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Energy Methods for the Analysis of Temperature Distributions and Thermal Stresses in Structures

Méthodes énergétiques pour l'étude de la répartition de la température et des contraintes à la suite des variations de température dans les ouvrages

Energie-Methoden zur Untersuchung von Temperaturverteilungen und Spannungen infolge Temperaturänderungen in Baukonstruktionen

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Introduction

In recent years structural analysts were confronted more often than before with the problem of predicting the effects of heat inputs on structures. This trend was brought about by such developments as atomic reactor technology, high speed aircraft and missile technology, and also by an increased use of welded connection in structures.

One of the most important thermal effects in structures consists in the occurence of thermal stresses, produced by heat inputs or temperature changes. Thus, the structural engineer is required to determine such thermal stresses, basing himself on the temperature distribution in the structure, which, in turn, has to be determined first from the given boundary conditions of the thermal problem. This latter task, as a rule, will be a rather unfamiliar one for the structural engineer, since the temperature distribution, which is governed by the empirical heat conduction equation, is of a type (diffusion) which does not lend itself to treatment by commonly employed methods, in particular energy methods, which have proved to be powerful tools in structural analysis.

This rather unpleasant feature of thermal stress analysis was removed recently by BIOT [1, 2, 3], who showed that by a suitable definition of two quantities, namely the thermo-elastic potential and the dissipation function, a variational formulation of either the coupled or separate problems of thermoelasticity and heat conduction becomes possible. A principle, complementary to BIOT's formulation, was established by the present writer [4].

The purpose of the present contribution is to show that a complete extention of the energy principles, available in isothermal structural analysis, to the case of thermo-elastic and temperature distribution problems is possible.

The existence of such an analogy, for the sake of brevity, will be demonstrated here with the example of a uniaxial state of stress and for the simplest possible loading, boundary conditions and material properties.

The three energy principles discussed in the sequel represent extentions of GREEN's principle for displacements (which yields equilibrium equations), CASTIGLIANO's principle for stresses (which yields, in the formulation used here, HOOKE's law) and REISSNER's generalized principle for displacements and stresses [5]. By way of introduction, these principles are restated first for the case of isothermal elasticity. The formulation and terminology are borrowed from a recent summary by REISSNER [6].

On Energy Theorems in Isothermal Elastic Structural Analysis

In isothermal elastic structural analysis, as exemplified by the onedimensional problem of an elastic bar of length l in compression (or extention), we consider first the following two classical energy theorems.

The principle of minimum strain energy states, that the equilibrium equations are obtained by setting the variation of the strain energy V, expressed in terms of displacements, equal to zero.

In our case

$$2V = \int_0^l E\epsilon^2 dx, \tag{1}$$

$$\delta V = \int_0^l E\epsilon\delta\epsilon\, dx = \int_0^l E\frac{\partial u}{\partial x}\delta\frac{\partial u}{\partial x}dx = -\int_0^l E\frac{\partial^2 u}{\partial x^2}dx + E\epsilon\delta u]_0^l \tag{2}$$

E is YOUNGS's modulus, $\epsilon = \partial u/\partial x$ is the strain and u is the displacement.

The term on the boundary vanishes because the displacement is not to be varied there, while the integrand yields the equation of equilibrium, in terms of displacement, which is, in the absence of any body forces, $\partial^2 u/\partial x^2 = 0$. This manner of deriving the equations of equilibrium in the three-dimensional case was first suggested by GREEN.

By contrast, in CASTIGIANO's method the strain energy V is expressed in terms of the stress σ, i. e.

$$V = \int_0^l \frac{\sigma^2}{2E}dx \tag{3}$$

and the variation yields

$$\delta V = \int_0^l \frac{\sigma}{E} \delta \sigma d x, \tag{4}$$

if HOOKE's law, $\sigma = E \epsilon$, is assumed to be valid. Integration by parts results in

$$\delta V = -\int_0^l u \frac{\partial \delta \sigma}{\partial x} d x + u \delta \sigma]_0^l. \tag{5}$$

The integrand vanishes because the equilibrium equation in terms of stresses, $\partial \sigma / \partial x = 0$, is assumed to be satisfied. If δV is set equal to zero, we obtain the usual form of CASTIGLIANO's principle, which states that the partial derivative of the strain energy with respect to an applied force equals the displacement of the point of application in the direction of the force.

For the present purposes we prefer to use a different version of CASTIGLIANO's principle. We consider not only the strain energy but also the work of external forces, as was done by REISSNER [6], i.e.

$$W = -\int_0^l \frac{\sigma^2}{2 E} d x + u \sigma]_0^l. \tag{6}$$

The variation of this expression results in

$$\delta W = \int_0^l -\frac{\sigma}{E} \delta \sigma d x + u \delta \sigma]_0^l \tag{7}$$

and after integration by parts

$$\delta W = \int_0^l \left(-\frac{\sigma}{E} + \frac{\partial u}{\partial x}\right) \delta \sigma d x + \int_0^l u \frac{\partial \delta \sigma}{\partial x} d x. \tag{8}$$

The second integral vanishes, because the equilibrium equation is again assumed to be satisfied, while the first integral yields $\sigma = E \epsilon$, i.e. HOOKE's law.

Since σ / E is the partial derivative of strain energy density with respect to the stress σ, HOOKE's law can be interpreted as resulting from CASTIGLIANO's principle. Now it is this derivative which yields the strain, through which the corresponding stress does work. Or, in other words, Hooke's law expresses CASTIGLIANO's principle, if applied to a unit volume of the material.

REISSNER [3] has unified the two separate principles of GREEN and CASTIGLIANO. He considered the strain energy in the form

$$V = \int_0^l \left(\epsilon \sigma - \frac{\sigma^2}{2 E}\right) d x \tag{9}$$

and assumed, in the variation process, the displacement and the stress as being independent from each other. Such variation, followed by partial integration, leads to

$$\delta V = \int_0^l \left[\left(\epsilon - \frac{\sigma}{E}\right)\delta\sigma - \frac{\partial\sigma}{\partial x}\delta u\right] dx + \sigma\,\delta u]_0^l. \tag{10}$$

If δV is set equal to zero, the coefficient of $\delta\sigma$ yields HOOKE's law, while the coefficient of δu yields the equilibrium equation.

On the basis of this unified principle, which furnishes both GREEN's and CASTIGLIANO's results, REISSNER was able to prove [6], that the former is a minimum, while the latter is a maximum principle.

The Basic Equations of Thermoelasticity and Heat Conduction

The classical problem of the coupled elastic and thermal fields in the uniaxial case is governed by the three equations

$$\begin{aligned} \sigma &= E\,\epsilon - E\,\alpha\,\theta, \\ \frac{\partial\sigma}{\partial x} &= 0, \\ k\frac{\partial^2\theta}{\partial x^2} &= c\frac{\partial\theta}{\partial t} + T_r\,E\,\alpha\frac{\partial\epsilon}{\partial t}. \end{aligned} \tag{11}$$

θ denotes here the excess temperature above a reference temperature T_r, α is the coefficient of thermal linear expansion, k is the heat conduction coefficient, c is the heat capacity per unit of volume, and t is the time.

It is customary to omit the "correction" term with $\partial\epsilon/\partial t$ in the heat conduction equation. This permits to solve first the temperature distribution problem, which is then independent of the elastic problem, and then, in a second step, to tackle the thermal stress problem on the basis of the first two equations. For purposes of the present discussion, no particular simplification is achieved by omission of this term, and, in fact, the development is more lucid if this term is retained. However, it is necessary to cast these basic equations into a different form, introducing in this course several new concepts.

The purely elastic stress, associated with elastic isothermal straining, is denoted by τ. We further define the "relative thermal displacement" h as the ratio of the time rate of heat flow to the reference temperature, the "thermo-elastic strain" γ by means of the equation

$$\gamma = -\frac{\partial h}{\partial x} + E\,\alpha\frac{\partial u}{\partial x} \tag{12}$$

and the "thermal force" g such that the product $g\,h$ is the work done by g in the "displacement" h.

With the aid of these four new quantities τ, h, γ and g, the basic eqs. (11) may be put into the form

$$
\begin{aligned}
&\epsilon-\frac{\tau}{E}=0, &&-\frac{\partial\tau}{\partial x}+E\alpha\frac{\partial\theta}{\partial x}=0,\\
&h-\frac{kg}{T_r p}=0, &&g-\frac{\partial\theta}{\partial x}=0,\\
&\gamma+\frac{c\theta}{T_r}=0.
\end{aligned}
\tag{13}
$$

The first two equations of the above set represent an obvious reformulation of HOOKE's law and the stress-equation of equilibrium, using the definition of stress τ. Eliminating g in the third equation with the use of the fourth and substituting h from the third into the last equation, the same form of the heat conduction equation is obtained as in the set (11). p represents the time operator $\partial/\partial t$ and may be treated as a constant.

Generalization of Reissner's Variational Principle for Stresses and Displacements

In the present thermodynamic system of variables we have to deal with 3 dynamic quantities, τ, θ and g and 2 kinematic quantities, u and h.

Following BIOT, we introduce his thermoelastic potential W in the form

$$W=\frac{\tau^2}{2E}+\frac{c\theta^2}{2T_r} \tag{14}$$

and his dissipation function D in the form

$$D=\frac{kg^2}{2pT_r}. \tag{15}$$

We now consider the energy expression

$$I=\int_0^l(\epsilon\tau-\gamma\theta+hg-W-D)\,dx \tag{16}$$

and assume the dynamic and kinematic variables to be independent from each other.

The variation of I, followed again by partial integration, leads to

$$
\begin{aligned}
\delta I=\int_0^l\Big[\Big(\epsilon-\frac{\tau}{E}\Big)\delta\tau+\Big(E\alpha\frac{\partial\theta}{\partial x}-\frac{\partial\tau}{\partial x}\Big)+\Big(h-\frac{kg}{pT_r}\Big)\delta g\\
+\Big(g-\frac{\partial\theta}{\partial x}\Big)\delta h-\Big(\gamma+\frac{c\theta}{T_r}\Big)\delta\theta\Big]\,dx+[(\tau-E\alpha\theta)\delta u+\theta\,\delta h]_0^l.
\end{aligned}
\tag{17}
$$

The integrated part vanishes, because neither displacement u nor h is varied at the ends $x=0, l$. If the integrand is to vanish, the coefficients of all varied quantities have to vanish, and this results in the five equations of the set (13).

We have thus generalized REISSNER's principle for stresses and displacements to the case of the coupled problem of thermoelasticity and heat conduction.

Generalization of Green's Principle for Displacements

The independent variables are now the two displacements u and h. We consider the expression

$$2 I_G = \int_0^l \left[E\epsilon^2 + \frac{T_r \gamma^2}{c} + \frac{T_r p h^2}{k} \right] dx. \tag{18}$$

The first two terms express the thermoelastic potential W in terms of kinematic variables, while the last term is the dissipation function D, also expressed in terms of the associated kinematic variable.

To perform the substitution from dynamic to kinematic variables in W and D, the first, third and fifth equations of the set (13) were employed.

The variation of I_G yields, after partial integration

$$\begin{aligned} \delta I_G = & \int_0^l \left[\left(-E\frac{\partial^2 u}{\partial x^2} - \frac{T_r}{c} E\alpha \frac{\partial \gamma}{\partial x} \right) \delta u + \left(\frac{T_r}{c}\frac{\partial \gamma}{\partial x} + \frac{T_r}{k} p h \right) \delta h \right] dx \\ & + \left[\frac{T_r}{c} E\alpha\gamma\,\delta u - \frac{T_r}{c}\gamma\,\delta h \right]_0^l. \end{aligned} \tag{19}$$

The integrand, set equal to zero, furnishes the second and fourth equations, generalizing thus GREEN's principle for displacements.

Generalization of Castigliano's Principle for Stresses

The independent variables are now the three dynamic quantities τ, θ and g. We consider the expression

$$2 I_c = -\int_0^l \left[\frac{\tau^2}{E} + \frac{c\,\theta^2}{T_r} + \frac{k g^2}{p T_r} \right] dx + 2\left[u\tau - E\alpha\theta u + \theta h \right]_0^l. \tag{20}$$

Again, the integrand is nothing but the negative of the sum of the thermodynamic potential and dissipation function, but now expressed in terms of dynamic quantities.

The variation of I_C may be written as

$$\delta I_c = \int_0^l \left[\left(-\frac{\tau}{E}+\epsilon\right)\delta\tau - \left(\frac{c\,\theta}{T_r}+E\,\alpha\,\epsilon-\frac{\partial h}{\partial x}\right)\delta\theta - \left(\frac{k g}{p T_r}-h\right)\delta g\right] dx$$
$$+\int_0^l \left[\left(\frac{\partial\delta\tau}{\partial x}-E\,\alpha\frac{\partial\delta\theta}{\partial x}\right)u+\left(\frac{\partial\delta\theta}{\partial x}-\delta g\right)h\right]dx. \tag{21}$$

The second integral vanishes because the equilibrium equations (second and fourth equations of the set (13)) are assumed to be satisfied, while the vanishing of the first integral yields the first, third and fifth equations of the basic set (13).

Concluding Remarks

Using the same line of thought as the one followed by REISSNER [6] it can be proved that in the extended variational theorem for "displacements" one is concerned with a minimum problem, while in the extended variational theorem for "stresses" one is concerned with a maximum problem. And just as in the case of isothermal elasticity, the variational theorem for stresses and displacements is no more than a stationary value problem. Details of this proof in the general case will be dealt with by the present author elsewhere.

BIOT's variational formulation [1, 2, 3] is recognized to be a "mixed" principle, in the sense that it yields two equations, one being part of extended GREEN's, and the other part of extended CASTIGLIANO's formulation.

All the energy theorems presented here for the special case of onedimensional problems can easily be formulated for the general, three-dimensional bodies and also for lumped systems, such as framed structures.

Suitable procedures for the application of the basic energy theorems to particular cases, as illustrated by BIOT [2] in heat flow analysis, are still to be developed.

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Summary

Some well-known energy theorems of structural analysis are generalized for the case when a part of the stresses is due to thermal effects. These methods assume that the temperature distribution is known. It is shown further that the temperature distribution in the structure itself may be determined by the use of analogous energy theorems, which can be established for both steady-state and transient conditions.

Résumé

L'auteur généralise quelques propositions connues de la méthode énergétique de la statique appliquée, pour le cas où une partie des contraintes dépend des fluctuations de la température. Ces méthodes de calcul supposent toutefois une répartition connue de la température. L'auteur montre que cette répartition peut être déterminée à l'aide de propositions analogues, aussi bien pour des conditions constantes que pour des conditions variables.

Zusammenfassung

Einige bekannte Sätze der Energiemethode der Baustatik werden verallgemeinert für den Fall, da ein Teil der Spannungen von Temperaturwirkungen abhängig ist. Diese Berechnungsmethoden setzen eine bekannte Temperaturverteilung voraus. Es wird weiterhin gezeigt, daß die Temperaturverteilung im System mit Hilfe analoger Sätze der Energiemethode bestimmt werden kann, und zwar sowohl für gleichbleibende als auch für veränderliche Bedingungen.