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Autor: Rawlings, B.

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A Computer Analysis of Structures under Impulsive Loading

Etude à la calculatrice électronique d'ouvrages soumis à des charges dynamiques

Untersuchung von Bauwerken unter stoßartiger Belastung mittels Rechengeräten

B. RAWLINGS
Department of Civil Engineering, University of Sydney

Introduction

The mathematical formulation of structural engineering problems for solution by electronic computer has occupied the attention of a large group of people over the past ten years. If attention is confined to framed structures, a convenient division may be made into elastic and plastic analyses of frames, either under static or dynamic loading conditions. Without endeavouring to give a comprehensive summary, mention should be made of the elastic, static procedures using either stiffness or flexibility approaches [1, 2, 3] which are based on the linear properties of structural elements, and the extension of the matrix stiffness approach by Livesley by piecewise-linear steps, to the analysis of structures containing plastic hinges, so that the behaviour at collapse follows as a limiting stage. Direct collapse analyses, based upon the assumptions of the rigid-plastic theory have also been programmed for a few specific classes of structure [4].

Turning to the behaviour of structures under dynamic loads, the matrix formulation of elastic response is well established [5] and will not be further discussed. In many cases however, it is necessary to examine the behaviour of a steel structure subjected to impulsive loading sufficiently large to cause severe permanent deformation. The simple mass-spring concept of a multistorey building which has found acceptance to date represents a first approximation; however the actual behaviour of individual members cannot be ascertained as the structure itself is not analysed in the process.

It is possible to achieve this end by making use of the rigid-plastic theory, in which it is assumed that no deformation occurs in any member until the (dynamic) full plastic moment of resistance of the section is exceeded. In this way the problem of solving the equations of elastic vibration of the various components of the structure is eliminated, and replaced by a relatively simple problem of rigid body mechanics. The limitations of validity of such an approach are discussed elsewhere [6] and will not be mentioned here.

In the present paper a method of analysis is presented which allows a rigid framed structure to be examined in terms of its deformation under any system of time-dependent applied forces. By formulating the equations in matrix form, the method becomes convenient for application to digital computation, as only standard matrix manipulations are involved.

The Behaviour of a Framed Structure Under Impulsive Loading

When a structure is deforming in accelerated motion under the action of impulsive loads it must satisfy simultaneously:

a) The requirements of dynamics, namely that each and every element of the structure is in equilibrium with the applied loads, internal reactions and inertia forces accompanying the motion.
b) The requirements of kinematics, namely that the displacements and their time derivatives are compatible with the assumed mode, and
c) The yield condition for the material, which requires that nowhere within the structure does there exist a condition of stress incompatible with the material strength. This third condition is made complex in the case of steel subject to dynamic loading by the fact that the yield strength is a function of strain-rate. In the analysis presented here, it will be assumed that the value of yield stress selected is constant over the range of strain-rates considered, an assumption which is normally quite close in practice.

In this analysis it is necessary first to formulate all of the dynamic equilibrium equations for the structure, and to express these in matrix form. These equations may be derived in two alternative and quite different ways; (1) by synthesising the structure from its component members, writing down the equations of motion of each, and making use of the conditions of equilibrium and displacement compatibility at the joints, or (2) by treating the structure as a whole, displacing it in each of its degrees of freedom and deriving the result by using Lagrange's Equations of Motion.

For convenience the second approach will be taken, and illustrated later in the paper. Consider a rigid framed structure acted upon by r applied time-dependent forces $|\boldsymbol{F}|$. It will be assumed that there are a number of possible collapse modes, and that there are p positions of peak moment where plastic hinges may develop. The number of degrees of freedom of the structure, when hinges have developed at all p points, is s.

The structure may now be treated as an assemblage of rigid links, joined at those sections where moment peaks exist, and arbitrary moments may be regarded as being externally applied to these joints.

The kinetic energy of the structure may then be written as

$$T = \sum_{s} \tfrac{1}{2} M \dot{q}_s^2 \tag{1}$$

and the work done by the moments and forces in a virtual displacement as

$$-\Delta V = -\sum_{p} m \Delta \theta - \sum_{r} F \Delta z , \tag{2}$$

where the m's are the moments applied at the peak moment positions and the displacements Δz and the rotations $\Delta \theta$ are linear functions of the co-ordinates q. Applying Lagrange's Equations,

$$\frac{\partial V}{\partial q_s} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) = 0,$$

s independent equations are derived, each being a linear relationship between the applied forces, peak moments and accelerations in each degree of freedom. These equations may be expressed in matrix form as

$$|\boldsymbol{F}| = |\boldsymbol{T}|\begin{vmatrix} m \\ \hdashline \ddot{\boldsymbol{z}} \end{vmatrix}, \tag{3}$$

where $|\boldsymbol{F}|$ is the vector of applied forces, $|\boldsymbol{T}|$ the matrix of terms associated with the configuration and inertia of the frame, and $\begin{vmatrix} \boldsymbol{m} \\ \hdashline \ddot{\boldsymbol{z}} \end{vmatrix}$ the vector of peak moments and accelerations.

Behaviour at Collapse

It will be observed that at this stage no account has been taken of the kinematic or yield conditions associated with any particular mode of collapse or size of member and, in order to obtain a solution in a particular case, this information must be incorporated in the analysis. If the applied loads are small and the various members of the frame are substantial enough to withstand these rigidly, no deformation will ensue so that $|\ddot{\boldsymbol{z}}| = 0$. The problem thus reduces to one in statics, and the loads to cause static collapse may be computed using the simple plastic theory. If, however the applied loads exceed those associated with the strength of the frame, collapse will occur in one or more modes. With each mode of deformation there is an associated kinematic condition which, in general simplifies the vector $|\ddot{\boldsymbol{z}}|$; also there is a yield condition for any given frame, wherein the magnitudes and signs of the plastic hinge moments are defined and these may be substituted into the vector $|\boldsymbol{m}|$. The remaining moments, where hinges have not formed are still undefined, but must, in fact be numerically less than the corresponding full plastic value, in order not to violate the yield condition. Thus for each mode of deformation the vector $\begin{vmatrix} \boldsymbol{m} \\ \hdashline \ddot{\boldsymbol{z}} \end{vmatrix}$ may be simplified to the form

$$\begin{vmatrix} \boldsymbol{m} \\ \hdashline \ddot{\boldsymbol{z}} \end{vmatrix} = |\boldsymbol{D}|\begin{vmatrix} m_p \\ \boldsymbol{m}_1 \\ \hdashline \ddot{\boldsymbol{z}}_1 \end{vmatrix}, \tag{4}$$

where $|\boldsymbol{D}|$ is a matrix which depends only upon the governing requirements of the mode, m_p is the dynamic full plastic moment of those hinges that have formed,

$|\boldsymbol{m}_1|$ is the vector of peak moments where hinges have not developed, and $|\ddot{\boldsymbol{z}}_1|$ is the vector of the independent accelerations associated with the mode.

Hence
$$|\boldsymbol{F}| = |\boldsymbol{T}|\,|\boldsymbol{D}| \begin{vmatrix} m_p \\ \boldsymbol{m}_1 \\ \hdashline \ddot{\boldsymbol{z}}_1 \end{vmatrix}, \tag{5}$$
$$= |\boldsymbol{W}| \begin{vmatrix} m_p \\ \boldsymbol{m}_1 \\ \hdashline \ddot{\boldsymbol{z}}_1 \end{vmatrix} \tag{6}$$
where
$$|\boldsymbol{T}|\ |\boldsymbol{D}| = |\boldsymbol{W}|. \tag{7}$$

The matrix must now be partitioned in the form
$$|\boldsymbol{W}| = |\boldsymbol{W}_1 \vdots \boldsymbol{W}_2| \tag{8}$$
in order to remove the column which relates to the term m_p.

Hence
$$|\boldsymbol{F}| = |\boldsymbol{W}_1|\, m_p + |\boldsymbol{W}_2| \begin{vmatrix} \boldsymbol{m}_1 \\ \hdashline \ddot{\boldsymbol{z}}_1 \end{vmatrix} \tag{9}$$
and
$$|\boldsymbol{F}| - |\boldsymbol{W}_1|\, m_p = |\boldsymbol{W}_2| \begin{vmatrix} \boldsymbol{m}_1 \\ \hdashline \ddot{\boldsymbol{z}}_1 \end{vmatrix}. \tag{10}$$

The matrix $|\boldsymbol{W}_2|$ will be square and non-singular in the case of a complete collapse mode so that the inverse matrix $|\boldsymbol{W}_2|^{-1}$ may be derived. Consequently
$$\begin{vmatrix} \boldsymbol{m}_1 \\ \hdashline \ddot{\boldsymbol{z}}_1 \end{vmatrix} = |\boldsymbol{W}_2|^{-1}\,[|\boldsymbol{F}| - |\boldsymbol{W}_1|\, m_p]. \tag{11}$$

This gives the equations governing the response in the particular mode and the values of the peak moments where hinges have not developed. Consequently it is possible, knowing $|\boldsymbol{F}|$, to determine any instant of time the acceleration of the frame (and thus the velocity and displacement by numerical integration) and the other peak moments, and to observe whether these violate the yield condition. Any violation of this would then necessitate subsequent analysis of behaviour in a new mode, with appropriate initial conditions for displacement, velocity and acceleration. If motion is occurring in a mode with more than one degree of freedom, kinematic bounds of validity must also not be violated. These, in general are governed by limiting conditions imposed upon the independent velocities, which must be tested.

Illustrative Example

For the purpose of illustrating the method a very simple example has been selected as shown in Fig. 1, the rigid bent $A\,B\,C\,D\,E\,F$ being subjected to applied forces H_B, H_C at B and C and to V_E at E. Masses M_B, M_C and M_E are attached at these points, the rest of the structure being assumed to have no inertia.

Taking as the generalised co-ordinates z_B, z_C and u_E as shown in Fig. 2 the kinetic energy

$$T = \tfrac{1}{2} M_B \dot{z}_B^2 + \tfrac{1}{2} M_C \dot{z}_C^2 + \tfrac{1}{2} M_E \dot{u}_E^2.$$

The work done in a virtual displacement (Fig. 3) is given by $-\Delta V$, where

$$\Delta V = H_B \Delta z_B + H_C \Delta z_C + V_E \Delta u_E + m_A \left(\frac{3 \Delta z_B}{h}\right) - m_B \left(\frac{6 \Delta z_B - 3 \Delta z_C}{h}\right)$$
$$- m_C \left(\frac{6 \Delta z_C - 3 \Delta z_B}{h}\right) + m_D \left(\frac{3 \Delta z_C}{h} + \frac{2 \Delta u_E}{l}\right) - m_E \left(\frac{4 \Delta u_E}{l}\right)$$
$$+ m_F \left(\frac{2 \Delta u_E}{l}\right),$$

where the m's are taken as positive if they develop tension on the inside fibres

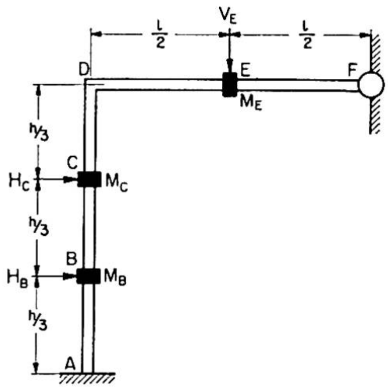

Fig. 1.

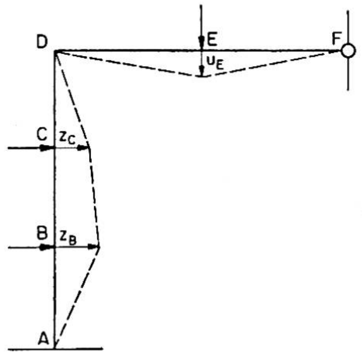

Fig. 2.

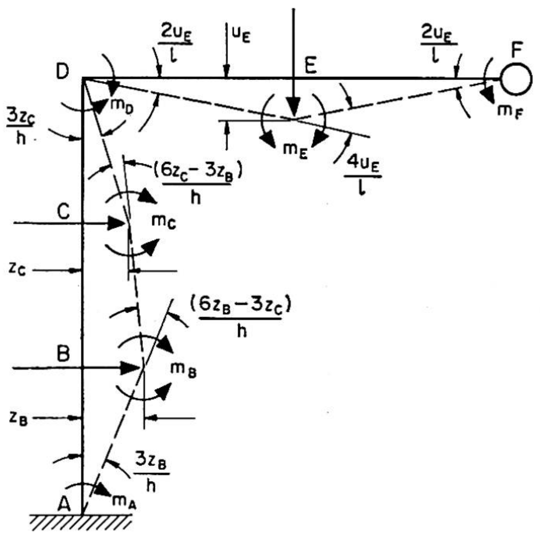

Fig. 3.

of the frame members. Applying Lagrange's Equations, by differentiating the above expressions with respect to z_B, z_C and u_E and their derivatives,

$$H_B + \frac{3}{h}\,(+m_A - 2\,m_B + m_C) - M_B\,\ddot{z}_B = 0,$$

$$H_C + \frac{3}{h}\,(+m_B - 2\,m_C + m_D) - M_C\,\ddot{z}_C = 0,$$

$$V_E + \frac{2}{l}\,(+m_D - 2\,m_E + m_F) - M_E\,\ddot{u}_E = 0.$$

In matrix notation as in Eq. (3) the expressions above become

$$\begin{vmatrix} H_B \\ H_C \\ V_E \end{vmatrix} = \left|\begin{array}{cccccc:ccc} -3/h & +6/h & -3/h & \cdot & \cdot & \cdot & M_B & \cdot & \cdot \\ \cdot & -3/h & +6/h & -3/h & \cdot & \cdot & \cdot & M_C & \cdot \\ \cdot & \cdot & \cdot & -2/l & +4/l & -2/l & \cdot & \cdot & M_E \end{array}\right| \left|\begin{array}{c} m_A \\ m_B \\ m_C \\ m_D \\ m_E \\ m_F \\ \hdashline \ddot{z}_B \\ \ddot{z}_C \\ \ddot{u}_E \end{array}\right|.$$

Consider now the case if the structure collapses in the mode shown in Fig. 4, and it is assumed that the members are of uniform section having a dynamic full plastic moment of resistance of m_p.

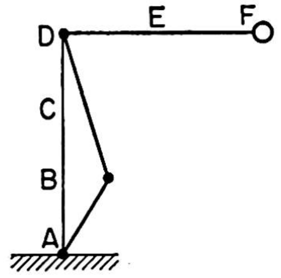

Fig. 4.

In this case the vector

$$|\boldsymbol{m}| = \begin{vmatrix} m_A \\ m_B \\ m_C \\ m_D \\ m_E \\ m_F \end{vmatrix} = \begin{vmatrix} -m_p \\ +m_p \\ m_C \\ -m_p \\ m_E \\ 0 \end{vmatrix}, \tag{12}$$

$$|\ddot{\boldsymbol{z}}| = \begin{vmatrix} \ddot{z}_B \\ \ddot{z}_C \\ \ddot{z}_D \end{vmatrix} = \begin{vmatrix} \ddot{z}_B \\ \frac{1}{2}\,\ddot{z}_B \\ 0 \end{vmatrix}, \tag{13}$$

so that

$$|\boldsymbol{D}| = \left|\begin{array}{cccc} -1 & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & +\frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot \end{array}\right|$$

and Eq. (5) becomes

$$\left|\begin{array}{c} H_B \\ H_C \\ V_E \end{array}\right| = \left|\begin{array}{cccccc|ccc} -3/h & +6/h & -3/h & \cdot & \cdot & \cdot & M_B & \cdot & \cdot \\ \cdot & -3/h & +6/h & -3/h & \cdot & \cdot & \cdot & M_C & \cdot \\ \cdot & \cdot & \cdot & -2/l & +4/l & -2/l & \cdot & \cdot & M_E \end{array}\right| \cdot$$

$$\cdot \left|\begin{array}{cccc} -1 & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & +\frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot \end{array}\right| \left|\begin{array}{c} m_p \\ m_C \\ m_E \\ \hline \ddot{z}_B \end{array}\right|$$

i.e.

$$\left|\begin{array}{c} H_B \\ H_C \\ V_E \end{array}\right| = \left|\begin{array}{ccc|c} +9/h & -3/h & \cdot & M_B \\ \cdot & +6/h & \cdot & \frac{1}{2} M_C \\ 2/l & \cdot & +4/l & \cdot \end{array}\right| \left|\begin{array}{c} m_p \\ m_C \\ m_E \\ \hline \ddot{z}_B \end{array}\right|.$$

Thus

$$\left|\begin{array}{c} H_B \\ H_C \\ V_E \end{array}\right| - \left|\begin{array}{c} 9/h \\ \cdot \\ 2/l \end{array}\right| m_p = \left|\begin{array}{cc|c} -3/h & \cdot & M_B \\ +6/h & \cdot & \frac{1}{2} M_C \\ \cdot & +4/l & \cdot \end{array}\right| \left|\begin{array}{c} m_C \\ m_E \\ \hline \ddot{z}_B \end{array}\right|$$

giving, from (11),

$$\left|\begin{array}{c} m_C \\ m_E \\ \hline \ddot{z}_B \end{array}\right| = \frac{l\,h}{6\,(M_C+4\,M_B)} \left|\begin{array}{ccc} (-2\,M_C/l) & (4\,M_B/l) & \cdot \\ \cdot & \cdot & (3/2\,h)\,(M_C+4\,M_B) \\ (24/l\,h) & (12/l\,h) & \cdot \end{array}\right| \cdot$$

$$\cdot \left[\left|\begin{array}{c} H_B \\ H_C \\ V_E \end{array}\right| - \left|\begin{array}{c} 9/h \\ \cdot \\ 2/l \end{array}\right| m_p \right].$$

Hence $\dot{z}_B$ and z_B may be determined at any stage of the motion, and m_C and m_E observed and tested to ensure that the applied forces H_B, H_C and V_E do not induce changes of mode during the deformation process.

As a second illustration, consider the possibility of deformation in the mode shown in Fig. 5. In this case there are two degrees of freedom, specified by the

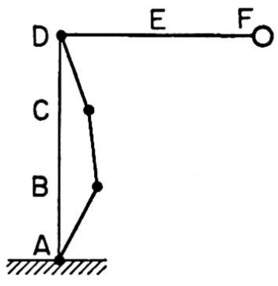

Fig. 5.

co-ordinates z_B and z_C; $m_A = -m_p$, $m_B = m_p$, $m_C = m_p$, $m_D = -m_p$, $m_F = 0$; $\ddot{u}_E = 0$. The mode equation for this case is

$$\begin{vmatrix} m_A \\ m_B \\ m_C \\ m_D \\ m_E \\ m_F \\ \hdashline \ddot{z}_B \\ \ddot{z}_C \\ \ddot{u}_E \end{vmatrix} = \left|\begin{array}{cc:cc} -1 & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hdashline \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{array}\right| \begin{vmatrix} m_p \\ m_E \\ \hdashline \ddot{z}_B \\ \ddot{z}_C \end{vmatrix}$$

and the final equation, after partitioning and inversion of the matrix $|\boldsymbol{W}_2|$ becomes

$$\begin{vmatrix} m_E \\ \hdashline \ddot{z}_B \\ \ddot{z}_C \end{vmatrix} = \begin{vmatrix} \cdot & \cdot & l/4 \\ 1/M_B & \cdot & \cdot \\ \cdot & 1/M_C & \cdot \end{vmatrix} \left[\begin{vmatrix} H_B \\ H_C \\ V_E \end{vmatrix} - \begin{vmatrix} 6/h \\ 6/h \\ 2/l \end{vmatrix} m_p \right].$$

Again, $\dot{z}_B$ and $\dot{z}_C$ may be found at any stage of the motion, and for the continued existence of the mode;

1. m_E must be numerically less than m_p,
2. $\dot{z}_B > \dot{z}_C/2$,
3. $\dot{z}_C > \dot{z}_B/2$.

Discussion

Although the principles have been illustrated only by a very simple example, in which the equations of motion may be derived readily without recourse to matrix techniques, the method may be applied to frames which are considerably more complex. Furthermore, the computation requires only the normal procedures of matrix manipulation which form a standard adjunct of many computer programmes. As mentioned earlier the formulation of the equilibrium equations may also be achieved by synthesising the structure from its component members, and this procedure may be completely programmed. However this necessitates a number of matrix manipulations and in many cases the manual analysis using Lagrange's Equations proves just as quick and convenient.

The analysis may be extended to cover the case of a rigid-linear strain-hardening material having the characteristics shown in Fig. 6, provided

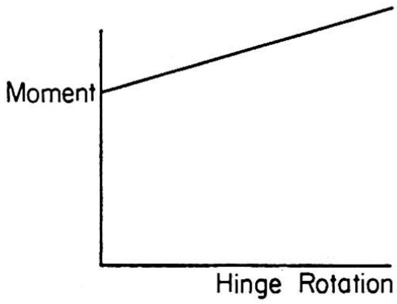

Fig. 6.

geometry changes under deformation are small. In this case, for each term in $\ddot{z}$ there will be an additional term in z, and the final equation will take the form

$$\begin{vmatrix} \boldsymbol{m}_1 \\ \ddot{\boldsymbol{z}}_1 \end{vmatrix} = |\boldsymbol{W}_2|^{-1}[|\boldsymbol{F}| - |\boldsymbol{W}_1| m_p - |\boldsymbol{K}| \, |\boldsymbol{z}_1|],$$

where $|\boldsymbol{K}|$ is a matrix of strain-hardening terms.

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Summary

The equations of motion are derived, governing the dynamic behaviour of a rigid-frame steel structure subjected to time-dependent loading of sufficient intensity to cause permanent deformation. The material is assumed to have rigid-plastic characteristics; consequently all elastic response is ignored. The analysis is formulated in matrix notation in a way enabling a problem to be examined by means of standard digital computer routines.

Résumé

L'auteur établit les équations de mouvement qui régissent le comportement dynamique d'un portique métallique soumis à des charges variables dans le temps et d'intensité suffisante pour provoquer des déformations permanentes. Se plaçant dans les conditions de la théorie rigide-plastique on omet les réactions élastiques. La notation matricielle du calcul permet l'emploi d'un calculateur numéral selon les méthodes courantes.

Zusammenfassung

Es werden die Bewegungsgleichungen für das dynamische Verhalten eines Stahlrahmens unter zeitabhängiger Belastung einer Intensität abgeleitet, die eine dauernde Verformung ergibt. Das Material soll starr-plastisch sein und deshalb wird keinerlei elastisches Verhalten berücksichtigt. Die Analyse erfolgt in Matrixschreibweise in der Art, daß ein Problem sich mittels normaler Digitalrechenverfahren untersuchen läßt.