4 Le pont comme construction dans l'espace sous la poussée du vent  
     a) Vue perspective du pont  
         Réactions horizontales et moments fléchissants de la poutre horizontale  
         conçue comme:  
     b) poutre continue sur toute la longueur du pont  
     c) poutre continue dans la partie médiane  
     d) poutres à deux appuis

## ZUSAMMENFASSUNG

Es werden 6 verschiedene Stufen von Optimierungsaufgaben angeführt: minimaler Baustoffaufwand, minimale Gestehungskosten, Optimierung von Typenreihen, optimaler Einsatz verschiedener Baustoffe, optimale Aufteilung der gegebenen Baukapazitäten für verschiedene Baumaterialien auf den Baubedarf und schliesslich deren günstigste Zusammensetzung. Es werden zwei Beispiele von Optimierungsaufgaben vom Bau der Bogenbrücke über die Moldau gebracht.

## SUMMARY

Six different problems of optimization are presented: structures with lowest weight, with lowest cost, optimization of type-ranges, optimal use of different building materials, optimal distribution of the disponible building capacity and finally for different materials the best combination of building capacities. The arch bridge over the Vltava-river is used, to demonstrate two examples of optimization.

## RÉSUMÉ

L'auteur présente 6 différents problèmes d'optimisation: minimum du poids, minimum du prix de revient, optimisation des rangées types, choix optimal parmi différents matériaux de construction, distribution optimale des capacités de construction possibles, la combinaison optimale des capacités de construction. Le pont en arc sur la Vltava sert comme exemple pour 2 solutions d'optimisation.

## **Optimisation of Structures**

Optimisation des structures

Optimierung von Tragwerken

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In Prof. J. Courbon's opinion "the optimisation of a structure consists of designing and constructing that structure at the lowest cost, with the object of fulfilling a well defined purpose. In particular the safety factor must be specified. ..."

A more general definition of optimisation is suggested, based not on the lowest cost but on the lowest overall expenses incurred during service life, including the initial cost of the structure. Such definition can be phrased as follows: the optimisation of a structure consists in designing and constructing that structure so as to minimize the sum of the initial cost plus the present value of the maintenance and operating expenses plus the expectation of all expenses incurred in case of accident or obsolescence<sup>(1)</sup>. The safety factor of each structural element has to be selected so as to fulfill such purpose.

The consequences of failure are not the same for all the elements of a given structure: for some of them failure can bring about complete collapse of the structure, whilst for other elements an easy repair will be possible without discontinuing service.

Why not therefore regard the safety factor as one of the parameters that the designer has to choose while respecting certain rules, just as he does with the materials, the type of structure, and the particular method of construction ?

As there is no value of the safety factor that will give a clear cut limit between safe and unsafe design, there is no reason for specifying its value once and for all, as most codes of practices do. The safety factor of each element of

(1) Expectation is meant here as the probability of the accidents multiplied by the amount of the expenses incurred.

a structure has instead to be chosen taking into consideration not only the initial cost but also the maintenance expenses and the possible consequences of accidents.

What has been said of the different elements of a structure does also apply to structures of the same type but used in different circumstances, the failure of some having much more costly consequences than the failure of others. This would be the case of a dam located respectively upstream or downstream a town.

In general a higher safety factor implies an increase in the initial cost of an element or of a structure, but it permits of a reduction of the maintenance expenses and of the risk of accidents. A lower safety factor, on the other hand, permits of a lower initial cost, but implies an increase in maintenance expenses and in the risk of accidents. In general it also permits of an easier and cheaper adaptation to changing service conditions. But no safety factor, even much larger than the ones usually adopted, will ever give absolute safety.

Instead of minimizing the initial cost of a structure by taking a safety factor specified beforehand, one can try to minimize a generalized cost consisting of initial cost plus maintenance expenses plus expectation of expenses to be incurred in case of possible accidents, all such costs being expressed in terms of the safety factor.

This will permit of a very general approach to the problem of the optimisation of a structure by taking into account not only the parameters mentioned in the report but also the safety factor.

As a good example of such optimisation procedure, the most economical height of some dykes that protect the Netherlands against sea invasions has been chosen by finding the elevation  $H$  such that

$$d \left[ \frac{C(H) + D e^{-\alpha(H-H_0)}}{dH} \right] = 0$$

where

- $H$  - elevation of the top of the dyke
- $C(H)$  - cost of the dyke as a function of  $H$
- $\alpha$  - a factor depending on the frequency with which certain sea levels are reached
- $H_0$  - lowest theoretical elevation of dyke top
- $D$  - expenses incurred in case of overtopping of the dyke



Interest rates are currently very high. It seems therefore worthwhile to take into account the time value of money by deferring expenditure as much as possible. This can be done by reducing the initial cost, even when this implies increased maintenance expenses and higher risks during the service life of the structure.

Let us call

$C$  - initial cost

$m$  - number of types of failure envisaged for a given structure.

$n$  - number of periods of time - usually years - of the service life of the structure.

$p_1^i, p_2^i \dots p_j^i \dots p_m^i$  - probability of each type of failure during each period of time  $i$ .

$D_1^i, D_2^i \dots D_j^i \dots D_m^i$  - direct expenses with repair costs for each type of failure, and, where applicable, cost of replacing the entire structure.

$I_1^i, I_2^i \dots I_j^i \dots I_m^i$  - indirect expenses incurred for each type of failure, such as indemnities to third parties.

$M_1, M_2 \dots M_i \dots M_n$  - maintenance and operation expenses during each period of the structure's service life.

$P_1, P_2 \dots P_i \dots P_n$  - survivalship of the structure, i.e. the probability of its being maintained in service during each period of time, taking into account all causes for discontinuance of service including accidents and obsolescence.

$r$  - interest rate.

Depending on the type of structure, the probability of failure  $p$ , the repair costs  $D$ , and the indirect expenses  $I$ , will either decrease or increase with time.

The expectation of all expenses involved in all types of failure can be evaluated for each period of time  $i$  by

$$E_i = \sum_{j=1}^m p_j^i (D_j^i + I_j^i)$$

The most economical structure will, on the long run, and taking into account the parameters just mentioned, be the one that renders the following sum a minimum:

$$\begin{aligned} & C + && \text{initial cost} \\ & + (M_1 + E_1) (1 + r)^{-1} + && \text{present value of all expenses} \\ & && \text{incurred during 1st year} \\ & + (M_2 + E_2) (1 + r)^{-2} P_1 + && \text{ditto for 2nd year} \\ & + (M_3 + E_3) (1 + r)^{-3} P_1 P_2 + && \text{ditto for 3rd year} \\ & + \dots \dots \dots \\ & + (M_n + E_n) (1 + r)^{-n} P_1 P_2 \dots P_{n-1} \longrightarrow \min \end{aligned}$$

Depending on the type of structure, maintenance and operation expenses  $M$  and the survivalship of the structure  $P$  can either increase or decrease with time. If there is no reason to envisage the discontinuation of service, e.g. because the structure will have to be rebuilt even in case of total collapse, then  $P_1, P_2 \dots P_n$  can be taken as unity in the above expression.

If  $p, D, I, M$  and  $P$  can be assumed to remain constant, and  $P$  taken as unity, the expression can be given the much simpler form

$$C + a_n (M + E) \rightarrow \min$$

where  $a_n$  is the "series present value factor", given by

$$a_n = (1 + r)^{-1} + (1 + r)^{-2} + \dots + (1 + r)^{-n}$$

Although the values to be introduced in the above expressions are difficult to evaluate, particularly the expectations, these expressions or a similar ones could permit of a quantitative evaluation of the safety factor leading to the most economical structure, on the long run. It will also enable the influence of the usable life of the structure to be considered quantitatively in design, the same applying to its maintenance and operation costs and to the possibility of its adaptation to foreseeable changes, as recommended by the reporter in the opening paragraph.

## SUMMARY

A more general definition than the one given at the beginning of the report is suggested, based not on the lowest cost of the structure for a specified safety factor, but on the lowest overall expenses incurred during the service life, namely initial cost, maintenance, operation and risks of both accidents and obsolescence. The safety factors of each structural element and of the whole structure have to be selected so as to fulfill such purpose.

Such definition can be phrased as follows: the optimisation of a structure consists in designing and constructing that structure so as to minimize the sum of the initial cost plus the present value of the maintenance and operating expenses plus the expectation of all expenses incurred in case of accident or obsolescence.

## RÉSUMÉ

On propose pour l'optimisation d'une structure une définition plus générale que celle présentée au début du rapport, ayant pour objet non pas la structure au moindre prix pour un coefficient de sécurité imposé, mais la structure la plus économique en service, compte tenu de tous les frais de service y compris ceux

d'entretien, et des risques d'accidents et d'obsolescence. Les coefficients de sécurité pour chaque élément et aussi pour l'ensemble de la structure, doivent être choisis conformément à cet objet.

A cette définition on peut donner la forme suivante: l'optimisation d'une structure consiste à concevoir et à réaliser la structure de façon à rendre minimum la somme de son prix avec la valeur actuelle des frais d'entretien et avec l'espérance mathématique des frais directs et indirects que puisse entraîner des accidents ou des modifications prévisibles des conditions de service.

## ZUSAMMENFASSUNG

Es wird eine erweiterte Definition des Begriffs "Optimierung eines Tragwerkes" vorgeschlagen, worin neben den Erstellungskosten auch die zu einem späteren Zeitpunkt noch zu erwartenden Kosten (Unterhalt und Risiken) mitberücksichtigt werden sollen. Dies wäre die wirtschaftlichste Lösung auf weite Sicht.

Eine solche Definition könnte folgendermassen lauten: Die "Optimierung eines Tragwerkes" besteht aus der Wahl des zweckmässigen Sicherheitskoeffizienten jedes Tragwerkes und aus der wirtschaftlichsten Verbindung zwischen Projekt - und Erstellungskosten mit den laufenden Unterhalts - und Bedienungskosten und den auf statistischer Vorhersage zu erwartenden Kosten gelegentlicher Unfälle und eventueller Änderungen.

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## Optimum Design for Structural Safety

Dimensionnement optimal pour la sécurité d'une construction

Optimierung für die Bausicherheit

### FRED MOSES

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### BACKGROUND

In recent years there have been developments in the area of optimum design of structures which concerned the sizing and proportioning of members for minimum weight or cost. The utilization of digital computers and advances in allied fields of Mathematical Programming and Operations Research led to the formulation of structural optimization as a problem in Mathematical Programming. Given a set of design variables such as depth, thickness, area, moment of inertia all denoted by a vector  $X_j$  the design problem becomes:

$$\text{Minimize } f(X_j) \quad (1)$$

$$\text{such that } g_j(X_j) \geq 0 \quad j=1,2,\dots, \text{Number of constraints} \quad (2)$$

$f(X_j)$  is a function of weight or cost to be minimized while  $g_j(X_j)$  are the design limitations on stress, stability and deflection or any practical fabrication or construction restrictions. It is necessary to be able to compute for any set of design variables  $X_j$  the stresses, deflections and stability associated with this design. Recent papers in the structural engineering literature have presented efficient optimum design techniques for a wide class of problems including plate girders, trusses, frames, stiffened plates and cylinders.<sup>1</sup> These works have used Mathematical Programming techniques such as linear programming, dynamic programming, gradient methods and unconstrained minimization.<sup>2</sup> The Mathematical Programming approach to design may be limited in that unlike Professor Courbon's paper entitled, "Optimization of Structures" it does not consider creative changes in design or even large deviations from an initial prescribed design geometry and topology.<sup>3</sup> Nevertheless, it has led to important economies in structural design and has been used in practice particularly for conventional type structures such as plate girder bridges and frames.

This paper considers the problem of optimization within the context of safety. It has been proposed by Professor Freudenthal that a rational approach

to safety must be based on reliability or probability of failure.<sup>4</sup> This leads to a new formulation of an optimum design problem which is:

$$\text{Minimize } f(X_i) \quad (3)$$

$$\text{such that } P_f(X_i) \leq P_f \text{ allowable} \quad (4)$$

$P_f \text{ allowable}$  is the minimum failure probability acceptable for the design and is assumed in this study as given.  $P_f(X_i)$  is a function which gives the overall failure probability of the structure as a function of the design variables to be determined. This constraint on  $P_f(X_i)$  is the only constraint used on structural behavior although other constraints may be based on construction or other requirements. Several methods for computing  $P_f(X_i)$  and utilizing it in an optimum design procedure are presented. Some factors which have motivated this approach to optimization include the following points:

- a) Safety as expressed in terms of probability of survival may actually be impaired in current deterministic optimum design programs. This is because existing design codes and safety factors are used to provide protection against stress, deflection and stability-type failure modes. These safety factors were developed over a period of time in practice and were not associated with structures which were optimized. Most mathematical programming optimum designs end up with a larger number of constraints on stress and deflection against their limit than an unoptimized design. It should be expected, therefore, that the probability of failure which is the probability that any failure mode occurs will be higher for an optimized design. An optimization procedure which uses overall structural failure probability as the behavior constraint should produce more balanced designs consistent with the development of rational safety.
- b) In order to reach more significant levels of structural optimization it is necessary to compare optimized structures of different configuration, material and geometry. Within this decision context a rational comparison is possible only if the structures have the same level of safety as expressed in terms of probability of failure.
- c) The use of new materials such as brittle composites with greater scatter in strength tests and new structural applications in environments with greater uncertainty suggest that there will be more emphasis in the future on designing for probability levels rather than using preassigned safety factors. It, therefore, seems appropriate to formulate the design optimization problem as in equations 3 and 4.
- d) Reliability based optimum design may actually facilitate the mathematical optimization problem by replacing the numerous limitations (on member stress

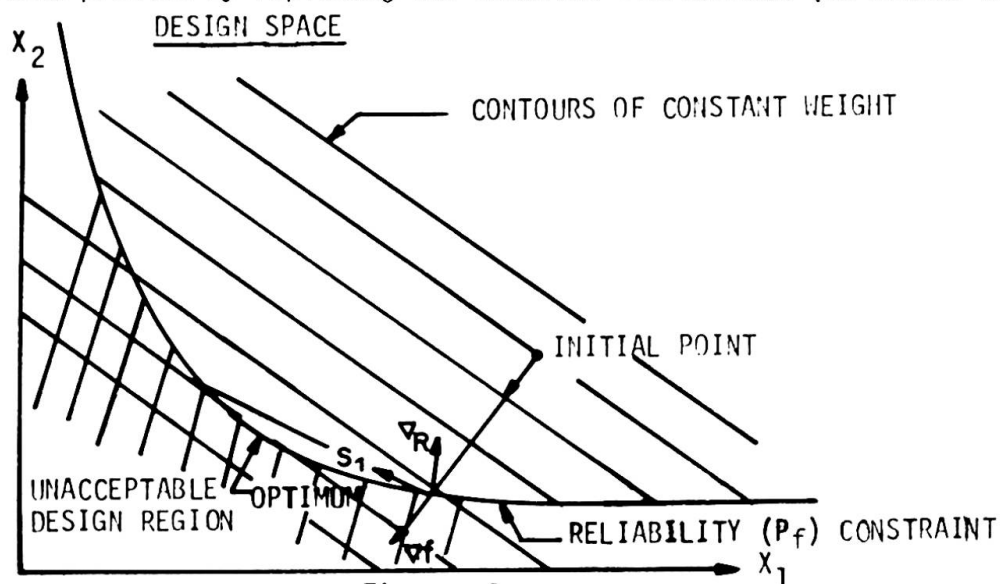


Figure 1

and deflections) in a deterministic design by a single constraint on overall structural failure. The mathematical and computational complexity, however, has been transformed from the minimization aspect to the analysis of failure probability. The problem of minimizing the weight subject to only one constraint as illustrated in Figure 1 with two variables can be handled by several techniques including linear programming approximations and useable feasible gradient moves. Figure 1 illustrates an algorithm for minimization by use of the gradients or normal vectors to the weight function and the reliability constraint. From an arbitrary starting point the design is changed in steps in the direction of the gradient to the weight until the reliability constraint is encountered. Subsequent changes in the design variables are made in a direction  $s_1$  which both reduces weight and avoids violating the reliability constraint. This is a useable feasible direction and methods for determining this direction are well known.<sup>5</sup> The design changes are continued until the constraint gradient and reliability gradient are colinear.

### RELIABILITY ANALYSIS AND OPTIMUM DESIGN

Most work in failure probability has concentrated on a problem in which all the strength variability was included in one member and all the load variability was included in one load. Freudenthal presented the probability of failure of this one member one load structure often called the fundamental case of structural reliability including the effect of frequency distributions, standard deviations for load and strength and the safety factor or ratio of mean values.<sup>6</sup> In considering the design or proportioning of members in multi-member multi-load structures, a model is needed to show the effect on failure probability of each of the individual members of the structure and their interaction with all load conditions. Two reliability applications of importance are presented herein. The first is multi-member "weakest link" structures discussed by Professor Freudenthal in which the structure fails if any single element fails under any load condition.<sup>4</sup> The second application is "redundant" structures such as limit designed frames in which failure is the occurrence of any collapse mode each involving more than one element yielding. In both cases loads and strengths are random variables described by known frequency distributions. The reliability analysis computes for a given design the overall failure probability of the structure. The optimum design problem is to proportion member sizes to have minimum weight or cost for a specified allowable failure probability.

### "WEAKEST LINK" STRUCTURES

The failure probability of a single member of strength  $R$  under a single load condition  $S$  can be determined from the following equation:<sup>6</sup>

$$P_f = \int_0^{\infty} \Pr \{S > \tau\} \Pr \{R = \tau\} d\tau = \int_0^{\infty} [1 - F_S(\tau)] f_R(\tau) d\tau \quad (5)$$

$\Pr$  should be read "probability that".  $F(\tau)$  is the distribution function and  $f(\tau)$  the density function. In extensions to multi-member structures under one load condition it has often been proposed that the overall failure probability could be obtained from the following equation:<sup>7,8,9</sup>

$$P_f = 1 - \prod_{i=1}^N [(1 - P_{fi})] \quad N = \text{Number of members} \quad (6)$$

$P_{fi}$  is the failure probability of the  $i$ th member and  $P_f$  the overall failure probability. If the individual  $P_{fi}$  are small as is usually the case, then equation 6 becomes:

$$P_f = \sum_{i=1}^N P_{fi} \quad (7)$$



Equation 6 ignores, however, the fact that the events corresponding to member failures are statistically correlated since the stresses in each member are completely correlated since they arise from the same loading. The member failures are not 100% correlated since the strengths are independent random variables. The consideration of correlation leads to a lower value of  $P_f$ . If there is only one load condition and  $N$  members or element strengths, then it is easy to verify that the equation that gives the failure probability is:

$$P_f = 1 - \int_0^{\infty} \prod_{i=1}^N [1 - F_{Ri}(a_i \tau)] f_p(\tau) d\tau \quad (8)$$

The constant  $a_i$  relates the forces or stress levels in member  $i$  to the load value of  $P=\tau$ . Equation 8 is valid for "weakest-link" structures which fail if any member fails. This includes determinate structures and those indeterminate structures with little "fail-safe" probability of survival available after the first member has yielded. This is true for structures with brittle members which can't carry any load after reaching yield load and for all structures for which the load variability greatly exceeds the strength variability. If the structure is subjected to  $M$  repeated application of the same loading condition then  $P_f$  can be computed by integrating on the density function of the worst load which is:

$$f_{P_{max}}(\tau) = M [F_p(\tau)]^{M-1} f_p(\tau) \quad (9)$$

The failure probability is then:

$$P_f = 1 - \int_0^{\infty} \prod_{i=1}^N [1 - F_{Ri}(a_i \tau)] f_{P_{max}}(\tau) d\tau \quad (10)$$

If all loads are not of the same load condition but represent distinct load conditions applied at different times then an exact solution for  $P_f$  requires an evaluation of a multiple integral based on the joint distribution function of the load conditions. Various bounds have been presented on the failure probability based on evaluating integrals which reflect the importance of statistical dependence between failure modes due to a single load condition on multiple members or a single member acted on by distinct load conditions.<sup>10,11</sup>

An important factor in reliability design of "weakest link" structures is whether the design constraint is based on equation 10 which is exact for the case shown or equation 6 which is an upper bound on  $P_f$  and ignores the statistical correlation between failure modes. A previous study showed that this correlation significantly affects the reliability analysis if the variability of the load random variable exceeds that of the strength as in structures designed to resist as their major loading wind and earthquakes.<sup>11</sup> In such instances, a lower bound on  $P_f$  which is the largest member failure probability may be used as the design constraint.

In studying optimum design of "weakest-link" structures two factors were under observation. One was the effect of statistical correlation between failure modes on the overall structural weight and the second is the influence of unequal individual member failure probabilities. It should be noted that all previous studies on reliability based optimums have used equation 6 as the basis for computing the failure probability and thus have ignored the correlation.<sup>7,8,9</sup> To study the effect of correlation, a design is found for a truss with one load condition. All members are assumed to have equal mean loads and, therefore, have the same optimum area. The consideration of the exact value of the  $P_f$  constraint in equation 8 including statistical correlation allows each member to be designed for a higher individual failure probability than if equation 6 were used as the constraint and correlation ignored. The higher individual failure probability

means a lower weight and the ratio of the two optimum weights based on equation 8 and equation 6 is plotted in Figure 2. With a normal frequency distribution of load and strength for the coefficients of variation shown the maximum weight saving reaches 7.3% for case 1 in a 50 member structure.

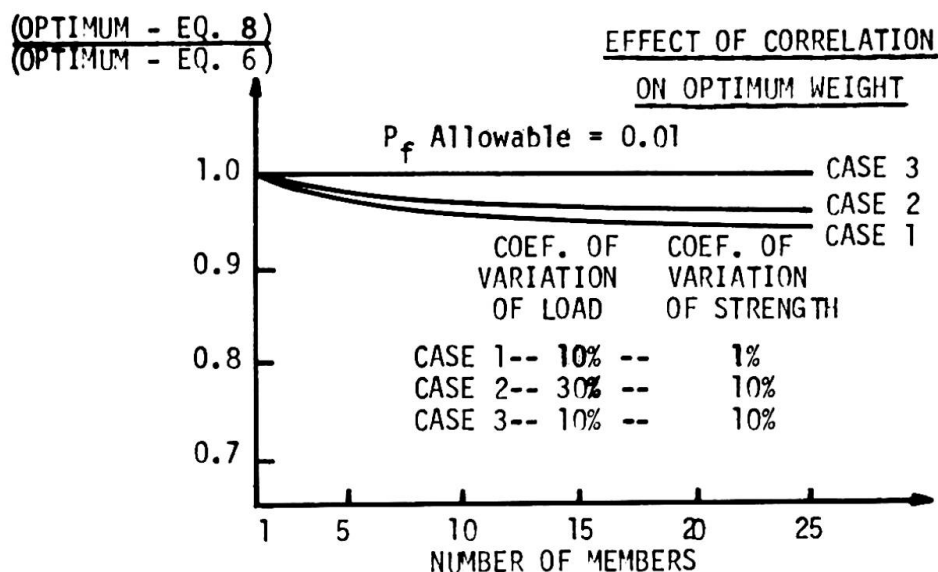


Figure 2

Another indication of the correlation factor becomes evident when the overall failure probability is written as:

$$P_f = a_1 P_{f1} + a_2 P_{f2} + \dots + a_i P_{fi} + \dots + a_N P_{fN} \quad (11)$$

$P_{fi}$  is the failure probability of the  $i$ th member and  $a_i$  is the percentage of its failure probability that this member contributes to the overall probability. The  $a_i$  shown in Figure 3 are computed either by sequentially integrating equation 8 for increasing number of members or by another technique discussed elsewhere.<sup>11</sup> If there were no statistical correlation all  $a_i$  would equal 1.0. If there was complete correlation between failure modes and the members were ordered with member 1 having the highest failure probability then  $a_1$  would be 1.0 and all other  $a$ 's equal zero.

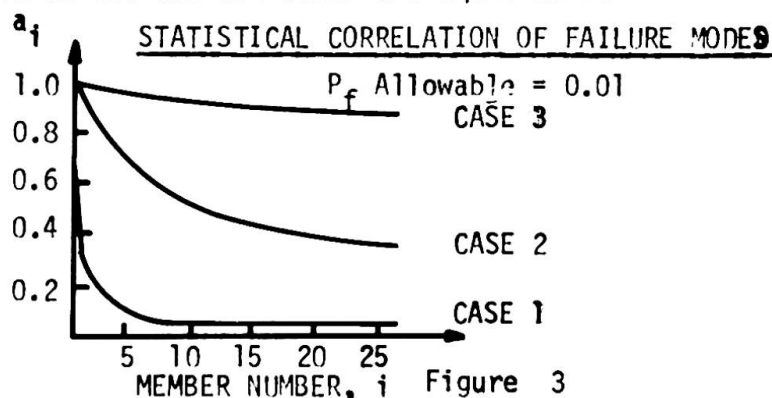


Figure 3

Figure 3 shows the potential weight saving with other frequency distributions and members with unequal lengths and mean applied loads since it indicates the correlation effect and the error introduced into the overall probability expression as the number of members in a structure increases. The magnitude of the weight saving can be obtained from plots of member failure probability vs. member size (or equivalently its safety factor) and is available from the work

on the single member single load fundamental case.<sup>6</sup> It is seen from Figure 3 that the correlation factor becomes more significant when the load variability exceeds the strength variability as in structures subject to wind and earthquake loading.

The second factor mentioned above for study is "weakest-link" structures with members of unequal mean loads. An optimum design in this case has unequal safety factor and failure probability for each member. In general minimum weight results if heavier members have higher than the average failure probability and lighter member with lower mean applied loads or stresses have less than one average member failure probability. It has been proposed that the following equation applies to an optimum design.<sup>8</sup>

$$\frac{\text{Weight of member } i}{\text{Total weight}} = \frac{P_{fi}}{P_{f \text{ allowable}}} \quad (12)$$

This equation was derived and is applicable to cases where equation 7 for  $P_f$  is used as the design constraint. Furthermore, it implies that the element  $P_{fi}$  depends only on the size of the  $i$ th element, or:

$$\frac{\partial P_f}{\partial x_i} = \frac{\partial P_{fi}}{\partial x_i} \quad (13)$$

In general, for a statically indeterminate structure this last equation is not applicable and:

$$\frac{\partial P_f}{\partial x_i} = \sum_{k=1}^N \frac{\partial P_{fk}}{\partial x_i} \quad (14)$$

That is the change in any member affects the force distribution and, therefore, the failure probability of every indeterminate member of the structure. It should be noted that the simplicity that equation 7 introduces into the optimization procedure is not lost when the correlation is included in the computation of the failure probability. This simplicity is needed since most optimization methods need the gradient to the constraint. Using equation 11, the components of the gradient of the reliability constraint can be computed as:

$$\frac{\partial P_f}{\partial x_i} = \sum_{k=1}^N a_k \frac{\partial P_{fk}}{\partial x_i} \quad (15)$$

It has been found from experience that the  $a_k$  which accounts for the correlation does not change much in a small region in which the gradient is determined from the partial derivatives using a finite difference technique. Thus the gradient based on  $N$  computations of the form shown in equation 15 need not be obtained from finite difference perturbations of equation 8 of  $N$  members and one load but rather from equation 5 which is for one member and one load. The computation time saved in this manner may be significant.

To illustrate an optimum design for a structure with unequal member sizes Table 1 is presented for a determinate truss with 10 members. The formulation of the design problem as a minimum weight design with a failure probability constraint as indicated in equations 3 and 4 was used. Equation 12 can be seen not to be valid at the optimum due to correlation between failure modes not considered in its derivation. Also shown in Table 1 is a design based on equal safety factor for each member such that the overall failure probability of the structure based on equation 7 is equal to the allowable value. The difference between the weight of the optimum design and the equal safety factor design is partly due to

the correlation factor but principally due to the Mathematical Programming technique discussed above which proportions the members in an optimum manner using only the single overall failure probability constraint.

TABLE 1 - 10 Member Example<sup>a</sup>

| Member | Mean Load Value | Equal Safety Factor  |                 | Optimum Design       |                        |
|--------|-----------------|----------------------|-----------------|----------------------|------------------------|
|        |                 | Area in <sup>2</sup> | P <sub>fi</sub> | Area in <sup>2</sup> | P <sub>fi</sub>        |
| 1      | 0.1P            | 0.274                | 0.0001          | 0.297                | 0.519x10 <sup>-4</sup> |
| 2      | 0.2P            | 0.547                | 0.0001          | 0.554                | 0.604x10 <sup>-4</sup> |
| 3      | 0.3P            | 0.817                | 0.0001          | 0.818                | 0.958x10 <sup>-4</sup> |
| 4      | 0.4P            | 1.09                 | 0.0001          | 1.09                 | 0.991x10 <sup>-4</sup> |
| 5      | 0.5P            | 1.37                 | 0.0001          | 1.35                 | 1.23 x10 <sup>-4</sup> |
| 6      | 0.6P            | 1.64                 | 0.0001          | 1.61                 | 1.61 x10 <sup>-4</sup> |
| 7      | 0.7P            | 1.92                 | 0.0001          | 1.86                 | 2.08 x10 <sup>-4</sup> |
| 8      | 0.8P            | 2.19                 | 0.0001          | 2.11                 | 2.65 x10 <sup>-4</sup> |
| 9      | 0.9P            | 2.46                 | 0.0001          | 2.35                 | 3.25 x10 <sup>-4</sup> |
| 10     | 1.0P            | 2.74                 | 0.0001          | 2.59                 | 3.91 x10 <sup>-4</sup> |

a Mean Load,  $\bar{P}=60,000$  lb.; C.V.(P) = 20%

Mean Yield stress,  $\bar{\sigma}_y=40,000$  psi; C.V. ( $\sigma_y$ ) = 5%

Density = 0.283 lb/in.<sup>3</sup>;

Length  $L_i=60$ " for all members.

Both P and  $\sigma_y$  have normal distributions

Weight based on equal safety design factor = 255.6 lb.

Weight based on optimum design = 248.6 lb.

$P_f$  allowable is 0.001

### "REDUNDANT" STRUCTURES

In many structures particularly those designed by limit or ultimate design methods several members or elements must simultaneously reach their capacity before the structure is failed. This is the case with some indeterminate trusses and also beams and frames in which mechanisms form at failure. It is assumed that each element strength is an independent random variable.<sup>12</sup>

A failure mechanism occurs if the contribution of load elements exceeds the strength elements for any particular collapse mode. If the contributions are linear this leads to an equation for reserve strength  $Z_j$  in a mode j of:

$$Z_j = \sum_{i=1}^n a_{ji} M_i - \sum_{k=1}^L b_{jk} P_k \quad \begin{array}{l} i=1, \dots, n - \text{Critical Elements} \\ k=1, \dots, L - \text{Loads} \\ j=1, \dots, m - \text{Collapse Modes} \end{array} \quad (16)$$

Equation 16 would for example govern a frame against the formation of a collapse mechanism. The overall failure probability of the structure is the probability that any  $Z_j$  is less than zero and can be written as:

$$P_f = \Pr \{Z_1 \leq 0\} + \Pr \{Z_2 \leq 0, Z_1 > 0\} + \Pr \{Z_3 \leq 0, Z_2 > 0, Z_1 \geq 0\} + \dots \quad (17)$$

A method presented elsewhere has been developed to compute the probability that each  $Z_j$  is less than zero, including the effect of statistical correlation between  $Z_j$  present because some load and strength terms appear in more than one collapse equation.<sup>13</sup>

As an example of an optimum redundant design consider a single bent frame with design variables corresponding to the plastic moment capacity of the beam and column. Figure 4 shows the deterministic constraints based on a safety factor approach and the constraint based on failure probability as expressed in equation 17. The weight function used which is shown linearized in Figure 4 is:

$$W = K \sum_{i=1}^R (M_i)^{2/3} L_i \quad (18)$$

$M_i$  is the plastic moment capacity,  $L_i$  the length of the member,  $R$  the number of members and  $K$  is a constant.

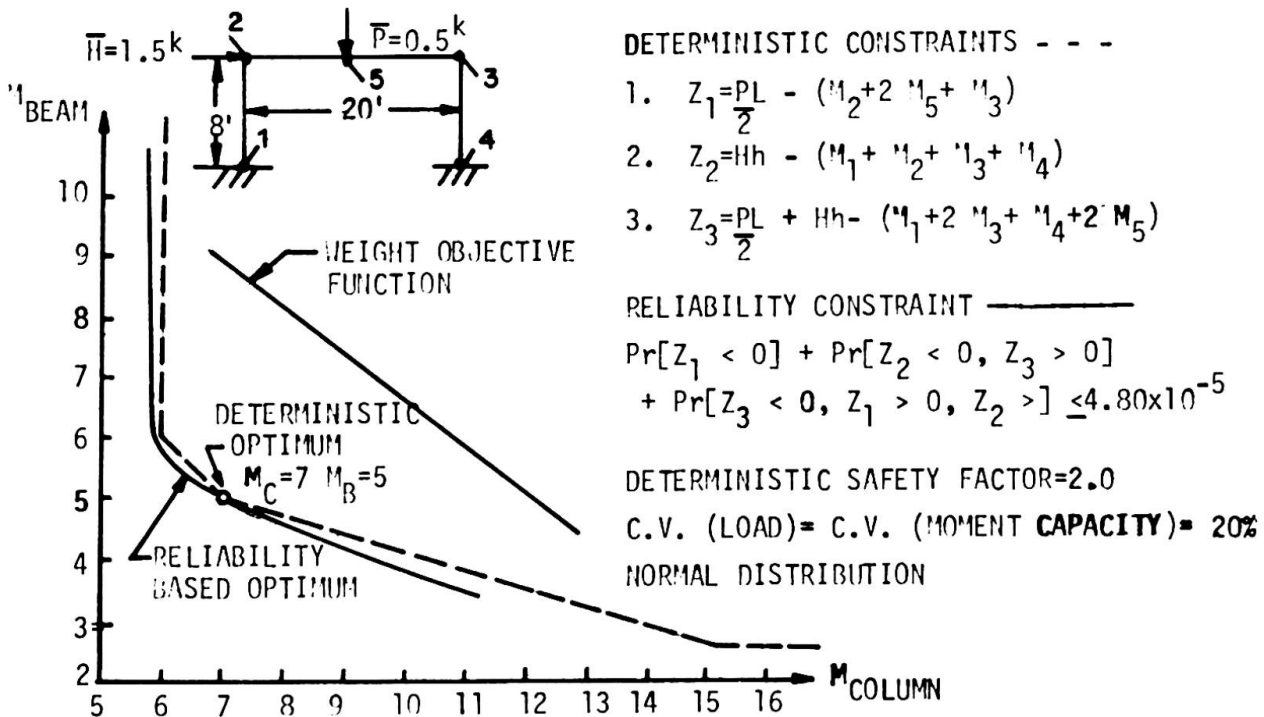


Figure 4 DESIGN SPACE FOR BENT FRAME

The allowable failure probability was set at  $4.80 \times 10^{-5}$  and was chosen based on a failure probability analysis of the optimum design obtained using the deterministic or nonstatistical constraints. Optimum design methods for frames to resist plastic collapse in the deterministic case are well known and since all constraints are linear, a linear programming technique is applicable. A result of a deterministic optimum design is that a theorem for this case shows that the number of collapse modes designed up against their limit in the optimum design equals the number of design variables which is the number of unknown member plastic moment capacities.<sup>14</sup> Although this is acceptable from a deterministic viewpoint and does not violate conventional safety factors, it must be viewed as unsafe from a reliability design viewpoint. The fact that an optimum design with a large number of design variables has an equally large number of failure modes designed to their limit must indicate that the failure probability is increased over a conventional unoptimized safety factor limited design. Consequentially, the replacement of the deterministic constraints by a single reliability constraint should lead to a more balanced optimum and also allows the failure probability to be specified in the constraint. An added benefit that may be seen from Figure 4 is that the design optimized with respect to failure probability is lower in weight than the deterministic optimum with the same failure probability. Further examples of optimum single story frames showed the optimum weight increased as both the allowable failure

probability was decreased and the coefficients of variation of load and strength was increased.<sup>15</sup>

To illustrate reliability based optimum design for larger redundant structures the frame shown in Figure 5 with six design variables was studied.

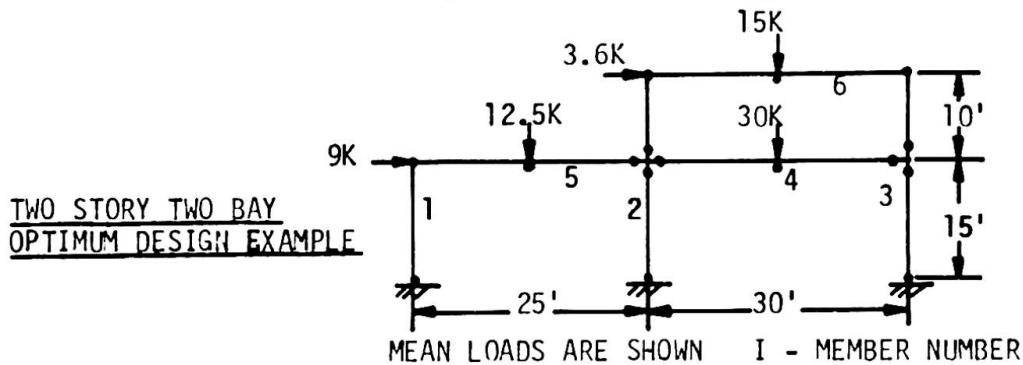


Figure 5

The resulting design moment capacities for the frame in Figure 5 is shown in Table 2 for designs with normal and log normal frequency distribution for load and strength of different coefficients of variation. It is interesting to observe as shown in Table 2 that at least for the examples studied one mode seems to dominate in having a relatively larger failure probability than the other modes.<sup>13</sup>

Table 2  
Optimum Design Results of Two Story  
Two Bay Frame Shown in Figure 4

| Example No. | Optimum Design Moment Capacities K-ft |      |      | C.V. Mom. Load % | $P_f$ Allowable | Weight Function | Frequency Distribution | Individual Collapse Mode Failure Probability in Order of Largest Value First |         |         |
|-------------|---------------------------------------|------|------|------------------|-----------------|-----------------|------------------------|------------------------------------------------------------------------------|---------|---------|
|             | 1                                     | 2    | 3    |                  |                 |                 |                        | 1                                                                            | 2       | 3       |
|             | 4                                     | 5    | 6    |                  |                 |                 |                        | 4                                                                            | 5       |         |
| 1           | 29.2                                  | 95.8 | 84.4 | 0.10             | 7.78(2)*        | 312.5           | Normal                 | 6.70(2)                                                                      | 6.87(3) | 4.50(3) |
|             | 175.0                                 | 73.2 | 74.4 | 0.20             |                 |                 |                        | 6.26(4)                                                                      | 3.16(4) |         |
| 2           | 27.8                                  | 96.3 | 84.4 | 0.10             | 7.80(2)         | 310.9           | Log Normal             | 4.85(2)                                                                      | 1.55(2) | 1.35(2) |
|             | 173.8                                 | 72.0 | 77.9 | 0.20             |                 |                 |                        | 2.21(3)                                                                      | 1.67(3) |         |
| 3           | 28.0                                  | 78.7 | 71.0 | 0.20             | 7.72(2)         | 297.3           | Normal                 | 4.94(2)                                                                      | 1.22(2) | 1.16(2) |
|             | 170.9                                 | 69.4 | 74.9 | 0.10             |                 |                 |                        | 2.81(3)                                                                      | 2.68(3) |         |
| 4           | 27.3                                  | 78.3 | 71.3 | 0.20             | 7.16(2)         | 293.5           | Log Normal             | 4.19(2)                                                                      | 1.64(2) | 1.09(2) |
|             | 166.4                                 | 65.1 | 74.9 | 0.20             |                 |                 |                        | 3.13(3)                                                                      | 2.25(3) |         |
| 5           | 29.1                                  | 87.8 | 72.3 | 0.15             | 7.52(2)         | 300.6           | Normal                 | 5.39(2)                                                                      | 1.10(2) | 9.89(3) |
|             | 170.3                                 | 68.0 | 74.1 | 0.15             |                 |                 |                        | 1.71(3)                                                                      | 1.69(3) |         |

\* Exponents of failure probability are shown in parenthesis (m) and should be read as  $10^{-m}$



## DISCUSSION AND CONCLUSIONS

- 1] The results presented indicate the feasibility of using reliability or probability of failure constraints in solving for optimum multi-member structural designs. By using Mathematical Programming methods to proportion member sizes a design is obtained which has an overall failure probability equal to an allowable value. This approach to design appears more rational than many current optimum design methods which use conventional code safety factors to restrict member dimensions based on stress and deflection limitations. As a result such optimum designs end up with many element constraints active which from a reliability viewpoint reduces its safety below a conventionally unoptimized design. Two examples presented include "weakest-link" structures for which any member failure constitutes failure of the structure and "redundant" structures which fail by forming collapse mechanisms after several members have simultaneously yielded.
- 2] It is seen from the examples presented that a reliability based optimum design does not have equal safety factor for all elements. In a "weakest-link" structure the heavier members have higher failure probability values than lighter members. This factor is influenced by the degree of statistical correlation between member failures which depends on the ratio of the variability or coefficient of variation of the load to the strength. In an optimum "redundant" structure such as a frame designed to resist formation of a collapse mechanism, the same safety factor is not present for each mechanism at the optimum design. Rather the Mathematical Programming method proportions each member to achieve minimum weight within the constraint of overall failure probability. For the frames studied it was observed that one particular mechanism in an optimum design dominates in its value of failure probability but it is not possible to choose beforehand which mechanism this will be.
- 3] An important factor influencing the magnitude of the optimum design as well as its member sizes will be the choice of load and strength frequency distributions and their parameters particularly the coefficients of variation. Curves of cost vs. statistical parameters show choice of frequency distribution is not too critical unless the distribution is highly skewed. Another important factor is the choice of an allowable failure probability. This should depend on the function of the structure as well as the failure consequences in social and economic terms and is not considered herein.<sup>16</sup>
- 4] The computation of failure probability for any frequency distribution as presented herein and its incorporation in an optimum design procedure should stimulate studies of random variables encountered in structural engineering to improve their description. Empirical studies are needed to provide reasonable frequency distributions for static strength, fatigue life, creep rate, floor loading as well as stochastic theories for dynamic phenomena associated with wind, highway and earthquake loading. The use of optimum design techniques illustrated herein should be useful in assessing the importance of changes in the parameters of these frequency distributions in terms of optimum cost or weight rather than in terms of predicted failure probabilities. Failure probabilities are usually expressed in quantities of  $10^{-2}$  to  $10^{-9}$  and small changes in frequency distribution may cause large changes in failure probability. However, the change in optimum weight associated with this change in frequency distribution may be of smaller magnitude.
- 5] A truly optimum design should consider the behavior of the structure over various types of loading conditions as well as possible strength deteriorations. In a more extensive approach under study an optimum design is to be found which considers all levels of failure including yielding, formation of cracks, large



deflections, instability and collapse. Although for some "weakest-link" structures yielding and collapse occur simultaneously, this is not true for most structures.<sup>17</sup> One approach to this problem would be to impose an allowable failure probability which depends on the damage for each failure level and to seek an optimum design which satisfies all probability of failure constraints. Another approach is to combine the constraints into one reliability constraint which would contain the probability of a level of failure occurring multiplied by a factor which includes the associated damage.

6] One question often raised in consideration of failure probability analysis is the meaningfulness of a statement that the failure probability is 0.001 since not enough data could conceivably exist to support this claim. It is easy, however, to see that a statement that  $P_f$  of structure A is 0.001 and of structure B 0.00001 is meaningful.<sup>18</sup> In the light of optimum design the  $P_f$  statements take on further meaning since they influence the overall structural cost. Furthermore, in a single design it would be possible to use various combinations of  $P_f$  allowables, coefficients of variation of load and strength to see how they influence the overall cost. The final structural cost would be based on the best estimate of these parameters in the light of previous experience, empirical studies of existing structures and other economic factors.

#### ACKNOWLEDGEMENT

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## SUMMARY

An optimization procedure is presented in which safety in terms of reliability or probability of failure is used as the controlling design limitation for finding minimum weight structures. Reliability analysis and optimum proportioning in multimember structures is given for "weakest-link" and redundant cases. Examples illustrate the effect on optimum weight of frequency distributions, coefficients of variation and allowable failure probability. Some aspects of the reliability analysis problem are discussed.

## RÉSUMÉ

Le rapport présente un procédé d'optimisation où la sécurité, en termes d'endurance et de probabilité de ruine, détermine le dimensionnement minimal d'une construction. Dans les structures à éléments multiples, l'analyse de la sécurité et le dimensionnement optimal sont déterminés par le membre le plus faible et par les conditions extrêmes exagérées. Des exemples expliquent l'effet de la répartition des fréquences, des coefficients de variation et de la probabilité de ruine acceptable sur le dimensionnement optimal. Quelques aspects du problème de l'analyse de la sécurité sont discutés.

## ZUSAMMENFASSUNG

In diesem Beitrag wird ein Optimierungsverfahren vorgestellt, in welchem der Sicherheitsbegriff in Zuverlässigkeits- oder Wahrscheinlichkeitsraten des Bruches zur Kontrollbegrenzung des minimalen Gewichtes gebraucht wird. Die Zuverlässigkeitsanalyse und die Optimierung (vielstägiger) hochgradiger Bauwerke ist für das schwächste Glied und extreme Fälle durchgeführt worden. Beispiele zeigen die Wirkung der Häufigkeitsverteilungen, der Streuungsmasse und der zulässigen Bruchwahrscheinlichkeit auf das minimale Gewicht. Einige Merkmale der Zuverlässigkeitsanalyse werden besprochen.

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## **Optimisation des structures par la considération des états limites plastiques**

Optimierung der Tragwerke unter Berücksichtigung der plastischen Grenzzustände

Optimisation of Structures on the Basis of the Plastic Behaviour of Material

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Une structure peut être considérée comme optimisée lorsque, conçue pour assurer correctement les services qui lui sont demandés, son exécution peut être effectuée pour le moindre prix.

La recherche d'une structure optimale est nécessairement fort complexe, compte tenu de toutes les données techniques et économiques.

Le poids de matière employée est un élément du prix de revient, mais la recherche du poids minimal n'est certainement pas la voie qui conduit au moindre prix.

L'optimisation dans la conception et le dimensionnement conduira à la recherche de la distribution la plus judicieuse de la matière.

Dans cette recherche fort complexe nous ne considérerons ici qu'un seul aspect : sécurité de la structure par rapport à la ruine par affaissement.

Cette étude laisse donc de côté tout ce qui concerne les problèmes de rupture fragile ou d'instabilité de forme ; il ne faut pas se dissimuler l'importance de ces problèmes d'instabilité lorsque l'on tient compte de la plastification de certains éléments.

Nous laissons également de côté la prise en compte de toutes les incertitudes concernant l'action des charges et le comportement de la matière. Il s'agit là d'un autre problème ; on peut cependant remarquer que la méthode la plus générale pour faire un contrôle en sécurité, s'appliquant tant dans le domaine élastique que dans le domaine plastique, est celle qui consiste à frapper les diverses sollicitations de coefficients de pondération (méthode préconisée par la Convention Européenne de la Construction Métallique).

Enfin l'étude se borne à l'équilibre statique. On suppose donc que l'application des charges se fait progressivement, et que la structure en se déformant sous l'action des charges est à tout instant en équilibre.

Traditionnellement les calculs de l'équilibre de la structure se font en supposant que la matière est parfaitement élastique.

Il faut cependant signaler que de tout temps on a eu conscience de l'importance des qualités plastiques en ce qui concerne la sécurité. Les plus anciens cahiers des charges ont toujours exigé un important allongement plastique pour les aciers de construction.

La ruine se produisant par affaissement, certains éléments sont nécessairement entrés en phase plastique. Le calcul élastique ne correspond donc pas au comportement réel de la matière en état limite.

Une première conséquence est que le calcul élastique ne conduit pas à une sécurité homogène.

Une section soumise à flexion possède une réserve de sécurité d'autant plus grande que le moment de saturation plastique est plus différent du moment de limite élastique. Plus une section est loin de l'optimisation élastique, plus grande est la réserve de capacité de flexion par plastification.

Une structure hyperstatique calculée élastiquement possède du fait des plastifications possibles une réserve de capacité de résistance que ne possède pas une structure isostatique calculée avec les mêmes critères.

Ces deux faits à eux seuls justifient déjà la prise en compte des états limites plastiques.

Mais le problème que nous voulons aborder est celui-ci : "La considération des états limites plastiques conduit-elle à une distribution plus judicieuse de la matière ?"

Nous simplifierons le problème en supposant :

a) que la matière est un corps élasto-plastique parfait (l'image correspond assez bien à l'acier de construction)

b) que les sections fléchies sont optimisées élastiquement, c'est-à-dire que toutes les fibres atteignent simultanément la limite élastique. Le moment de limite élastique est alors égal au moment de saturation plastique.

Nous considérerons des systèmes hyperstatiques.

Nous ferons souvent appel aux polygones d'écoulement (en employant la méthode de RJANITSYN pour les systèmes de poutres).

Pour un système de charges donné l'état limite nous permettra de déterminer l'état d'autocontrainte.

Le problème pour les charges variables sera de savoir si cet état d'autocontrainte est définitif (structure adaptée) dans le cas contraire on aura cumul de déformations sous mises en charges successives.

Nous aurons donc à considérer :

- 1) des systèmes de charge invariables dans le temps (poids propre)
- 2) des systèmes de charge variables en intensité ou en position.

#### 1) SYSTEMES DE CHARGE INVARIABLES

Dans une structure isostatique les moments de flexion et efforts tranchants en tous points sont déterminés par le système de charge, et sont indé-

pendants de la distribution de la matière. L'optimisation élastique conduit nécessairement au poids minimal de matière. Si la distribution de matière s'écarte de celle obtenue par optimisation élastique la plasticité n'apporte aucune réserve de capacité.

Dans une structure hyperstatique les moments et efforts tranchants en tous points dépendent non seulement du système de charge mais également des réactions hyperstatiques et donc de la distribution de la matière.

Il n'est donc pas évident que l'optimisation élastique qui impose un choix des réactions hyperstatiques conduise au poids minimal.

Nous considérerons deux types de structure :

- a) un système réticulé
- b) un système de poutres.

#### a) système réticulé.

Nous supposons que toutes les barres sont rigoureusement centrées et que nous pouvons négliger la rigidité des attaches

La barre d'indice  $i$  est soumise à un effort normal  $N_i$  fonction de  $n$  inconnues hyperstatiques  $R_0, \dots, R_n$

Le potentiel élastique est :

$$W = 1/2 \sum_i \frac{N_i^2 l_i}{E s_i}$$

Si la structure est optimisée élastiquement, on a :

$$N_i / s_i = \sigma_e$$

$$W = 1/2 \sum_i \frac{N_i l_i \sigma_e}{E}$$

les  $n$  inconnues hyperstatiques sont définies par

$$\partial W / \partial R_0 = 0 \dots \partial W / \partial R_n = 0$$

soit :

$$\sum_i \frac{\partial N_i l_i}{\partial R_0} = 0 \dots \sum_i \frac{\partial N_i l_i}{\partial R_n} = 0 \quad (1)$$

Supposons que nous cherchions une autre distribution de matière avec  $n$  réactions  $R'_0, \dots, R'_n$  obtenues par adaptation plastique et conduisant à l'emploi du volume minimal de matière

La structure adaptée devra être optimale élastique pour avoir le volume minimal

Les  $N_i$  seront fonction de  $R'_0, \dots, R'_n$

On aura  $N_i / s_i = \sigma_e$

Le volume sera :

$$V = \sum_i s_i l_i = 1/\sigma_e \sum_i N_i l_i$$

L'optimisation correspond aux  $n$  inconnues  $R'_0, \dots, R'_n$  satisfaisant à :

$$\partial V / \partial R'_0 = 0 \dots \partial V / \partial R'_n = 0$$

d'où :

$$\sum_i \frac{\partial N_i l_i}{\partial R'_0} = 0 \dots \sum_i \frac{\partial N_i l_i}{\partial R'_n} = 0 \quad (2)$$



Le système (2) est identique au système (1), et donc les  $n$  valeurs optimales sont bien les  $n$  valeurs  $R_0 \dots R_n$  correspondant à la distribution de matière déterminée par optimisation élastique.

Pour le système réticulé l'optimisation élastique correspond au volume minimal de matière.

#### b) système de poutres.

Nous supposons un système où seuls les moments de flexion sont à prendre en considération.

Le moment de flexion en un point d'abscisse  $s$  est fonction de  $n$  moments hyperstatiques  $M_0 \dots M_n$ , ce moment sera

$$M(s, M_0 \dots M_n)$$

Le potentiel élastique est

$$W = \frac{1}{2} \int_s \frac{M^2(s, M_0 \dots M_n)}{EI(s)} ds$$

Les  $n$  moments  $M_0 \dots M_n$  sont obtenus par :

$$\frac{\partial W}{\partial M_0} = 0 \dots \dots \frac{\partial W}{\partial M_n} = 0 \quad (3)$$

L'optimisation élastique pour la poutre de demi-hauteur  $v(s)$  conduit à

$$Mv/I = \pm \sigma_e$$

Dans la structure optimisée élastiquement :

$$W = \frac{1}{2} \frac{\sigma_e^2}{E} \int_s \frac{I(s)}{v(s)} ds \quad (4)$$

L'optimisation élastique conduit ainsi à distribuer la matière compte tenu des  $n$  inconnues déterminées par le système (3).

Est-il possible d'avoir  $n$  valeurs  $M'_0 \dots M'_n$  obtenues par adaption plastique et conduisant à une distribution de matière donnant un volume minimal pour les membrures ?

On aura  $M(s, M'_0 \dots M'_n)$

En tous points

$$Mv/I = \pm \sigma_e$$

$$\frac{M^2 v^2}{I^2} = \sigma_e^2$$

Si  $S$  est la section des 2 membrures au point d'abscisse  $s$

$$I = S v^2$$

Et le volume total sera :

$$V = \int_s S ds$$

$$V = \int_s \frac{I(s)}{v^2 ds} ds \quad (5)$$

Les  $n$  moments  $M'_0 \dots M'_n$  seront définis par :

$$\frac{\partial V}{\partial M'_0} = 0 \dots \dots \frac{\partial V}{\partial M'_n} = 0 \quad (6)$$

On voit en vertu de (4) et (5) que ce système (6) est identique au système (3) donc :  $M_0 = M'_0 \dots\dots\dots M_n = M'_n$

Comme dans le cas du système réticulé, l'optimisation élastique conduit au volume minimal de matière.

En définitive donc, que le système soit isostatique ou hyperstatique, la distribution de matière déterminée par optimisation élastique conduit au volume minimal de matière. L'optimisation absolue n'est d'ailleurs pas réalisable. Mais, contrairement à ce qui se passe pour les systèmes isostatiques, si une structure hyperstatique n'est pas optimisée élastiquement, l'adaptation plastique permet une augmentation de la charge ultime d'autant plus importante que la distribution de matière est plus éloignée de celle déterminée par optimisation élastique. C'est-à-dire que, de même que l'adaptation plastique améliore d'autant plus le rendement d'une section en flexion que cette section est plus mal conditionnée pour la flexion (écart entre le moment de limite élastique et le moment de saturation plastique), de même l'adaptation plastique d'une structure hyperstatique améliore d'autant mieux la limite d'une charge de distribution donnée que la répartition de la matière s'écarte davantage de celle obtenue par optimisation élastique.

Nous prendrons à titre d'exemple le cas d'une poutre continue à deux travées égales sous l'action d'une charge uniforme.

Soit  $p$  la charge par unité de longueur. Dans tout ce qui suit nous prendrons une convention de signe très couramment utilisée dans l'étude des poutres : les moments seront comptés positifs dans le sens inverse des axes.

#### a) Poutre optimisée élastiquement

Il est facile de voir que la distribution de matière avec optimisation élastique conduit à un volume de membrures

$$V_0 = \frac{\sqrt{2}-1}{6\sqrt{2}} \frac{p l^3}{\sigma_e}$$

#### b) Poutre à inertie constante

En calcul élastique le volume de ces membrures est

$$V_e = \frac{1}{8} \frac{6\sqrt{2}}{\sqrt{2}-1} V_0 = 2.56 V_0$$

En calcul plastique (formation d'une 2ème rotule en travée)

$$V_p = (3 - 2\sqrt{2}) \times \frac{6\sqrt{2}}{\sqrt{2}-1} V_0 = 1.76 V_0$$

On voit ainsi sur cet exemple l'intérêt de la prise en compte de la plastification pour une poutre non optimisée élastiquement.

Sur l'exemple choisi on a d'ailleurs une adaptation plastique assez importante, ainsi que l'on peut s'en rendre compte en cherchant l'état d'auto-contrainte qui resterait dans la poutre si on la déchargeait.

En considérant le polygone d'écoulement on voit (en posant  $x = tl$ ) que le moment d'autocontrainte sur l'appui B est positif et égal à :

$$M_d = \frac{8t-3}{4t^2} M_0$$

L'adaptation se limitant au moment de la formation d'une deuxième rotule plastique dans la section  $t = \sqrt{2} - 1$  on trouve :

$$M_a = + 0,456 M_0$$

On voit ainsi pour une structure hyperstatique tout l'intérêt pour la recherche d'une optimisation économique de s'écarter de l'optimisation élastique en simplifiant la structure à condition de tenir compte de l'adaptation plastique sous l'action de charges permanentes.

## 2) CHARGES MOBILES

Le cas de charges mobiles est plus complexe tant en ce qui concerne l'optimisation élastique que l'adaptation plastique définitive, c'est-à-dire celle qui rend le système adapté rigoureusement élastique sous l'action des charges mobiles et qui assure donc qu'il n'y aura pas de cumul de déformations sous le passage des charges mobiles.

Pour trouver la distribution de matière conduisant à l'optimisation élastique, il faudra chercher la courbe enveloppe des moments fléchissants maximaux dans chaque section d'abscisse  $s$ .

Soit  $\mathcal{M}(s)$  cette loi de la courbe enveloppe. En optimisation élastique on devra avoir si  $I(s)$  et  $v(s)$  sont l'inertie et la demi-hauteur de la section définie par  $s$  :

$$I(s) = \frac{\mathcal{M}(s)v(s)}{\sigma_e}$$

Pour chaque position de la charge on aura une loi des moments  $M(s, M_0, \dots, M_n)$  où  $M_0, \dots, M_n$  sont les moments sur appuis du système hyperstatique, définis si  $W$  est le potentiel élastique avec la loi  $M(s, M_0, \dots, M_n)$  par :

$$\frac{\partial W}{\partial M_0} = 0, \dots, \frac{\partial W}{\partial M_n} = 0$$

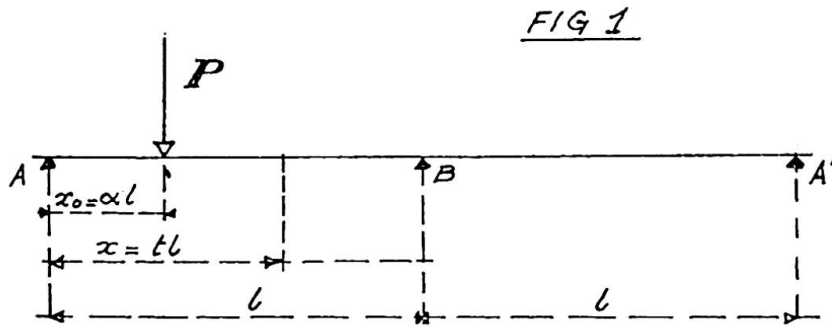
La loi  $\mathcal{M}(s)$  dépend ainsi de la distribution de la matière. L'optimisation élastique est possible par itération en partant par exemple d'une loi  $I(s) = \text{Cte}$ .

Mais il n'est plus possible de démontrer simplement que l'optimisation élastique dans le cas de charges mobiles conduit au poids minimal de matière.

Nous allons étudier l'optimisation élastique, et l'adaptation plastique d'une poutre non optimisée élastiquement sur le cas simple déjà envisagé plus haut d'une poutre continue à deux travées égales sous le passage d'une charge mobile unique.

### a) Poutre optimisée élastiquement

Soit la charge mobile  $P$  agissant dans la section d'abscisse  $x_0 = \alpha l$ , la section courante étant  $x = tl$



Le moment sur appui B pour la charge P en  $x_0 = \alpha l$  sera :

$$M_B = -Pl \frac{\int_0^{\alpha} (1-\alpha) t^2 \frac{dt}{I(t)} + \int_{\alpha}^1 \alpha t (1-t) \frac{dt}{I(t)}}{2 \int_0^1 t^2 \frac{dt}{I(t)}}$$

Pour faire un calcul numérique nous divisons la poutre de A à B en n parties égales, chacun de ces n tronçons ayant une inertie  $I_1, I_2, \dots, I_n$

Nous plaçons la charge successivement à l'extrémité d'un tronçon (le  $q^0$ ) soit :  $\alpha = q/n$

$$M_B = \frac{(1-\alpha) \sum_{p=1}^{p=q} \frac{1}{I_p} (p^2 - p + 1/3) + \alpha \sum_{p=q+1}^{p=n} [(n+1)p - p^2 - \frac{3n+2}{6}]}{\sum_{p=1}^{p=n} \frac{1}{I_p} (p^2 - p + 1/3)} \times \left(-\frac{Pl}{2}\right)$$

Ceci permet de calculer le moment en chaque tronçon noté p pour une charge placée en q, on obtient

$$M_p^q = K Pl$$

Pour chaque valeur de p on désignera par  $\mathcal{M}_p$  la valeur maximale de  $M_p^q$  et on écrira

$$I_p = \frac{\mathcal{M}_p v}{\sigma_e}$$

En partant d'une loi  $I_p = \text{Cte}$  on peut par approximations successives trouver les valeurs optimales  $I_p$  de  $I_p$

Le calcul a été fait au Centre de calcul de l'Institut Français du Pétrole.

On a pris  $n = 10$ , à la 6<sup>e</sup> approximation la correction était de l'ordre de  $10^{-4}$

Les inerties  $I_p$  qui ont été déterminées sont

$$I_p = \frac{A Pl v}{\sigma_e}$$

Les valeurs de A optimisées sont les suivantes :

| p | 1        | 2        | 3        | 4        | 5        |
|---|----------|----------|----------|----------|----------|
| A | 0,088467 | 0,154082 | 0,197291 | 0,218701 | 0,219136 |

| p | 6        | 7        | 8        | 9        | 10       |
|---|----------|----------|----------|----------|----------|
| A | 0,199769 | 0,162440 | 0,110592 | 0,053356 | 0,067942 |

On a alors le volume  $V_0$  des membrures d'une poutre optimisée élastiquement:

$$V_0 = \sum_{i=1}^{10} 2S_i \cdot \frac{l}{10} = \frac{Plv}{\sigma_e} \sum_{i=1}^{10} A = 0,1471 \frac{Pl^2}{v\sigma_e}$$

#### b) Poutre à inertie constante

En période élastique le moment maximal en valeur absolue se produit toujours au point d'application de la charge P point noté  $x_0 = \alpha l$

Ce moment est :

$$M = \frac{\alpha(1-\alpha)(4-\alpha(1+\alpha))Pl}{4}$$

Si  $M_0$  est le moment de limite élastique égal par hypothèse au moment de saturation  $M_0$  plastique, la charge admissible au point défini par  $\alpha$  sera

$$P = \frac{4}{\alpha(1-\alpha)(4-\alpha(1+\alpha))} \frac{M_0}{l}$$

La poutre étant à inertie constante on aura

$$M_0 = \frac{I\sigma_e}{v} = 25v\sigma_e$$

La charge admissible pour toute position sera celle donnée par la valeur de  $\alpha$  rendant maximale l'expression

$$\alpha(1-\alpha)[4-\alpha(1+\alpha)]$$

Soit pour  $\alpha = 0,43$

Ce qui donne la valeur maximale de P

$$P = 4,819 \frac{M_0}{l}$$

Le volume  $V_e$  des membrures d'une poutre de portée l sera

$$V_e = 2Sl = \frac{M_0 l}{v\sigma_e} = \frac{1}{4,819} \frac{Pl^2}{v\sigma_e} = 0,2076 \frac{Pl^2}{v\sigma_e}$$

Nous allons chercher le comportement plastique de cette poutre à inertie constante sous le passage de la charge P.

Pour étudier la plastification de la poutre nous considérerons les polygones d'écoulement pour chaque position  $x_0 = \alpha l$  de la charge P, en utilisant la méthode de Rjanitsyn.

D'après les calculs indiqués précédemment (dans la recherche de l'optimisation élastique) le moment élastique sur l'appui B pour une charge P placée en  $x_0 = \alpha l$  sera :

$$M_B = - \frac{Pl}{4} \cdot \alpha (1 - \alpha^2)$$

Si  $M'$  désigne le moment en phase élastique on a :

$$0 \leq t \leq \alpha \quad M' = Pl(1 - \alpha)t - \frac{Pl}{4} \cdot \alpha (1 - \alpha^2) t$$

$$0 \leq t \leq 1 \quad M' = Pl\alpha(1 - t) - \frac{Pl}{4} \alpha (1 - \alpha^2) t$$

$$1 \leq t \leq 2 \quad M' = -\frac{Pl}{4} \alpha (1 - \alpha^2) (2 - t)$$

Nous désignerons toujours par  $M_0$  le moment de limite élastique ou de saturation plastique

$$M_0 = \sigma_e I/y$$

Nous supposerons après adaptation plastique qu'il y a en B un moment d'autocontrainte

$$\eta M_0$$

Les moments M dans la poutre sont alors :

$$0 \leq t \leq 1 \quad M = M' + \eta M_0 t$$

$$1 \leq t \leq 2 \quad M = M' + \eta M_0 (2 - t)$$

Nous poserons  $\frac{Pl}{4} \alpha (1 - \alpha^2) = \xi M_0$

$\xi$  caractérise l'état de charge et  $\eta$  l'autocontrainte

$$0 \leq t \leq \alpha \quad M = \left[ \left( \frac{4}{\alpha + \alpha^3} - 1 \right) \xi t + \eta t \right] M_0$$

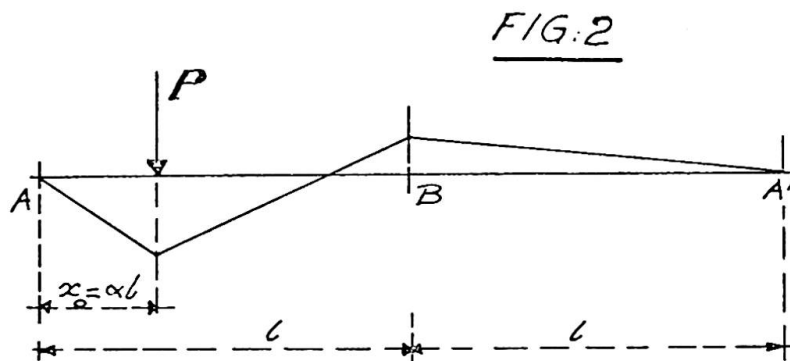
$$\alpha \leq t \leq 1 \quad M = \left[ \left( \frac{4}{1 - \alpha^2} (1 - t) - t \right) \xi + \eta t \right] M_0$$

$$1 \leq t \leq 2 \quad M = \left[ -(2 - t) \xi + \eta (2 - t) \right] M_0$$

Supposant  $\alpha$  donné (charge P en  $x_0 = \alpha l$ ) la condition limite de plasticité s'écrit en tout point :

$$-M_0 \leq M \leq +M_0$$

Pour cette charge P en  $x_0$  le diagramme des moments de flexion est toujours le suivant :



Il suffit donc d'écrire les conditions :

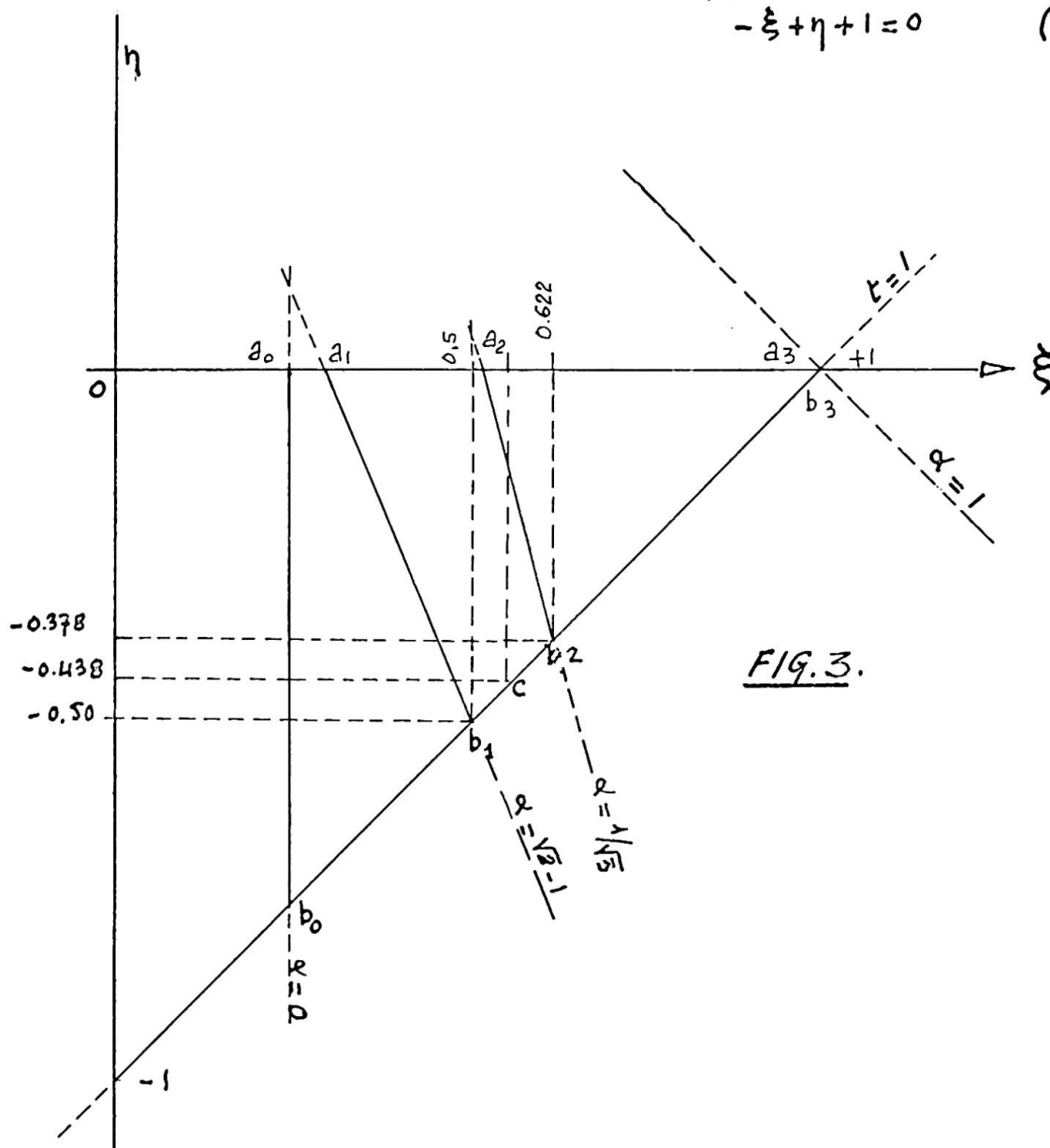
$$M(x_0) = +M_0$$

$$M_B = -M_0$$

C'est-à-dire pour les limites :

$$\left(\frac{4}{1+\alpha} - \alpha\right)\xi + \alpha\eta - 1 = 0 \quad (t=\alpha)$$

$$-\xi + \eta + 1 = 0 \quad (t=1)$$





On a ainsi une famille de polygones d'écoulement en fonction de  $\alpha$  délimités par les 2 droites d'équations ci-dessus pour  $t = \alpha$  et  $t = 1$

La droite ( $t = \alpha$ ) coupe l'axe  $O\xi$  en un point  $a$ , ce point varie de  $a_0$  ( $\xi = 0,25$  pour  $\alpha = 0$ ) à  $a_3$  ( $\xi = 1$  pour  $\alpha = 1$ ).

Cette droite ( $t = \alpha$ ) coupe la droite ( $t = 1$ ) en un point  $b$  variant de  $b_0$  (pour  $\alpha = 0$ ) à  $b_3$  (pour  $\alpha = 1$ ).

Appliquant  $P$  au point  $x_0 = \alpha l$  et partant d'un état naturel en faisant croître  $P$  depuis la valeur  $0$  le point représentatif se déplace sur l'axe  $O\xi$ . Lorsque ce point arrive à  $a$  (intersection de la droite  $t = \alpha$  avec  $O\xi$ )  $P$  atteint la charge limite élastique qui correspond à

$$\xi = \frac{1+\alpha}{4-\alpha(1+\alpha)}$$

La charge limite élastique  $P_e(\alpha)$  est alors :  $P_e(\alpha) = \frac{4}{\alpha(1-\alpha)(4-\alpha(1+\alpha))} \frac{M_0}{l}$

Si  $P$  continue à croître  $\xi$  augmente il y a plastification dans la section  $x_0 = \alpha l$ , le point représentatif décrit la droite  $t = \alpha$  à partir de  $a$ . Lorsque ce point arrive en  $b$  (intersection des droites  $t = \alpha$  et  $t = 1$ ) on atteint le moment limite en  $B$ , à cette valeur de  $\xi$  correspond la charge ultime plastique  $P_u(\alpha)$ , on a alors

$$\xi = \frac{(1+\alpha)^2}{4}$$

d'où :

$$P_u(\alpha) = \frac{1+\alpha}{\alpha(1-\alpha)} \frac{M_0}{l}$$

Si on déchargeait alors la poutre il y aurait en  $B$  un moment d'autocontrainte

$$M_B = - \left( 1 - \frac{(1+\alpha)^2}{4} \right) M_0$$

Si l'on cherche la charge mobile ultime  $P_u$ , cette charge doit être la plus petite valeur de  $P_u(\alpha)$ ,  $\alpha$  variant de 0 à 1.

Ce minimum se produit pour

$$\frac{d}{d\alpha} \frac{1+\alpha}{\alpha(1-\alpha)} = 0$$

Soit :  $\alpha = \sqrt{2} - 1$

On a alors :

$$P_u = 5,828 M_0/l$$

(RJANITSYN a donné le même résultat obtenu par une autre méthode).

Si cette charge  $P_u$  arrivée au point  $x = (\sqrt{2}-1)l$  on supprimait cette charge on aurait en  $B$  un moment d'autocontrainte déterminée par l'ordonnée du point  $b_1$  c'est-à-dire  $\eta = -0,5$

$$M_B = -0,5 M_0$$

Si la charge  $P_u$  va au-delà de  $x = (\sqrt{2}-1)l$ ,  $\xi$  continue à croître, car  $\xi$  est donné par :

$$\frac{1}{M_0} \frac{Pl}{4} \alpha(1-\alpha^2) = 1,461 \alpha(1-\alpha^2)$$

$\xi$  passe donc par un maximum pour la valeur maximale de  $\alpha(1-\alpha^2)$  ce qui se produit pour  $\alpha = \frac{1}{\sqrt{3}}$

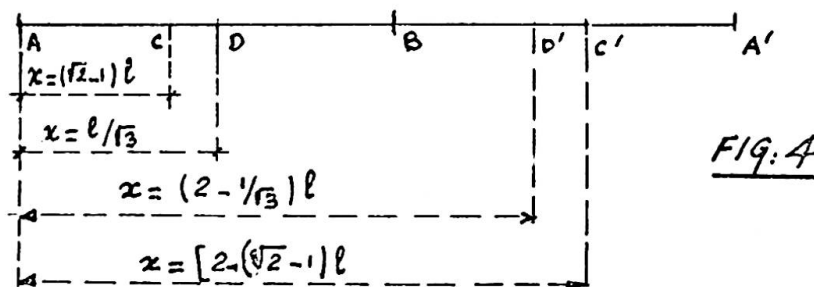
On a alors  $\xi = 1.461 \times \frac{1}{\sqrt{3}} \times \frac{2}{3} = 0.562$

Le point représentatif s'est déplacé sur la droite  $t = 1$  (plastification sur l'appui B) de  $b_1$  à c.

L'ordonnée de c sur cette droite  $t = 1$  est

$$-1 + 0,562 = -0,438$$

Lorsque la charge mobile  $P_u$  se déplace à partir de l'appui A, il y a plastification dans la section C ( $x = (\sqrt{2}-1)l$ ) au moment où la charge arrive en C et le moment d'autocontrainte en B est alors  $-0,5 M_o$ ; lorsque cette charge arrive en D ( $x = l/\sqrt{3}$ ) il y a plastification sur l'appui B et le moment d'autocontrainte en B n'est plus alors que  $-0,438 M_o$ ; si la charge continue vers A', de D' ( $x = (2 - 1/\sqrt{3})l$ ) à C' ( $x = [2, (\sqrt{2}-1)]l$ ) il y a plastification des sections de D' à C' et le moment d'autocontrainte en B remonte à  $-0,5 M_o$ .



Chaque passage de la charge  $P_u$  sur toute la poutre provoque de nouvelles déformations permanentes, il y a  $P_u$  cumul de déformations.

La charge  $P_u$  n'est admissible que si elle n'est susceptible d'être appliquée qu'un nombre très limité de fois au cours de la vie de l'ouvrage.

La charge ultime  $P'_u$  admissible pour un nombre illimité de passages est inférieur à  $P_u$ . On peut  $P'_u$  déterminer cette charge en imposant la condition que le moment d'autocontrainte en B atteigne une valeur définitive après un premier passage.

On trouve le  $\eta_o$  maximal constant

$$\eta_o = -0.448$$

correspondant à une charge ultime

$$P'_u = 5,742 \frac{M_o}{l} < P_u = 5,828 \frac{M_o}{l}$$

Si l'on veut maintenant comparer l'économie de volume des membrures d'une travée de portée  $l$  pour une charge mobile  $P$  on écrira :

$$M_0 = \sigma_e I / \sqrt{v}$$

$$I = 2 S v^2$$

et le volume des membrures pour une travée de portée  $l$  sera

$$V = 2Sl = M_0 / \sqrt{\sigma_e} \cdot l.$$

Avec la charge ultime  $P'_u = 5,742 M_0 / l$

On a le volume :

$$V_p = \frac{1}{5.742} \frac{Pl^2}{\sqrt{\sigma_e}} = 0.1742 \frac{Pl^2}{\sqrt{\sigma_e}}$$

Ainsi pour le cas de la charge mobile  $P$  on peut comparer les résultats obtenus pour le volume des membrures

- poutre optimisée élastiquement  $V_o = 0.1471 \frac{Pl^2}{\sqrt{\sigma_e}}$
- pour une poutre d'inertie constante calculée élastiquement  $V_e = 0.2076 \frac{Pl^2}{\sqrt{\sigma_e}} = 1.411 V_p$
- pour une poutre à inertie constante calculée plastiquement  $V_p = 0.1742 \frac{Pl^2}{\sqrt{\sigma_e}} = 1.184 V_o$

On voit ainsi pour la charge mobile, comme pour le cas de la charge permanente, que si l'optimisation élastique conduit au volume minimal, il y a un gain considérable, lorsque la distribution de matière est loin de celle déterminée par optimisation élastique, à prendre en compte l'état limite plastique.

#### Combinaison de la charge permanente et de la charge mobile.-

On pourra reprendre l'étude en considérant à nouveau les frontières d'écoulement sous l'effet d'une charge permanente  $p$  par unité de longueur et d'une charge mobile  $P$ .

On posera :

$$pl = KP$$

On aura pour chaque valeur de  $K$  des familles de frontières d'écoulement en fonction de  $\alpha$

Le développement de cette discussion est trop long pour trouver sa place ici.

On peut pour chaque valeur de  $K$  déterminer la charge ultime admissible par la recherche du moment d'autocontrainte donnant une adaptation définitive.

Voici quelques résultats :

- pour  $K = 0,1$  (charge mobile égale à 10 fois le poids permanent)

$$\eta_0 = -0,409$$

- pour  $K = 1$  (charge mobile égale au poids permanent)

$$\eta_0 = -0,15$$

- pour  $K = 10$  (charge mobile égale au dixième du poids permanent)

$$\eta_0 = + 0,318$$

Lorsque la charge mobile est faible il y a adaptation plastique sur l'appui central et le moment d'autocontrainte sur appui est positif.

Si l'on fait croître la valeur de la charge mobile par rapport au poids permanent, le moment d'autocontrainte sous la charge ultime diminue.

Pour une certaine valeur de  $K\eta_0 = 0$ , il n'y a pas d'adaptation, pour cette valeur de  $K$  la poutre à inertie constante est optimale élastiquement.

Si la valeur de la charge mobile croît encore par rapport au poids permanent, l'adaptation plastique se produit sous la charge ultime en travée, et le moment d'autocontrainte sur appui central est négatif.

Enfin à partir d'une certaine valeur de  $K$  il y a risque de cumul de déformations plastiques.

### Conclusion

On voit en définitive que quels que soient les systèmes de charges, permanentes ou variables, pour une structure constituée en matériau pouvant, tel que l'acier doux, être considéré comme élasto-plastique, la distribution de matière déterminée par optimisation élastique conduit toujours au poids minimal.

L'optimisation élastique rigoureuse est souvent irréalisable, en outre elle conduit très généralement à une structure onéreuse (sauf cas exceptionnels où la structure simple est en même temps optimale élastiquement, comme on a vu que c'était le cas pour une poutre continue à 2 travées égales pour un certain rapport entre la charge et la charge permanente).

Lorsque la structure s'écarte de l'optimal élastique elle a, si elle est hyperstatique, une réserve de capacité pour la charge ultime d'affaissement en calcul élastique que n'a pas la structure isostatique.

On doit donc tendre à réaliser des systèmes hyperstatiques à condition de baser la sécurité sur la charge ultime plastique, ce qui permet de simplifier la structure avec une majoration de poids relativement faible, et donc d'optimiser la structure sur le critère de l'économie de réalisation.

Dans le cas de charges mobiles non exceptionnelles le calcul plastique peut conduire au danger du cumul des déformations.

Le critère alors à considérer est l'état d'autocontrainte qui doit être constant dans toutes les situations.

On peut donc établir une programmation sur la base du calcul élastique pour chercher la distribution de matière la plus économique.

L'état d'autocontrainte peut être déterminé sans tracer de diagrammes par des calculs élastiques.

L'état d'autocontrainte est un critère à la fois pour l'optimisation et pour la vérification du non cumul de déformations.

## RÉSUMÉ

Pour un matériau élasto-plastique, la distribution de matière déterminée par optimisation élastique conduit, pour les charges permanentes et variables, au poids minimal que la structure soit iso-statique ou hyperstatique.

L'avantage de la structure hyperstatique, si on envisage l'état limite plastique, est de pouvoir réaliser économiquement en simplifiant et s'écartant de l'optimal élastique.

On détermine, par des calculs élastiques, l'état d'auto-contrainte qui sera un critère d'optimisation et de vérification de non cumul des déformations.

## ZUSAMMENFASSUNG

Für einen elastisch-plastischen Werkstoff führt die durch elastische Optimierung erhaltene Materialverteilung sowohl unter ständiger Last als auch bei veränderlichen Lasten zum minimalen Gewicht, gleichwohl ob das Tragwerk statisch bestimmt oder unbestimmt ist.

Statisch unbestimmte Tragwerke bieten den Vorteil, dass sie bei Berücksichtigung des plastischen Grenzzustandes durch Vereinfachung und unter Abweichung vom elastischen Optimum wirtschaftlich ausgeführt werden können.

Es wird durch elastische Berechnung der Eigenspannungszustand ermittelt, welcher als Kriterium der Optimierung und der Nichtüberlagerung der Verformungen dient.

## SUMMARY

For an ideal elastic-plastic material, the optimisation based on elastic calculations of a statically determined or indetermined structure provides the minimum weight material distribution.

When computing a statically indetermined structure on the basis of plastic behaviour, it is possible to simplify the optimum elastic material distribution in a more economical way.

The residual stresses remaining in the structure after unloading is a criteria of optimisation and a test against increasing permanent deformations with every loading cycle.

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## Decision Theory Approach to Structural Optimization

Optimisation des constructions à l'aide de la "théorie des décisions"

Stellungnahme der Entscheidungstheorie zur Bauwerksoptimierung

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Some of the optimization problems mentioned by M. Cambon are similar in principle to those treated in the theory of decision making under uncertainty that has been developed in recent years. Such theories are of considerable potential value in design since they provide a logical framework in which engineering judgement can be applied realistically and consistently.

### 1. FUNDAMENTALS OF DECISION THEORY

The theory of decision under uncertainty deals with the problem of choosing "optimum" courses of action when the results of action involve uncertain future events and a variety of possible consequences. To satisfy the basic requirements of the theory, a designer must establish four sets of basic data before decision:

- (i) a set of feasible actions  $a_i$  from which he must choose one,
- (ii) a set of possible outcomes  $o_{ij}$  resulting from the choice of any action  $a_i$ ,
- (iii) a probability measure specifying the probability  $p_{ij}$  that  $o_{ij}$  will occur if he chooses any action  $a_i$ ,
- (iv) a set of value losses  $L_{ij}$  specifying the relative undesirability of all outcomes  $o_{ij}$ .

Extensions to these basic requirements have been made when experiments are performed prior to final action [1]. Provided that a designer's values follow



certain rules of consistency [2], the relative desirability of any action is given by its total expected loss

$$T_i = \sum_j p_{ij} L_{ij} \quad (1.1)$$

and the action with least total expected loss is the optimum action.

In structural design alternative actions may involve different structural forms, materials, construction methods, and member sizes. Outcomes may involve initial costs, construction delays, appearance, serviceability and safety.

Prior to design outcomes cannot be predicted because of uncertainties in costs, material behaviour, foundation conditions and future loadings. In general there are insufficient data to establish statistical distributions and a designer must use judgement to establish subjective probability measures. It has been shown [3] that provided several consistency requirements are satisfied, subjective probability measures obey the conventional rules of probability. However subjective probabilities change as new information becomes available. The formal mechanism for modifying subject distributions is Bayes' Theorem [4]. Applications of subjective probabilities have been made to the strength of concrete structures [5] and the prediction of earthquake probabilities.

In order to formulate an optimization criteria, the relative value of all outcomes must be expressed on a single scale. Some outcomes such as construction costs and time delay involve monetary losses which can be measured directly. The value of other losses must be measured on the same scale as monetary losses from a set of elementary decisions. Thus, for example, structural collapse may only involve economic losses of the order of initial costs but must be assigned much higher relative values for moral, social, and professional reasons.

To illustrate the essentials of a decision theory approach, consider a simple example involving the preliminary design of a structure in which two basic schemes are under consideration. It is assumed that both schemes would be designed to provide the same safety and serviceability. The only variables under consideration are appearance, construction costs and construction delay.

To establish relative values of appearance the following basic decision is made. If both schemes were completed at the same time, the first scheme would be preferable if its initial costs were not more than \$25,000 greater than those of the second scheme. Denoting the losses associated with aesthetics of the schemes as  $A_1$  and  $A_2$  the basic decision implies that  $A_2 = A_1 + \$25,000$ . After further investigation, a designer might estimate the range of possible initial costs and time delays and their respective probabilities for the two schemes as shown in Table 1.

TABLE 1

| Scheme | Outcome Losses and Probabilities |       |                    |       |                    |       |
|--------|----------------------------------|-------|--------------------|-------|--------------------|-------|
|        | Initial Costs (\$000)            | Prob. | Time Delay (\$000) | Prob. | Appearance (\$000) | Prob. |
| I      | 90                               | 0.1   | 0                  | 0.2   | $A_1$              | 1     |
|        | 100                              | 0.3   | 5                  | 0.3   |                    |       |
|        | 110                              | 0.3   | 10                 | 0.4   |                    |       |
|        | 120                              | 0.2   | 15                 | 0.1   |                    |       |
|        | 130                              | 0.1   |                    |       |                    |       |
| II     | 80                               | 0.1   | 0                  | 0.4   | $A_1 + 25$         | 1     |
|        | 90                               | 0.4   | 5                  | 0.4   |                    |       |
|        | 100                              | 0.4   | 10                 | 0.2   |                    |       |
|        | 110                              | 0.1   |                    |       |                    |       |

The numbers in this example were chosen to illustrate the common problem of conflicting objectives. The first scheme is more aesthetically pleasing than the second but probably would cost more and is less likely to be completed on time. The total expected loss for the first scheme is  $A_1 + \$116,000$  while the second scheme has an expected loss of  $A_1 + \$124,000$ . Hence the first scheme is preferable to the second.

## 2. APPLICATIONS TO SAFETY ANALYSIS

Although decision theory can be applied to any situation in which a designer can formulate the necessary sets of basic data, existing studies have concentrated on design strength decisions. In many cases, alternative actions can be grouped into schemes with alternatives in a scheme essentially identical in appearance, functional efficiency and ease of construction. Optimization within such a scheme involves the choice of member sizes and proportions on the basis of initial costs and safety alone. Solution of such restricted optimization problems is the purpose of standard design procedures.

Each alternative in a scheme is associated with a variety of possible resistances and future loads. Since future conditions are uncertain, probability measures must be applied to the families of possible loads and resistances. Such probabilities must be at least partially subjective to allow for uncertainties in workmanship, methods of analysis, and load idealization. Using well known methods of analysis, load and resistance distributions can be combined to obtain failure probabilities [6].

If attention is restricted to one class of failure, the decision table for optimization within a scheme takes the form shown in Table II.

TABLE II

| Alternatives | Outcome Losses and Probabilities |             |         |             |
|--------------|----------------------------------|-------------|---------|-------------|
|              | Success                          | Probability | Failure | Probability |
| $a_1$        | $S_1$                            | $(1 - P_1)$ | $F_1$   | $P_1$       |
| $a_2$        | $S_2$                            | $(1 - P_2)$ | $F_2$   | $P_2$       |
| "            | "                                | "           | "       | "           |
| "            | "                                | "           | "       | "           |
| $a_n$        | $S_n$                            | $(1 - P_n)$ | $F_n$   | $P_n$       |

Where  $S_i$  and  $F_i$  are the success and failure losses if alternative  $a_i$  is chosen.

In view of the definition of a basic scheme, it is reasonable to assume the following loss relationships.

$$S_i = I_i + S \quad (2.1)$$

$$F_i = I_i + S + F \quad (2.2)$$

where  $I_i$  is the initial cost of alternative  $a_i$ . The constants  $S$  and  $F$  are success and failure losses in excess of initial costs for the scheme. With these losses, the optimum alternative has a total loss given by:

$$T_{opt} = \min (I_i + p_i F) \quad (2.3)$$

except for the constant  $S$  that is irrelevant to decision. A similar optimization rule has been suggested by other authors [7]. However in this development, the failure loss  $F$  is not simply the capitalized economic losses associated with failure but may be, and generally is, greater than such losses.

The relationship between total losses and a safety factor  $n$  can be deduced qualitatively for many common structures. Since initial costs generally increase relatively slowly with increasing  $n$ , and failure probabilities decreases relatively quickly, total losses decrease quickly, reach a minimum and then increase slowly eventually merging with initial costs. This behaviour is shown symbolically in Figure 1. In practice, costs and failure probabilities would be discontinuous functions of  $n$  but the trends indicated can be expected in many situations.

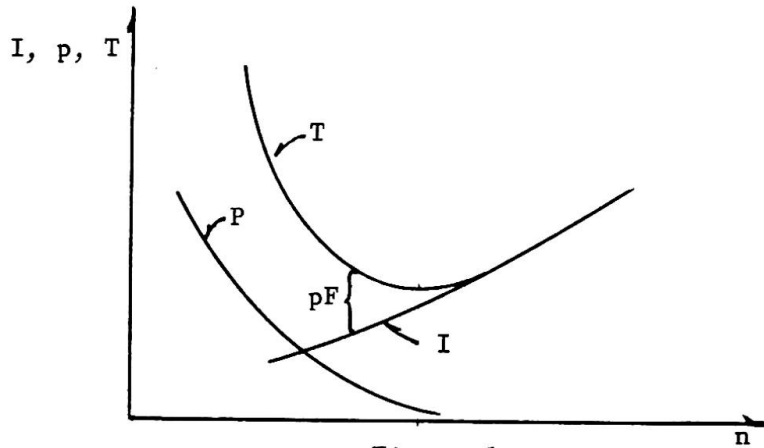


Figure 1

The behaviour of total losses for large  $n$  and low failure probability is of particular interest. If, for example, an alternative has an initial cost of  $\$1 \times 10^6$ , failure losses equal to 100 times initial costs and a failure probability of  $1 \times 10^{-6}$ , its total loss is equal to  $\$1,000,000 + \$100$ . Obviously the loss of  $\$100$  associated with failure is negligible in comparison to the initial cost.

A formal basis for neglecting failure losses may be established by stating that small variations in initial costs are irrelevant. Such a statement implies that initial costs must be rounded off in forming expected losses. To ensure consistency, failure losses must be rounded to the same level of precision. In the preceeding example, if initial costs are rounded to the nearest  $\$1,000$ , any failure probability less than  $5 \times 10^{-6}$  results in zero failure loss. In other words, any failure probability less than  $5 \times 10^{-6}$  is effectively zero.

With a restriction on sensitivity to initial costs, the general behaviour of initial costs and total losses shown in Figure 1 becomes that shown in Figure 2.

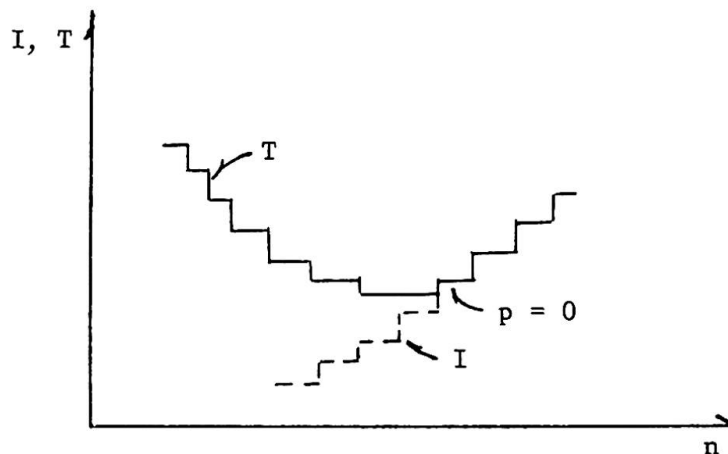


Figure 2

A number of alternatives may have equal initial costs and failure losses, and a number of alternatives may have effectively zero failure probability. The least costly alternative with zero failure loss is always either equally or more safe and costly than the theoretically optimum alternative. Computations under a variety of conditions indicate that an order of magnitude estimate of failure losses may be sufficient for practical applications [8].

An alternative formulation based on the probabilities and values associated with strength alone has been applied by Burnes to the choice of capacity reduction factors in reinforced concrete design [9]. In this study, subjective probability measures and the values associated with various real strengths were obtained from several individuals. After modification of initial probability measures by the use of Bayes' Theorem and experimental data, optimum code values of reduction factors were obtained.

Both of the preceeding formulations admit design based on the traditional concept of absolute safety. Effective absolute safety can result from insensitivity to initial costs and small failure losses. True absolute safety can be achieved if a designer believes that certain conditions cannot occur and assigns zero probability to such conditions. However, there can be no objective procedure for establishing true absolute safety.

### 3. USES AND LIMITATIONS OF DECISION THEORY

The unfortunate fact that experimental evidence is rarely a sufficient basis for design is well known in structural engineering practice. Inevitably objective fact must be combined with intuition and experience. However, in conventional design decisions rational analysis is confined to the study of physical phenomena and judgement is applied in a more or less arbitrary manner. In a decision theory approach judgement is explicitly recognized as an essential component of design and an attempt is made to analyze its operation. In effect, the art of engineering is assumed to be an intuitive science.

Any formulation recognizing engineering judgement obviously can not yield optimum designs in an absolute sense. All optimization is relative to the alternatives considered, the existing state of technology and powers of individual intuition. However, one can hope to discover the rules of good judgement and to establish a logical framework in which judgement can be used to best advantage. In particular, statements of judgement can be reduced to their simplest and most meaningful form and logical consistency can be achieved.

In the simple theory developed to date, the decision process has been abstracted into two basic components - prediction and value systems. In

addition, two fundamentals postulates have been adopted. The first is that a rational individual operates as an intuitive probability analyzer when making predictions. The second is that in an uncertain situation a rational individual operates as an intuitive expected loss minimizer with constant values. The first of these postulates is almost certainly correct in principle. However, subjective probabilities are seldom numerically precise. The second postulate is more questionable and must be verified experimentally.

Possibly the greatest potential use of decision theory is a vehicle for stating opinions in group decisions. At the present time there is often considerable dispute over the numerical factors to be used in standard design procedures. Such differences of opinion probably arise because individuals have different estimates of probability distributions and different value systems. By decomposing the choice of such factors into a large number of separate small statements of opinion, it may be possible to isolate the differences between individuals and establish reasonable agreement.

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## L'optimisation des ouvrages courants sur routes et autoroutes

Optimierung gebräuchlicher Brücken über Strassen und Autobahnen

Optimization of Usual Structures over Roads and Highways

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### 1. CONSIDERATIONS PRELIMINAIRES.

Le problème de l'optimisation des ouvrages d'art courants nous est apparu vers 1960 du jour où a démarré la réalisation du réseau français d'autoroutes de liaison; ce programme, à lui seul, implique la construction annuelle de 200 ouvrages, dont 160 présentent entre eux des analogies étroites.

La pratique de l'étude et du contrôle des ouvrages d'art nous a permis d'acquérir un certain nombre de certitudes.

Tout d'abord, la solution la meilleure ne se trouve pas dans la réalisation d'un nombre très limité de modèles prêts à l'emploi, les raisons en étant les suivantes :

- un ouvrage doit s'adapter exactement à son site; organe de franchissement d'une route il doit en épouser le tracé et sa conception doit être adaptée aux particularités géotechniques du lieu.

- la réalisation systématique d'un nombre limité de modèles conduirait à des distorsions géométriques (sur la longueur des travées de rive par exemple) telles que les économies faites sur les études et la réalisation ne compenseraient pas les dépenses supplémentaires en matériaux et les défauts d'aspect.

La recherche d'une solution judicieuse ne peut ignorer certaines considérations :

- le groupement en lots des ouvrages et leur dispersion (à titre d'exemple, l'Administration des Ponts et Chaussées a lancé en 1964 un appel d'offres, qui portait sur un lot de 69 ouvrages répartis sur 2 autoroutes).

- la dimension des Entreprises pressenties;

- les facilités d'approvisionnement en matériaux et leur qualité;

- les délais et les moyens dont on dispose pour les études et le contrôle des travaux;

- le degré d'économie recherché (dépenses éventuelles pour l'aspect et la longévité).

Ces considérations conduisent à procéder à une standardisation dans laquelle on trouve les avantages suivants :

- possibilité d'organiser les chantiers d'ouvrages d'art sous une forme échelonnée avec des équipes spécialisées;

- faculté d'amortir des matériels particuliers tels que des coffrages métalliques;

- accélération des études et facilité du contrôle;

- enfin et surtout possibilité d'améliorer au cours des années un type d'ouvrage que l'on réalise à de très nombreux exemplaires.

La "standardisation" se trouve donc à mi-chemin entre la conception isolée et le prêt à l'emploi.

## 2. UNE PREMIERE APPROCHE DE L'OPTIMUM DANS LE CADRE DE LA STANDARDISATION.

La standardisation consiste à dégager un certain nombre de types parmi lesquels on peut ranger la plupart des ouvrages courants. Mais comme certaines parties entières d'ouvrages sont communes à plusieurs types - les appuis intermédiaires, ou encore les joints de chaussées - on peut dégager les notions "d'ouvrages-types" et même "d'éléments-types".

Les études sur la standardisation d'un élément ou d'un ouvrage type se traduisent par la mise au point d'un dossier-pilote. Ce dernier comprend, dans le cas d'un tablier ou d'un ouvrage monolithique, trois parties :

- les pièces-pilotes servant au dégrossissage du projet ;
- les bordereaux et les pièces explicatives nécessaires à la commande du calcul et éventuellement des dessins automatiques;
- un ou plusieurs exemples d'application à des fins purement pédagogiques.

L'ensemble pèse de 2 à 3 kilogrammes.

La liste des dossiers-pilotes figurés au tableau ci-dessous fait état du développement actuel de notre standardisation.

L'ensemble de ces dossiers-pilotes sont couronnés par deux documents généraux :

- un catalogue qui rend compte de l'avancement de cette standardisation.
- un manuel d'estimation par une méthode de prix composés.

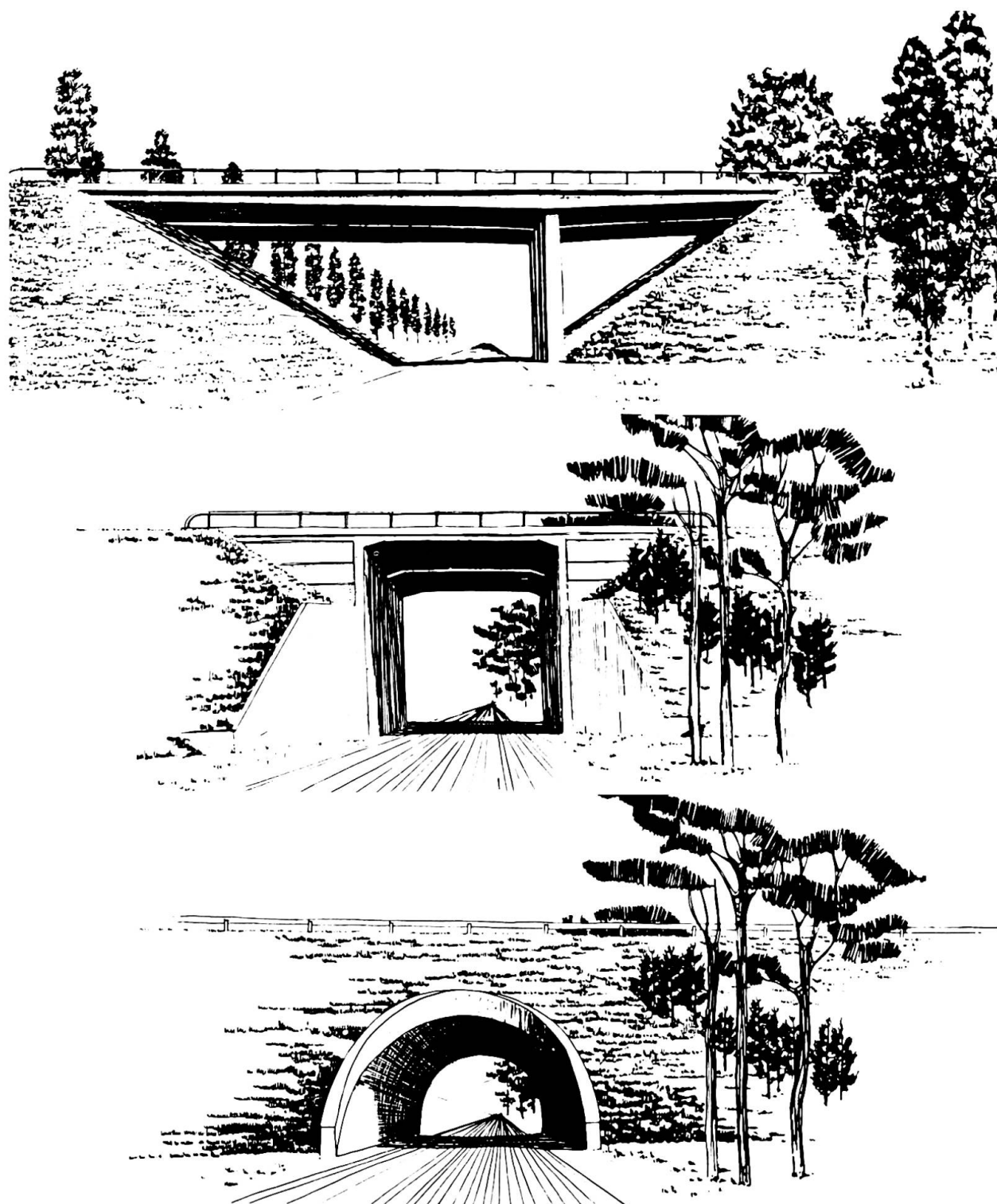
Un troisième, traitant de l'esthétique, est en préparation (fig.1)

Nous nous proposons de montrer comment l'utilisation des pièces-pilotes du dossier-pilote V.I.P.P., qui traite des Viaducs à travées Indépendantes en Poutres préfabriquées de béton Précontraint, permet de s'approcher efficacement de la solution la meilleure.

### 2.1. LE CHOIX DE LA STRUCTURE.

D'une façon générale, un chapitre du catalogue situe ce type de tablier parmi l'ensemble des tabliers-types.

| Classe d'élément-type                         | Élément-type ou ouvrage-type                                                                               | Sigle  | Niveau d'automatisation.<br>cf 8 3.2 |
|-----------------------------------------------|------------------------------------------------------------------------------------------------------------|--------|--------------------------------------|
| Fondations                                    | Fondation des ouvrages courants sur semelles                                                               | FOCS   | -                                    |
|                                               | Fondations profondes (en préparation)                                                                      |        |                                      |
| Appuis                                        | Piles et palées                                                                                            | PP     | 1                                    |
|                                               | Culées types                                                                                               | CT     | -                                    |
| Tabliers de passages supérieurs et inférieurs | Travées solidaires à poutres de hauteur constante de béton armé                                            | PSI-BA | 3                                    |
|                                               | Travée indépendante à poutres de hauteur constante de béton armé                                           | TI-BA  | 3                                    |
|                                               | Travées solidaires en dalle pleine d'épaisseur constante de béton précontraint                             | PSI-BP | 4                                    |
|                                               | Travées solidaires en dalle pleine d'épaisseur constante de béton armé                                     | PSI-DA | 3                                    |
|                                               | Travées solidaires en dalle élégie de hauteur variable de béton précontraint                               | PSI-DE | 3                                    |
|                                               | Plaques en dalle d'épaisseur constante de béton précontraint (par la méthode des réflexions biharmoniques) | MRB-BP | 1                                    |
|                                               | Plaques en dalle d'épaisseur constante de béton armé (par la méthode des réflexions biharmoniques)         | MRB-BA | 2                                    |
|                                               | Travées solidaires ou indépendantes en ossature mixte                                                      | OM     | 2                                    |
|                                               | Viaduc à travées indépendantes à poutres préfabriquées de béton précontraint                               | VIPP   | 2                                    |
| Ouvrages monolithiques                        | Passage inférieur en cadre fermé de béton armé                                                             | PICF   | 3                                    |
|                                               | Passage inférieur en portique ouvert de béton armé                                                         | PI-PO  | 3                                    |
|                                               | Passage supérieur à béguille de béton précontraint                                                         | PS-BQ  | 1                                    |
| Équipements                                   | Garde-corps, glissières, corniches, grilles centrales                                                      | GC     | -                                    |
|                                               | Joint de chaussée, appareils d'appui, dalle de transition, évacuation des eaux.                            | JADE   | -                                    |
|                                               | Surfaçage des tabliers, étanchéité, couches de roulement                                                   | STER   | -                                    |



Figures 1.

Extrait d'un guide esthétique pour ouvrages courants actuellement en préparation. Le franchissement d'une route à deux voies est traité sous trois formes différentes qui font toutes appel à des ouvrages-types du S.E.T.R.A. Le guide précise leur coût respectif.

La notice générale du dossier-pilote précise plus particulièrement le domaine d'application technique et économique du V.I.P.P.

La pièce intitulée "dimensionnement du tablier" indique les quantités de bétons, d'aciers et de coffrages qui ont été utilisées pour la réalisation d'un certain nombre d'ouvrages du même type - sept au total - et débouche sur des résultats statistiques.

La méthode des prix composés évoquée plus haut, permet d'apprécier rapidement et très simplement une estimation avec une marge d'erreur de quelques % à indice de prix donné.

## 2.2. LE CHOIX DES DISPOSITIONS CONSTRUCTIVES.

Le dossier permet de traiter un large éventail de dispositions constructives. Leurs descriptions sont suivies de discussions et finalement de recommandations sur leur choix de façon à permettre une approche statistique de la solution optimale.

Les questions sont envisagées sous les aspects les plus divers.

Ainsi la sécurité des usagers est-elle envisagée dès le début à propos de la disposition des poutres de rive, lesquelles ne doivent pas coïncider avec l'emplacement des glissières pour ne pas rendre impossible la fixation de celles-ci sur platine.

Les problèmes liés à la manutention des poutres et à leur assemblage sont évoqués à propos de la forme de l'amorce des entretoises.

La volonté d'intégrer exactement l'ouvrage dans son site se traduit par l'énoncé d'une série d'adaptations en relation avec la courbure en plan du tracé, avec le biais des lignes d'appuis par rapport à l'axe de la chaussée, et avec une variation du dévers de la chaussée (fig. 2 ).

Par souci esthétique, des variantes pour la forme des poutres de rives sont suggérées, et cinq types d'appuis intermédiaires sont proposés.

## 2.3. LES PROCÉDES D'EXECUTION.

La notice générale énonce un certain nombre de règles dont le respect au niveau de l'exécution est nécessaire à la réussite; elles concernent l'installation du chantier, la constitution des coffrages la disposition des armatures, la mise en oeuvre du béton et des différentes phases de précontrainte, la manutention des poutres, l'exécution du hourdis et des entretoises, les opérations de finition.

## 3. L'OPTIMISATION DES ETUDES D'OUVRAGES D'ART COURANTS PAR LE TRAITEMENT AUTOMATIQUE DES INFORMATIONS

L'incidence du coût, de la qualité des études et du contrôle sur l'optimum n'a pas été évoquée de façon explicite dans le Rapport préliminaire malgré son importance relative dans le prix de revient final, qui est de l'ordre de 5 à 10 %.

De surcroît, considérant que l'on dispose d'une masse d'Ingénieurs généralement limitée, il est souhaitable de délivrer au maximum ces derniers des tâches purement mécaniques comme l'établissement manuel d'une note de calcul détaillée pour leur permettre de reporter leur activité sur les problèmes où leur intervention est irremplaçable : la conception, l'exécution et son contrôle.

L'effort que nous avons consenti en matière d'automatisation des études est extrêmement important parce qu'il est poursuivi de

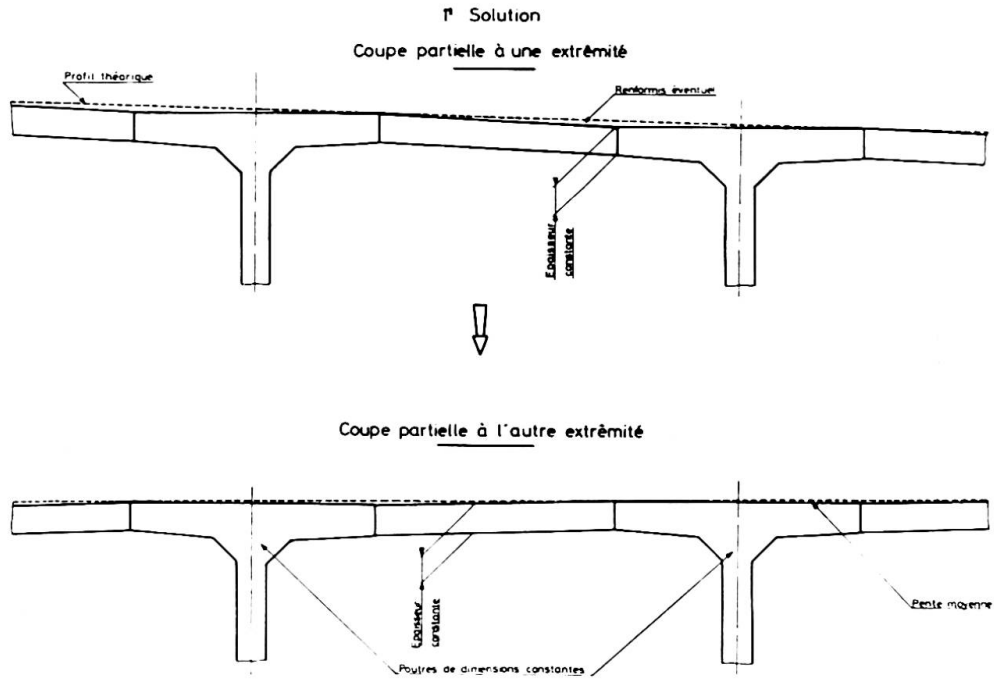


Figure 2. Extrait du sous-dossier 1 "Pièces-pilotes" du dossier-pilote VIPP 67. L'une des trois dispositions constructives qui tiennent compte d'un dévers variable de chaussée.

## COUPE TRANSVERSALE

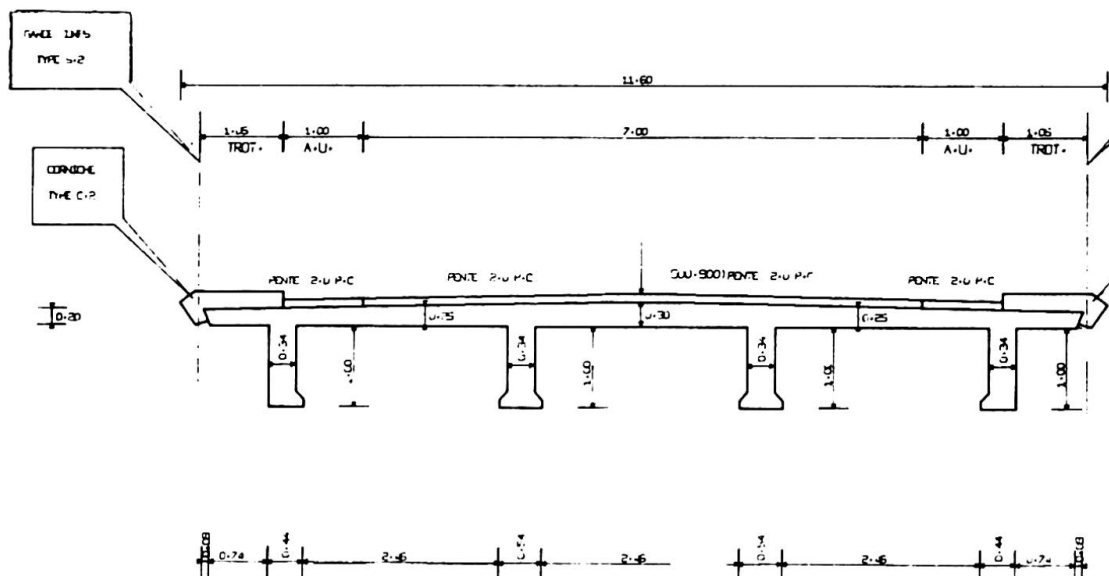


Figure 3. Coupe transversale de tablier à poutres sous chaussée automatiquement dessinée par traceur de courbes.



façon homogène et appliqué à l'ensemble des structures-types.

Le but poursuivi est l'automatisation des différentes tâches qui ne sont pas reconnues comme conceptuelles.

### 3.1. LES ETUDES D'AVANT-PROJET DETAILLE.

Ainsi désigne-t-on le dégrossissage du projet. Ce dégrossissage comprend les opérations suivantes :

- le choix du type d'ouvrage,
- le prédimensionnement,
- les dessins d'implantation et de coffrage.

Les deux premières opérations sont accomplies par l'Ingénieur comme indiqué aux § 2.1. et 2.2.

Les travaux de dessins consistent à démarquer certaines pièces de la troisième partie où l'on trouve un modèle d'application complètement traité. Cette opération a été accomplie manuellement jusqu'à présent; nous l'avons automatisée dans le cas de la représentation de passages à travées multiples et nous comptons l'étendre au cas des cadres fermés, des portiques ouverts et des viaducs du type V.I.P.P. (fig. 3)

Ce mode d'étude aura atteint sa pleine efficacité le jour où les "entrées" ne devront plus être préétablies dans leur ensemble ni être formulées sous formes numériques, mais où le dialogue avec l'ordinateur sera possible y compris sous forme graphique par utilisation d'une unité d'affichage en time-sharing.

### 3.2. L'ETUDE DEFINITIVE.

A ce stade, qui est généralement préalable à l'appel d'offres, il est souhaitable de dimensionner et d'implanter exactement l'ouvrage et d'assortir cette définition du schéma des armatures principales.

L'automatisation de toutes ces tâches est une entreprise de longue haleine que nous accomplissons par étapes.

Niveau 1 : automatisation des calculs de résistance des matériaux de la structure supposée prédimensionnée (calcul des lignes et surfaces enveloppes des efforts et des contraintes; cf les figures de la référence (4)).

Niveau 2 : automatisation des armatures de B.A. et de B.P. et établissement d'un avant-métré.

Niveau 3 : automatisation du dimensionnement complet.

Niveau 4 : tracé automatique des schémas de coffrage et de ferrailage.

Le niveau d'automatisation des programmes associés aux dossiers pilotes de structures-types est indiqué tableau 1. Le programme V.I.P.P. est automatisé jusqu'au niveau 2, non compris toutefois le dimensionnement des armatures transversales (fig.4)

## 4. LES RECHERCHES D'OPTIMUM SUR OUVRAGES-TYPES EN B.P. PAR LE SYSTEME "EUGENE".

Malheureusement, la standardisation des ouvrages a jusqu'à présent été limitée dans le domaine des structures par des difficultés de calcul. Quant aux recherches sur les procédures de dimensionnement et de tracé des câbles, préliminaires au traitement auto-



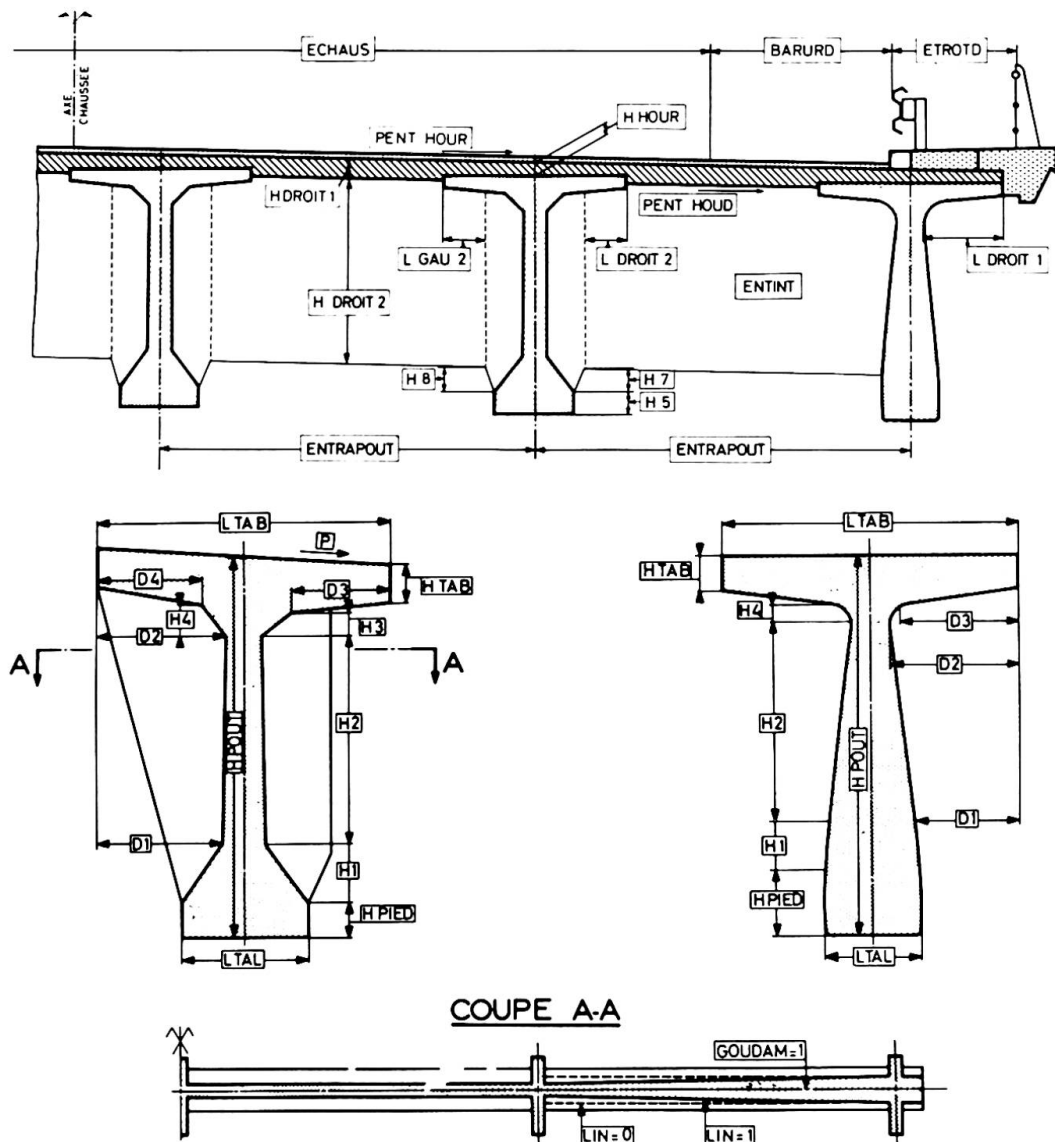


Figure 4. Extrait du sous dossier 2 "Calcul électronique" du dossier-pilote VIPP 67. Les variables géométriques intervenant dans la définition d'une travée de pont du type VIPP.

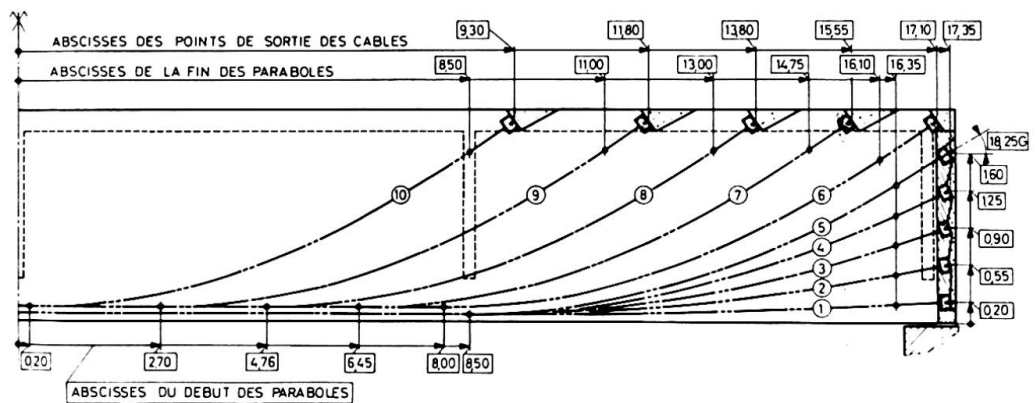


Figure 5. Extrait du sous-dossier 3 "Exemple d'application" du dossier-pilote VIPP 67. Résultats des recherches automatiques sur le dimensionnement et le tracé des câbles.

matique, elles sont déplorables de lenteur parce que menées de façon entièrement manuelle : ainsi, la mise au point d'un dossier - pilote et du programme associé poussé jusqu'au niveau 4 demande de quatre à six ans sous forme d'une version tous les deux ans.

"EUGENE", système d'analyse élastique des structures hautement complexes et d'optimisation des structures en béton précontraint aura, entre autres mérites, celui de lever ces deux hypothèses.

#### 4.1. STRUCTURES ELASTIQUES CALCULABLES PAR "EUGENE".

Les structures dont l'analyse élastique est envisageable doivent appartenir à l'ensemble des corps composés des corps élémentaires suivants :

- des poutres à caractéristiques géométriques et mécaniques, à rayon de courbure et torsion constants ou variables; il est tenu compte de l'influence d'une réaction élastique du milieu environnant aussi bien vis-à-vis des déplacements que des rotations, ainsi que d'une déformabilité à l'effort tranchant;

- des plaques homogènes isotropes à bords parallèles, rectilignes, tous deux libres ou tous deux simplement appuyés ou tous deux encastrés;

- des plaques homogènes isotropes à bords parallèles circulaires et libres;

- des plaques infinies homogènes et isotropes.

Les coefficients d'influence de ces corps ont été étudiés en faisant abstraction des conditions d'appui par les techniques mathématiques les plus modernes (transformation de Laplace et méthode de réflexions dans le cas des plaques).

L'application systématique du principe de réciprocité fait que les assemblages entre corps élémentaires, les efforts extérieurs et les déformations sont présentés et traités de façon identique. On peut donc englober ces trois notions sous le terme d'efforts-déformations.

Les natures des efforts-déformations envisageables sont : force-déplacement, couple-rotation, déformateur-déformation pure.

Leurs domaines de définition doivent être limités par des polygones pouvant dégénérer en segments ou en points.

Leurs distributions doivent être de la forme exponentielle - polynôme.

Les lignes et surfaces d'influence sont exploitées jusqu'à la détermination des enveloppes de contraintes induites par le poids propre et par les surcharges réglementaires.

#### 4.2. OPTIMISATION DES OUVRAGES EN BETON PRECONTRAINTE PAR LES METHODES DE RECHERCHE OPERATIONNELLE.

L'optimisation en question porte sur la détermination des variables continues qui interviennent dans le dimensionnement d'un ouvrage précontraint.

Il est bien entendu que cette recherche ne peut être envisagée qu'une fois réglés les problèmes de conception.

##### 4.2.1. LES VARIABLES ENVISAGEES.

Nous trouvons, tout d'abord, les variables attachées au dimen-

sionnement des coffrages. Précisons, toutefois, que la recherche correspondante respecte la forme générale et que son objet vise seulement à parfaire certaines dimensions.

Les variables qui ont été retenues pour la définition de la précontrainte sont le couple d'excentrement et les forces d'ancrage. On distingue les variables attachées aux précontraintes respectivement isostatiques et hyperstatiques.

#### 4.2.2. LES CONTRAINTES ENVISAGEES.

Moyennant ce choix des variables, nous avons démontré que la recherche de l'optimum pouvait relever de la programmation linéaire sous réserve d'introduire la notion de "déformateur" (1).

Les différentes classes de contraintes automatiquement prises en compte sont :

- les tensions admissibles (compression et cisaillement),
- les couvertures de béton admissibles,
- les courbures admissibles des câbles.

Mais il est possible d'introduire d'autres contraintes.

#### 4.2.3. LA FONCTION ECONOMIQUE.

Elle permet de rechercher aussi bien un optimum économique que technique (minimisation d'une dimension géométrique).

#### 4.2.4. LE SOFTWARE.

La procédure, qui est schématisée sur l'organigramme, fait appel au programme linéaire OPHELIE II et au générateur de rapport en clair OPHEDIT mis aux point à la Société d'Informatique Appliquée et exploitables sur CDC 6.600. (fig.6)

#### 4.2.5. LES DIFFERENTES PROCEDURES D'EXPLOITATION D'EUGENE (cf. ORGANIGRAMME).

L'exécution des procédures numérotées sur l'organigramme (1) (3) (4) et (6) permet de dégrossir un projet en donnant une approximation de la précontrainte et du tracé correspondant. (fig. 7, 8, 9a et 9b).

Le dimensionnement exact d'un ouvrage particulier se fait par appel aux procédures (1) (2) (3) (4) et (6). Dans le cadre d'une justification détaillée, on peut vérifier la validité de la linéarisation des pertes par un recyclage sur (7) (fig. 9c).

L'exécution de l'ensemble des procédures, y compris l'analyse post-optimale, permet de connaître l'incidence sur le coût final de différentes options prises au niveau de la conception, ce coût pouvant représenter un objectif économique ou simplement technique.

C'est donc bien un outil de recherche très complet applicable aussi bien à l'étude des ouvrages exceptionnels qu'à l'étude systématique des ouvrages-types.

### 5. CONCLUSIONS.

L'optimisation des ouvrages courants ne va pas sans moyens de dégrossissage ni sans moyens perfectionnés d'études approfondies.

Un moyen de dégrossissage rapide et sûr des paramètres à variation discontinue existe dans la standardisation des ouvrages-types.

Deux moyens d'étude approfondie sont préconisés qui font appel

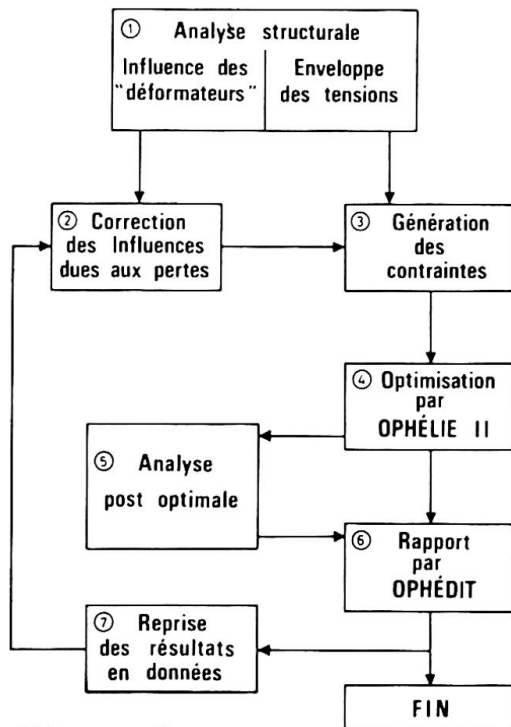


Figure 6

Figure 7

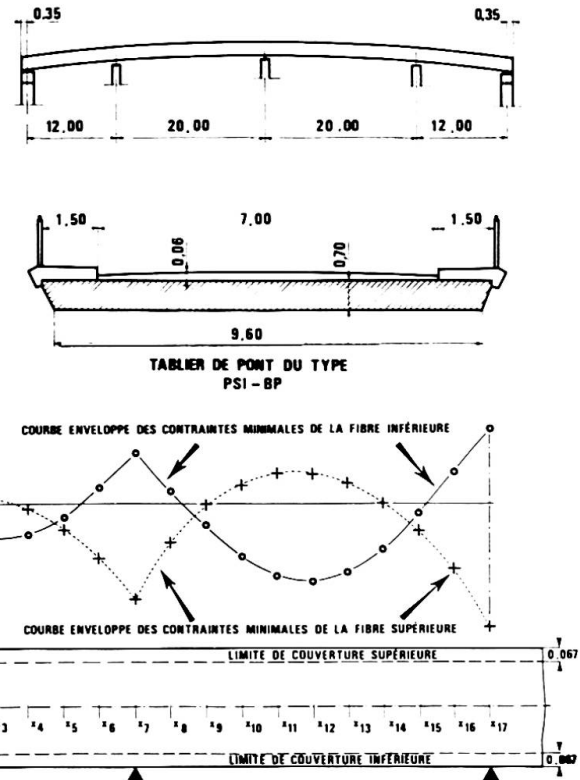


Figure 8

Figure 6. Organigramme d'EUGENE, système d'analyse élastique et d'optimisation des structures spatiales en béton précontraint.

Figure 7. L'exemple d'application: un PSI-BP à 4 travées solidaires.

Figure 8. Les données de l'exemple d'application: enveloppe des tensions, limites de couverture, arrêt des câbles.

Figure 9a. Tracé optimal non compte tenu des courbures admissibles des câbles et des pertes; force optimale: 2171 t.

Figure 9b. Tracé optimal non compte tenu des pertes; force optimale: 2171 t.

Figure 9c. Tracé optimal compte tenu des pertes; force optimale: 2411 t.

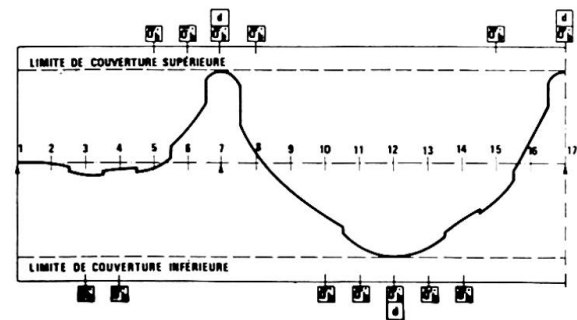


Figure 9a

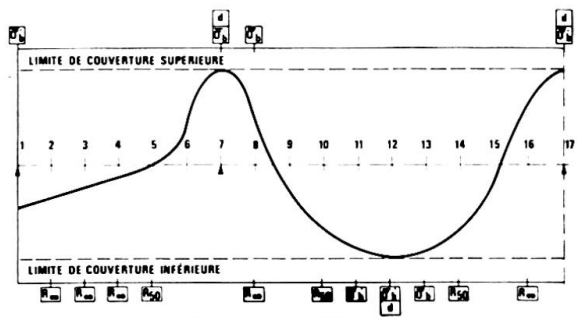


Figure 9b

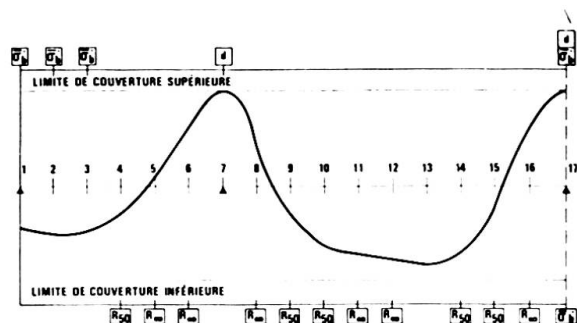


Figure 9c

l'un et l'autre au traitement automatique des informations et à la recherche d'un optimum sur un ensemble de variables continues :

- les études par programmes de calcul électronique entièrement automatiques applicables aux ouvrages-types réalisés en grand nombre;
- les programmes d'analyse structurale, sous réserve qu'ils soient d'une grande généralité et qu'ils débouchent sur des programmes de recherche opérationnelle.

Ces méthodes ont été progressivement introduites en l'espace d'une décennie dans la définition des ouvrages d'art courants du réseau routier et autoroutier français. Rapidement elles se sont révélées efficaces, et le développement des moyens à traiter l'information les rendra de plus en plus fécondes. Nous pensons qu'elles ont vraiment valeur générale.

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#### RÉSUMÉ

Pour définir les ouvrages routiers courants tout en approchant d'un optimum, deux moyens sont successivement appliqués: la standardisation et l'exploitation de programmes de calcul entièrement automatiques. Un système de programmation pour l'analyse élastique des structures complexes (poutres et plaques) et pour l'optimisation de la précontrainte nous permettra de vérifier les structures standardisées et d'accélérer la formulation de programmes de calcul entièrement automatiques.

#### ZUSAMMENFASSUNG

Um die gebräuchlichen Strassen- und Autobahnbrücken in Annäherung an ein Optimum festzulegen, werden zwei Mittel, eines aus dem andern folgernd, angewandt: Die Vereinheitlichung und die Ausschöpfung des Elektrorechners. Ein Programmsystem für die elastische Berechnung verwickelter Tragsysteme (Träger mit Platte) sowie für die Optimierung der Vorspannung wird uns erlauben, die vereinheitlichten Tragwerke zu kontrollieren und die Ausarbeitung der völlig automatischen Programme zu beschleunigen.

## SUMMARY

In order to define the usual bridges over the roads and motorways and come near to a given optimum, two methods are used successively: the "standardization" and the utilization of wholly automatic programs for computers. A programming system for the elastic analysis of sophisticated structures (beams with slabs) and for the optimization of pre-stressed structures will allow us to diversify the standardized structures and to speed up the formulation of wholly automatic programs.

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## **Optimal Design of Reinforced Concrete Beams and Frames**

Dimensionnement optimal des poutres et portiques en béton armé

Optimale Bemessung der Stahlbetonbalken und -rahmen

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The following optimization problem is solved in the paper: design a structure of given geometry for maximum efficiency vs. elastic design, so that under any possible load combination certain specified minimum load factors be guaranteed against both the collapse of the structure and the first yield of its critical sections.

By linearizing the merit function and developing a method to generate all limit equilibrium constraints the problem is solved with the help of linear programming and computer techniques.

The principles involved and corresponding optimal solutions are illustrated by the examples of a reinforced concrete continuous beam and frame.

## **INTRODUCTION**

In the last few years, limit design (as opposed to limit analysis) methods have been developed enabling limit equilibrium and serviceability conditions to be explicitly used in the mathematical formulation of the problem.

Such methods, called "serviceability" (as opposed to "compatibility") methods, have been extensively applied to reinforced concrete continuous beams [1], [2], [3] and [4]\*. More recently the validity of these methods has been

\* Figures within brackets refer to the list of references at the end of the paper.

investigated for a variety of basic assumptions and design criteria [5] and their extension was developed for frame design [6], [7]. A comprehensive discussion of the features of serviceability vs. compatibility methods has been presented by reference to their use in building structures [8].

The possibility of considering a variety of design criteria was earlier recognized [1], [5] and suggestions for optimizing relevant merit functions have subsequently been given [6], [7].

The problem of optimal design for reinforced concrete frames has been formulated in [6], but two major difficulties have been recognized in the actual solution of the problem: 1) the explicit expression of suitable merit functions and 2) the formulation of the limit equilibrium constraints for all possible modes of structural collapse.

Some views on the first problem are offered in papers by MASSONNET and SAVE [9] and by ANDERHEGGEN and THÜRLIMANN [10], in which the total cost or the steel volume are suggested as merit functions, respectively. The authors tested, with favorable results, the use of an "efficiency index" (defined as the ratio of the steel consumption by limit vs. elastic designs,  $v = V_0/V_E$ ) as merit function [7].

The second problem has been amply investigated [11] and a systematic procedure has been developed for generating all possible modes of collapse and predicting the most critical combinations [12].

As a result, by using mathematical programming techniques, it is now possible, and this paper illustrates how, to produce direct designs of reinforced concrete beams and frames such that 1) optimum, 2) limit equilibrium and 3) serviceability criteria be satisfied simultaneously.

It should be noted that an optimal solution verifying the three conditions above would still have to be checked for compatibility. While it is possible to add the compatibility conditions to those already considered and to attempt the solution of the more general programming problem of optimization with compatibility constraints, this will be left for a separate investigation.

#### THE OPTIMAL DESIGN PROBLEM

Consider a reinforced concrete beam or frame with given geometry and moments of inertia, to resist a system of known loads varying between prescribed limits.

It will be assumed that:

- 1) Reinforced concrete can be idealized as an elasto-plastic material with limited ductility;
- 2) All possible loading conditions are considered;
- 3) Limit equilibrium and serviceability are basic design criteria, with compatibility to be separately investigated;
- 4) The optimum criterion is to accomplish the minimum volume of longitudinal reinforcement.

It will also be assumed that bending action prevails, that shear and axial forces are negligible and that inelastic rotations are concentrated at critical sections as in the simple plastic theory.

Let  $j = 1, 2, \dots, s$  be the critical sections of the structure and  $M_{pj}^+, M_{pj}^-$  its corresponding positive and negative plastic moments, as in general, reinforced concrete sections may have unequal flexural resistance in the two bending directions.

A design solution is found when a set of  $M_{pj}^+$  and  $M_{pj}^-$  is derived for all critical sections of the structure such that the following conditions are satisfied:

- 1) Optimum: the design will provide the minimum volume of flexural reinforcing steel.
- 2) Limit Equilibrium: the structure will resist any loading combination of an intensity less than the prescribed ultimate load  $W_u$  and may collapse plastically for any load  $W^+ \geq W_u$ .
- 3) Serviceability: the critical sections of the structure will remain well within the elastic range for any combination of working loads and hence will have a safety against yield not less than a prescribed minimum value,  $\lambda_1$ . It can be shown that provision of adequate yield safety will ensure satisfactory serviceability, i.e. acceptable cracking, stresses and deflections.

Let  $W_u$ ,  $W$  and  $\lambda_o$  be the ultimate load, the service load and the overall load factor, respectively, when proportional loading is assumed, i.e.  $W_u = \lambda_o W$ . Let  $M_{pj}^\pm$  be the maximum (minimum) elastic envelope moment at section  $j$ , and denote  $\lambda_{1j}$  as the yield load factor of section  $j$ , i.e. a plastic hinge will occur at this section under some particular scheme of loading at a load level  $\lambda_{1j} W$ .

It can be shown [4] that the design plastic moment for section  $j$  is proportional to the corresponding elastic envelope moment for the ultimate load:

$$M_{pj} = x_j \lambda_o M_j \quad (1)$$

The scale factor  $x_j$  is called the yield safety parameter [3] and is defined by the ratio of yield to ultimate load factors:

$$x_j = \lambda_{1j} / \lambda_o \quad (2)$$

It is noted that high  $x_j$  values correspond to superior serviceability and low  $x_j$  values correspond to superior economy. An optimal solution will determine  $x_j$  values that accomplish the largest overall steel savings consistent with adequate serviceability. This is taken to mean that for any section, under any loading condition, the yield load factor  $\lambda_{1j}$  is not smaller than a specified lower bound  $\lambda_1 \geq 1$  (the case  $\lambda_1 = 1$  corresponds to the formation of a plastic hinge at service loads). For a section to become a plastic hinge in some possible mechanism it is necessary that  $\lambda_{1j} \leq \lambda_o$ . Therefore the serviceability constraints become:

$$\lambda_1 / \lambda_o \leq x_j \leq 1 \quad (j = 1, 2, \dots, s) \quad (3)$$

A limit equilibrium condition is associated with each possible mode of plastic collapse (i) of the structure. This expresses the relationship between the energy dissipated in the plastic hinge rotations  $U_i$  and the external work  $E_i$  of the ultimate load in displacements corresponding to the mode of collapse i:

$$U_i \geq E_i \quad (4)$$

Relation (4) indicates that the structure collapses in the  $i^{\text{th}}$  mechanism at either the prescribed load factor  $\lambda_0$  if  $U_i = E_i$  or at a larger load factor  $\lambda^+ > \lambda_0$  if  $U_i > E_i$ .

Assuming  $p$  possible mechanisms exist and expressing  $U_i$  in terms of the plastic moments (1) and  $E_i$  in terms of the specified ultimate load, the equilibrium constraints become after simplifications:

$$\sum_{j=1}^s a_{ij} x_j \geq c_i \quad (i = 1, 2, \dots, p) \quad (5)$$

where  $a_{ij}$  and  $c_i$  are some non-negative constants.

As the optimal solution corresponds by definition to the minimum steel volume  $V$ , the merit function is:

$$V = \int \frac{M_{pj} ds}{K_j} = \int \frac{\lambda_0}{K_j} M_j x_j ds = \min ! \quad (6)$$

where the integral is extended over the entire length of the structure and  $K_j$  is a constant depending on the section geometry and materials properties.

In summary the optimal design problem consists of determining the  $x_j$  values for  $s$  critical sections of the structure so that the optimum criterion (6), the limit equilibrium constraints (5) and the serviceability constraints (3) be satisfied.

This is easily identified as a typical programming problem to which the standard algorithms of mathematical programming are applicable.

#### MERIT FUNCTION, CONSTRAINTS AND OPTIMAL SOLUTION

Merit Function. The concrete sizes being fixed, the amount of steel reinforcement on which the design plastic moments depend is the only variable in the feasible design solutions. In current detailing practice the reinforcement provided at critical sections is maintained constant over certain lengths of the members before it is bent or terminated to conform with bond, anchorage and/or shear requirements. As a result the resisting moments provided follow a step-wise diagram, which can be idealized by a set of rectangles of depth  $M_{pj}$  and of length  $l_j$ , the distance over which  $M_{pj}$  is maintained constant [8], [10].

With these assumptions and notations the steel volume associated with each plastic moment can be expressed as:

$$V_j = A_{sj} l_j = M_{pj} l_j / K_j \quad (7)$$

Therefore the merit function (6) expressing the total steel volume is linearized in the form:

$$V = \sum_j M_{pj} l_j / \sqrt[3]{\gamma_j} = \sum_j x_j \lambda_0 M_j l_j / \sqrt[3]{\gamma_j} \quad (8)$$

where  $\gamma_j$  are the known ratios of moments of inertia for sections  $j$  to a reference moment of inertia for the structure.

**Constraints.** A major difficulty of the plastic analysis and design of highly indeterminate frames consists of writing explicitly the equilibrium conditions (5) representing the limit equilibrium constraints of the optimization problem. In the paper this is done by an extension of the theory of "combination of mechanisms" due to Neal and Symonds [13], which is fully developed elsewhere [11], [12].

The procedure is based on the fact that in the derivation of the optimal solution only a limited number,  $m$ , of equilibrium conditions (collapse modes) need be considered, which is called the critical or active set of equilibrium constraints.

The active set of constraints is identified by the fact that 1) it contains  $m = s - n$  collapse mechanisms, ( $n$  is the degree of statical indeterminacy of the structure) 2) which are linearly independent, 3) have only one degree of freedom, and 4) correspond to the lowest load factors of any possible mechanism. A computer programme which generates automatically all relevant combinations of mechanisms (COMECH) and which identifies the critical set of constraints based on the above mentioned criteria has been developed [11], [12]. A flow-diagram of the COMECH programme is shown in Fig. 1.

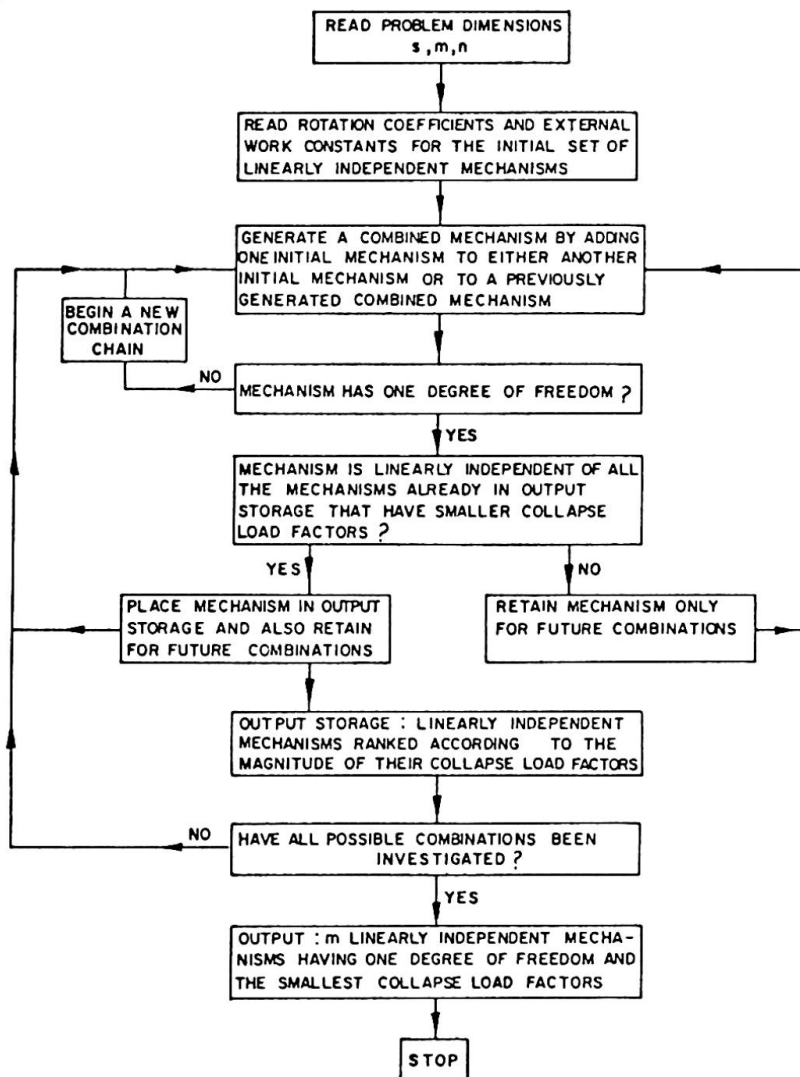


FIG. 1

Optimal Solution. With the linear merit function (8), the serviceability constraints (5) and the equilibrium constraints determined by COMECH, the simplex algorithm can be used to solve the resulting linear programming problem of optimal design [14]. A standard computer programme for the simplex algorithm is used to solve this problem.

The coefficients of the variables, the external work constants and the bounds on the variables are given as input and the following data are obtained as output:

- 1) The optimal design plastic moments for all critical sections;
- 2) The efficiency index of the optimal design,  $v$ , and
- 3) The effective ultimate safety of the active collapse modes.

### EXAMPLES

Example 1. Given is the five-span continuous beam with the geometry and loading

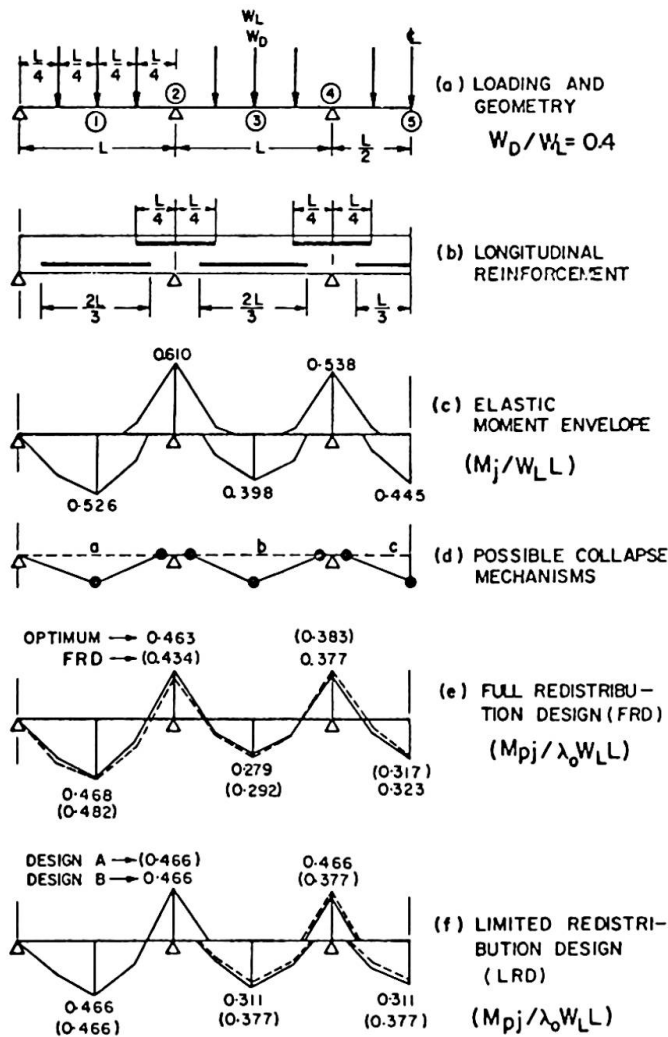


FIG. 2

in Fig. 2a. Live loads are 2.5 times higher than the dead loads and can be applied to any or all of the spans. Dead and live load factors of 1.5 and 1.8 respectively are assumed, conforming to the American practice. An overall load factor  $\lambda_0 = 1.715$  is implied. A minimum yield load factor  $\lambda_1 = 1.2$  is specified and therefore the minimum  $x_j$  value permitted is  $x_j = \lambda_1/\lambda_0 = 1.2/1.715 = 0.7$ .

With the conventional arrangement of the reinforcement as in Fig. 2b, the elastic moment envelope coefficients in Fig. 2c and assuming  $K_j = K = \text{const.}$ , the merit function is given by expression (9).

All possible modes of collapse are indicated in Fig. 2d and are labelled by (a), (b) and (c). The corresponding limit equilibrium conditions are given by expressions (10).

The serviceability constraints are given by expression (11).

Therefore the problem statement is:

minimize:

$$KV = \frac{2}{3} 0.526 x_1 + \frac{1}{2} 0.610 x_2 + \frac{2}{3} 0.398 x_3 + \frac{1}{2} 0.538 x_4 + \frac{1}{3} 0.445 x_5 \quad (9)$$

subject to:

$$1.052 x_1 + 0.610 x_2 \geq 1.4 \quad (10a)$$

$$0.610 x_2 + 0.796 x_3 + 0.538 x_4 \geq 1.4 \quad (10b)$$

$$1.076 x_4 + 0.890 x_5 \geq 1.4 \quad (10c)$$

and:

$$0.7 \leq x_j \leq 1 \quad (j = 1, 2, \dots, 5) \quad (11)$$

The set of  $x_j$  values corresponding to the optimal design is given in the last column of Table 1.

It is to be noted that the optimal solution corresponds to conditions (10) becoming equalities, i.e. to mechanisms (a), (b) or (c) occurring at the pre-scribed load factor  $\lambda_0 = 1.715$ . Therefore this is a full redistribution design. The corresponding bending moment diagram is illustrated by the full lines in Fig. 2e.

For the sake of comparison the elastic solution and three additional limit design solutions are provided as follows:

- a full redistribution design (FRD) based on [3], column 5 in Table 1 and bending moment diagram in dotted lines, Fig. 2e.
- two limited redistribution designs (LRD) labelled A and B respectively, with equal design moments at a number of sections for convenience of steel placing. These solutions are given in Table 1, columns 3 and 4, and the corresponding bending moments are illustrated in Fig. 2f.

TABLE 1

EXAMPLE 1: DESIGN SOLUTIONS

$$x_j = M_{pj} / \lambda_0 M_j$$

| Section (j) | ELASTIC DESIGN | LRD:A | LRD:B | FRD   | OPTIMAL DESIGN |
|-------------|----------------|-------|-------|-------|----------------|
| 1           | 1.000          | 0.887 | 0.887 | 0.916 | 0.889          |
| 2           | 1.000          | 0.763 | 0.763 | 0.712 | 0.760          |
| 3           | 1.000          | 0.946 | 0.781 | 0.732 | 0.700          |
| 4           | 1.000          | 0.700 | 0.866 | 0.712 | 0.700          |
| 5           | 1.000          | 0.848 | 0.700 | 0.712 | 0.726          |

Lower serviceability limit  $x_j = 0.700$

In Table 2 a summary is given of the effective safety against collapse for the various designs. If  $\lambda_1^i$  is the actual load factor for a particular design and collapse mode  $i$  the ratio  $\lambda_1^i / \lambda_0$  is indicative of the relative conservatism of various solutions.

In the same table the efficiency index is given for all solutions studied. As expected, it is noted that the larger is the relative safety against collapse the less economical is the design. The optimal solution is evidently the most efficient while providing exactly the required safety in all possible modes of failure. Note that the full redistribution design (FRD) is very close to the efficiency of the optimal solution.



TABLE 2

EXAMPLE 1: STRUCTURAL SAFETY AND EFFICIENCY

| Mechanism<br>(i) | ELASTIC<br>DESIGN | $\lambda'_i/\lambda_o$ |       |       | OPTIMAL<br>DESIGN |
|------------------|-------------------|------------------------|-------|-------|-------------------|
|                  |                   | LRD:A                  | LRD:B | FRD   |                   |
| a                | 1.189             | 1.000                  | 1.000 | 1.000 | 1.000             |
| b                | 1.389             | 1.142                  | 1.111 | 1.000 | 1.000             |
| c                | 1.405             | 1.075                  | 1.111 | 1.000 | 1.000             |
| $v = v_0/v_E$    | 1.000             | 0.828                  | 0.813 | 0.767 | 0.766             |

**Example 2.** The frame in Fig. 3a [13] is to be designed for any possible combination of the applied live loads, assuming zero dead load. With an overall load factor  $\lambda_o = 1.8$  and a minimum yield load factor  $\lambda_1 = 1.2$ , the minimum permissible value of  $x_j = 1.2/1.8 = 0.667$ .

The reinforcing details are provided in Fig. 3b and the elastic moment envelope is indicated in Fig. 3c. A merit function of the form (8) is obtained, which is not reproduced here for the sake of brevity.

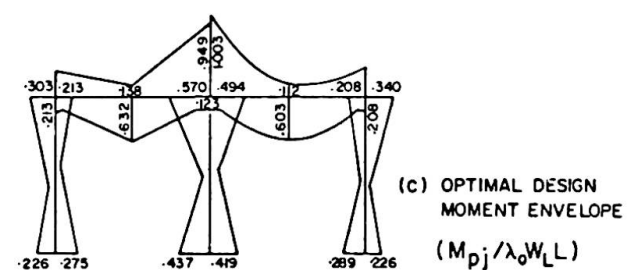
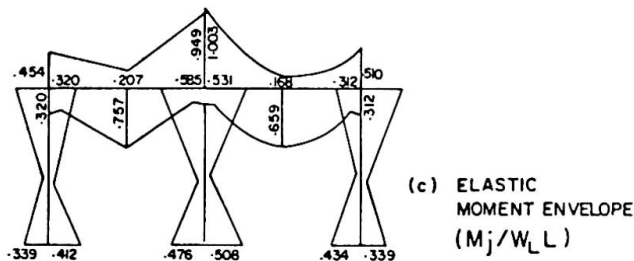
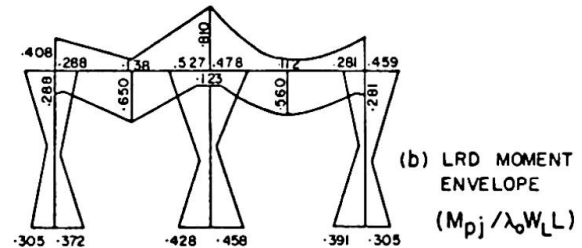
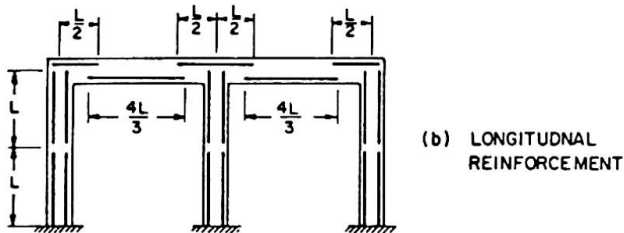
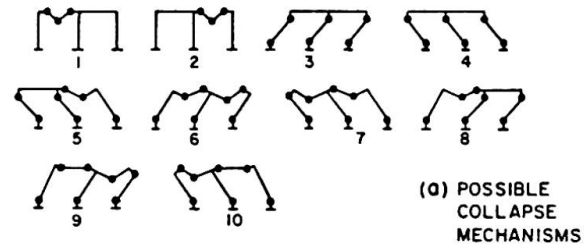
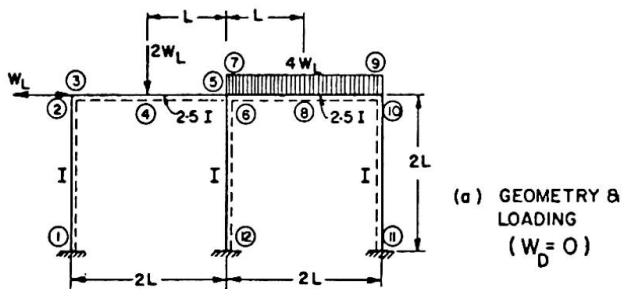


FIG. 3

FIG. 4

Of all the potential collapse modes, the COMECH programme [11] identifies the 10 mechanisms in Fig. 4a, which correspond to the critical or active set of limit equilibrium constraints.

By using the simplex computer programme the optimum solution is found and the corresponding  $x_j^\pm$  values are listed in the last columns of Table 3. Also listed are the elastic solution ( $x_j^\pm = 1 = \text{const.}$ ) and a limited redistribution design based on [7].

The B.M. diagrams associated with these designs are represented in Figs. 4b and 4c, which are obtained by scaling down the bending moments in Fig. 3c with the corresponding  $x_j$  values in Table 3.

TABLE 3

## EXAMPLE 2: DESIGN SOLUTIONS

$$x_j = M_{pj} / \lambda_o M_j$$

| Section<br>(j) | ELASTIC<br>DESIGN |         | LIM. RED. DESIGN<br>LRD |         | OPTIMAL<br>DESIGN |         |
|----------------|-------------------|---------|-------------------------|---------|-------------------|---------|
|                | $x_j^+$           | $x_j^-$ | $x_j^+$                 | $x_j^-$ | $x_j^+$           | $x_j^-$ |
| 1              | 1.000             | 1.000   | 0.900                   | 0.900   | 0.667             | 0.667   |
| 2              | 1.000             | 1.000   | 0.900                   | 0.900   | 0.667             | 0.667   |
| 3              | 1.000             | 1.000   | 0.900                   | 0.900   | 0.667             | 0.667   |
| 4              | 1.000             | 1.000   | 0.858                   | 0.667   | 0.835             | 0.667   |
| 5              | 1.000             | 1.000   | 0.667                   | 0.853   | 0.667             | 1.000   |
| 6              | 1.000             | 1.000   | 0.900                   | 0.900   | 0.931             | 0.975   |
| 7              | 1.000             | 1.000   | 0.667                   | 0.807   | 0.667             | 1.000   |
| 8              | 1.000             | 1.000   | 0.850                   | 0.667   | 0.915             | 0.667   |
| 9              | 1.000             | 1.000   | 0.900                   | 0.900   | 0.667             | 0.667   |
| 10             | 1.000             | 1.000   | 0.900                   | 0.900   | 0.667             | 0.667   |
| 11             | 1.000             | 1.000   | 0.900                   | 0.900   | 0.667             | 0.667   |
| 12             | 1.000             | 1.000   | 0.900                   | 0.900   | 0.824             | 0.919   |

Lower serviceability limit  $x_j = 0.667$

The effective collapse safety of the elastic, LRD and optimal solutions is indicated in Table 4, along with the corresponding efficiency indices. It should be noted that the optimal design enables 8 modes of collapse at the prescribed ultimate load with an overall steel reduction of about 20% vs. the elastic solution. The limited redistribution design generates only 2 modes of failure at  $\lambda_o W$  with a steel saving of 14% vs. the elastic design.

TABLE 4

## EXAMPLE 2: STRUCTURAL SAFETY AND EFFICIENCY

$$\lambda_i' / \lambda_o$$

| Mechanism<br>(i) | ELASTIC<br>DESIGN | LRD   | OPTIMAL<br>DESIGN |
|------------------|-------------------|-------|-------------------|
| 1                | 1.46              | 1.26  | 1.26              |
| 2                | 1.42              | 1.20  | 1.28              |
| 3                | 1.31              | 1.18  | 1.00              |
| 4                | 1.31              | 1.18  | 1.00              |
| 5                | 1.16              | 1.00  | 1.00              |
| 6                | 1.17              | 1.01  | 1.00              |
| 7                | 1.17              | 1.00  | 1.00              |
| 8                | 1.19              | 1.04  | 1.00              |
| 9                | 1.21              | 1.06  | 1.00              |
| 10               | 1.25              | 1.09  | 1.00              |
| $v = v_o / v_E$  | 1.000             | 0.860 | 0.803             |

## CONCLUSIONS

Optimal solutions can be derived for reinforced concrete beams and frames for minimum steel consumption, with adequate safety against both the structural collapse of structures and the first yield of their critical sections.

With the assumptions adopted in the paper the optimal design becomes a linear programming problem, which can be solved by using digital computers.

Examples of optimal design presented indicate savings of 20 - 23% in steel consumption vs. the elastic solutions based on the ultimate strength design for the sections.

While the techniques described are straightforward when applied using a digital computer, they appear prohibitive for hand calculation in design offices.

However standard optimal solutions may be computed and tabulated for typical beams in the same way as in [4].

Data in Tables 2 and 4 confirm that the full redistribution design (FRD) is nearly as efficient as the optimal design, a result which has been anticipated in some previous studies [7], [8]. Because of this feature and of the relative simplicity of serviceability methods it appears that approaching full redistribution is a realistic and practical objective in the limit design of reinforced concrete frames.

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## NOTATION

|                |                                                                                                                                |
|----------------|--------------------------------------------------------------------------------------------------------------------------------|
| $a_{ij}, c_i$  | constants defining the internal and external work in the limit equilibrium equations                                           |
| $A_{sj}$       | flexural reinforcement of section j                                                                                            |
| i              | index referring to the mode of collapse (mechanism)                                                                            |
| j              | index referring to critical sections of the structure                                                                          |
| $K_j$          | a constant which depends on the section geometry and materials properties and defines the flexural reinforcement of section j. |
| $\ell_j$       | total distance over which $M_{pj}$ prevails                                                                                    |
| m              | number of independent mechanisms                                                                                               |
| $M_j$          | elastic moment envelope value at section j                                                                                     |
| $M_{pj}^{\pm}$ | design plastic moments at section j                                                                                            |
| n              | degree of statical indeterminacy of the structure                                                                              |
| p              | total number of possible collapse modes (mechanisms)                                                                           |
| s              | total number of critical sections of the structure                                                                             |
| $V_E$          | steel volume required by elastic design                                                                                        |
| $V_O$          | steel volume required by optimal design                                                                                        |
| $v = V_O/V_E$  | efficiency index of the structural design                                                                                      |

|                                |                                                                     |
|--------------------------------|---------------------------------------------------------------------|
| $W$                            | service (dead + live) loads                                         |
| $W_D, W_L$                     | dead and live service loads, respectively                           |
| $W_u = \lambda_o W$            | specified ultimate load                                             |
| $\lambda_o$                    | specified ultimate load factor                                      |
| $\lambda_1$                    | specified yield load factor                                         |
| $\lambda_i'$                   | effective ultimate load factor in the $i^{\text{th}}$ collapse mode |
| $\lambda_{1j}$                 | minimum effective yield load factor of section $j$                  |
| $x_j = \lambda_{1j}/\lambda_o$ | yield safety parameter of section $j$                               |

## SUMMARY

Optimal solutions can be derived for reinforced concrete beams and frames for minimum steel consumption, with adequate safety against both the structural collapse of structures and the first yield of their critical sections. With the assumptions adopted in the paper the optimal design becomes a linear programming problem, which can be solved by using digital computers.

While the techniques described are straightforward when applied using a digital computer, they appear prohibitive for hand calculation in design offices. However standard optimal solutions may be computed and tabulated for typical beams in the same way as in [4].

## RÉSUMÉ

Il est possible d'arriver à un dimensionnement optimal des poutres et portiques en béton armé pour un minimum d'armature, avec sécurité adéquate contre la ruine totale de la structure et la première rupture dans une section critique. L'économie contre la méthode élastique est de l'ordre de 20-23 %.

La technique de calcul décrite est directe pour une calculatrice électronique, mais plutôt difficile pour le calcul manuel. Cependant des solutions-standard pourraient être préparées en tables pour des poutres typiques, comme dans [3]. D'ailleurs une méthode approchée (table 4) donne d'assez bons résultats.

## ZUSAMMENFASSUNG

Die optimale Bemessung der Stahlbetonbalken und -rahmen ist bei gleicher Sicherheit gegen Traglast sowie erstem plastischen Gelenk der gefährdeten Querschnitte für ein Minimum an Bewehrung möglich. Die Ersparnis gegenüber der elastischen Verfahren beträgt 20-23 %.

Während die beschriebene Methode auf Digital-Computern einfach anzuwenden ist, erscheint sie für die Handrechnung nicht empfehlenswert. Wie auch immer, die standardisierten Optimumslösungen mögen in der gleichen Weise wie in [3] berechnet und tabelliert werden. Im übrigen ergibt ein Näherungsverfahren (Tafel 4) hinreichend genaue Ergebnisse.

## Ein optimales Flächentragwerk für Dachkonstruktionen

An Optimum Plate Structure for Roofing

Une structure bi-dimensionnelle optimale pour toitures

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### 1. Aufgabenstellung

Es war die Aufgabe gestellt, für eine industrielle Serienfertigung ein großflächiges, montagefähiges Dachelement für hallenartige Bauwerke mit einem mechanisierten Fertigungsverfahren so zu entwickeln, daß der Absatz durch die optimale Erfüllung der nachfolgend diskutierten Forderungen gesichert wird.

### 2. Einflußfaktoren

#### 2.1. Funktionelle und gestalterische Forderungen

Der Witterungsschutz ist nicht mehr die alleinige Aufgabe eines Daches. Die vielfältigen Funktionen, die Bauwerke für die Industrie, den Handel, den Verkehr und die Gesellschaft erfüllen müssen, bedingen sehr unterschiedliche Forderungen an die Dachausbildung. Die Ausführung einer individuellen Dachkonstruktion für eine spezielle Bauaufgabe ermöglicht die hierbei gestellten funktionellen und gestalterischen Forderungen maximal zu erfüllen. Soll aber eine Dachkonstruktion entsprechend der obigen Aufgabenstellung nicht nur die funktionellen und gestalterischen Forderungen befriedigen, sondern ein Optimum aus allen hier genannten Einflußfaktoren darstellen, dann müssen alle möglichen funktionellen und gestalterischen Forderungen derjenigen Bauwerke, für die die großflächigen Dachelemente eingesetzt werden sollen, in die Betrachtung der Optimierung mit einbezogen werden. Wie C o u r b o n im Vorbericht schreibt, sind wir leider noch nicht in der Lage alle Einflußfaktoren einer Tragwerksoptimierung so aufzubereiten, daß mittels eines elektronischen Rechners eine exakte Optimierung erfolgen kann.

In Tafel 1 sind die funktionellen und gestalterischen Forderungen geordnet zusammengestellt, die nach einer "ingenieurmäßigen Optimierung", d.h., einer allein von Erfahrungen und Erkenntnissen abgeleiteten Einschätzung, die mit den neuentwickelten Dachelementen ausgeführten Dachkonstruktionen erfüllen müssen, können oder nicht befriedigen brauchen.



Tafel 1 Funktionelle und gestalterische Forderungen

| die voll erfüllt werden müssen                                                                                                                                                                                                                                                                                                                                                                                      | die durch Variationen erfüllt werden müssen                                                                                                                                                                                                                                                                                                                                                                 | die ausgeschlossen bleiben                                                                                                                                                                                           |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <ul style="list-style-type: none"> <li>-Schutz gegen Niederschläge</li> <li>-Entwässerung</li> <li>-Begehbarkeit</li> <li>-Anpassungsfähig an alle rechteckigen Grundrisse, ein- und mehrschiffig</li> <li>-Ausbildung relativ großer sützenfreier Räume</li> <li>-Gute architektonische Gestaltung des Gesamtbauwerkes</li> <li>-Ansprechende Innengestaltung</li> <li>-Geringster Unterhaltungsaufwand</li> </ul> | <ul style="list-style-type: none"> <li>-Geringer Wärmeaustausch mit der Umwelt (1)</li> <li>-Anordnung von Oberlichten (2)</li> <li>-Anhängen einer Zwischendecke (3)</li> <li>-Anhängen von Ausrüstungen normaler Größenordnung (3)</li> <li>-Ausbildung von kleinen und mittleren Durchbrüchen (4)</li> <li>-Schallisolierend (5)</li> <li>-Befriedigung brandschutztechnischer Bedingungen(6)</li> </ul> | <ul style="list-style-type: none"> <li>-Anpassung an vom Rechteck abweichende Grundrisse</li> <li>-Anhängen von schweren Lasten</li> <li>-Ausbildung großer Durchbrüche</li> <li>-Keine Schneesackbildung</li> </ul> |

(1) Variation der Wärmedämmstoffe und ihre Dicke

(2) Konstruktive Möglichkeiten und Zusatzbelastung vorsehen

(3) Befestigungsmöglichkeiten und Zusatzbelastung durch geringfügige Veränderung der Normalelemente vorsehen

(4) Tragfähigkeit auch mit Aussparungen sichern

(5) Anordnung geeigneter Isolierschichten

(6) In Sonderfällen durch Anordnung einer Brandschutzzwischen-  
decke

## 2.2. Forderungen aus den Grundsätzen der Industrialisierung des Bauwesens

Die Veränderung des gesamten Bauwesens von der handwerklichen Fertigung zur industriemäßigen Großproduktion bedingt, daß die den Witterungsunbilden ausgesetzte, schwere körperliche Arbeit weitestgehend reduziert und durch Maschinen übernommen wird; daß ein großer Teil der Arbeitsprozesse von den ortsveränderlichen Baustellen in stationäre, hochmechanisierte Werke verlegt wird, wo serienmäßig weitgehend komplettierte Bauelemente für die Montage vorgefertigt werden, um in kürzester Zeit eine gestellte Bauaufgabe erfüllen zu können. In Tafel 2 sind die entsprechenden Forderungen an ein industriemäßig gefertigtes Dachelement aufgeführt.



Tafel 2 Forderungen der industriemäßigen Großproduktion

| die voll erfüllt werden müssen                                                          | die erfüllt werden können       |
|-----------------------------------------------------------------------------------------|---------------------------------|
| - Mechanisierte Fertigung in stationären Produktionsstätten                             | - Komplettiert mit Wärmedämmung |
| - Rationelle Serienfertigung variabler Elemente                                         | - Komplettiert mit Dachhaut     |
| - Kürzeste Lieferfristen                                                                | - Unterseite sichtflächenfertig |
| - Arbeitsprozesse auf der Baustelle für Montage und Verbindung weitestgehend reduzieren |                                 |

### 2.3. Eigenschaften und Preise der verfügbaren Baumaterialien

Die Baumaterialien beeinflussen die Optimierung einer Dachkonstruktion direkt durch die feststehenden Zahlengrößen, wie Festigkeitswerte, Verformungsgrößen, spezifisches Gewicht, durch ihre Dauerbeständigkeit, ihr Verhalten im Feuer und die sehr zeitvariablen Preise und Unterhaltungskosten. Die Auswahl der optimalsten Baustoffe aus der sich ständig erweiternden Palette der Grundbaustoffe und ihrer vielgestaltigen Kombinationsmöglichkeiten ist zur Zeit nur durch vorliegende Erfahrungen möglich. In vielen Fällen können jedoch eine Anzahl von Baustoffen aussortiert werden, die die vorgegebenen Bedingungen nicht ausreichend erfüllen.

Unter Beachtung der wirtschaftlich noch nicht befriedigend gelösten dauerbeständigen Konservierung der Stahlkonstruktionen und ihrer schnellen und großen Verformung im Feuer wurde der Stahlbau ausgeschlossen. Auch das Holz mußte hier trotz der Fortschritte in der Unbrennbarmachung gestrichen werden. Der Einsatz von Kunststoffen für tragende Konstruktionen war durch die noch zu geringen Erfahrungen ihres Dauerverhaltens und ihrer sehr hohen Preise unmöglich. Die bestehenden Preisrelationen nach Tafel 3 bedingten neben den anderen statisch-konstruktiven Vorteilen die Wahl des Spannbetons für die hier gestellte Aufgabe.

Tafel 3 Preisrelationen Bewehrungsstahl zu Beton

|            | Stahl im Mittel pro<br>m <sup>3</sup> Beton bei Flächen-<br>tragwerken [1]<br>kg | Verhältnis der<br>Materialkosten<br><br>Stahl pro m <sup>3</sup> /m <sup>3</sup> Beton |
|------------|----------------------------------------------------------------------------------|----------------------------------------------------------------------------------------|
| Stahlbeton | 190 ÷ 230                                                                        | 1,29/1,00 ÷ 1,57/1,00                                                                  |
| Spannbeton | 100 ÷ 130                                                                        | 0,79/1,00 ÷ 1,02/1,00                                                                  |

### 2.4. Statisch-konstruktive Forderungen

Die Gebrauchswerteigenschaften und die Lebensdauer werden entscheidend durch die Erfüllung der statisch-konstruktiven Forderungen beeinflusst. Vordringlich muß eine ausreichende Standesicherheit des Bauwerkes gewährleistet werden trotz, aus ökonomischen Gründen bedingten, minimalen Einsatzes von Baustoffen. Entsprechende Sicherheiten müssen auch für die Transport- und Montagezustände der Dachelemente vorhanden sein. Ein weiterer

gewichtiger Faktor ist die Dauerbeständigkeit unter den unterschiedlichsten Einflüssen. Es dürfen innerhalb der Nutzungszeit keine Veränderungen eintreten, die die Funktion des Bauwerkes einschränken oder gar zu Schäden führen. Tafel 4 enthält die statisch-konstruktiven Forderungen für eine ausreichende Sicherheit und Lebensdauer. Die Optimierung der Querschnittsbildung ist hierbei nach einer "delikatsten Formgebung" sehr genau möglich [2] [3].

Tafel 4 Statisch-konstruktive Forderungen

| Eigenschaften des Bauteils                           | Statisch-konstruktive Forderungen                                                                                                                                                                                       |
|------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Tragfähigkeit                                        | <ul style="list-style-type: none"> <li>- groß</li> <li>- wenig schwankend durch Fertigungstoleranzen</li> <li>- Sicherheit gegen Bruch <math>\gamma = 2,0 \div 2,5</math></li> <li>- keine Stabilitätsbrüche</li> </ul> |
| Verformbarkeit                                       | <ul style="list-style-type: none"> <li>- begrenzt</li> <li>- geringe Auswirkungen von Schwinden und Kriechen sowie bei Temperaturänderungen</li> </ul>                                                                  |
| Rißbildung                                           | <ul style="list-style-type: none"> <li>- unerwünscht, durch Vorspannen vermeiden oder auf Haarrisse begrenzen</li> </ul>                                                                                                |
| Querschnittsgestaltung                               | <ul style="list-style-type: none"> <li>- großes Trägheitsmoment für beide Hauptachsen bei geringster Querschnittsfläche</li> <li>- kleines Verhältnis der abgewinkelten oberen Fläche zur Grundfläche</li> </ul>        |
| Verhalten bei Transport und Montage                  | <ul style="list-style-type: none"> <li>- unempfindlich, keine größeren Rißerscheinungen, Abplatzungen oder bleibende Verformungen</li> </ul>                                                                            |
| Verbindungen untereinander und mit anderen Bauteilen | <ul style="list-style-type: none"> <li>- einfach und schnell ausführbar</li> <li>- dauerbeständig und wartungsfrei</li> </ul>                                                                                           |
| Korrosionsbeständigkeit                              | <ul style="list-style-type: none"> <li>- ausreichend durch dichten Beton mit entsprechender Betondeckung für Spannstahl <math>\geq 25</math> mm für die schlaffe Bewehrung <math>\geq 10</math> mm</li> </ul>           |
| Feuerwiderstand                                      | <ul style="list-style-type: none"> <li>- mindestens 1/2 Stunde nach vorgeschriebener Brandbelastung ohne zusätzliche Maßnahmen</li> </ul>                                                                               |

## 2.5. Fertigungsbedingte Forderungen

Eine mechanisierte Serienfertigung mit hoher, gleichbleibender Qualität stellt an die Gestaltung der Elemente bestimmte Forderungen, wenn mit einem Minimum an Kosten produziert werden soll. In Tafel 5 sind die auf Grund von umfangreichen Untersuchungen ermittelten fertigungspezifischen Forderungen an ein dünnwandiges und großflächiges Dachelement niedergeschrieben. Als Fertigungsverfahren kommt für derartig großflächige Fertigeteile nur ein Standverfahren auf Matrizen in Frage. Wegen der ungewöhnlichen Geometrie und Größe dieser Konstruktionen können andere Verfahren, wie beispielsweise die Aggregatfließfertigung, nicht angewandt werden.

Tafel 5 Fertigungsbedingte Forderungen

| Fertigungsbedingte Forderungen                                                                                                                                                       | Die Erfüllung der Forderung ermöglicht:                                                                                                       |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------|
| - Unveränderlicher Querschnitt über die ganze Länge (keine Querrippen)                                                                                                               | - Einsatz hintereinander laufender Aggregate nach dem Prinzip der Gleitfertigung zum Aufbringen, Verteilen, Verdichten und Glätten des Betons |
| - Dicke über die Breite unveränderlich oder nur wenig unterschiedlich (keine Längsrippen)                                                                                            | - Mechanisierung aller Arbeitsprozesse                                                                                                        |
| - Querneigung im Querschnitt kleiner als 40°                                                                                                                                         | - Einfache Ausbildung der Matrice                                                                                                             |
| - Keine oder nur eine geringe Längskrümmung (30 cm Stich bei 18 m Gesamtlänge)                                                                                                       | - Gleichbleibende intensive Verdichtung des Betons über die ganze Breite                                                                      |
| - Keine Auflagerverstärkungen                                                                                                                                                        | - Gleichmäßige Betonqualität im gesamten Querschnitt                                                                                          |
| - Unkomplizierte Bewehrung mit angemessener Betondeckung (Fertigungstoleranzen berücksichtigen)                                                                                      | - Verzicht auf eine Doppelschalung                                                                                                            |
| - Betongüte $\leq$ B 450                                                                                                                                                             | - Einsatz geschweißter Bewehrungsmatten                                                                                                       |
| - Variation der Elemente zur Aufnahme unterschiedlicher Nutzlasten nur durch Änderung der Anzahl und Durchmesser der Bewehrungsseile sowie einer gleichmäßigen Verstärkung der Dicke | - Mechanisierung des gesamten Bewehrungsbaues                                                                                                 |
| - Fertigungs-ausrüstung einfach und wenig störanfällig                                                                                                                               | - Gerade Spanngliedführung, wenn keine Längskrümmung vorhanden ist.                                                                           |
|                                                                                                                                                                                      | - Fertigung beliebiger Längen auf einer Matrice                                                                                               |
|                                                                                                                                                                                      | - Geringer Arbeitszeitaufwand für die Vorfertigung und den Einbau der Bewehrung                                                               |
|                                                                                                                                                                                      | - Einhaltung der Qualitätsforderungen hinsichtlich der Betondeckung                                                                           |
|                                                                                                                                                                                      | - Anwendung der Gleitfertigung mit relativ einfachen Aggregaten                                                                               |
|                                                                                                                                                                                      | - Fertigung aller Variationen auf der gleichen Matrice mit dem gleichen Fertigungsaggregat                                                    |
|                                                                                                                                                                                      | - Geringe Investkosten                                                                                                                        |
|                                                                                                                                                                                      | - Geringe laufende Kosten                                                                                                                     |

## 2.6. Transport- und Montagebedingungen

Für einen wirtschaftlichen Transport und eine zügige Montage der Dachelemente ist die Erfüllung folgender Bedingungen von grosser Bedeutung.

- Breiten- und Längenabmessungen müssen sowohl Straßen- und Eisenbahntransport ermöglichen
- Quer- und Längsschnitt müssen eine günstige Stapelung gestatten (möglichst viele Elemente übereinander)
- Die Elemente müssen einfach anzuschlagen und ohne weitere Hilfsmittel montierbar sein (ohne Traverse, ohne Hilfsrüstungen)

-Die Masse der Fertigteile muß auf die im Einsatz befindlichen Hebezeuge abgestimmt sein.

### 3. Beschreibung des Flächentragwerkes

Beim Entwurf und der Konstruktion des im Bild 1 dargestellten vorgespannten trapezförmigen Faltwerkträgers (VT-Falte) wurden alle vorgenannten Einflußfaktoren einbezogen, um ein optimales Tragwerk zu erhalten [4]. Dementsprechend wurde ein großflächiges Dachelement entwickelt, das infolge seiner Anpassungsmöglichkeiten vorteilhaft im Industrie-, Gesellschafts- und Landwirtschaftsbau eingesetzt werden kann. Durch seine variable Längengestaltung mit Stützweiten von 12,0 bis 18,0 m,

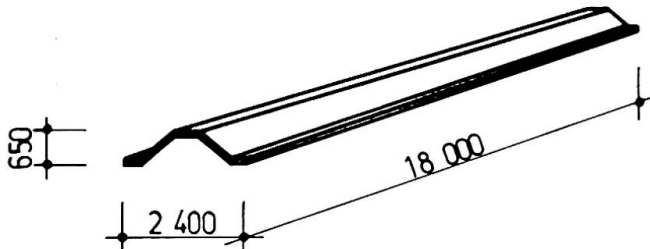


Bild 1 VT-Falte 18  
Standardelement

| Belastungsschema | Bereich | Eigenlast<br>$g$ (kp/m <sup>2</sup> ) | Flächenlast<br>$p_F$ (kp/m <sup>2</sup> ) | Randlast<br>$P_R$ (kp/m) |
|------------------|---------|---------------------------------------|-------------------------------------------|--------------------------|
|                  | A       | 155                                   | 130                                       | 0                        |
|                  |         | 155                                   | 100                                       | 20                       |
|                  | B       | 155                                   | 250                                       | 0                        |
|                  |         | 155                                   | 100                                       | 90                       |
|                  | C       | 180                                   | 320                                       | 0                        |
|                  |         | 180                                   | 100                                       | 120                      |

Bild 2 Grenzwerte der Belastung

die unterschiedliche Belastbarkeit (Flächenlasten, Randlasten und Einzellasten) und die große Tragfähigkeit bei Vermeidung jeglicher Stabilitätsgefährdung konnten die maßgebenden funktionellen Forderungen, wie Anpassung an alle rechteckigen Grundrisse, Anordnung von Oberlichtern, Anbringung von Einzellasten (bis 500 kp), Anhängen von Zwischendecken, Ausbildung von Durchbrüchen usw. allesamt erfüllt werden. Es können ein- und mehrschiffige Hallen als Kalt- und Warmdach mit Außen- oder Innenentwässerung und beliebigem rechteckigem Grundriß bis 18 m Spannweite ausgeführt werden. Gewisse Schwierigkeiten bereitet das Anbringen von Oberlichtern, da die Fuge im Tiefpunkt der Falten liegt und damit Probleme der einwandfreien Abdichtung der Oberlichte eintreten. Auch die gestalterischen Forderungen sind durch die geometrisch einfache Querschnittsgestaltung mit glatter Unterseite ohne Rippen und Verstärkungen und durch die Möglichkeit der Anordnung von Dachüberständen bis 2,5 m weitgehend befriedigt. Durch die Anwendung der beschränkten Vorspannung in Verbindung mit dem Einsatz von geschweißten Bewehrungsmatten aus St IVb wurde der Stahleinsatz auf ein Minimum reduziert und der Preis des Elementes niedrig gehalten. Die erreichten Materialkennwerte betragen für die Standardausführung

$$\text{Beton (B 450)} = 0,064 \text{ m}^3/\text{m}^2 \quad / \quad \text{Stahl} = 6,00 \text{ kg/m}^2.$$

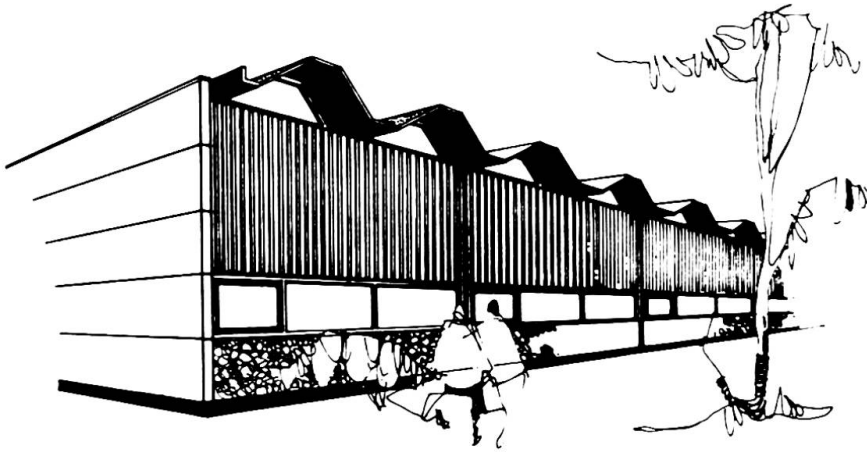


Bild 3 Perspektive eines Gebäudes mit VT-Falten

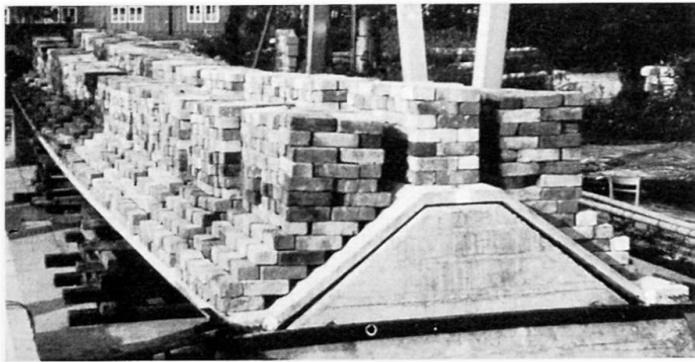


Bild 4 Testversuch (Ziegelbelastung)

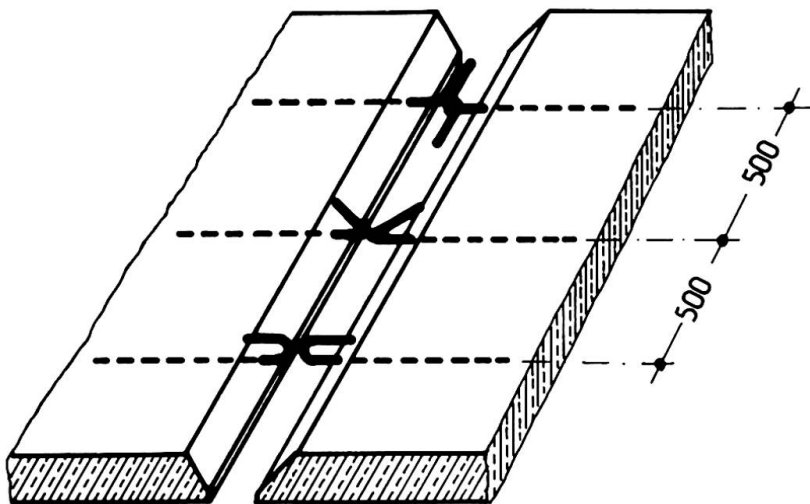


Bild 5 Fugenausbildung

Die statisch-konstruktiven Forderungen konnten vor allem durch Einsatz von Spannbeton und die Wahl des nach unten geöffneten Querschnittes erfüllt werden. Da die Längsränder der Falte unter Belastung Zugspannungen erhalten, war von vornherein ein Versagen des Bauteils infolge fehlender Stabilität des freien Randes ausgeschlossen. Daraus resultiert letztendlich die hohe Tragfähigkeit der Falte und ihre Unempfindlichkeit bei Transport und Montage. Die Vorspannung schränkte vor allem die Durchbiegung auch unter Beachtung von Kriechen und Schwinden ein und verhinderte das Auftreten von Rissen in sämtlichen Gebrauchszuständen. Zur Erfüllung des Korrosionsschutzes, vor allem des Spannstahles, ist ein dichter Beton von 30 mm vorgesehen. Die Fertigungstoleranz zur Mindestbetondeckung beträgt 5 mm. Diese Betondeckung kann im Bedarfsfall



auch auf 35 mm verstärkt werden, wobei zusätzlich noch der Vorspannungsgrad "volle Vorspannung" möglich ist. Für die schlaffe Bewehrung beträgt die theoretische Betondeckung 13 mm, wobei eine Fertigungstoleranz von 3 mm zugelassen wird. Die gewählte Querschnittsform wiederum ergibt bei geringem Materialeinsatz ein hohes Widerstandsmoment in senkrechter und waagerechter Richtung und trägt somit wesentlich zum guten Tragverhalten bei. Außerdem wirken sich Fertigungstoleranzen in der Elementendicke und der Lage der Mattenbewehrung nur geringfügig auf das Gesamttragvermögen aus, so daß auch in der Serienproduktion nahezu gleichbleibende Bruchsicherheiten die Falte charakterisieren. Die Verbindung der Falte mit der Unterkonstruktion erfolgt in einfacher Weise durch Verlegen in eine Mörtelfuge, während die Verbindung der Falten untereinander durch Ausbetonieren der Längsfuge geschieht, in die aus der Falte herausstehende Rundeisen hineinragen.

#### 4. Fertigung des Flächentragwerkes

Im Ergebnis der Optimierung vorgenannter Einflußfaktoren wurde das nachfolgend beschriebene Fertigungsverfahren einschließlich seiner Ausrüstungen entwickelt und in die Produktion eingeführt[5].

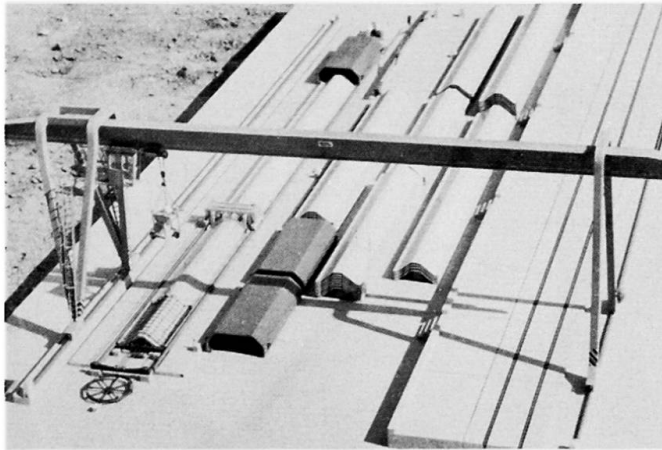


Bild 6 Modellaufnahme einer Fertigungsstätte für VT-Falten

Die Ausrüstung für die Fertigung von VT-Falten besteht aus einer Matrice mit zugehörigen Dampfhauben, einem Verdichtungs- und Glättaggregat mit Zugvorrichtung, einer Einzeldrahtspannpresse und geeigneten Hebezeugen. Die Fertigung kann sowohl in einer Halle als auch mit provisorischem Witterungsschutz im Freien erfolgen. Die Stahlbetonmatrice mit einer Gesamtlänge bis zu 80 m besteht aus hohlkastenförmigen Einzelteilen. Der Zusammenbau erfolgt mit Hilfe von Spanngliedern. Das Verdichtungs- und Glättaggregat besteht aus einem Fahrwerk mit Abstreich-, Rüttel- und Glättbohle.

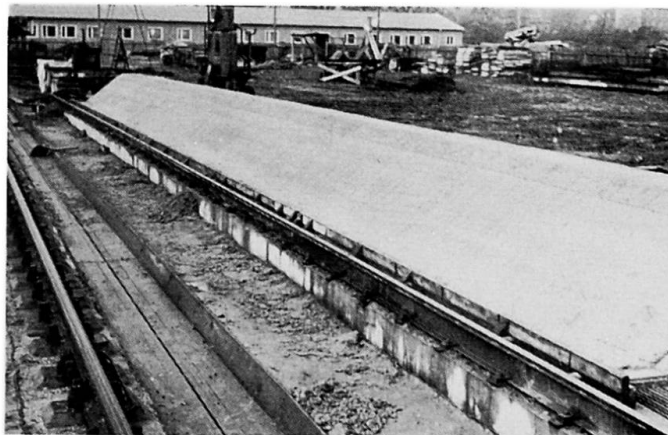


Bild 7 Stahlbeton-Matrice zur Fertigung von VT-Falten

Die eigentliche Fertigung beginnt mit der Vorbereitung der Bewehrung. Nach dem Abkanten der Bewehrungsmatten und dem Ölen der Matrice werden zunächst die im Auflagerbereich der Falten unterliegenden Matten und die Spaltzugbewehrung verlegt. Danach wird die Spannbewehrung ausgelegt und verankert. Anschließend wird die 12 cm hohe Seitenschalung befestigt und die oberen Bewehrungsmatten ausgelegt, mit Abstandhal-

tern versehen und die Spannbewehrung vorgespannt. Der Beton wird mit einem am Kran hängenden Kübel auf die Matrize aufgebracht und vorläufig noch von Hand vorverteilt. Das Fertigungsaggregat - mit einer Geschwindigkeit von 0,7 m/min über die Matrize gezogen - verteilt und verdichtet den Beton und glättet die Oberfläche. Nach der Fertigung wird das Element mit Hauben abgedeckt und mittels Dampf schnellerhärtet

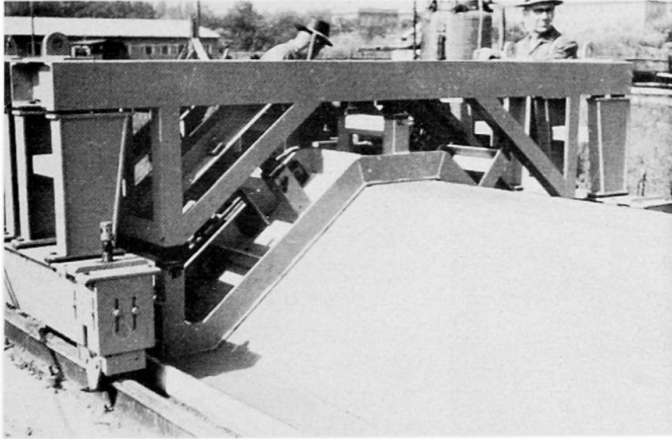


Bild 8 Aggregat zum Verteilen, Verdichten und Glätten des Betons

## 5. Durchgeführte Versuche

Zur Optimierung der Querschnittsform bei Ausschöpfung aller vorhandenen Tragreserven wurden eine Anzahl Testversuche und anschließend Tragfähigkeitsversuche des ausgewählten Elementes durchgeführt. Auf Modellversuche wurde nicht zurückgegriffen, da sie bei der Dünnwandigkeit des Elementes verbunden mit der gewünschten Genauigkeit der Versuche keine ausreichenden Ergebnisse liefern. Die Testversuche sollten in erster Linie entscheiden, ob das Element mit nach oben oder nach unten geöffneten Querschnitt ausgeführt werden sollte und ob man auf eine Längskrümmung der Falte verzichten kann. Der nach oben offene Querschnitt zeigte eine sehr große Empfindlichkeit gegenüber nicht exakter Lage der Mattenbewehrung und gab keine Garantie für eine gleichbleibende Güte des Elementes in der Serienproduktion. Außerdem lassen sich die Querbiegemomente dieses Querschnittes nur durch eine nicht erwünschte Längskrümmung auf die Werte des nach unten offenen Querschnittes ohne eine Längskrümmung reduzieren. Weiterhin liegen hierbei die freien Ränder im Druckbereich und sind dadurch stabilitätsgefährdet. Es wurde deshalb der unter 3 beschriebene nach unten offene Querschnitt ohne Längskrümmung des Elementes ausgewählt.

Durch Vorversuche wurde festgestellt, daß bei einer starren, nicht mitschwingenden Matrize (Stahlbeton oder Stahl) bei reichlicher Bewehrung in den Hochpunkten des Querschnittes, geeigneter Betonkonsistenz und Betondicken von 40 bis 70 mm ein Abrutschen des Betons bei Oberflächenverdichtung bis zu einer Neigung von  $40^\circ$  nicht auftritt und gute Betonqualitäten (B 450) und eine hohe Dichtigkeit erreicht werden können. Bei Fertigungsversuchen mit Faltenträgern zeigte sich, daß ein nach unten offener Querschnitt auch hinsichtlich seiner Fertigung günstigere Voraussetzungen bietet. Das gilt besonders für den Einbau und die Einhaltung der Lage der Bewehrung. Außerdem wird hierbei die Spannbewehrung gradlinig geführt und der Aufwand für die Ausrüstung ist wesentlich geringer als für einen Faltenträger mit nach oben offenem Querschnitt und einer Längskrümmung. Die zahl-



reichen Fertigungsverfahren wurden unter unterschiedlichen Bedingungen durchgeführt. Dabei wurde das Verfahren und die Ausrüstung ständig verbessert, so daß schließlich eine produktionsreife Anlage mit optimalen Kennwerten zur Verfügung steht.

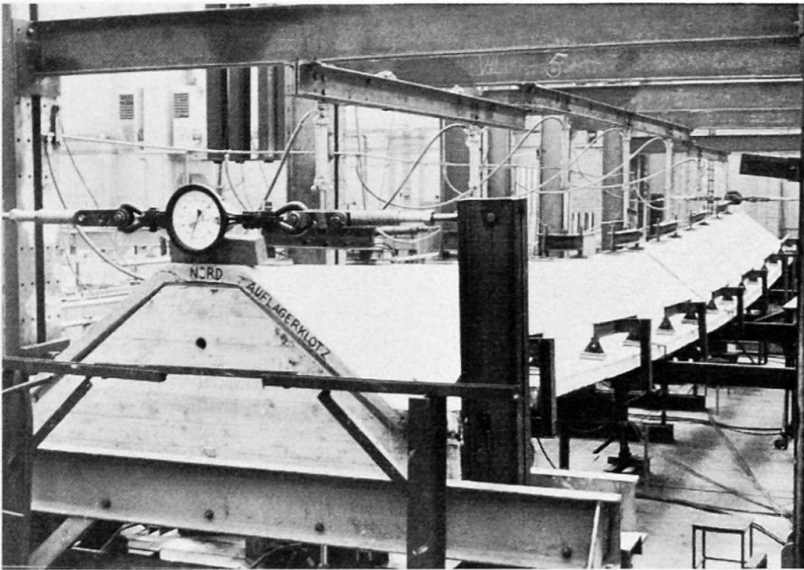


Bild 9 Tragfähigkeitsversuch  
VT-Falte 18



Bild 10 Transport einer VT-Falte

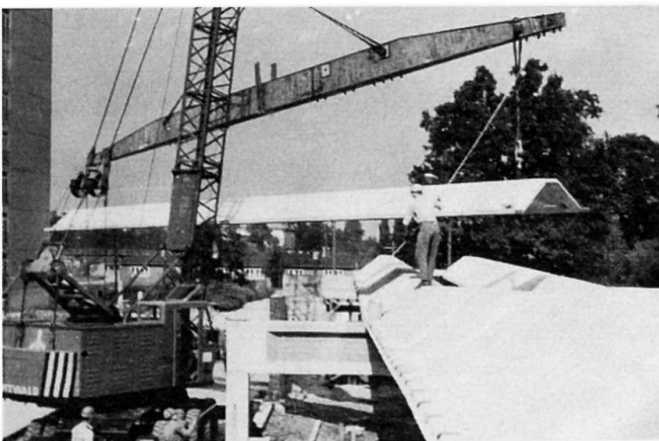


Bild 11 Montage von VT-Falten

Aus den Tragfähigkeitsversuchen konnte die exakte Tragfähigkeit des Elementes ermittelt und alle vorhandenen Reserven ausgeschöpft werden. So wurden die aufnehmbaren Flächenlasten, die aufnehmbaren Linienlasten pro Faltwerksrand und die möglichen Einzellasten ermittelt. Weiterhin wurden Größe, Form und Zahl der möglichen Aussparungen geprüft, maximale Dachüberstände ermittelt, die aufnehmbaren Torsionsmomente (einseitige Belastung) festgestellt und alle zugehörigen Durchbiegungen und Verformungen gemessen. Im Gegensatz zu den Testversuchen, bei denen die Belastung mittels Ziegelsteinen erfolgte, sind bei den Tragfähigkeitsversuchen hydraulische Lasteintragungen mit exakten Messungen durchgeführt worden. Zur Ermittlung der Spannungen im Element wurden Dehnungsmessungen des Betons, des schlaffen Stahls und des Spannstahles vorgenommen.

Bei durchgeführten Transportversuchen erwies sich die VT-Falte als sehr unempfindlich gegenüber Schwingungen und zeigte keinerlei Risse oder Verformungen, obwohl einesteils mit hohen Geschwindigkeiten (60 km/h) und andererseits auch über unwegsames Baugelände gefahren wurde. Die Montageversuche, die in Zusammenhang mit der Errich-

tung eines kleinen Experimentalbaues durchgeführt wurden, zeigten durch die Obergurtaufhängung der Falte eine sehr stabile Lage der Falte während der Montage und gestatteten ein ungleiches und ruckartiges Anheben und Absetzen sowie andere ungünstige Montagebeanspruchungen, ohne daß Beschädigungen oder Verformungen der Elemente eingetreten sind. Gleichzeitig ergab sich durch diese Unempfindlichkeit der Falte und durch die einfache Auflagerung auf Mörtelfuge eine zügige Montage. Innerhalb einer Stunde konnten 3 Falten mit insgesamt 140 m<sup>2</sup> Grundrißfläche montiert werden.

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#### ZUSAMMENFASSUNG

Nach Diskussion und systematischer Zusammenstellung aller wesentlichen Einflußfaktoren auf die Entwicklung eines großflächigen, montagefähigen, industriell zu fertigenden Dachelementes erfolgte eine "ingenieurmäßige Optimierung" durch Auswertung von Erfahrungen, Erkenntnissen und zielgerichteten Tragfähigkeits-, Fertigungs-, Transport- und Montageversuchen. Der Diskussionsbeitrag nennt die Einflußfaktoren und erläutert kurz die durchgeführten Versuche. Das Ergebnis, ein optimales Flächentragwerk und ein dazu entwickeltes Fertigungsverfahren werden beschrieben.

## SUMMARY

The "Engineering optimisation" took place after discussing and compiling all the essential factors influencing the development of an industrially producible and erectable large size roof unit through the analysis of experience, knowledge and aimed experiments to determine its ultimate strength and suitability for prefabrication, transportation and erection. The paper mentions the influencing factors and illustrates briefly the experiments carried out. The results, an optimum plate structure and a prefabrication process developed for it, are described.

## RÉSUMÉ

Après discussion et rapprochement systématique de tous les facteurs essentiels dans le développement d'un élément de toiture avec une grande surface de couverture à monter et à fabriquer industriellement, on procéda à une optimisation "dans le style de l'ingénieur", c.-à-d. en se servant des expériences, des connaissances, des essais de charge adéquats, des essais de fabrication, de transport et de montage. La contribution à la discussion indique les facteurs influents et explique en peu de mots les essais exécutés. Le résultat, une structure bi-dimensionnelle optimale et un procédé de fabrication pour la-dite construction, sont décrit.

Remarques de l'auteur du rapport introductif  
Bemerkungen des Verfassers des Einführungsberichtes  
Comments by the author of the introductory report

J. COURBON

Les très intéressantes communications que nous avons entendues au sujet du thème I c " Optimisation des Structures " me paraissent pouvoir être divisées en deux classes, l'une relative aux recherches théoriques, l'autre relative aux applications pratiques.

Il est intéressant de noter que trois des quatre rapports d'intérêt théorique sont basés sur les notions de calcul des probabilités.

M. Vasco Costa propose pour l'optimisation des structures une définition, plus générale que celle donnée dans le rapport préliminaire, tenant compte des frais de service et d'entretien et des risques d'accidents. Il en résulte qu'optimiser une structure consiste à rendre minimum la somme de son prix, de ses frais d'entretien et de l'espérance mathématique des frais entraînés par des accidents ou des modifications prévisibles des conditions de service. Cette définition me paraît tout à fait en accord avec le rapport préliminaire qui signalait que l'évaluation du prix d'une structure devait tenir compte de sa durée de vie, de son entretien, et de la possibilité de son adaptation aux modifications prévisibles du service demandé.

Pour M. Moses, l'optimisation est également basée sur la notion de probabilité de ruine. Parmi les nombreuses idées contenues dans son rapport, retenons la distinction entre les structures dont la ruine est entraînée par celle d'un seul élément (weakest link structures), et les structures hyperstatiques dont la ruine exige celle de plus d'un élément. Il en résulte que les différents éléments d'une structure optimisée ne doivent pas présenter le même facteur de sécurité, lorsqu'on les considère individuellement. Ce point me paraît essentiel pour l'économie des structures, car il permet de conserver un coefficient de sécurité d'ensemble suffisant. Par exemple dans un pont suspendu, on peut sans danger outrepasser les règlements en vigueur pour dimensionner la poutre de rigidité, tandis qu'il faut montrer une très grande prudence en ce qui concerne les suspentes.

M. Turkstra rattache l'optimisation des structures à la théorie de la décision. Cette théorie basée sur la notion des probabilités d'évènements consécutifs à une décision, permet, dans l'élaboration d'un projet d'évaluer les conséquences des nombreux choix laissés au projeteur. Cette théorie qui donne en quelque sorte à l'Ingénieur la possibilité de "mesurer son jugement", me paraît de nature à développer son imagination et à l'écarter de la routine.

M. Lorin montre que, pour un matériau élastoplastique, la distribution de matière déterminée par optimisation élastique conduit, pour les charges permanentes et variables au poids minimal de la structure. Ce résultat n'est exact que parce qu'il s'agit de systèmes de forces extérieures données il serait en défaut dans le cas de déformations imposées telles que celles pouvant résulter de tassements d'appuis, et dans le cas de matériaux tels que le béton dont la déformation n'est pas régie par des lois élastiques. L'avantage des structures hyperstatiques est de pouvoir résister économiquement, même lorsqu'on les simplifie et qu'on les éloigne ainsi de l'idéal élastique ; cette idée paraît pleine d'intérêt au point de vue de l'optimisation.

Il est naturel que les communications relatives aux applications pratiques soient très diverses, puisque le souci d'optimiser les structures constitue le but recherché par les constructeurs.

M. Faltus applique les principes de base qui gouvernent l'optimisation des structures au cas d'un très grand ouvrage métallique ; un pont en arc de 330 m de portée sur la rivière Vltava.

MM. Cohn et Grierson ont fait porter leurs recherches sur les poutres et les portiques en béton armé. En prenant pour critère d'optimisation le minimum d'armatures, ce qui dans ce cas particulier paraît très raisonnable, la question se ramène à un problème de programmation linéaire qui peut aisément être résolu à l'aide d'un ordinateur. Des exemples montrent que l'on peut ainsi obtenir des économies de l'ordre de 20 pour cent par rapport aux errements habituels.

MM. J. C. Leray et Langello nous ont montré que, dans le cas d'ouvrages routiers courants, donc nombreux, d'une part la standardisation des ouvrages, d'autre part l'exploitation de programmes de calcul entièrement automatiques étaient des facteurs essentiels d'économie, donc d'optimisation. Les résultats obtenus dans ce domaine par le Service d'Etudes Techniques des Routes et des Autoroutes constituent la démonstration tangible des idées développées par M. Leray.

Enfin MM. Schmidt et Kühn nous ont parlé d'un problème beaucoup plus particulier : préfabriquer industriellement un élément de toiture de grande surface. Il convient de noter le rôle essentiel joué par l'expérimentation (fabrication, essais de résistance, transport et montage) dans cette recherche. Il est évident que des avant-projets de l'élément, même nombreux et divers, n'auraient pas permis de conclure judicieusement.

Les communications que nous avons entendues montrent l'ampleur du problème de l'optimisation des structures. Qu'il me soit permis de remercier, au nom de l'Association Internationale des Ponts et Charpentes, les éminents professeurs et ingénieurs qui, en nous exposant les résultats de leurs recherches, nous ont montré que, sans diminuer la sécurité, il est possible de réaliser de substantielles économies. Je souhaite cependant que les idées développées aujourd'hui continuent à se développer, et finalement se traduisent pour les praticiens par des prescriptions d'application facile figurant dans les futurs Règlements.

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## CONCLUSIONS / SCHLUSSFOLGERUNGEN / CONCLUSIONS

J.R. ROBINSON

Président de la Commission de travail I

L'étude critique des critères de sécurité et de leurs fondements conceptuels a mis en lumière tout le terrain gagné par la conception probabiliste. La plupart des ingénieurs de notre domaine des Ponts et Charpentes ont pris conscience de ce que l'action de construire ne peut s'accomplir dans une sécurité absolue et admettent l'idée de risque de ruine admissible. L'opposition à cette idée s'est pourtant manifestée au Congrès en s'appuyant sur de sérieuses objections. Ces objections semblent néanmoins céder à une analyse approfondie et le support le plus rationnel de l'activité de l'ingénieur semble être justement l'idée que son résultat est aléatoire. Cette idée tout d'abord l'incite à redoubler d'attention pour éviter les fautes de conception et d'exécution. Les fautes sont des causes de ruine dont le nombre est petit -heureusement- et l'influence grande. Par cela même elles échappent au calcul des probabilités. Dans le domaine qui appartient à ce dernier les contributions présentées ont ajouté aux moyens mathématiques existants. Par exemple, il est possible, quand les conditions de linéarité nécessaires sont remplies, de considérer des processus stochastiques et non pas seulement des variables aléatoires. C'est ainsi que la sécurité d'un pont suspendu à haubans vis à vis de sa flexion latérale sous l'effet du vent a pu être appréciée par le calcul de sa réponse à la turbulence du vent.

D'autre part les essais sur modèles prennent une grande importance pour compléter les moyens mathématiques : modèles élastiques pour étudier le comportement en service ; modèles structuraux dont le matériau est le même que celui de l'ouvrage pour étudier les mécanismes de ruine ; modèles géomécaniques, qui permettent d'inclure dans l'étude l'interaction de l'ouvrage et du sol.

Il n'y a plus aujourd'hui d'opposition entre le calcul en plasticité et le calcul dans l'hypothèse de l'élasticité. Les théories de l'élasticité, de la plasticité et de la viscosité se complètent, l'une ou l'autre convenant selon les phénomènes en jeu. Le choix de la théorie adéquate est souvent affaire de bon sens et peut s'appuyer dans des cas douteux sur des considérations de probabilité. Quelques résultats très généraux ont été mis en lumière au Congrès. Tout d'abord les comportements élastique, plastique et visqueux des matériaux interviennent avec leurs dispersions propres dans un phénomène complexe. On peut, alors, prévoir que les charges de flambement de longue durée présentent une dispersion plus grande que les charges de flambement de courte durée, en raison de la dispersion particulièrement forte que manifestent les coefficients de viscosité des matériaux. Par ailleurs, l'hypothèse du fluage linéaire semble caractériser convenablement le comportement viscoélastique du béton sous contraintes de service. Elle permet de généraliser la constatation faite sur quelques cas particuliers que, parmi les ponts en béton précontraint, les ouvrages hyperstatiques présentent des déformations différées beaucoup plus faibles que les ouvrages isostatiques, avantage décisif pour les premiers.

L'optimisation des structures s'est révélée relever de deux sortes de processus bien différents : processus discontinus et processus continus. L'optimisation par processus discontinu est faite des choix de l'ingénieur, type de structure, matériaux, assemblages, procédés d'exécution, etc.. Elle est l'exercice même de son art. Le jugement de l'ingénieur est ainsi une composante essentielle de l'art de projeter. La théorie de la décision peut alors fournir la charpente logique sur laquelle ce jugement peut s'appuyer au mieux pour exercer ses choix. L'optimisation par processus continu suit la première et demande l'intervention du calcul automatique. Elle est intéressante quand des séries nombreuses d'ouvrages ou de nombreux éléments d'un même ouvrage sont en cause. Elle conduit logiquement à l'automatisation des projets. Enfin l'optimisation peut déborder la technique et déboucher sur l'économique dans les économies planifiées, en particulier.