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## IVa

### **Analysis of Bridge Structures Comprising Two Continuous Curved Main Box Girders, Whose Supports are Staggered or not, and That are Connected by Cross Beams having Flexural but not Torsional Rigidity**

Calcul des structures comprenant deux poutres caisson maîtresses continues et courbes, à supports décalés ou non, et reliées par des traverses sans rigidité torsionnelle

Berechnung von Brücken mit zwei durchlaufenden, gekrümmten, kastenförmigen Hauptträgern, deren Auflager beweglich oder fest sind, und die mit biegesteifen, jedoch drillweichen Querträgern verbunden sind

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#### Assumptions

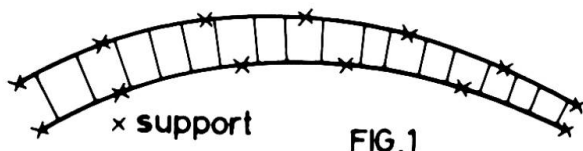
- 1) The structure behaves elastically.
- 2) The supports of the box girders are unyielding and the intermediate supports provide vertical reactions only.
- 3) The loads act on the two box girders.
- 4) The formula's for uniform torsion are valid. It is well known that the errors resulting from this assumption are small in the case of box girders.

The two main girders may or may not have the same number of spans. Their flexural rigidity  $EI$  and torsional rigidity  $GC$  may be variable. The flexural rigidity of the cross beams may be infinite or finite. If the distance  $a$  between the box girders varies, the rate of variation must be small enough for the transverse beams to be practically perpendicular to the girders. The supports of the girders may or may not coincide with the locus of the shear center of their cross sections, which we shall henceforth call the center line. The end supports may or may not allow flexural or torsional rotation of the ends of the box girders. The loads may act on or off the center line of the main girders.

#### Nodes

In figure 1 each box girder is represented by its center line.

We first consider the box girder on the outside of the curve, together with one half of each tie beam (fig. 2). Along its center line nodes are introduced :



at each support, at each junction with a connecting beam, at the point of application of every concentrated external load or moment,

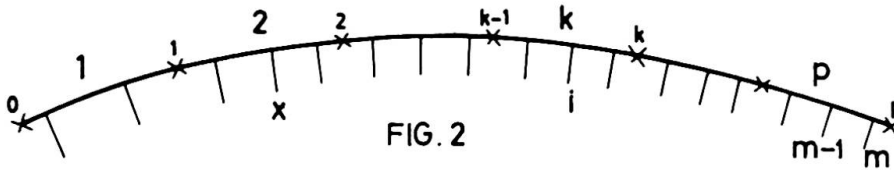


FIG. 2

at the boundaries between zones of constant curvature, or of constant flexural or torsional rigidity, or of constant distributed load,

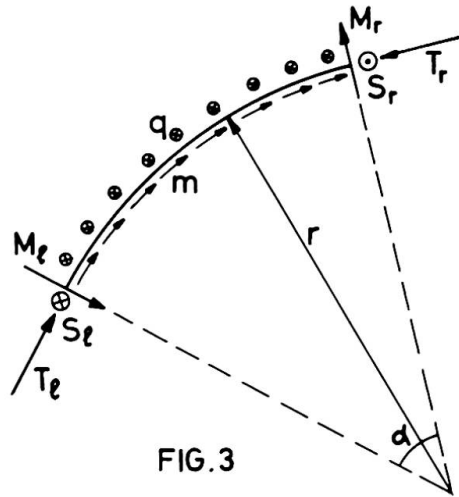


FIG. 3

and at all points of the girder where it is desired to know the stress resultants or displacement components. Hence each girder element between two successive nodes has or is assumed to have a constant radius  $r$  (fig. 3) and constant rigidities  $EI$  and  $GC = \frac{EI}{\rho}$ , and it carries or is assumed to carry a uniform downward load  $q$  along its center line and a constant moment  $m$  (per unit length) about the center line.  $m$  is taken positive when it acts in the direction of the rotation of a corkscrew that moves forward in the direction of the arrow in figure 3.

Element transmission matrix

The forces at the left end of the element considered as a free body are the shear force  $S_l$ , the bending moment  $M_l$  and the torque  $T_l$ .

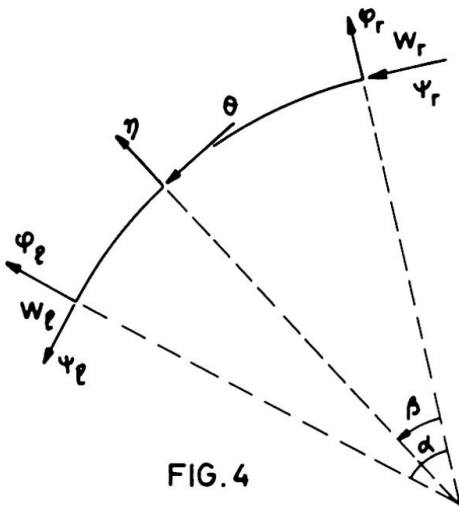


FIG. 4

Similar forces act at the right end of the element.  $S_l$  is considered positive when it acts downwards on the element,  $S_r$  when it acts upwards. The positive direction of the bending moments and torques is defined by the corkscrew rule, as is that of the rotations  $\phi$  in the vertical plane tangent to the center line and  $\psi$  in the plane perpendicular to that line, both rotations being represented in figure 4 by arrows perpendicular to the plane of rotation. The vertical (downward) displacements at the ends of the element are denoted by  $w_l$  and  $w_r$ .

Statics provides the following relationships between the internal forces at the right end and at the left

end of the element :

$$S_r = S_l + \alpha qr \tag{1}$$

$$M_r = M_l \cos \alpha - T_l \sin \alpha + S_l r \sin \alpha + r(qr - m)(1 - \cos \alpha) \tag{2}$$

$$T_r = M_l \sin \alpha + T_l \cos \alpha + S_l r(1 - \cos \alpha) - r(qr - m) \sin \alpha + \alpha qr^2 \tag{3}$$

One obtains the bending moment  $M$  and torque  $T$  in the section defined by the angle  $\beta$  by replacing  $\alpha$  by  $\alpha - \beta$  in the expressions 2 and 3. The curvature  $\eta$  and the twist per unit length  $\theta$  at the same point are given by  $\eta = \frac{M}{EI}$  and  $\theta = \frac{T}{GC} = \frac{\rho T}{EI}$

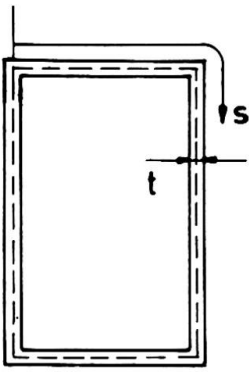


FIG. 5

with C given by Bredt's formula  $C = \frac{4A^2}{\oint \frac{ds}{t}}$  when

the box girder is monocellular (fig. 5 - A : cross-sectional area bordered by the center line of the girder walls).

Geometry, as applied to small angles and deflections, allows the following relationships between the vertical and rotational displacements at both ends of the girder element to be written :

$$\begin{aligned} \varphi_r &= \varphi_\ell \cos\alpha - \psi_\ell \sin\alpha + \int_0^\alpha \cos\beta \cdot \eta r d\beta - \int_0^\alpha \sin\beta \cdot \theta r d\beta \\ &= \varphi_\ell \cos\alpha - \psi_\ell \sin\alpha + \frac{r}{EI} \int_0^\alpha M \cos\beta d\beta - \frac{\rho r}{EI} \int_0^\alpha T \sin\beta d\beta \end{aligned} \quad (4)$$

$$\begin{aligned} \psi_r &= \varphi_\ell \sin\alpha + \psi_\ell \cos\alpha + \int_0^\alpha \sin\beta \cdot \eta r d\beta + \int_0^\alpha \cos\beta \cdot \theta r d\beta \\ &= \varphi_\ell \sin\alpha + \psi_\ell \cos\alpha + \frac{r}{EI} \int_0^\alpha M \sin\beta d\beta + \frac{\rho r}{EI} \int_0^\alpha T \cos\beta d\beta \end{aligned} \quad (5)$$

$$\begin{aligned} w_r &= w_\ell + \varphi_\ell r \sin\alpha - \psi_\ell r (1 - \cos\alpha) + \int_0^\alpha r \sin\beta \cdot \eta r d\beta - \int_0^\alpha r (1 - \cos\beta) \cdot \theta r d\beta \\ &= w_\ell + \varphi_\ell r \sin\alpha - \psi_\ell r (1 - \cos\alpha) + \frac{r^2}{EI} \int_0^\alpha M \sin\beta d\beta - \frac{\rho r^2}{EI} \int_0^\alpha T (1 - \cos\beta) d\beta \end{aligned} \quad (6)$$

Substituting the expressions of M and T into 4, 5 and 6, and performing the integrations, one finds three equations, which may be assembled with 1, 2 and 3 into the matrix equation

$$\begin{bmatrix} w_r \\ \varphi_r \\ \psi_r \\ M_r \\ T_r \\ S_r \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & r \sin\alpha & -r(1 - \cos\alpha) & b_{14} & b_{15} & b_{16} & & b_{17} \\ 0 & \cos\alpha & -\sin\alpha & b_{24} & b_{25} & b_{26} & & b_{27} \\ 0 & \sin\alpha & \cos\alpha & b_{34} & b_{35} & b_{36} & & b_{37} \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha & r \sin\alpha & & r(qr - m)(1 - \cos\alpha) \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha & r(1 - \cos\alpha) & & -r(qr - m)\sin\alpha + \alpha qr^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & & \alpha qr \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1 \end{bmatrix} \times \begin{bmatrix} w_\ell \\ \varphi_\ell \\ \psi_\ell \\ M_\ell \\ T_\ell \\ S_\ell \\ 1 \end{bmatrix}$$

or  $V_r = B V_\ell$  , with (7)

$$\begin{aligned} b_{14} &= \frac{r^2}{2EI} [(1 + \rho)\alpha \sin\alpha - 2\rho(1 - \cos\alpha)] & b_{15} &= \frac{r^2}{2EI} (1 + \rho)(\alpha \cos\alpha - \sin\alpha) \\ b_{24} &= \frac{r}{2EI} [(1 - \rho)\sin\alpha + (1 + \rho)\alpha \cos\alpha] & b_{25} &= -\frac{r}{2EI} (1 + \rho)\alpha \sin\alpha \\ b_{34} &= \frac{r}{2EI} (1 + \rho)\alpha \sin\alpha & b_{35} &= \frac{r}{2EI} [(1 + \rho)\alpha \cos\alpha - (1 - \rho)\sin\alpha] \\ b_{16} &= -\frac{r^3}{2EI} [(1 + \rho)(\alpha \cos\alpha - \sin\alpha) + 2\rho(\alpha - \sin\alpha)] \\ b_{26} &= \frac{r^2}{2EI} [(1 + \rho)\alpha \sin\alpha - 2\rho(1 - \cos\alpha)] & b_{36} &= -\frac{r^2}{2EI} (1 + \rho)(\alpha \cos\alpha - \sin\alpha) \end{aligned}$$

$$\begin{aligned}
 b_{17} &= \frac{r^3}{EI} \left[ (1+\rho)(1-\cos\alpha - \frac{\alpha}{2}\sin\alpha)(qr-m) + \rho(1-\cos\alpha - \frac{\alpha^2}{2})qr \right] \\
 b_{27} &= -\frac{r^2}{EI} \left[ \frac{1}{2}(1+\rho)(\alpha\cos\alpha - \sin\alpha)(qr-m) + \rho(\alpha - \sin\alpha)qr \right] \\
 b_{37} &= \frac{r^2}{EI} \left[ (1+\rho)(1-\cos\alpha - \frac{\alpha}{2}\sin\alpha)(qr-m) + \rho(1-\cos\alpha)m \right]
 \end{aligned}$$

One finds the column vector  $V_r$ , made up of the displacement components and stress resultants pertaining to the right end of the girder element by premultiplication of the column vector  $V_l$  pertaining to the left end by the element transmission matrix B.

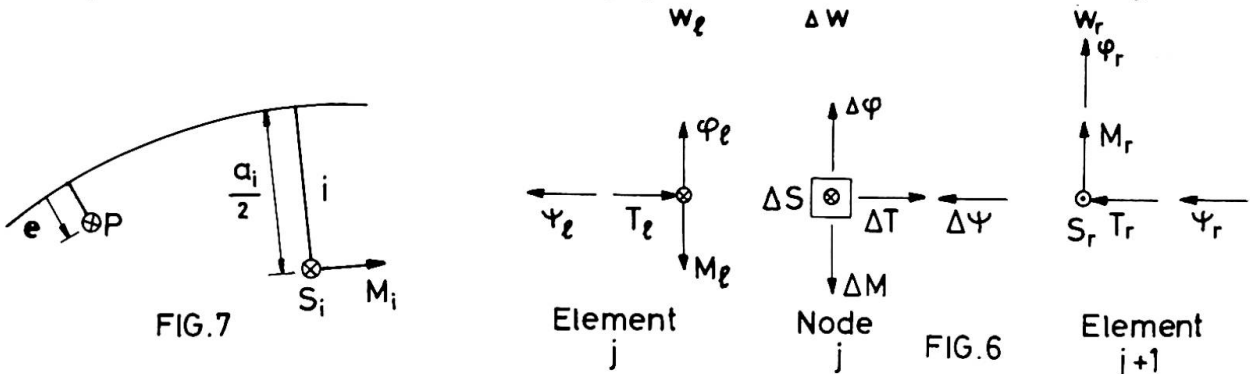
Boundary vector and boundary matrix

A boundary vector L and a boundary matrix D are associated with each node at which an external load acts on the box girder. This applies in particular to the junctions of the girder with the tie beams, but not to its supports.

Discontinuities  $\Delta w, \Delta\phi$  or  $\Delta\psi$  in the displacement components  $w, \phi$  and  $\psi$  at the node may also be included in the vector

$$L = \begin{bmatrix} \Delta w \\ \Delta\phi \\ \Delta\psi \\ \Delta M \\ \Delta T \\ \Delta S \\ 0 \end{bmatrix} \text{ and in the matrix } D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \Delta w \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta\phi \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta\psi \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta M \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta T \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta S \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

The positive directions of the  $\Delta$ -quantities are shown in figure 6.



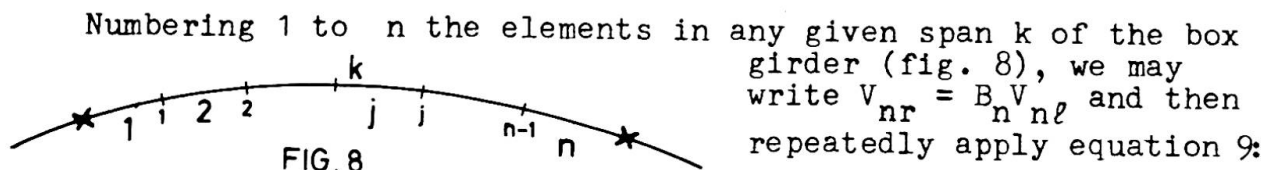
If the load consists of a downward force P acting with the eccentricity e (positive toward the center of curvature) with respect to the center line (fig. 7) :  $\Delta S = P$  and  $\Delta T = Pe$ . At the junction of the girder with the cross beam i :  $\Delta S = S_i$  and  $\Delta T = M_i + \frac{a_i}{2} S_i$  ( $S_i$  and  $M_i$  : shearing force and bending moment at mid span of the transverse beam i).

Denoting by  $L_j$  the boundary vector and by  $D_j$  the boundary matrix associated with the node j, that separates the girder elements j and j+1, and observing that the quantities on the right hand, resp. the left hand side of figure 6 are the components of the vector  $V_{j+1,l}$  relating to the left end of the element j+1, resp. of the vector  $V_{j,r}$  relating to the right end of the element j, one sees from considerations of geometry and equilibrium that

$$V_{j+1,\ell} = L_j + V_{jr}$$

Using equation 7, we obtain  $V_{j+1,\ell} = L_j + B_j V_{j\ell}$  (9)

Span transmission matrix



$$V_{nr} = B_n V_{n\ell} = B_n (L_{n-1} + B_{n-1} V_{n-1,\ell}) = B_n (L_{n-1} + B_{n-1} (L_{n-2} + B_{n-2} V_{n-2,\ell}))$$

$$= \dots = B_n (L_{n-1} + B_{n-1} (L_{n-2} + B_{n-2} (\dots (L_2 + B_2 (L_1 + B_1 V_{1\ell}))))))$$

Since  $D_j V_{j\ell} = L_j$  ( $j=1,2,\dots,n-1$ ), the above equation may be transformed into  $V_{nr} = B_n (D_{n-1} + B_{n-1} (D_{n-2} + B_{n-2} (\dots (D_2 + B_2 (D_1 + B_1)))))) V_{1\ell}$

or  $V_{nr} = U_k V_{1\ell}$  (10)

with the span transmission matrix  $U_k$  defined by

$$U_k = B_n (D_{n-1} + B_{n-1} (D_{n-2} + B_{n-2} (\dots (D_2 + B_2 (D_1 + B_1))))))$$
 (11)

Now denoting by  $V_{kr}$  and  $V_{k\ell}$  the vectors  $V$  pertaining to the right end and to the left end of the span  $k$ , equation 10 is identical with  $V_{kr} = U_k V_{k\ell}$  (12)

The product of any two matrices  $B$  has zero elements and unit elements in the same places as the matrices  $B$  themselves. Hence, the span transmission matrix is of the type

$$U_k = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} & u_{26} & u_{27} \\ 0 & u_{32} & u_{33} & u_{34} & u_{35} & u_{36} & u_{37} \\ 0 & 0 & 0 & u_{44} & u_{45} & u_{46} & u_{47} \\ 0 & 0 & 0 & u_{54} & u_{55} & u_{56} & u_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & u_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If all the loads on the box girder, including the shearing forces  $S_i$  and bending moments  $M_i$  in the tie beams, and the displacement discontinuities  $\Delta w$ ,  $\Delta \phi$  and  $\Delta \psi$ , if any, are known, equation 11 yields the numerical value of the 24 elements  $u$  of every one of the  $p$  matrices  $U_k$  ( $p$  is the number of spans of the box girder - fig.2).

Expanding equation 12, we obtain

$$w_{kr} = w_{k\ell} + u_{12} \phi_{k\ell} + u_{13} \psi_{k\ell} + u_{14} M_{k\ell} + u_{15} T_{k\ell} + u_{16} S_{k\ell} + u_{17}$$
 (14)

$$\phi_{kr} = u_{22} \phi_{k\ell} + u_{23} \psi_{k\ell} + u_{24} M_{k\ell} + u_{25} T_{k\ell} + u_{26} S_{k\ell} + u_{27}$$
 (15)

$$\psi_{kr} = u_{32} \phi_{k\ell} + u_{33} \psi_{k\ell} + u_{34} M_{k\ell} + u_{35} T_{k\ell} + u_{36} S_{k\ell} + u_{37}$$
 (16)

$$M_{kr} = u_{44} M_{k\ell} + u_{45} T_{k\ell} + u_{46} S_{k\ell} + u_{47}$$
 (17)

$$T_{kr} = u_{54} M_{k\ell} + u_{55} T_{k\ell} + u_{56} S_{k\ell} + u_{57}$$
 (18)

$$S_{kr} = S_{k\ell} + u_{67}$$
 (19)

Girder transmission matrix

1) All supports coincide with the center line of the girder

Then  $w_{kr} = w_{kl} = 0$  for all values of  $k$ . We obtain  $\varphi_{kr}$ ,  $\psi_{kr}$ ,  $M_{kr}$  and  $T_{kr}$  as functions of  $\varphi_{kl}$ ,  $\psi_{kl}$ ,  $M_{kl}$  and  $T_{kl}$  only by adding 14 successively to 15, 16, 17 and 18, after multiplying 14 with respectively  $-\frac{u_{26}}{u_{16}}$ ,  $-\frac{u_{36}}{u_{16}}$ ,  $-\frac{u_{46}}{u_{16}}$  and  $-\frac{u_{56}}{u_{16}}$ . The resulting expressions may be written as the matrix equation

$$\begin{bmatrix} \varphi_{kr} \\ \psi_{kr} \\ M_{kr} \\ T_{kr} \\ 1 \end{bmatrix} = \begin{bmatrix} u_{22} - \frac{u_{26}}{u_{16}}u_{12} & u_{23} - \frac{u_{26}}{u_{16}}u_{13} & u_{24} - \frac{u_{26}}{u_{16}}u_{14} & u_{25} - \frac{u_{26}}{u_{16}}u_{15} & u_{27} - \frac{u_{26}}{u_{16}}u_{17} \\ u_{32} - \frac{u_{36}}{u_{16}}u_{12} & u_{33} - \frac{u_{36}}{u_{16}}u_{13} & u_{34} - \frac{u_{36}}{u_{16}}u_{14} & u_{35} - \frac{u_{36}}{u_{16}}u_{15} & u_{37} - \frac{u_{36}}{u_{16}}u_{17} \\ -\frac{u_{46}}{u_{16}}u_{12} & -\frac{u_{46}}{u_{16}}u_{13} & u_{44} - \frac{u_{46}}{u_{16}}u_{14} & u_{45} - \frac{u_{46}}{u_{16}}u_{15} & u_{47} - \frac{u_{46}}{u_{16}}u_{17} \\ -\frac{u_{56}}{u_{16}}u_{12} & -\frac{u_{56}}{u_{16}}u_{13} & u_{54} - \frac{u_{56}}{u_{16}}u_{14} & u_{55} - \frac{u_{56}}{u_{16}}u_{15} & u_{57} - \frac{u_{56}}{u_{16}}u_{17} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times$$

or, denoting the two column matrices in this equation by  $W_{kr}$  and  $W_{kl}$ , and the square matrix

by  $F_k$  :  $W_{kr} = F_k W_{kl}$  (21)

$$\times \begin{bmatrix} \varphi_{kl} \\ \psi_{kl} \\ M_{kl} \\ T_{kl} \\ 1 \end{bmatrix} \quad (20)$$

Neither of the quantities  $\varphi$ ,  $\psi$ ,  $M$  and  $T$  varies suddenly at any intermediate support. Therefore  $W_{kl} = W_{k-1,r}$  ( $k=2,3,\dots,p$ ) and

$$\begin{aligned} W_{pr} &= F_p W_{pl} = F_p F_{p-1} W_{p-1,l} = F_p F_{p-1} F_{p-2} W_{p-2,l} = \dots \\ &= F_p F_{p-1} F_{p-2} \dots F_2 F_1 W_{1l} \end{aligned}$$

or  $W_{pr} = Z W_{1l}$  (22)

with the girder transmission matrix defined by

$$Z = F_p F_{p-1} F_{p-2} \dots F_2 F_1 \quad (23)$$

$Z$  is a  $5 \times 5$  matrix ; its fifth row consists of four zero elements and one unit element.

2) Some or all of the supports are located off the center line of the box girder

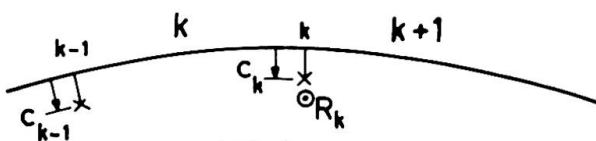


FIG. 9

The eccentricity of support  $k$ , between the spans  $k$  and  $k+1$ , towards the center of curvature of the girder is denoted by  $c_k$  (fig. 9). The downward movement of point  $k$  on the

center line is  $w_{kr} = c_k \psi_{kr} = c_k \psi_{k+1,l}$ , and that of point k-1 is  $w_{kl} = c_{k-1} \psi_{kl}$ . Equation 14 thus becomes

$$c_k \psi_{kr} = u_{12} \phi_{kl} + (c_{k-1} + u_{13}) \psi_{kl} + u_{14} M_{kl} + u_{15} T_{kl} + u_{16} S_{kl} + u_{17} \quad (24)$$

We multiply both members of 16 with  $c_k$  and subtract from 24 to obtain an equation that we solve for  $S_{kl}$ :

$$S_{kl} = - \frac{u_{12} - c_k u_{32}}{u_{16} - c_k u_{36}} \phi_{kl} - \frac{c_{k-1} + u_{13} - c_k u_{33}}{u_{16} - c_k u_{36}} \psi_{kl} - \frac{u_{14} - c_k u_{34}}{u_{16} - c_k u_{36}} M_{kl} - \frac{u_{15} - c_k u_{35}}{u_{16} - c_k u_{36}} T_{kl} - \frac{u_{17} - c_k u_{37}}{u_{16} - c_k u_{36}} \quad (25)$$

Substitution of this expression for  $S_{kl}$  in 15, 16, 17 and 18 yields a set of four equations that may be written in matrix form:

$$\begin{bmatrix} \phi_{kr} \\ \psi_{kr} \\ M_{kr} \\ T_{kr} \\ 1 \end{bmatrix} = \begin{bmatrix} h_{22} & h_{23} & h_{24} & h_{25} & h_{27} \\ h_{32} & h_{33} & h_{34} & h_{35} & h_{37} \\ h_{42} & h_{43} & h_{44} & h_{45} & h_{47} \\ h_{52} & h_{53} & h_{54} & h_{55} & h_{57} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \phi_{kl} \\ \psi_{kl} \\ M_{kl} \\ T_{kl} \\ 1 \end{bmatrix} \quad (26)$$

or 
$$W_{kr} = H_k W_{kl} \quad (27)$$

with 
$$h_{ij} = u_{ij} - \frac{u_{1j} - c_k u_{3j}}{u_{16} - c_k u_{36}} u_{16} \quad (i=2,3,4,5 ; j=2,3,4,5,7) \quad (28)$$

on the understanding that  $u_{42} = u_{43} = u_{52} = u_{53} = 0$  and that for  $j=3$   $c_{k-1} + u_{13}$  must be substituted for  $u_{13}$  in the numerator.

Obviously all the elements  $u$  appearing in the matrices  $U_k$  and  $F_k$  and in the equations 25 and 28 relate to span  $k$  of the girder, although this is not specified explicitly in the notation. The matrices  $H_k$  and  $F_k$  are identical when  $c_{k-1} = c_k = 0$ .

Equilibrium of the infinitely short portion of the girder resting on support  $k$  (fig. 10) requires  $S_{k+1,l} = S_{kr} - R_k$  and  $T_{k+1,l} = T_{kr} - c_k R_k$ . Elimination of  $R_k$  leads to

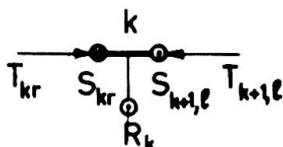


FIG.10

$T_{kr} - T_{k+1,l} = c_k (S_{kr} - S_{k+1,l})$ . For  $T_{kr}$  we substitute in this equation the expression included in equation 26. For  $S_{kr}$  we substitute, in accordance with equation 19,

$u_{67} + S_{kl}$ , with expression 25 substituted for  $S_{kl}$ . For  $S_{k+1,l}$  we substitute expression 25, written for span  $k+1$  instead of span  $k$ ; we differentiate the elements  $u$  pertaining to span  $k+1$  from those pertaining to span  $k$  by denoting the former by  $u'$ ; the quantities  $\phi_{k+1,l}$ ,  $\psi_{k+1,l}$  and  $M_{k+1,l}$  appearing in the expression for  $S_{k+1,l}$  are equal to  $\phi_{kr}$ ,  $\psi_{kr}$  and  $M_{kr}$ , and consequently they can be written as functions of  $\phi_{kl}$ ,  $\psi_{kl}$ ,  $M_{kl}$  and  $T_{kl}$  by means of equation 26.



The operations described finally yield an equation that we solve for  $T_{k+1,l}$  :  $T_{k+1,l} = \bar{h}_{52}\varphi_{kl} + \bar{h}_{53}\psi_{kl} + \bar{h}_{54}M_{kl} + \bar{h}_{55}T_{kl} + \bar{h}_{57}$  (29)

the quantities  $\bar{h}_{5j}$  being defined by

$$\left(1 + c_k \frac{u'_{15} - c_{k+1}u'_{35}}{u'_{16} - c_{k+1}u'_{36}}\right) \bar{h}_{5j} = h_{5j} + c_k \frac{u_{1j} - c_k u_{3j}}{u_{16} - c_k u_{36}} - \frac{c_k}{u'_{16} - c_{k+1}u'_{36}} \cdot \left[ h_{2j}(u'_{12} - c_{k+1}u'_{32}) - h_{3j}(c_k + u'_{13} - c_{k+1}u'_{33}) - h_{4j}(u'_{14} - c_{k+1}u'_{34}) \right] \quad (30)$$

on the understanding that  $u_{13} + c_{k-1}$  must be substituted for  $u_{13}$  and

that, for  $j=7$ ,  $-c_k \left( u_{67} + \frac{u'_{17} - c_{k+1}u'_{37}}{u'_{16} - c_{k+1}u'_{36}} \right)$  must be added to the second member of equation 30.

The first three relationships contained in 26 may be assembled with equation 29 into the matrix equation

$$\begin{bmatrix} \varphi_{k+1,l} \\ \psi_{k+1,l} \\ M_{k+1,l} \\ T_{k+1,l} \\ 1 \end{bmatrix} = \begin{bmatrix} h_{22} & h_{23} & h_{24} & h_{25} & h_{27} \\ h_{32} & h_{33} & h_{34} & h_{35} & h_{37} \\ h_{42} & h_{43} & h_{44} & h_{45} & h_{47} \\ \bar{h}_{52} & \bar{h}_{53} & \bar{h}_{54} & \bar{h}_{55} & \bar{h}_{57} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \varphi_{kl} \\ \psi_{kl} \\ M_{kl} \\ T_{kl} \\ 1 \end{bmatrix} \quad (31)$$

or 
$$W_{k+1,l} = \bar{H}_k W_{kl} \quad (32)$$

The matrices  $\bar{H}_k$  and  $F_k$  are identical when  $c_{k-1} = c_k = 0$ .

Now we successively apply equation 27 for  $k=p$  and equation 32 for  $k=p-1, p-2, \dots, 2, 1$  :

$$\begin{aligned} W_{pr} &= H_p W_{pl} = H_p \bar{H}_{p-1} W_{p-1,l} = H_p \bar{H}_{p-1} \bar{H}_{p-2} W_{p-2,l} = \dots \\ &= H_p \bar{H}_{p-1} \bar{H}_{p-2} \dots \bar{H}_2 \bar{H}_1 W_{1l} \end{aligned}$$

or 
$$W_{pr} = Z W_{1l} \quad (22)$$

with the girder transmission matrix defined by

$$Z = H_p \bar{H}_{p-1} \bar{H}_{p-2} \dots \bar{H}_2 \bar{H}_1 \quad (33)$$

### End support conditions

We expand matrix equation 22 :

$$\begin{aligned} \varphi_{pr} &= z_{11}\varphi_{1l} + z_{12}\psi_{1l} + z_{13}M_{1l} + z_{14}T_{1l} + z_{15} \\ \psi_{pr} &= z_{21}\varphi_{1l} + z_{22}\psi_{1l} + z_{23}M_{1l} + z_{24}T_{1l} + z_{25} \\ M_{pr} &= z_{31}\varphi_{1l} + z_{32}\psi_{1l} + z_{33}M_{1l} + z_{34}T_{1l} + z_{35} \\ T_{pr} &= z_{41}\varphi_{1l} + z_{42}\psi_{1l} + z_{43}M_{1l} + z_{44}T_{1l} + z_{45} \end{aligned} \quad (34)$$

### 1) Both end supports coincide with the center line of the girder

Various boundary conditions at the left end support may occur :

a) The end of the girder rotates freely about its tangent and also in the vertical plane containing the tangent :  $T_{1l} = M_{1l} = 0$

- b) The end of the girder cannot rotate in either direction :  
 $\psi_{1\ell} = \varphi_{1\ell} = 0$
- c) The end of the girder rotates freely in one of those directions, but is fixed in the other direction :  
 either  $\psi_{1\ell} = 0$  and  $M_{1\ell} = 0$   
 or  $T_{1\ell} = 0$  and  $\varphi_{1\ell} = 0$  (unlikely combination).
- One of these pairs of conditions obtains at the right end support.

Whatever the combination of end support conditions may be, two terms are zero in the right hand member of each equation 3<sup>4</sup>. Moreover, the left hand member of two equations is also zero. These may be solved for the two unknown components of the vector  $W_{1\ell}$ .

## 2) The end supports are located off the center line of the girder

In this case practically the only boundary conditions imaginable are : free rotation of the girder ends about their tangent and in the vertical plane containing the tangent. The following relationships then obtain :

$$M_{1\ell} = 0 \quad \text{and} \quad T_{1\ell} = -c_o R_o = +c_o S_{1\ell} \quad \text{at the left end support, and}$$

$$M_{pr} = 0 \quad \text{and} \quad T_{pr} = +c_p R_p = +c_p S_{pr} \quad \text{at the right end support.}$$

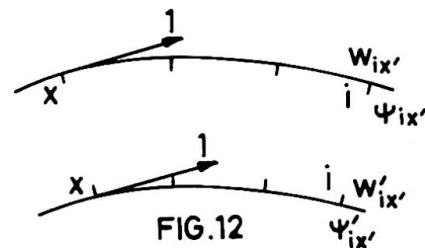
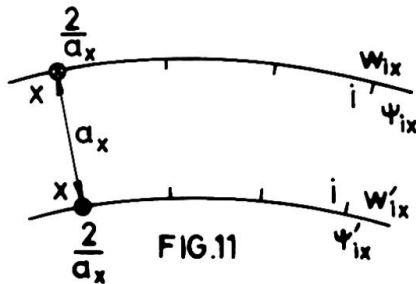
We write equation 25 for  $k=1$ , multiply by  $c_o$ , replace the left hand member  $c_o S_{1\ell}$  by  $T_{1\ell}$ , and thus obtain a relationship containing only  $\varphi_{1\ell}$ ,  $\psi_{1\ell}$  and  $T_{1\ell}$ . A second such relationship is provided by the third equation 3<sup>4</sup>, with  $M_{1\ell} = M_{pr} = 0$ . To obtain a third, we premultiply both members of equation 12, written for  $k=p$ , by  $U_p^{-1}$  :  $V_{p\ell} = U_p^{-1} V_{pr}$ , extract from this equation the expressions for  $w_{p\ell}$  and  $\psi_{p\ell}$  as functions of  $w_{pr} = c_p \psi_{pr}$ ,  $\varphi_{pr}$ ,  $\psi_{pr}$ ,  $M_{pr} = 0$ ,  $T_{pr}$  and  $S_{pr}$ , write that the former expression is equal to  $c_{p-1}$  times the latter expression, substitute  $\frac{T_{pr}}{c_p}$  for  $S_{pr}$  in the resulting equation, and so arrive at a relationship between  $\varphi_{pr}$ ,  $\psi_{pr}$  and  $T_{pr}$ , that we transform into a relationship  $\varphi_{1\ell}$ ,  $\psi_{1\ell}$  and  $T_{1\ell}$  by using the first, second and fourth equations 3<sup>4</sup>. Thus we finally have three equations that we may solve for  $\varphi_{1\ell}$ ,  $\psi_{1\ell}$  and  $T_{1\ell}$ .

## Displacement components and stress resultants

Whether the end supports coincide with the center line of the girder or not, we know vector  $W_{1\ell}$  completely after having performed the computations just described. We may now calculate  $W_{k\ell}$  for all other values of  $k$  by repeatedly using equation 32. Equation 25 then yields the numerical value of  $S_{k\ell}$  for  $k=1,2,\dots,p$ . Since  $w_{k\ell} = +c_{k-1} \psi_{k\ell}$ , we know all the vectors  $V_{k\ell}$  and are able to compute the vector  $V$  pertaining to any node in any span by means of equation 9.

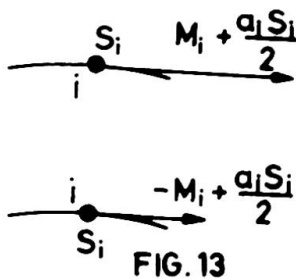
Deformation induced by direct load - Deformation influence coefficients - Actual deflections and rotations

The above analysis is used to calculate the deflections  $w$  of the outside box girder and the rotations  $\psi$  about the tangent at the nodes coinciding with the  $m$  cross beams (fig. 2) for  $2m+1$  different loading conditions. We denote by  $w_{i0}$  ( $i=1,2,\dots,m$ ) the deflection and by  $\psi_{i0}$  the rotation at the junction with cross beam  $i$ , produced by the given external loading on the box girder ;  
 by  $w_{ix}$  ( $i=1,2,\dots,m ; x=1,2,\dots,m$ ) the deflection and by  $\psi_{ix}$  the rotation produced at the same point by a downward force  $\frac{2}{a_x}$  applied at the junction with cross beam  $x$  (fig. 11) ;  
 by  $w'_{ix}$ , the deflection and by  $\psi'_{ix}$ , the rotation produced at the same point by a unit moment acting at the junction with cross beam  $x$ , as shown in figure 12.



The corresponding quantities for the inside girder are calculated likewise. They are denoted by the same symbols, marked off by an ' (figures 11 and 12).

Maxwell's reciprocal theorem shows that the calculated deflection and rotation influence coefficients must satisfy the relations  $a_x w_{ix} = a_i w_{xi}$  ;  $a_x \psi_{ix} = -2w_{xi}$  ;  $\psi_{ix} = \psi_{xi}$  .



The action of any cross beam  $i$  on the outside main girder consists of the downward force  $S_i$  and the moment  $M_i + \frac{a_i S_i}{2}$  , as shown in figure 13 ; its action on the inside girder consists of the upward force  $S_i$  and the moment  $-M_i + \frac{a_i S_i}{2}$  .

The actual deflection  $w_1$  and rotation  $\psi_1$  of the outside main girder at its junction with cross beam  $i$  under the influence of the given loading acting on the complete structural system are given by

$$w_1 = w_{i0} + \sum_{x=1}^m \left[ \frac{a_x}{2} w_{ix} S_x + w_{ix} \left( M_x + \frac{a_x S_x}{2} \right) \right]$$

$$= w_{i0} + \sum_{x=1}^m \left[ (w_{ix} + w'_{ix}) \frac{a_x S_x}{2} + w_{ix} M_x \right] \quad (35)$$

$$\begin{aligned} \psi_1 &= \psi_{10} + \sum_{x=1}^m \left[ \frac{a_x}{2} \psi_{1x} S_x + \psi_{1x} \left( M_x + \frac{a_x S_x}{2} \right) \right] \\ &= \psi_{10} + \sum_{x=1}^m \left[ (\psi_{1x} + \psi_{1x}') \frac{a_x S_x}{2} + \psi_{1x}' M_x \right] \end{aligned} \quad (36)$$

The actual deflection  $w_1'$  and rotation  $\psi_1'$  of the inside main girder are likewise

$$w_1' = w_{10}' + \sum_{x=1}^m \left[ (-w_{1x}' + w_{1x}') \frac{a_x S_x}{2} - w_{1x}' M_x \right] \quad (37)$$

$$\psi_1' = \psi_{10}' + \sum_{x=1}^m \left[ (-\psi_{1x}' + \psi_{1x}') \frac{a_x S_x}{2} - \psi_{1x}' M_x \right] \quad (38)$$

Deformation of the cross beams

The loads acting on any tie beam  $i$  and emanating from the main girders are shown in figure 14. They bring about the end deflections  $w_1$  and  $w_1'$  and end rotations  $\psi_1$  and  $\psi_1'$ .

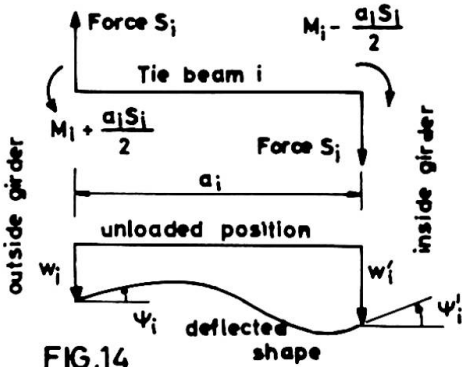


FIG.14

Assuming that the braces between the box girders are full-webbed beams and not trusses, and that the shear deformation is therefore negligible with respect to the flexural deformation, one easily derives the following relations :

$$\psi_1 - \psi_1' = \frac{a_1 M_1}{EI_1} \quad (39)$$

$$\psi_1 + \psi_1' + 2 \frac{w_1' - w_1}{a_1} = \frac{a_1^2 S_1}{6EI_1} \quad (40)$$

( $I_1$  : moment of inertia of the tie beam  $i$ ).

Similar relations between the shear force, the bending moment and the displacement components may be derived for trussed connections between the main girders.

Obtaining the values of the unknowns  $S_x$  and  $M_x$

Substituting the expressions 35 to 38 for  $w_1$ ,  $\psi_1$ ,  $w_1'$  and  $\psi_1'$  into 39 and 40 we find

$$\sum_{x=1}^m \left[ (\psi_{1x} + \psi_{1x}' + \psi_{1x}' - \psi_{1x}') \frac{a_x S_x}{2} + (\psi_{1x}' + \psi_{1x}') M_x \right] - \frac{a_1 M_1}{EI_1} = \psi_{10}' - \psi_{10} \quad (41)$$

( $i=1,2,\dots,m$ )

$$\begin{aligned} \sum_{x=1}^m \left\{ \left[ \psi_{1x} + \psi_{1x}' - \psi_{1x}' + \psi_{1x}', - \frac{2}{a_1} (w_{1x}' + w_{1x}' + w_{1x}' - w_{1x}') \right] \frac{a_x S_x}{2} \right. \\ \left. + \left[ \psi_{1x}', -\psi_{1x}', - \frac{2}{a_1} (w_{1x}' + w_{1x}') \right] M_x \right\} \\ - \frac{a_1}{3EI_1} \cdot \frac{a_1 S_1}{2} = -\psi_{10} - \psi_{10}' + \frac{2}{a_1} (w_{10} - w_{10}') \quad (i=1,2,\dots,m) \end{aligned} \quad (42)$$

This set of  $2m$  simultaneous algebraic equations may be solved

for the  $2m$  unknowns  $\frac{a_x S_x}{2}$  and  $M_x$ . The matrix to be inverted is of the order  $2m$ , while the structure is  $2m+p+p'+4$  times statically indeterminate ( $p'$ : number of spans of the inside girder), if the ends of the main girders are free to rotate.

When the shearing forces and bending moments in the bracing are known, it is easy to determine the final internal stress resultants and displacement components for the box girders, either by superposing for each one  $2m+4$  cases of loading already analysed, or by analysing each box girder separately under its full loading, including the forces and moments emanating from the bracing.

The complexity of the structural system considered is such that the calculation is hardly feasible without a computer.

#### Remark about the location of the connecting beams

It was assumed implicitly in the above that the braces do not meet the box girders at the supports. Yet, it is almost natural for some tie beams to be connected with the main girders at supported cross sections. This situation can be handled in the analysis by assuming that the junction of any such cross beam is located beside, but close to the support, leaving an infinitely short girder element between itself and the support.

#### SUMMARY

The structural system described in the title is analysed by applying the transmission (or reduction) method to the curved main girders separately, thus obtaining deformation components and influence coefficients, and by using the force method to find the shear forces and bending moments in the connecting beams.

#### RÉSUMÉ

Les deux poutres maitresses courbes sont d'abord étudiées séparément par la méthode de transmission ou de réduction, ce qui fournit les composantes de la déformation de ces poutres, ainsi que les coefficients d'influence de ces composantes. Ensuite les efforts tranchants et les moments fléchissants dans les traverses sont déterminés au moyen de la méthode des forces.

#### ZUSAMMENFASSUNG

Die zwei gekrümmten Kastenträger werden zunächst gesondert mit dem Übertragungsverfahren studiert. Diese Berechnung liefert die Verformungskomponenten, sowie Einflusszahlen für diese Komponenten. Nachher werden die Querkraft und das Biegemoment in den Querbalken mit dem Kraftverfahren ermittelt.