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Vb1

L'auscultation dynamique des ponts à la S. N. C. F.

Dynamische Untersuchungen der Brücken der S. N. C. F.

Dynamic research of bridges of the S. N. C. F.

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La question des effets dynamiques subis par les ponts se présente à l'ingénieur sous un double aspect : scientifique et pratique; le premier comprend sans doute le second, mais son étude est actuellement trop peu avancée pour que leurs objectifs ne se distinguent pas nettement par les délais dans lesquels ils peuvent être atteints.

Difficultés de l'étude scientifique des effets dynamiques

L'étude scientifique de la question vise à la connaissance précise des conditions dans lesquelles se produisent les effets dynamiques, de leur grandeur, de la façon dont ils interviennent dans la sécurité des constructions; elle nécessite beaucoup de recherches tant théoriques qu'expérimentales. Jusqu'ici, les études théoriques poussées n'ont pas manqué, les expérimentations assez vastes non plus, la comparaison des résultats obtenus par les deux voies a été tentée. Cependant, il est encore impossible d'évaluer à priori avec quelque précision, pour un élément donné d'une poutre de pont, la différence entre les effets maxima provoqués par une charge en stationnement et les effets maxima résultant du passage de la même surcharge animée d'une vitesse donnée. Sans doute, quelques points ont été éclaircis; des recherches très remarquables conduites avec soin et persévérance, simultanément sur les plans théorique et expérimental, ont permis de reconnaître, par exemple, l'influence des forces périodiques résultant de l'action des contrepoids d'équilibrage des locomotives : les mesures de flèche obtenues concordent avec ce que la théorie permet de prévoir. Mais il s'agit seulement de l'effet d'une cause particulière, que tous les ingénieurs ne considèrent pas comme prépondérante; les résultats solidement établis ne concernent que les flèches des ouvrages suffisamment longs pour que les mouvements résultant des conditions d'entrée des charges aient le temps de s'amortir avant que le maximum des déformations soit atteint; moins sûrs pour les contraintes dans les membrures, ils restent sujets à caution pour les contraintes dans les treillis; ils supposent d'ailleurs une vitesse constante, des chemins de roulement, du matériel et des convois assez bien constitués pour que des mouvements parasites ne s'amorcent pas.

En fait, interviennent simultanément, dans les effets dynamiques et pour des fractions non négligeables, diverses causes dont on sait mal tenir compte; ainsi, le passage d'une surcharge donnée à une même vitesse ne détermine pas exactement les mêmes effets en raison des conditions toujours incomplètement définies, dans lesquelles le convoi se présente à l'entrée du pont. Ce ne sont pas là des objections théoriques : elles ont été très nettement vérifiées et vivement commentées au cours des essais effectués de 1929 à 1931 sous les auspices de l'U. I. C.

En négligeant même, momentanément, les difficultés résultant de l'indétermination ou de la complexité des conditions initiales, l'étude scientifique des effets dynamiques s'est trouvée freinée par le défaut d'appareils enregistreurs absolument fidèles, notamment pour la mesure des contraintes. Les progrès considérables effectués dans ce domaine pendant la guerre, la mise au point des extensomètres à fil résistant, permettent d'escompter la solution prochaine de cette difficulté; mais la formation des opérateurs et l'exécution des nombreuses mesures nécessaires entraîneront, pour toute observation systématique en campagne, des délais et des frais importants. La seule répétition des observations faites jusqu'ici avec des appareils plus rustiques demandera déjà un temps très long.

D'autre part, l'étude complète des effets dynamiques ne saurait se borner à la détermination des déformations élastiques provoquées par le passage des charges. Il faudra évidemment se rendre compte si les indications d'un extensomètre peuvent être interprétées de même pour des charges statiques et pour des charges dynamiques, si une même extension instantanée correspond effectivement à une même contrainte et dans quelle mesure on peut assimiler, dans l'étude de la sécurité, deux contraintes effectives de même grandeur, l'une fixe, l'autre instantanée. Il semble bien que les observations sur la fatigue des métaux ne s'appliquent pas directement aux éléments de ponts, vu les conditions de variations des efforts et les fréquences auxquelles ils sont soumis. Une étude pertinente de cette seule question exigera aussi de longs délais.

L'étude du problème des effets dynamiques sur les ponts ne paraît susceptible de progresser largement qu'en y employant les méthodes modernes de la recherche scientifique : recours à des équipes spécialisées, chargées de tâches déterminées, et coordination de leurs efforts. Chaque groupe de recherche devrait comprendre des ingénieurs de formations diverses : mécaniciens capables de reconnaître dans l'infinité des processus vibratoires possibles ceux qui doivent effectivement se rencontrer dans un élément d'un tablier complexe et pour quelles conditions de présentation des charges, physiciens connaissant suffisamment les propriétés des matériaux pour se rendre compte de l'influence des variations d'efforts ou de déformation sur les contraintes admissibles dans les ponts, techniciens susceptibles de concevoir et réaliser des appareils donnant sans altération les quantités à mesurer, opérateurs rompus au maniement des appareils et à leur installation dans les parties les moins accessibles d'une charpente. Les résultats n'apparaîtront sans doute que peu à peu, à condition de ne pas distraire les équipes constituées de la tâche laborieuse qui leur aura été confiée et de les encourager dans leurs recherches sans les chicaner trop s'il ne sort pas très vite de leurs travaux une formule définitive d'impact ne comportant que deux ou trois paramètres.

Nécessité de connaître pratiquement certains aspects des effets dynamiques

En attendant que de telles études aient tiré au clair l'importance et le rôle des effets dynamiques dans les ponts et que des lois chiffrées permettent de les utiliser pratiquement, les ingénieurs qui reçoivent la charge de conserver de vieux ponts ou d'en établir de nouveaux se demandent toujours, comme leurs prédécesseurs d'il y a cent ans: si un convoi passe sur l'ouvrage à une vitesse V, quelle est, en chaque point, la majoration de contrainte par rapport à la contrainte statique? Beaucoup jugent la question un peu académique : sous le régime général des coefficients de majoration réglementaires soit explicites (quand les contraintes admises sont élevées), soit implicites (quand les contraintes admises sont faibles), il ne semble pas (ponts suspendus insuffisamment rigides mis à part) que se soient manifestés des incidents graves résultant d'évaluations trop optimistes. De très anciens ouvrages, bien construits, d'après les contraintes-limites en usage il y a quatre-vingts ans, ont supporté sans renforcement des accroissements considérables des charges et des vitesses des convois de chemin de fer. L'on peut donc soutenir que, si la connaissance précise des effets dynamiques contribue à la satisfaction de l'esprit de l'ingénieur, son influence sur la sécurité et sur l'économie demeure modéré. Mais c'est là une façon de voir trop simpliste.

Le fait que des ouvrages tiennent ne prouve pas que l'évaluation empirique des effets dynamiques ait été assez correcte, mais seulement que les erreurs qui en résultent ne portent pas la probabilité de ruine de ces ponts à une valeur nettement supérieure à celle qui se rencontre dans les constructions courantes. Cette conclusion n'est d'ailleurs pas générale : il arrive, par exemple, qu'au-dessus d'une certaine vitesse, des rivets s'ébranlent, des fissures se propagent, des barres de treillis se mettent à vibrer ou à flamber d'une façon inquiétante. Il vaut mieux avoir reconnu cette circonstance par une étude ou par des essais sur le pont et avoir imposé une limitation de vitesse, plutôt que d'en être informé par des incidents de service.

D'un autre côté, les formules empiriques de majoration peuvent être améliorées; si l'on peut établir qu'un type d'ouvrage est particulièrement favorable à l'atténuation des effets dynamiques, il est normal d'appliquer, à ce type, un coefficient particulier qui permettra, sans réduire la sécurité, de réaliser des économies.

Aussi, la S. N. C. F., sans négliger la question scientifique et générale des effets dynamiques sur les ponts, s'est-elle beaucoup préoccupée d'obtenir des renseignements immédiatement utilisables et des réponses rapides à des questions pratiques touchant la sécurité de ses ouvrages ou l'économie des projets. L'ordre de grandeur des effets dynamiques correspond-il toujours à celui qui résulte des formules réglementaires? Le matériau et le type des ouvrages ont-ils une influence telle qu'elle justifie la préférence

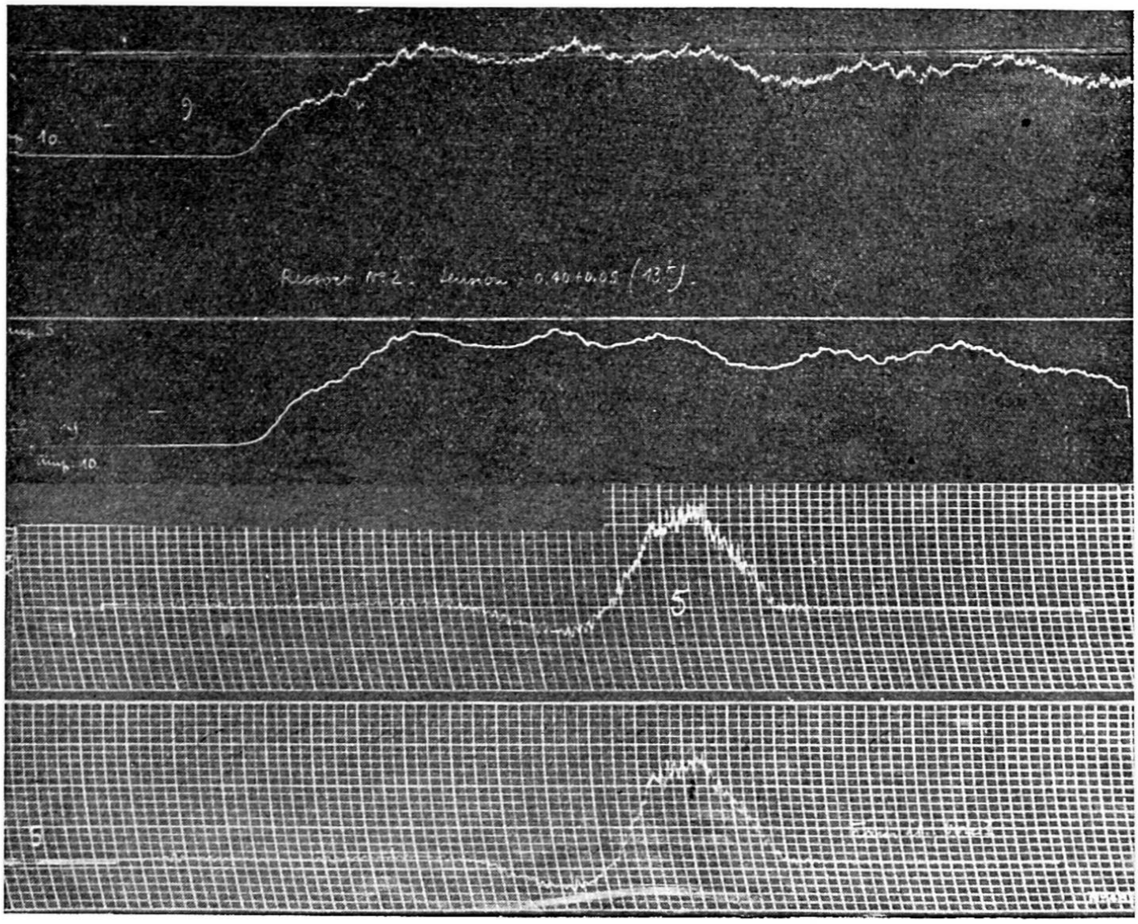


Fig. 1. Différences obtenues dans deux enregistrements de flèches effectués chacun avec deux fleximètres de type classique.

accordée à certains types ou à certaines dispositions particulières? Dans quels ouvrages ou quels éléments d'ouvrages y a-t-il lieu de craindre des mouvements anormaux ou des déformations importantes au passage des convois? Les mouvements que l'on constate sur tel pont sont-ils véritablement excessifs? A quel taux la vitesse doit-elle être réduite pour les rendre acceptables?

Pour répondre objectivement à ces questions ou à d'autres analogues, il faut faire appel à l'expérience. Mais il n'est pas indispensable, pour le but pratique poursuivi, de disposer de moyens aussi importants que ceux envisagés plus haut. L'utilisation d'appareils moins parfaits se justifie quand la simplicité de leur mise en place et de leur manœuvre permet de les confier à des spécialistes d'ouvrages d'art qui, seuls, seront capables de les poser rapidement. Les mesures peuvent ainsi être multipliées dans des conditions diverses sans frais exagérés. De fait, la S. N. C. F. en effectue fréquemment sous des surcharges circulant systématiquement à des vitesses diverses. En dehors des réponses qu'elles apportent, moyennant une interprétation critique convenable, aux problèmes particuliers, ces expériences constituent une documentation permettant de reconnaître certaines lois empiriques. Nous donnerons, au cours du Congrès, un aperçu des résultats obtenus dans cette voie.

Appareils utilisés

Les appareils classiques, fleximètres et extensomètres à enregistrement mécanique, peuvent, pour des observations à objectif pratique et limité, fournir déjà des éléments de comparaison et de classement.

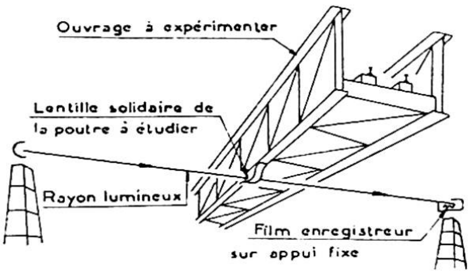


Fig. 2. Fleximètre optique.

Toutefois, pour obtenir des indications plus sûres ou pour procéder à d'autres investigations, de nouveaux appareils ont été mis au point par la S. N. C. F. ou sont en cours de construction : fleximètre, rouligraphe, accéléromètre.

Fleximètre

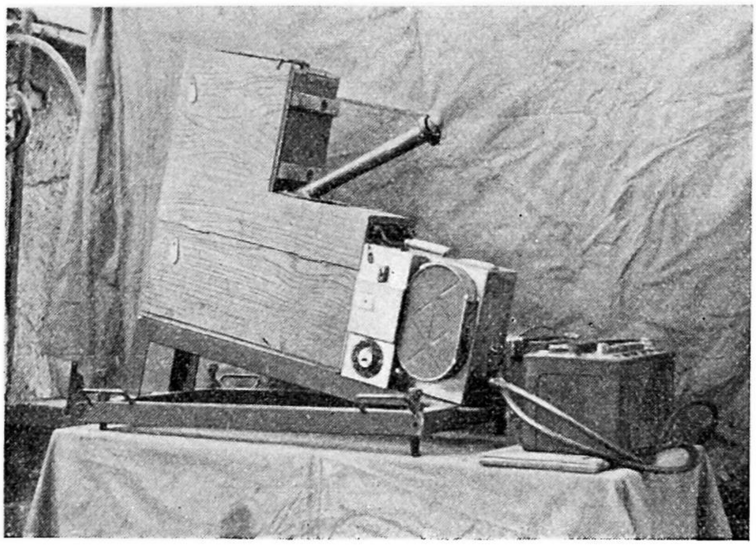
La justification de la création d'un nouvel appareil réside dans la constatation que deux fleximètres amplificateurs de type classique, attachés en des points très voisins et posés dans des conditions paraissant correctes, ne donnent pas exactement le même enregistrement; dans certains cas, rares il est vrai, les graphiques accusent des différences nettes, tant pour l'amplitude que pour l'allure et l'amortissement des oscillations (fig. 1).

L'origine de ces différences réside évidemment dans les vibrations parasites introduites par la transmission, d'où le désir de supprimer celle-ci.

Le fleximètre optique (fig. 2) comprend une lentille achromatique rendue solidaire de la membrure inférieure de la poutre étudiée. Une source lumineuse envoie sur la lentille au travers d'une mince fente horizontale, un pinceau de rayons qui donne une image sur une fente verticale placée devant un dérouleur de papier sensible. Le déplacement du point lumineux et, par suite, celui de la lentille, s'enregistre en une courbe continue.

On emploie assez souvent une lentille de 8 mètres de distance focale, la source lumineuse et l'enregistreur étant placés respectivement à 16 mètres de part et d'autre de la poutre; l'amplification est alors très voisine de 2, mais elle peut être augmentée, le cas échéant, ainsi que la luminosité des images, en prenant une lentille de distance focale différente ou en faisant varier l'éloignement de la source et celui de l'enregistreur.

A défaut de circonstances locales favorables, la source et l'enregistreur sont amenés au niveau de la lentille en les disposant sur des échafaudages robustes placés assez loin du pont pour ne pas être influencés par les trépidations du sol. En opérant à la tombée du jour, on peut se con-



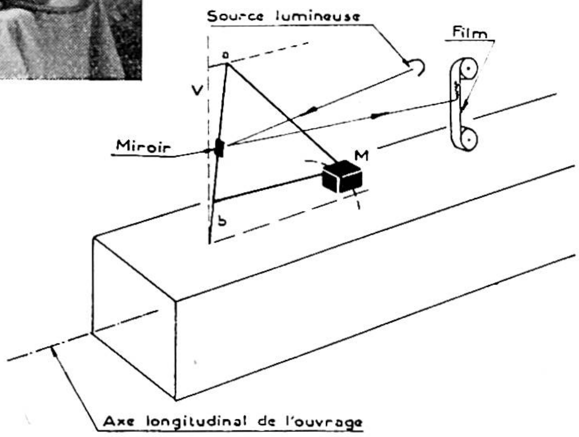


Fig. 3 et **4.** Rouligraphe.

tenter d'une source peu intense, ce qui facilite l'installation en pleine campagne.

La lentille étant montée sur un support robuste, solidement fixé au pont, il n'est pas à craindre qu'elle soit le siège de vibrations propres susceptibles d'apporter des perturbations dans l'enregistrement. Les causes d'erreur possibles paraissent limitées au trouble que pourrait apporter, à la propagation des rayons lumineux, une atmosphère hétérogène et variable; cette cause ne semble pas à considérer pendant la courte durée d'un passage.

L'amplification exacte est déterminée, pour chaque essai, en mesurant le déplacement de l'image qui correspond à un déplacement vertical connu de la source.

ROULIGRAPHE

Cet appareil a pour objet principal de déceler et mesurer les vibrations horizontales, peu étudiées jusqu'ici, mais dont l'effet est parfois très sérieux; nous en donnerons des exemples au cours du Congrès. Il utilise le principe que l'on trouve dans la conception de nombreux sismographes, et suivant lequel une masse pendulaire relativement lourde demeure pratiquement fixe quand on ne la soumet qu'à des impulsions de cadence rapide par rapport à sa fréquence d'oscillation; cette dernière est rendue suffisamment lente en utilisant un pendule d'axe quasi-vertical.

Les appareils anciens, basés sur le même système, comportaient des frottements importants résultant, soit des glissières, soit du système d'enregistrement attelé sur la masse; pour pouvoir vaincre ceux-ci, une force de rappel importante était nécessaire. Il en résulte alors une période d'oscillation trop courte pour que les déplacements du pont laissent la masse pratiquement fixe. Cette imperfection ne permettait guère de les utiliser, et avec beaucoup de réserves, que pour des comparaisons.

Dans le rouligraphe (fig. 3 et 4), on a, au contraire, réduit systématiquement les frottements : la masse pendulaire est suspendue par une tige et s'appuie par une bielle, toutes deux munies d'articulations à billes; l'enregistrement est purement optique. A cet effet, un miroir cylindrique solidaire de l'axe de rotation reçoit d'une source solidaire du bâti de l'appareil un rayon lumineux qu'il renvoie sur la fente horizontale d'un enregistreur dans lequel se déroule, d'un mouvement continu et à des vitesses réglables, un film photographique de 35 mm.

Dans ces conditions, on peut obtenir sans difficulté des périodes de trois secondes avec des frottements insignifiants (en oscillation libre, l'amplitude est réduite de moitié après une trentaine d'oscillations).

L'appareil enregistre, soit en vraie grandeur, soit avec amplification, les déplacements transversaux de cadence inférieure à la seconde, mais, par son principe même, il enregistre également les mouvements angulaires autour d'un axe horizontal perpendiculaire à la direction des déplacements, d'où le nom de « rouligraphe ».

Pratiquement, il est facile, le cas échéant, de séparer les deux mouvements dont les périodes sont très différentes.

Accéléromètre

L'emploi d'un accéléromètre conjugué avec un fleximètre ou mesureur d'oscillations (horizontales ou verticales) permet de se faire une idée de la confiance à accorder aux deux enregistrements et de mettre en évidence les vibrations rapides à faible amplitude qui sont peu visibles dans les graphiques d'oscillation.

De nombreux accéléromètres ont été imaginés ou essayés dans les dernières années. La S. N. C. F. dispose notamment d'un accélérographe qui a été signalé au Congrès de 1936 (*Rapport Final*, p. 645), dans lequel les variations de pression résultant des accélérations agissent sur un quartz piezo-électrique. Cet appareil nécessite toutefois, à proximité de l'ouvrage, une installation électrique un peu encombrante; aussi, pour les essais fréquents utilise-t-on un détecteur de vibration à induction et, pour les fréquences basses et d'amplitudes notables, a-t-on prévu un accéléromètre mécanique (à enregistrement optique). Ce dernier est constitué par une barre de torsion encastrée aux extrémités, portant en son milieu une traverse à laquelle est fixée une masse qui peut être éloignée ou rapprochée

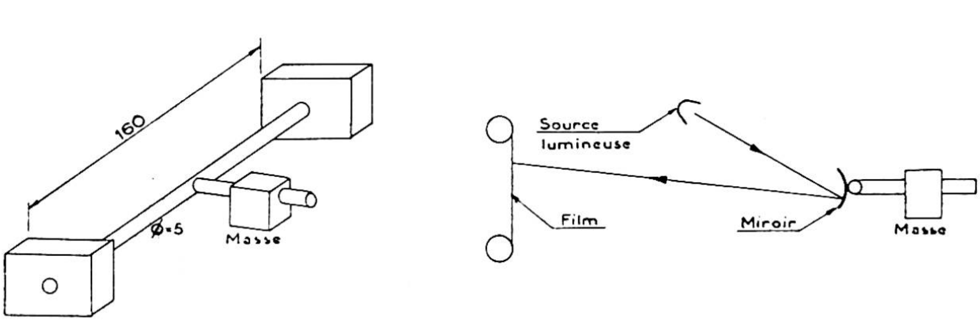


Fig. 5. Accéléromètre.

de la barre de torsion; l'appareil est muni d'un dispositif d'amortissement électro-magnétique.

Résumé

Malgré d'importants travaux théoriques et expérimentaux, on connaît encore très mal l'importance des effets dynamiques auxquels sont soumis les divers éléments des ponts et quelle est leur influence sur la sécurité. Quand on envisage une étude scientifique de la question, on se heurte à l'impossibilité de tenir compte simultanément des nombreux paramètres susceptibles d'intervenir dans le phénomène. La théorie reste hésitante tant qu'on ne se borne pas à des cas schématiques et beaucoup d'expériences demeurent confuses, même lorsqu'on ne discute pas la fidélité des enregistrements. Pour élucider vraiment la question, il faudra sans doute beaucoup de travail à des équipes spécialisées, les unes dans les calculs, les autres dans la construction d'appareils, les autres dans leur mise en place et leurs observations, etc., et coordonner tous ces travaux.

Faut-il jusqu'à ce que des résultats indiscutables aient été obtenus par les chercheurs, se contenter, pour la vérification des ouvrages anciens et l'étude des ouvrages nouveaux, d'utiliser sans trop y croire, les coefficients de majoration en usage dans les divers pays? On ne peut alors répondre aux questions qui se posent fréquemment aux ingénieurs responsables des ponts. Faut-il limiter la vitesse sur tel ouvrage et à quel taux? Y a-t-il intérêt, pour réduire les effets dynamiques, à adopter tel type d'ouvrage ou tel mode de pose ? Pour donner une solution pratique à des questions analogues, la connaissance très précise du phénomène d'impact n'est pas nécessaire, puisqu'il s'agit principalement de comparaisons de quelques effets mesurables. Il est loisible d'utiliser les équipes habituelles chargées de l'auscultation des ponts et d'employer des appareils n'enregistrant que les aspects les plus simples des manifestations dynamiques et la S. N. C. F. procède souvent à de telles observations à fins limitées et pratiques. On donnera, au cours du Congrès, des indications sur les constatations faites et certains résultats généraux mis en évidence.

Pour ces observations, la S. N. C. F. a systématiquement recours à des appareils de conception, de mise en place et d'utilisation très simples, dont l'amplification et l'enregistrement sont normalement obtenus par voie optique. Le rapport donne des indications sur des appareils nouveaux conçus dans cet esprit : fleximètre, enregistreur d'oscillations horizontales ou de rotations d'axe horizontal (rouligraphe), accéléromètre vertical.

Zusammenfassung

Trotz zahlreichen theoretischen und experimentellen Untersuchungen ist die Bedeutung der dynamischen Einflüsse auf Brückenbauten und die dadurch beeinflusste Sicherheit noch wenig bekannt. Wenn man eine wissenschaftliche Untersuchung dieser Frage ins Auge fasst, so stösst man auf die Unmöglichkeit, gleichzeitig die zahlreichen Parameter, welche die verschiedenen Erscheinungen charakterisieren, zu berücksichtigen. Die

Theorie bleibt auf unsicherer Grundlage, wenn man sich nicht auf schematisierte Fälle beschränkt und zahlreiche Versuche bleiben unklar, auch wenn über die Genauigkeit der Untersuchungen kein Zweifel besteht. Um die Frage von Grund auf zu klären, wird es zweifellos einer grossen Arbeit von ausgewählten Fachleuten bedürfen, welche einerseits in der Berechnung, andererseits im Bau von Versuchseinrichtungen, andere noch in der Auswertung der Versuche und schliesslich in der Zusammenfassung sämtlicher Ergebnisse spezialisiert sind.

Es scheint sehr fragwürdig zu sein, mangels fester Grundlagen den einfachen Weg der Verwendung von Stosszuschlägen zu beschreiten. Letztere sind nicht in der Lage, auf immer wiederkehrende Fragen der Brückenbeanspruchung eine klare Antwort zu erteilen. Eine für die Praxis geeignete Lösung verlangt nicht notwendigerweise eine genaue Kenntnis des Stossvorganges; denn es handelt sich hauptsächlich um den Vergleich von einigen messbaren Wirkungen. Dafür genügt es, die gewöhnlichen Wege mit den üblichen Bestand von Fachleuten einzuschlagen, welche mit Hilfe von Versuchseinrichtungen nur die einfachsten Erscheinungsformen der dynamischen Beanspruchungen bestimmen, wie es meistens auch die S. N. C. F. tut. Während des Kongresses werden darüber Angaben gemacht und gewisse allgemeine Ergebnisse hervorgehoben werden.

Für diese Beobachtungen hat die S. N. C. F. systematisch Aufnahmegeräte einfacher Art verwendet, deren Uebersetzung sowie Registrierung normalerweise auf optischem Wege erfolgt. Der vorliegende Bericht enthält Angaben über neue Versuchseinrichtungen, die zu diesem Zwecke erstellt wurden : Durchbiegungsmesser, Registrierapparate für horizontale sowie für Drehschwingungen mit horizontaler Axe (rouligraphe) sowie auch vertikale Beschleunigungsmesser.

Summary

In spite of much theoretical and experimental research, little is known of the importance of dynamic influences on bridge-building and consequential safety. If we go into scientific research of this question we are confronted with the impossibility of taking into consideration the numerous parameters which characterise the various phenomena. Theory remains on an unreliable basis unless we restrict ourselves to graphic instances, and numerous tests remain unconvincing, even where there is no doubt as to the accuracy of the research work. To elucidate the question from start to finish will necessitate much work by chosen specialists who are experts, on the one hand, in calculation and, on another hand, in the construction of experimental apparatus, whilst yet others specialise in summing up the tests and, finally, in making a synopsis of the whole results.

Owing to lack of reliable rudiments, it appears very doubtful whether we can adopt the simple method of using thrust increases. It is not possible for the latter to provide a clear answer to the constantly recurring questions of bridge stresses. A solution that is suitable in practice does not necessarily require a precise knowledge of the processus of thrust : for it is principally a matter of a comparison of a few effects that can be measured. For that, it suffices to follow the general procedure with the

usual aid of experts who, with the aid of testing apparatus, only ascertain the simplest outward forms of dynamic stresses, as is mostly done by the French National Railways. Accounts of this will be made during the Congress and certain general results emphasized.

For these observations the French National Railways systematically used recording apparatus of a simple kind, the results being interpreted and recorded optically. The present report contains particulars of new testing apparatus created for this purpose: deflection indicator, recording apparatus for horizontal and rotary vibrations with a horizontal axis (rouligraph), and a vertical acceleration indicator.

Vb2

Sollicitations dynamiques de poutres sous charges mobiles

Ueber die dynamischen Beanspruchungen von Trägern infolge beweglicher Lasten

A study of dynamic influences of moving loads on girders

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An investigation into the dynamic influences of moving loads on bridges is being made at the Institution of Structural Engineering and Bridge Building at the Royal Institute of Technology, Stockhom, Sweden. To begin with, the simplest cases are thoroughly studied, and then the various effects are added one by one in order to determine their separate and cumulative action. The investigation comprises both theoretical and experimental studies.

In the years 1943 to 1946 the main part of the work was done by Rolf Lerfors, C. E., and from the end of 1946 the investigation has been continued by the Author.

In the first place, I studied the case of a concentrated load moving at a constant speed along a girder of uniform section. Methods of solution for this case have been given by several authors, but these methods are either very laborious or else so roughly approximative that the solution is too inaccurate. Therefore, I have tried to simplify and to rationalize the arithmetical computations in a method due to Prof. Inglis (1), which I consider to be one of the most reliable methods available for this purpose. This method was simplified so that it was possible to calculate a great number of cases and thus to form an estimate of the dynamic increment in all practical cases.

It is very important that, if damping is left out of account, the dynamic

(1) Inglis, *A Mathematical Treatise on Vibrations in Railway Bridges*, Cambridge, 1934.

increment can be shown to be completely determined by two dimensionless quantities defined as follows :

$$\alpha = \frac{\text{velocity of load}}{2 \times \text{natural frequency of girder} \times \text{length of girder}}$$

$$\nu = \frac{\text{mass of load}}{\text{mass of girder}}$$

For practical purposes, the approximate limits of these quantities are

$$0 < \alpha < 0{,}15$$

$$0 < \nu < 5$$

For $\nu = 0$, Timoshenko (2) has shown that the dynamic increment in centre deflection is approximately given by

$$\frac{\alpha}{1 - \alpha} .$$

In the more difficult case where $\nu \neq 0$, a great many methods were studied, and, as has been mentioned above, that due to Inglis was found to be most suitable. He expresses the concentrated load per unit length of span by the Fourier series

$$p = \frac{2\,P}{l} \sum_{i=1}^{\infty} \sin i\varphi \cdot \sin \frac{i\pi x}{l} .$$

The notations are given in fig. 1.

In the computations, Inglis uses only the first term of this series.

The deflection is assumed to be

$$y = q\,(\varphi) \cdot \sin \frac{\pi x}{l}$$

and we obtain, for the determination of $q(\varphi)$, the differential equation

$$\frac{d^2q}{d\varphi^2}\left(\frac{1}{2\,\nu} + \sin^2\varphi\right) + \frac{dq}{d\varphi}\sin 2\,\varphi + q\left(\frac{1}{2\,\nu\alpha^2} - \sin^2\varphi\right) = \frac{y_{00}}{2\,\nu\alpha^2} \cdot \sin\varphi$$

where y_{00} denotes the static deflection for $\varphi = \frac{\pi}{2}$.

Inglis solves the above equation by means of two series, viz., a series for a forced oscillation and a series for a free oscillation. The coefficients of these series are determined as usual.

The second series is by far the most difficult. Its determination involves much work.

It can be shown that the free oscillation is in very close agreement with the expression

$$q_{\text{free}} = \frac{A}{\sqrt{1 + 2\,\nu \sin^2\varphi}} \sin\left[\sqrt{\frac{1}{\alpha^2} - \nu} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 + 2\,\nu\sin^2\varphi}}\right]$$

which is comparatively simple to calculate, especially after tabulating the expressions

$$\frac{1}{\sqrt{1 + 2\,\nu \sin^2\varphi}} \quad \text{and} \quad \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 + 2\,\nu\sin^2\varphi}} .$$

(2) See, for example, TIMOSHENKO, *Vibration Problems in Engineering*, New-York, 1937.

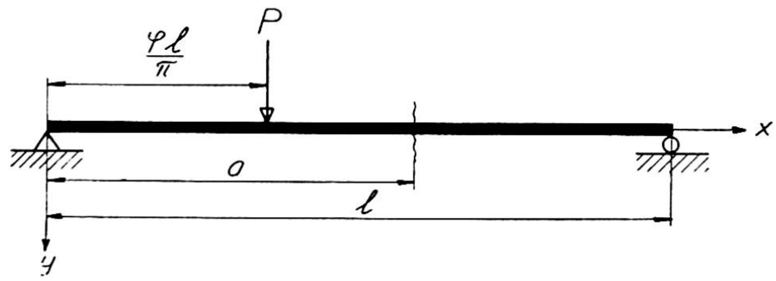

Fig. 1.

The integral can be determined by means of tables of elliptic integrals.

By using this formula, the time required for computing one case is reduced from about fifteen hours to a few hours.

By means of this simplified method, I have calculated and plotted a diagram showing the dynamic increment in deflection as a function of α and ν. It has been convenient to define this dynamic increment by

$$\varepsilon_d = \frac{(q_{\text{dyn}} - q_{\text{stat}})_{\max}}{q_{\text{stat}\,\max}}$$

where q_{dyn} has the same significance as $q(\varphi)$ above, and q_{stat} corresponds to $\alpha = 0$ and has the value

$$q_{\text{stat}} = y_{00} \cdot \sin \varphi \, .$$

For small values of α and ν $(\alpha^2 \nu < 0.01)$, I got a direct expression for ε, i.e.

$$\varepsilon_d = \alpha^2 \left[\frac{1 + 2\nu}{1 - \alpha^2} + \frac{2\nu}{1 - 9\alpha^2}\right] + \left[1 + \alpha^2 \left(\frac{1 + 2\nu}{1 - \alpha^2} - \frac{6\nu}{1 - 9\alpha^2}\right)\right] \frac{\alpha}{\sqrt{1 - \alpha^2 \nu}\sqrt{1 + 2\nu}}$$

For higher values, I have first calculated $q(\varphi)$ and then ε.

These results were used for plotting the diagram shown in fig. 2.

When q is known, the force exerted by the load on the girder and the acceleration forces acting on the girder itself can be calculated on the same assumptions as before. Consequently, the moments and the shearing forces acting on the girder can also be computed. With the notations given in fig. 1, these values are

$$M_a = M_{0a} \cdot \frac{\frac{\varphi l}{\pi}}{a y_{00} \sin \varphi} \left[q + \alpha^2 \frac{d^2 q}{d\varphi^2} \left(1 - \frac{2\, l \sin \frac{\pi a}{l}}{\pi (l - a)} \frac{\sin \varphi}{\varphi} \right) \right]$$

$$R_a = R_{0a} \frac{\frac{\varphi l}{\pi}}{a y_{00} \sin \varphi} \left[q + \alpha^2 \frac{d^2 q}{d\varphi^2} \left(1 + \frac{2\, l}{a \pi} \cos \frac{\pi a}{l} \sin \varphi \right) \right] .$$

M_a denotes the moment at the distance a from the left end of the girder, and M_{0a} designates the static moment produced when the load is applied at this section.

R_a and R_{0a} denote the corresponding shearing forces.

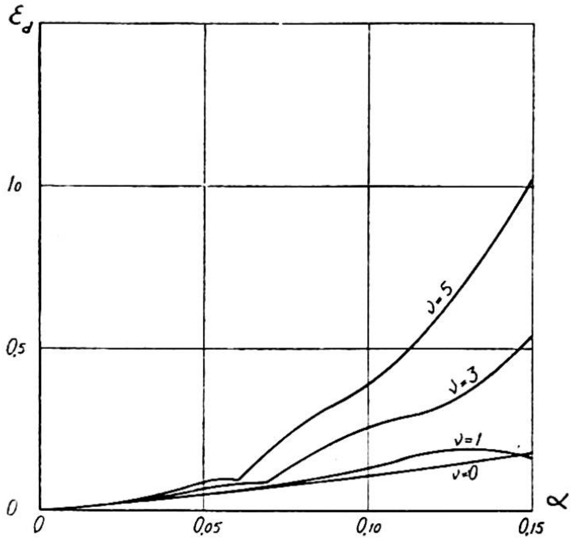

Fig. 2. Dynamic increment in centre deflection as a function of α and ν.

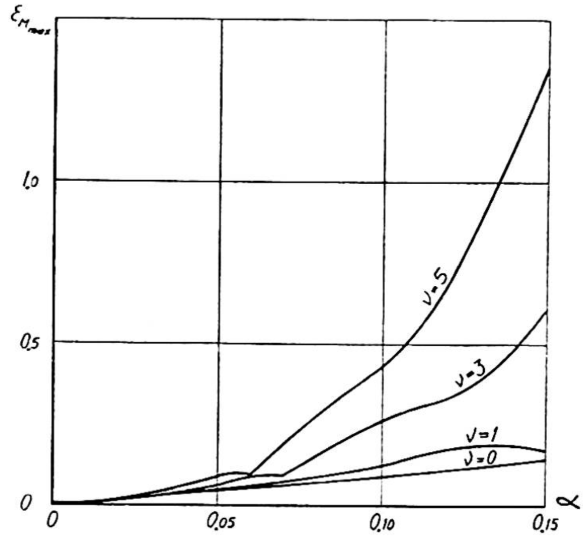

Fig. 3. Maximum dynamic increment in bending moment as a function of α and ν.

In the same way as before, we define

$$\varepsilon_{M\,a} = \frac{\left(M_{a_{dyn}} - M_{a_{stat}}\right)_{max}}{M_{a_{stat_{max}}}}$$

$$\varepsilon_{R\,a} = \frac{\left(R_{a_{dyn}} - R_{a_{stat}}\right)_{max}}{R_{a_{stat_{max}}}} .$$

The maximum value of ε for a given ν and α is of great interest. This maximum value is obtained when $\frac{\varphi l}{\pi} = a$ and is

$$\varepsilon_{M\,max} = \left\{ \frac{1}{y_{00} \sin \varphi} \left| q + \alpha^2 \frac{d^2 q}{d\varphi^2} \left(1 - \frac{2 \sin^2 \varphi}{\varphi(\pi - \varphi)}\right)\right]\right\}_{max} - 1$$

$$\varepsilon_{R\,max} = \left\{ \frac{1}{y_{00} \sin \varphi} \left[q + \alpha^2 \frac{d^2 q}{d\varphi^2} \left(1 + \frac{\sin 2\varphi}{\varphi}\right)\right]\right\}_{max} - 1 .$$

The values of $\varepsilon_{M\,max}$ are shown in fig. 3. $\varepsilon_{R\,max}$ differs very little from $\varepsilon_{M\,max}$ and is always less than the latter value.

To verify the theoretical results, model tests are being made. The test set-up is shown in fig. 4.

The girder is made of steel and has the approximate dimensions $5 \times 50 \times 1\,100$ mm. The load is a ball of the type used in ball-bearings, which rolls along a track on the girder. The bending stresses are measured by resistance strain gauges at several sections of the girder, and are recorded by an oscillograph which also indicates the time and the instants at which the load passes through definite points.

So far, tests have been made for $\nu = 3.5$ only. Fig. 5 shows some of the oscillograph records obtained in the tests, compared with the corresponding dynamic influence lines computed theoretically. In fig. 6 the test

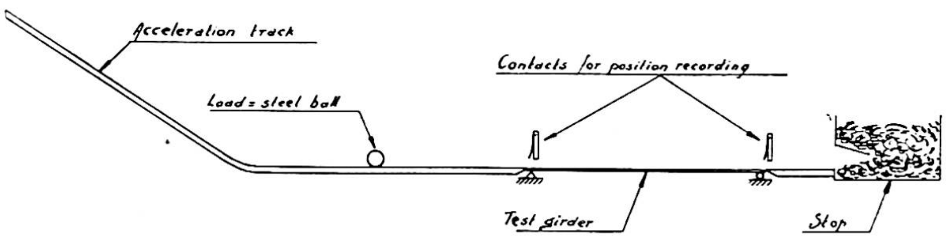

Fig. 4. Set-up used for laboratory tests.

values of ε expressed as function of α for several sections are compared with the computed values.

The agreement between the theoretical and experimental results shows that the theory can be regarded as fairly accurate. Nevertheless, further tests must be made before the first chapter of the investigation can be completed. After that, the questions relating to damping, spring-borne masses, etc., will be studied.

I hope that we shall be able to publish a more detailed account of the results later on. In the meantime, suggestions or questions are welcome.

The Author expresses his gratitude to Professor Georg Wästlund having stimulated the research described in this article.

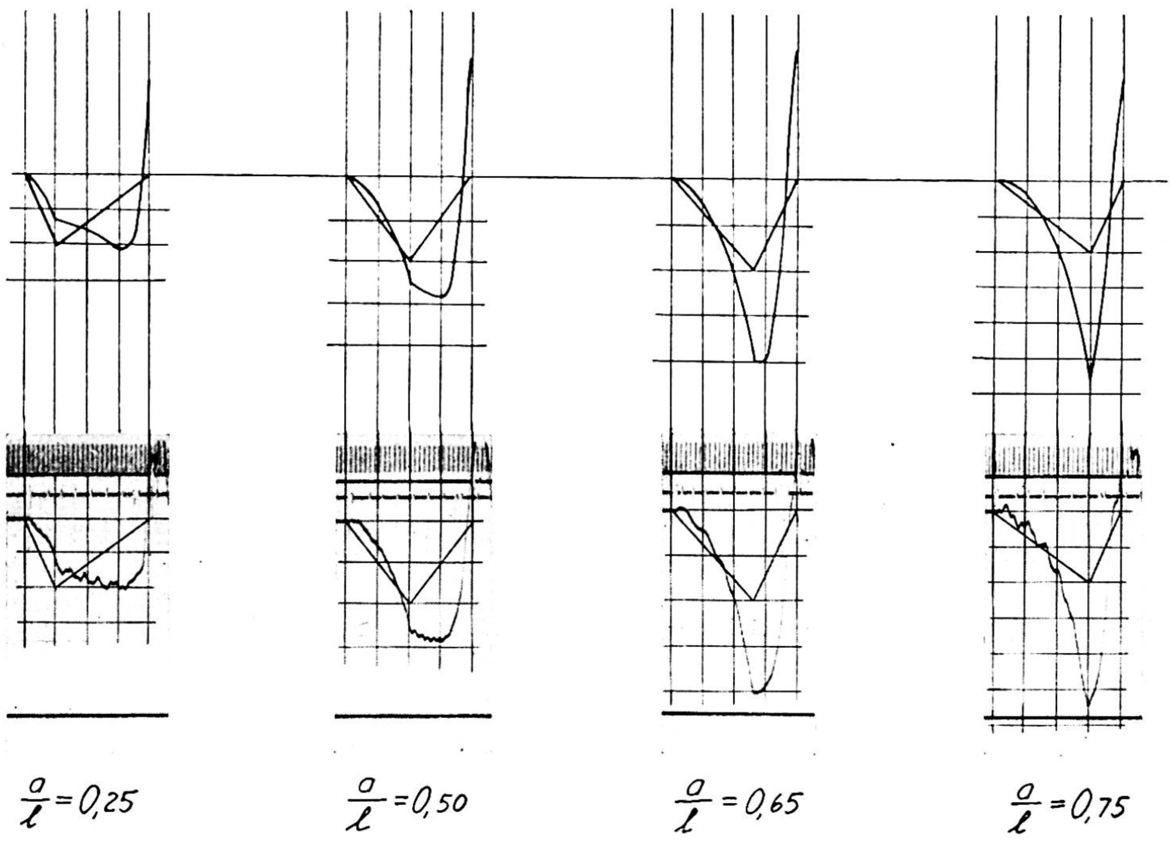

Fig. 5. Theoretically computed dynamic influence lines for bending stresses compared with test values $\nu = 3.5$, $\alpha = 0.2$.

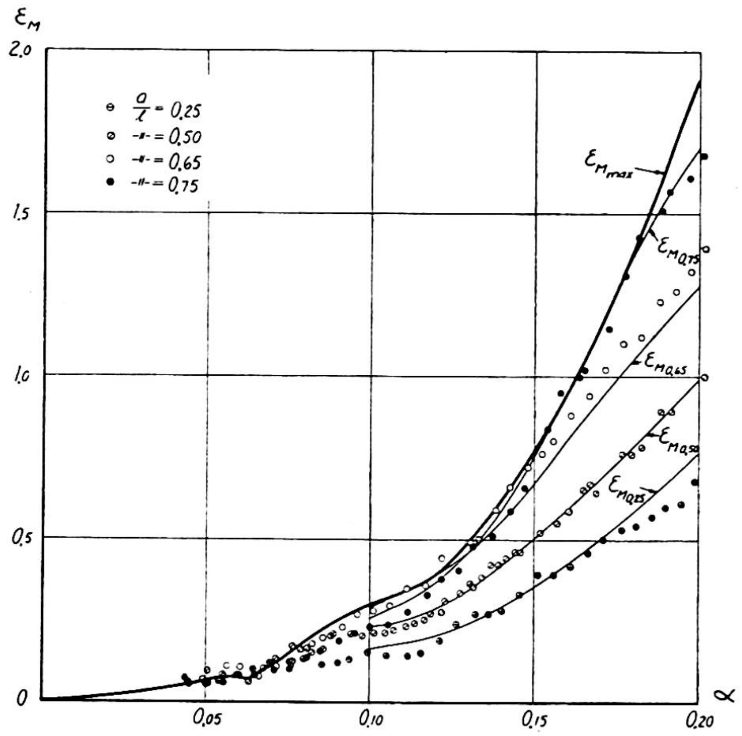

Fig. 6. Theoretical curves showing $\varepsilon_{M\max}$ and ε_{Ma} as functions of α compared with test values of ε_M^v. $\nu = 3.5$.

Résumé

L'Institut de Construction des Bâtiments et des Ponts de l'Ecole Royale Polytechnique à Stockholm poursuit actuellement une étude sur les influences dynamiques exercées par les charges mobiles sur les ponts. La première partie de cette étude traite le cas d'une charge concentrée unique qui roule à une vitesse constante le long d'une poutre à section uniforme. En ce cas, on peut démontrer que l'augmentation dynamique ε est complètement déterminée par deux quantités définies comme suit :

$$\alpha = \frac{\text{vitesse de la charge}}{2 \times \text{fréquence naturelle de la poutre} \times \text{longueur de la poutre}}$$

$$\nu = \frac{\text{masse de la charge}}{\text{masse de la poutre}}$$

Au moyen d'une méthode imaginée par M. le Professeur Inglis, qui a été légèrement modifiée et complétée par l'auteur du présent rapport, on a exprimé ε par une fonction de α et ν. La figure 3 montre l'augmentation dynamique maximum des moments fléchissants.

On a constaté une bonne concordance entre les résultats des calculs théoriques et ceux des essais effectués sur modèle, ainsi qu'il ressort des figures 5 et 6.

Zusammenfassung

Im Institut für Hochbau und Brückenbau an der Kgl. Technischen Hochschule in Stockholm wird gegenwärtig eine Untersuchung über die dynamischen Einflüsse der beweglichen Lasten auf Brücken durchgeführt. Deren erster Teil behandelt den Fall einer Einzellast, die sich mit konstanter

Geschwindigkeit längs eines Trägers von gleichbleibendem Querschnitt bewegt. Es zeigt sich, dass der dynamische Zuschlag ε in diesem Falle durch zwei Grössen vollständig bestimmt ist, die wie folgt definiert werden :

$$\alpha = \frac{\text{Geschwindigkeit der Last}}{2 \times \text{Eigenschwingungszahl des Trägers} \times \text{Länge des Trägers}}$$

$$\nu = \frac{\text{Masse der Last}}{\text{Masse des Trägers}}$$

Mit Hilfe eines von Prof. Inglis angegebenen Verfahrens, das vom Verfasser etwas abgeändert und ergänzt wurde, kann ε als Funktion von α und ν ausgedrückt werden. Abb. 3 zeigt die Höchstwerte des dynamischen Zuschlags zu den Biegungsmomenten.

Bei Modellversuchen wurde weitgehende Uebereinstimmung zwischen den theoretisch errechneten Werten und den Versuchsergebnissen festgestellt, siehe Abb. 5 und 6.

Summary

An investigation into the dynamic influences of moving loads on bridges is being made at the Institution of Structural Engineering and Bridge Building, Royal Institute of Technology, Stockholm, Sweden. The first part of this investigation deals with the case where a single concentrated load moves at a constant speed along a girder of uniform section. In this case, it can be shown that the dynamic increment ε is completely determined by two quantities defined as follows :

$$\alpha = \frac{\text{velocity of load}}{2 \times \text{natural frequency of girder} \times \text{length of girder}}$$

$$\nu = \frac{\text{mass of load}}{\text{mass of girder}}$$

By means of a method due to Prof. Inglis, which has been slightly modified and amplified by the Author, ε has been expressed by a function of α and ν. Fig. 3 shows the maximum dynamic increment in bending moment.

The results of model tests were found to be in good agreement with the theoretical results (see figs. 5 and 6).

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Equation différentielle pour le calcul des vibrations produites dans les constructions portantes par les charges mobiles

Differentialgleichung für die Schwingungsberechnung von Tragkonstruktionen infolge beweglicher Lasten

Differential equation for calculation of vibrations produced in load-bearing structures by moving loads

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This paper is a contribution to the theoretical study of the problem of forced vibrations in load-bearing structures of finite extent subjected to any arbitrary boundary conditions. The vibrations are assumed to be produced by one or several non-elastically applied loads, and possibly also transverse forces devoid of mass, which move on the structure with a constant velocity.

A usual treatment of similar problems consists in deducing a differential equation which represents the motion of the load-bearing structure and the load, and in finding a formal solution to this equation by means of a series expansion. This method has been applied by many authors to a beam which is hinged and freely supported at both ends, and is acted upon by a single moving load. Among these authors, the following deserve to be mentioned in this connection. Kryloff (1) neglects the influence exerted by the mass of the load on the natural vibration of the system, and thus obtains a simple formula for the deformation at any arbitrary subsequent instant. Inglis (2) expresses the deformation and the load by a Fourier series, in which he disregards all terms except the first. In other words, he imagines the concentrated load to be replaced by a load distri-

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(1) Kryloff, A. N., *Mathematische Annalen*, Vol. 61, 1905. See also Timoshenko, S.; *Vibration Problems in Engineering*, U. S. A., 1928.

(2) Inglis, C. E., *A Mathematical Treatise on Vibrations in Railway Bridges*, Cambridge. 1934.

buted over the whole beam according to a sine function having a half wave length which is equal to the length of the beam and an amplitude which varies in accordance with the same sine function. Schallenkamp ([3]) deals in a similar manner with the vibration of the load in a vertical direction only. Looney ([4]) tackles the problem by means of the calculus of differences on the assumption that the natural vibration can be represented by a sine function, and disregards all harmonics except the fundamental. Contrary to Inglis, Rinkert ([5]) takes into account a finite number of terms in the series expressing the concentrated load.

A procedure that is commonly used by most of the investigators referred to in the above is to obtain the solution of the differential equation by means of sine series in which each term can be regarded as an analytical expression of one of the types of vibrations performed by a beam which is hinged and freely supported at both ends. If the beam is subjected to an impact caused by a force which is devoid of mass, vibrations will be produced, the frequencies of which are determined solely by the properties of material, dimensions, and boundary conditions of the beam, and each of these frequencies corresponds to a definite type of vibration. On the other hand, if the force acts in conjunction with a mass which takes part in the vibration, for instance, if the beam is submitted to a moving load, then different conditions will arise. In this case, a continuous change in the position of the mass on the beam gives rise to a continuous variation in each frequency and in the corresponding type of vibration. If we try to find a formal solution to the differential equation of vibration produced by a moving load by using series of functions, which are not variable with time, this implies an attempt to calculate a resulting vibration at any arbitrary gauge point by the aid of a limited number of terms. In reality, this motion is composed of several entirely separate vibrations, and an intricate analysis of frequencies will be required in order to segregate these vibrations.

The use of formal solutions entails insufficient accuracy in the calculation of stresses. Even if the series used for solving the differential equation is found to be convergent, it is not certain that the second derivative will approach a correct value, or will be convergent at all. This is due to the fact that a given initial substitution used in solving the differential equation will prove successful depending on the extent to which the first term of the series agrees with the actual, total deformation. Consequently, the fact that the boundary conditions are satisfied by all terms of the series alone is not sufficient. An analogous statement has been made by Courant ([6]) regarding variational and buckling problems. Therefore, as the type of vibration varies continuously in the case of moving loads, there must always be an uncertainty in the calculation of stresses.

The purpose of this paper is to demonstrate a method for a more general study of this problem under any arbitrary boundary conditions.

([3]) SCHALLENKAMP, A., *Schwingungen von Trägern bei bewegten Lasten* (*Ingenieur-Archiv*, 1937).

([4]) LOONEY, Ch. T. G., *Impact on Railway Bridges* (*University of Illinois, Bulletin* No. 19, Vol. 42, 1944).

([5]) RINKERT, A., *Vibrations of a Beam with Hinged Ends under Action of a Load Moving with Constant Speed.* Examination Work at the Institution of Structural Engineering and Bridge-building at the Royal Institute of Technology, Stockholm 1945 (in Swedish).

([6]) COURANT, R., *Variational Methods for the Solution of Problems of Equilibrium and Vibrations* (*Bulletin of the American Math. Soc.*, Vol. 49, No. 1, Jan. 1943).

In contradistinction from the earlier investigations, we shall take into account the variation in the type of natural vibration due to the change in the position of the load. Each term of the series used in solving the differential equation corresponds to the type of natural vibration performed at a given instant, and is therefore dependent on the position of the load, and hence on time. Accordingly, the state of vibratory motion is known at any instant, and the increments in moment and in stress due to each natural vibration can therefore also be calculated. Just as most other investigators who have dealt with this problem, we neglect the influence of damping, which has been discussed by Holzer (7) and Sezawa (8), among others. The deformations are assumed to be so small that the effect of rotatory inertia and of shear can be disregarded. These questions have been studied by Timoshenko (9), Goens (10), Pickett (11) and others.

Consider free harmonic vibration of a system which is compound in the range G and consists of a load-bearing structure in conjunction with one or several stationary masses. In this case, we can deduce a harmonic differential equation of the following well-known type, which is independent of the time factor

$$L[\varphi_n] + \rho \cdot \lambda_n \cdot \varphi_n = 0 \tag{1}$$

where φ_n is a characteristic function which represents the n-th type of deformation of the system within the range G, and λ_n is the corresponding characteristic value (12). $L[\varphi_n]$ is a linear differential expression formed with respect to the space coordinates x, y, z, and is defined within the same range, and ρ is a given dimensionless function which represents the relative mass density distribution of the system. Eq. (1) is often called Euler's differential equation.

If the section of the load-bearing structure is uniform, the caracteristic functions and the corresponding natural frequencies can be computed from the above differential equation by means of the methods given by Den Hartog (13) or Kármán and Biot (14), and others. For this purpose, a general solution of the differential equation is found for the ranges between the boundaries and the point of application of the load. By applying the boundary conditions, we obtain a frequency equation, and by means of this equation we can calculate a set of roots, each of which corresponds to a definite form of the characteristic function. The lowest root value corresponds to the fundamental frequency. Berg (15) has carried out this calculation for a hinged, freely supported beam, and has tabulated the values of the characteristic function as a function of the position of the

(7) Holzer, H., *Zeitschrift für angew. Math. und Mech.*, V. 8, p. 272, 1928.
(8) Sezawa, K., *Zeitschrift für angew. Math. und Mech.*, V. 12, p. 275, 1932.
(9) Timoshenko, S., *On the Correction for Shear of the Differential Equation for Transverse Vibration of Prismatic Bars* (*Philosophical Magazine*, Ser. 6, Vol. 41, p. 744 and Vol. 43, p. 125).
(10) Goens, E., *Ueber die Bestimmung des Elastizitätsmoduls von Stäben mit Hilfe von Biegungsschwingungen* (*Annalen der Physik*, 5. Ser., Vol. 11, p. 649, 1931).
(11) Pickett, G., *Equations for Computing Elastic Constants from Flexural and Torsional Resonant Frequencies of Vibration of Prisms and Cylinders* (*American Society for Testing Materials*, Vol. 45, 1945).
(12) Courant, R. and Hilbert, D., *Methoden der mathematischen Physik*, Band 1, Berlin, 1924, Kap. V.
(13) Den Hartog, J. P., *Mechanical Vibrations*, New York and London, 1940.
(14) v. Kármán, T. and Biot, M. A., *Mathematical Methods in Engineering*, New York and London, 1940.
(15) Berg, Owe, *Biegungsschwingungen eines in beiden Enden unterstützten punktförmig belasteten Balkens* (*Zeitschr. angew. Math. Mech.*, Bd 24, Nr 1, 1944).

load and definite given ratios between the masses of the load and the beam.

For the majority of practical purposes, some approximate method can usually be applied. For instance, Lord Rayleigh ([16]) assumes that the characteristic function approximates to the static deformation curve produced by the weight of the load, provided that the inertia forces are neglected, and calculates the frequency by means of the energy method. Ritz ([17]) puts the characteristic function equal to a series, and determines the coefficients of expansion and the frequencies by the aid of Hamilton's variational principle. Galerkin ([18]) applies a variational method by which, however, one arrives at exactly the same expression as is obtained by the method devised by Ritz. Gran Olsson ([19]) has found that with the aid of the principle of virtual displacements the same results may be obtained as by the methods of Ritz and Galerkin. Finally, Grammel ([20]) takes the integral equation of the system as a starting-point and determines the frequency from the relation between an assumed deformation and the nucleus.

All these methods give a sufficiently accurate upper limit of the natural frequency. The deviations of the approximate expression of the characteristic function from its exact form have scarcely any influence on the ultimate value, and the error is therefore usually inconsiderable. If greater importance is attached to the form of the characteristic function, as is often the case in the determination of stresses, or if the rigidity, section area and the mass density of the load-bearing structure are variable, it is convenient to use Vianello's ([21]) approximate method for plotting the characteristic function graphically by means of a funicular polygon. Great accuracy can be obtained by applying this procedure several times. This problem has also been dealt with by Inglis ([22]).

The characteristic functions φ_1, φ_2, φ_3, ..., etc., form a complete orthogonal system which satisfies the conditions for orthogonality, viz.,

$$\left.\begin{aligned} \int_G \rho \cdot \varphi_r \cdot \varphi_n d\tau &= 0\,, & r &\neq n \\ \int_G \rho \cdot \varphi_n^2 d\tau &= \text{constant}\,, & r &= n \end{aligned}\right\} \qquad (2)$$

where G denotes the limit range of the integral and τ is used to designate one or several coordinates in space. If the constant is put equal to unity, the characteristic functions are termed normalised.

Furthermore, the theorem of expansion of characteristic functions states that any stepwise finite arbitrary function f which statisfies the same

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([16]) Lord Rayleigh, *The Theory of Sound*, Vol. 1, 2nd Ed., pp. 111 and 287. *Phil. Mag.*, Vol., 47, p. 566, 1899 and Vol. 22, p. 255, 1911.

([17]) Ritz, W., *Ueber eine neue Methode zur Lösung gewissen Variationsprobleme der mathematischen Physik*. (*J. f. reine und angew. Math.*, Bd 135, pp. 1-61, 1909. — *Gesammelte Werke*, p. 265, Paris, 1911).

([18]) Galerkin, B. G., *Expansion in Series for Solving Some Equilibrium Problems for Plates and Beams* (*Wjestnik Ingenerow Petrograd*, 1915, Heft 19, in Russian).

([19]) Gran Olsson, R., *Die Anwendung des Prinzips der virtuellen Arbeit bei der Lösung von Knickproblemen* (*Det Kongelige Norske Videnskabers Selskab, Forhandlinger* Bd XVII, Nr 46).

Gran Olsson, R., *The Principle of Virtual Displacement Applied in Approximate Solutions of Eigenvalue Problems* (*Dixième Congrès des Mathématiciens Scandinaves*, Copenhague, 1946).

([20]) Grammel, R., *Ein neues Verfahren zur Lösung Technischer Eigenwertprobleme* (*Ing.-Arch.*, Bd 10/1939, pp. 35-46). See also : Lösch, Fr., *Berechnung der Eigenwerte linearer Integralgleichungen* (*Zeitschr. angew. Math. Mech.*, Bd 24, Nr 1. 1944).

([21]) Vianello, L., *Grafische Untersuchung der Knickfestigkeit gerader Stäbe* (*Z. V. D. I*, 1898, Juli-Dez., p. 1436).

([22]) Inglis, C. E., *Natural Frequencies and Modes of Vibration in Beams of Non-uniform Mass and Section* (*Trans. I. N. A.*, Vol. LXXI, p. 145, 1929).

boundary conditions as the characteristic functions φ_n and has a self-adjungated finite linear differential expression $L[\varphi_n]$ can be expressed by an absolutely and uniformly convergent series composed of these characteristic functions, viz.,

$$f = \sum_{n=1}^{\infty} c_n \cdot \varphi_n$$

where

$$c_n = \int_G \rho \cdot f \cdot \varphi_n d\tau \,. \tag{3}$$

For simplification, we shall deduce, in the first place, the differential equation for a single moving load having a mass which cannot be neglected.

It is convenient to imagine the masses of the load-bearing structure and of the load as a compound system whose natural vibration is unambiguously determined by the position of the load in a steady state of vibratory motion and varies with the position of the load, whereas the weight of the load is regarded as an external vertical force devoid of mass applied at the centre of gravity of the moving mass. In the treatment of the problem it makes no difference whether a force devoid of mass, e.g. a pulsating force, is added to the weight of the load, and the problem is thus reduced to the determination of the vibration produced by the resultant external force.

Since the system has an infinite number of degrees of freedom for every position of the moving mass, the deformation from the position of equilibrium at any arbitrary instant $t = t_j$ can be expressed by a series comprising all those degrees of freedom which come into play at that instant. Accordingly, we put

$$w = \sum_{n=1}^{\infty} q_n(t_j) \cdot \varphi_{nj}(t_j, s) \tag{4}$$

where w denotes the deformation from the position of equilibrium, and φ_{nj} corresponds to the n-th characteristic function determined in a steady state of vibratory motion with the moving mass in the j-th position on the structure. For instance, φ_{nj} can be imagined to be composed of trigonometric and hyperbolic functions in which the arguments are also variable with time. The quantity q_n is an unknown factor which varies with time only, and is termed a generalised coordinate.

The maximum kinetic and potential energies of the system, which are denoted by T and V respectively, are determined at the instant when the load is at the point s_j.

For the load-bearing structure alone, we obtain

$$T_b = \frac{m}{2} \int_G \left(\frac{\partial w}{\partial t_j}\right)^2 d\tau$$

where m denotes the mass per unit length τ of the structure.

The velocity of the moving mass at any arbitrary point is

$$\frac{\partial w}{\partial t_j} = \sum_{n=1}^{\infty} (\dot{q}_n \cdot \varphi_{nj} + q_n \cdot \dot{\varphi}_{nj})$$

where

$$\dot{q}_n = \frac{dq_n}{dt_j} \quad \text{and} \quad \dot{\varphi}_{nj} = \frac{\partial \varphi_{nj}}{\partial t_j} .$$

Therefore, we get

$$T_b = \frac{m}{2} \int_G \left[\sum_{n=1}^{\infty} (\dot{q}_n \cdot \varphi_{nj} + q_n \cdot \dot{\varphi}_{nj}) \right]^2 d\tau .$$

In calculating the corresponding increase in kinetic energy due to the moving mass having the weight P, we must take into account its curvilinear motion whose component in the direction w is determined by

$$w_P(s_j) = \sum_{n=1}^{\infty} q_n(t_j) \cdot \varphi_{nj}(t_j, s_j)$$

where s_j is also a function of time. Then the velocity of the load in the direction w can be written

$$\frac{\partial w_P}{\partial t_j} = \sum_{n=1}^{\infty} [\dot{q}_n \cdot \varphi_{nj}(s_j) + q_n \cdot \dot{\varphi}_{nj}(s_j)]$$

and the amount contributed by this velocity to the kinetic energy of the system is

$$T_P = \frac{P}{2g} \left[\sum_{n=1}^{\infty} [\dot{q}_n \cdot \varphi_{nj}(s_j) + q_n \cdot \dot{\varphi}_{nj}(s_j)] \right]^2 .$$

The potential energy of the system is equal to the sum of all inertia forces times the respective displacement of these forces. Since the inertia forces are proportional to the deformation, we consider their mean value, and the amount V_b contributed by the load-bearing structure to the potential energy of the system is determined by the expression

$$V_b = \frac{m}{2} \int_G \frac{\partial^2 w}{\partial t_j^2} \cdot \widehat{w} d\tau$$

$$= \frac{m}{2} \int_G \left(\sum_{n=1}^{\infty} q_n \cdot \widecheck{\omega_n^2}(s_j) \cdot \varphi_{nj} \right) \cdot \left(\sum_{n=1}^{\infty} q_n \cdot \varphi_{nj} \right) d\tau$$

where $\omega_n(s_j)$ is the frequency of the n-th natural vibration.

The corresponding increase in potential energy V_p due to the load is

$$V_p = \frac{P}{2g} \left(\sum_{n=1}^{\infty} q_n \cdot \omega_n^2(s_j) \cdot \varphi_{nj}(s_j) \right) \cdot \left(\sum_{n=1}^{\infty} \widehat{q_n} \cdot \widecheck{\varphi_{nj}}(s_j) \right) .$$

Using the notation

$$\int_G \varphi^2_{nj} d\tau + \frac{P}{mg} \cdot \varphi^2_{nj}(s_j) = H_n(s_j)$$

and observing that the terms of the form

$$\int_G \varphi_{rj} \varphi_{nj} d\tau + \frac{P}{mg} \cdot \varphi_{rj}(s_j) \cdot \varphi_{nj}(s_j) ; \qquad r \neq n$$

are zero according to the conditions for orthogonality, see Eq. (2), we can write the total maximum kinetic and potential energies of the system

$$T = \frac{m}{2} \int_G \left[\sum_{n=1}^{\infty} (\dot{q}_n \cdot \varphi_{nj} + q_n \cdot \dot{\varphi}_{nj}) \right]^2 d\tau$$

$$+ \frac{P}{2g} \left[\sum_{n=1}^{\infty} [\dot{q}_n \cdot \varphi_{nj}(s_j) + q_n \cdot \dot{\varphi}_{nj}(s_j)] \right]^2$$

$$V = \frac{m}{2} \sum_{n=1}^{\infty} q_n^2 \cdot \omega_n^2 (s_j) \cdot H_n(s_j) \ . \tag{5}$$

These values are inserted in Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial (T - V)}{\partial q_r} = Q_r$$

when Q_r denotes the force corresponding to q_r and is termed the generalised force. We then obtain

$$\ddot{q}_r \cdot H_r(s_j) + \int_G \varphi_{rj} \sum_{n=1}^{\infty} (2\,\dot{q}_n \cdot \dot{\varphi}_{nj} + q_n \cdot \ddot{\varphi}_{nj})\, d\tau$$

$$+ \frac{P}{mg} \cdot \varphi_{rj}(s_j) \cdot \sum_{n=1}^{\infty} [2\,\dot{q}_n \cdot \dot{\varphi}_{nj}(s_j) + q_n \cdot \ddot{\varphi}_{rj}(s_j)] + q_r \cdot \omega_r^2 (s_j) \cdot H_r(s_j) = \frac{Q_{rj}}{m} \ . \tag{6}$$

The generalised force Q_{rj} is determined so that the work done by the external force in the case of variation in the generalised coordinate q_r divided by this variation should be equal to Q_{rj}.

In this case, the work is equal to $P(s_j) \cdot \delta q_r \cdot \varphi_{rj}(s_j)$, where $P(s_j)$ denotes the weight of the load, possibly with the addition of an external force devoid of mass, which is applied at the same point. We then obtain

$$Q_{rj} = P(s_j) \cdot \varphi_{rj}(s_j) \ .$$

Eq. (6) gives a system of linear inhomogeneous differential equations of the second order having an infinite number of terms and variable coefficients. This equation can be regarded as the complete differential equation of forced vibration produced in a load-bearing structure by a moving non-elastic mass. It is very difficult to find an exact solution of this equation. In order to avoid this difficulty, we must resort to simplifications.

The influence of the curvilinear motion is so slight that it can be regarded as a correction, at least in normal structures met with in practice and subjected to ordinary permissible loads. In such cases it is obvious that the error will be negligible if the correction consists in disregarding the influence of all vibrations except the r-th. If the characteristic functions are normalised so that $\int_G \varphi_{rj}^2 \cdot d\tau = 1$, the variation in the form of the characteristic function must be so small that it could be neglected, with the result that both $\dot{\varphi}_{rj}$ and $\ddot{\varphi}_{rj}$ would become equal to zero. If the velocity of the moving mass is assumed to be constant and equal to v, we get

$$\dot{\varphi}_{rj}(s_j) = v \cdot \overset{\mathrm{I}}{\varphi}_{rj}(s_j) \quad \text{and} \quad \ddot{\varphi}_{rj}(s_j) = v^2 \cdot \overset{\mathrm{II}}{\varphi}_{rj}(s_j)$$

where

$$\overset{\mathrm{I}}{\varphi}_{rj}(s_j) = \frac{d\varphi_{rj}(s_j)}{ds_j} \quad \text{and} \quad \overset{\mathrm{II}}{\varphi}_{rj}(s_j) = \frac{d^2\varphi_{rj}(s_j)}{ds_j^2}$$

Consequently, the complete differential equation takes the simplified form

$$\ddot{q}_r \cdot \mathrm{H}_r(s_j) + \dot{q}_r \cdot \dot{\mathrm{H}}_r(s_j) + q_r \cdot [\omega_r^2(s_j) \cdot \check{\mathrm{H}}_r(s_j) + \mathrm{N}_r(s_j)] = \frac{\mathrm{P}(s_j) \cdot \varphi_{rj}(s_j)}{m} \tag{8}$$

where

$$\mathrm{H}_r(s_j) = 1 + \frac{\mathrm{P}}{mg} \cdot \varphi_{rj}^2(s_j)$$

$$\dot{\check{\mathrm{H}}}_r(s_j) = \frac{2v}{mg} \cdot \mathrm{P} \cdot \overset{\mathrm{I}}{\varphi}_{rj}(s_j) \cdot \varphi_{rj}(s_j)$$

$$\check{\mathrm{N}}_r(s_j) = \frac{v^2}{mg} \cdot \mathrm{P} \cdot \overset{\mathrm{II}}{\varphi}_{rj}(s_j) \cdot \varphi_{rj}(s_j)$$

$$s_j = v \cdot t_j$$

It makes no great difference whether one or several masses move with the same velocity on the load-bearing structure. The variation in mass distribution with time influences the form and the frequency of the characteristic function in a similar manner. It is convenient to keep the notations φ_n and ω_n unchanged, but they are used to denote the new characteristic function and the corresponding natural frequency. Accordingly, the j-th position of the load at any arbitrary instant $t = t_j$ signifies the location of k loads having the respective weights P_1, P_2, P_3, ..., P_k, and $\mathrm{P}_k(s_{kj})$ denotes the weight P_k, possibly with the addition of an external force devoid of mass, which is applied at the point s_{kj}. The new differential equation can be written in the form

$$\ddot{q}_r \cdot \mathrm{H}_r(s_j) + \dot{q}_r \cdot \dot{\mathrm{H}}_r(s_j) + q_r \cdot [\omega_r^2(s_j) \cdot \mathrm{H}_r(s_j) + \mathrm{N}_r(s_j)]$$

$$= \frac{1}{m} \sum_{n=1}^{k} \mathrm{P}_k(s_{kj}) \cdot \varphi_{rj}(s_{kj}) \tag{9}$$

where

$$\mathrm{H}_r(s_j) = 1 + \frac{1}{mg} \cdot \sum_{n=1}^{k} \mathrm{P}_k \cdot \check{\varphi}_{rj}^2(s_{kj})$$

$$\dot{\mathrm{H}}_r(s_j) = \frac{2v}{mg} \cdot \sum_{k=1}^{k} \mathrm{P}_k \cdot \overset{\mathrm{I}}{\varphi}_{rj}(s_{kj}) \cdot \varphi_{rj}(s_{kj})$$

$$\mathrm{N}_r(s_j) = \frac{v^2}{mg} \cdot \sum_{k=1}^{k} \mathrm{P}_k \cdot \overset{\mathrm{II}}{\varphi}_{rj}(s_{kj}) \cdot \varphi_{rj}(s_{kj})$$

$$s_j = v \cdot t_j$$

Eq. (9) is an ordinary linear inhomogeneous differential equation of the second order with variable coefficients.

If the fact that the characteristic function varies with time is completely disregarded, then Eq. (8) corresponds to a formal solution of the problem under consideration. In that case, the functions φ_r and H_r are referred to the non-loaded structure, and $\omega_r^2(s_j) \cdot H_r(s_j)$ is constant and equal to the square of the natural frequency of this structure. For instance if we put

$$\varphi_r = \sqrt{\frac{2}{l}} \cdot \sin \frac{r\pi}{l} x ,$$

which is identical with the normalised characteristic function of the freely supported beam (l = length of beam, $0 \leqslant x \leqslant l$, and r = integers 1, 2, 3, ...), then we obtain for $r = 1$, the same differential equation as that deduced by Inglis. Therefore, it follows that his results can be regarded as a special case.

The tests made lately by A. Hillerborg ([23]) seem to indicate that the variation in the form of the characteristic function with the position of the load could be neglected in the case of the freely supported beam. It remains to be demonstrated whether this variation may also be disregarded in dealing with other structures or under different boundary conditions. This question can be examined theoretically on the basis of the present investigation.

An approximate solution of the differential equation (9) is briefly deduced in what follows.

By putting the right-hand member of Eq. (9) equal to zero, we obtain a homogeneous differential equation. After dividing by $H_r(s_j)$, this equation can be written

$$\ddot{q}_r + \frac{\dot{H}_r(s_j)}{H_r(s_j)} \cdot \dot{q}_r + \left[\omega_r^2(s_j) + \frac{N_r(s_j)}{H_r(s_j)}\right] \cdot q_r = 0 \tag{10}$$

and can be interpreted as a differential equation of free amplitude-modulated and frequency-modulated vibration.

To solve this equation, we put

$$q_r = e^{-\int \beta_r dt} (A \sin \int \varpi_r dt + B \cos \int \varpi_r dt) \tag{11}$$

where A and B are arbitrary constants, whereas s_r and ϖ_r are arbitrary functions of time which express the damping and the frequency of vibration respectively. If this expression is inserted in eq. (10) the two relations

$$\beta_r = \frac{1}{2}\left[\frac{\dot{\varpi}_r}{\varpi_r} + \frac{\dot{H}_r}{H_r}\right] \tag{12a}$$

$$\varpi_r^2 = \omega_r^2 + \frac{N_r}{H_r} - \dot{\beta}_r - \beta_r \cdot \frac{\dot{H}_r}{H_r} + \beta_r^2 \tag{12b}$$

must hold good in order that the above expression should satisfy the differential equation (10).

If we assume that the vibration sets in at the intsant $t = t_i$, the value of $_{\exp}[-\int \beta_r dt]$ at any arbitrary subsequent instant $t = t_j$ can be calculated by integration. By using eq. (12a), we then obtain

$$exp\left[-\int_{t_i}^{t_j} \beta_r dt\right] = \sqrt{\frac{\varpi_r(s_i)}{\varpi_r(s_j)} \cdot \frac{\omega_r(s_j)}{\omega_r(s_i)}} . \tag{13}$$

([23]) Hillerborg, A. L., *A Study of Dynamic Influences of Moving Loads on Girders (International Association for Bridge and Structural Engineering. Congress 1948 at Liège. Preliminary Publication)*.

If the natural frequency is determined by Rayleigh's method, it can be demonstrated that

$$\sqrt{\frac{H_r(s_i)}{H_r(s_j)}} \approx \frac{\omega_r(s_j)}{\omega_r(s_i)} . \tag{14}$$

Since the modulated frequency ϖ_r does not differ from the natural frequency of the system to any notable extent, $\frac{\varpi_r(s_i)}{\varpi_r(s_j)}$ can be put approximately equal to $\frac{\omega_r(s_i)}{\omega_r(s_j)}$. Then

$$exp\left[-\int_{t_i}^{t_j} \beta_r dt\right] \approx \sqrt{\frac{\omega_r(s_j)}{\omega_r(s_i)}} \tag{15}$$

that is to say, the free vibration is damped approximately in proportion to the square root of the natural frequency.

The difference between the right-hand and left-hand members of eq. (14) is so small that derivation can be admitted without involving any considerable error. We get

$$\frac{\dot{\omega}_r}{\omega_r} \approx -\frac{1}{2}\left(\frac{\dot{H}_r}{H_r}\right) .$$

Noticing that

$$\frac{\dot{\varpi}_r}{\varpi_r} \approx \frac{\dot{\omega}_r}{\omega_r}$$

and inserting this expression in eq (12*a*) gives

$$\beta_r \approx \frac{1}{4}\left(\frac{\dot{H}_r}{H_r}\right) .$$

Using this relation it can be deduced that the frequency of vibration is approximately determined by

$$\varpi_r^2 = \omega_r^2 + \frac{1}{2} \cdot \frac{N_r}{H_r} - \frac{1}{16}\left(\frac{\dot{H}_r}{H_r}\right)^2 \cdot \frac{H_r + 1}{H_r - 1} . \tag{16}$$

Consequently

$$q_r = \sqrt{\frac{\varpi_r(s_i)}{\varpi_r(s_j)} \cdot \frac{\omega_r(s_j)}{\omega_r(s_i)}} \cdot \left(A \sin \int_{t_i}^{t_j} \varpi_r \, dt + B \cos \int_{t_i}^{t_j} \varpi_r dt\right) \tag{17}$$

can be regarded as a general solution of this equation (10). The arbitrary constants A and B are determined by the initial conditions at the instant $t = t_i$.

The solution of eq. (9) can now also be found by means of generally known methods. If we assume that the first of several consecutive loads travelling on the load-bearing structure passes over the first support at the instant $t = 0$, the solution can be written in the form

$$q_r = \frac{1}{m}\int_0^{t_j}\left[\frac{\sum_{k=1}^{k} P_k(s_{ki}) \cdot \varphi_{ri}(s_{ki})}{H_r(s_i) \cdot \varpi_r(s_i)} \cdot \frac{\omega_r(s_j)}{\omega_r(s_i)} \cdot \sqrt{\frac{\varpi_r(s_i)}{\varpi_r(s_j)}} \cdot \sin \int_{t_i}^{t_j} \varpi_r dt\right] dt_i . \tag{18}$$

This expression can also be simplified. If the static deformation of the load-bearing structure due to the loads in the i-th position is calculated with the help of series composed of the functions φ_{ri} and φ_{rj}, it can readily be demonstrated that

$$\frac{\varphi_{ri}(s_{ki})}{H_r(s_i)\cdot\omega_r^2(s_i)} \approx \frac{\varphi_{rj}(s_{ki})}{H_r(s_j)\cdot\omega_r^2(s_j)} . \tag{19}$$

By using this relation and putting $m\cdot H_r(s_j)\cdot\omega_r^2(s_j)=\lambda_r(s_j)$ the solution of the differential equation (9) can finally be written

$$q_r=\frac{1}{\lambda_r(s_j)}\cdot\frac{\omega_r(s_j)}{\sqrt{\varpi_r(s_j)}}\int_0^{t_j}\left[\frac{\omega_r(s_i)}{\sqrt{\varpi_r(s_i)}}\left(\sum_{k=1}^{k}P_k(s_{ki})\cdot\varphi_{rj}(s_{ki})\right)\cdot\sin\int_{ti}^{t_j}\varpi_r\,dt\right]dt_i \tag{20}$$

and the problem under consideration can thus be regarded as theoretically solved.

The present work will be continued in a manner to allow a comparison between experimentally obtained results and theoretical values calculated with the aid of the developed theory.

Résumé

Le calcul théorique des déformations et des efforts causés dans les constructions portantes par les charges mobiles peut aussi être effectué dans le cas des constructions relativement compliquées. Dans la présente étude, ce problème est traité d'une manière plus générale que dans les travaux précédents, en faisant usage des fonctions caractéristiques, de sorte que l'équation différentielle établie dans le présent rapport est applicable aux conditions aux limites quelconques et à plusieurs charges mobiles inélastiques avec addition éventuelle d'autres forces dépourvues de masse. L'étude tient compte de la variation de la forme de la vibration naturelle avec la position de la charge. Une solution approximative de l'équation différentielle est présentée. L'auteur examine aussi d'autres méthodes applicables à l'étude de ce problème.

Zusammenfassung

Die Formänderungen und Spannungen, die in Tragkonstruktionen infolge einer oder mehrerer beweglichen Lasten entstehen, können auch bei verhältnismässig verwickelten Konstruktionen theoretisch berechnet werden. Zum Unterschied von früheren Untersuchungen, wird diese Frage in der vorliegenden Arbeit einer allgemeineren Behandlung unterzogen, und zwar mit Hilfe von Eigenfunktionen, so dass die aufgestellte Differentialgleichung für beliebige Randbedingungen sowie auch für mehrere bewegliche, nichtfedernde Lasten gegebenenfalls in Verbindung mit anderen massenlosen Kräften gültig ist. Die Änderung der Eigenschwingungsform mit der Lage der Last wird ebenfalls berücksichtigt. Eine angenäherte Lösung der Differentialgleichung ist gegeben. Auch andere Verfahren, die auf diese Frage anwendbar sind, werden besprochen.

Summary

The deformations and stresses produced in load-bearing structures by one or several moving loads can be calculated theoretically, even in the case of relatively complicated structures. In contradistinction from previous investigations, the present paper deals with this problem in a more general manner with the help of characteristic functions, so that the differential equation deduced in the paper holds good for any arbitrary boundary conditions and for several non-elastic moving loads, possibly acting in conjunction with other forces devoid of mass. The variation in the form of the natural vibration with the position of the load is also taken into account. An approximate solution of the differential equation is presented. Other methods for studying the problem are discussed.

Vb4

Vibrations amorties des portiques

Gedämpfte Schwingungen von Rahmenträgern

Damped oscillation of frame girders

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Les lois de la vibration amortie des constructions hyperstatiques ne sont pas élucidées du point de vue expérimental; le plus souvent on suppose que l'amortissement de la vibration est proportionnel à la vitesse du mouvement. Dans cet article, traitant du problème de la vibration amortie des portiques, nous admettons cette hypothèse facilitant l'analyse mathématique, sans discuter son exactitude. Nous simplifierons encore le problème en ne considérant que la vibration transversale forcée des portiques ayant des barres à section constante bien que la même méthode puisse être appliquée pour des barres à section variable et aussi pour les autres types de vibrations.

Résolution de la vibration amortie des systèmes en portique par la méthode de déformation

Dans ses travaux précédents l'auteur de ce mémoire a appliqué pour le calcul de la vibration non amortie la méthode de déformation ([1, 2]). On peut s'en servir pour le calcul de la vibration forcée amortie.

Pour le mouvement transversal amorti de l'élément de longueur d'une barre à section constante on a la relation

$$\mu dx \frac{\partial^2 v(x,t)}{\partial t^2} + b dx \frac{\partial v(x,t)}{\partial t} + \mathrm{EJ} \frac{\partial^4 v(x,t)}{\partial x^4} dx = 0 \qquad (1)$$

où $v(x, t)$ signifie le déplacement de la section x dans le temps t;

(1) *Stahlbau*, 1943, p. 5.
(2) *Mémoires de l'A. I. P. C.*, 8e volume : *Solution des pylônes d'antenne haubanés.*

μ la masse de l'unité de longueur de la barre, et
b l'amortissement par l'unité de longueur pendant la vitesse unité.

Dans la suite on remplace b par ω_b (ayant la forme de la fréquence angulaire) qui résulte de l'équation

$$b = 2\,\mu\omega_b\,. \tag{2}$$

En se limitant à la vibration forcée et harmonique, l'équation (1) est satisfaite par l'expression complexe

$$v\,(x,t) = [v\,(x) + i\bar{v}\,(x)]\sin\omega t + [\bar{v}\,(x) - iv\,(x)]\cos\omega t = [\bar{v}\,(x) - iv\,(x)]e^{i\omega t} \tag{3}$$

Après la substitution de (3) à (1) on obtient l'équation différentielle ordinaire

$$-\mu\omega^2\left(1 - 2i\frac{\omega_b}{\omega}\right)[v\,(x) + i\bar{v}(x)] + \mathrm{EJ}\,\frac{d^4\,[v\,(x) + i\bar{v}\,(x)]}{dx^4} = 0 \tag{4}$$

dont la solution générale est

$$v\,(x) + i\bar{v}(x) = \mathrm{C}_1\cos h\,(\Lambda + i\bar{\Lambda})\frac{x}{l} + \mathrm{C}_2\sin h\,(\Lambda + i\bar{\Lambda})\frac{x}{l}$$
$$+ \mathrm{C}_3\cos(\Lambda + i\bar{\Lambda})\frac{x}{l} + \mathrm{C}_4\sin(\Lambda + i\bar{\Lambda})\frac{x}{l} \tag{5}$$

où

$$\Lambda + i\bar{\Lambda} = l\sqrt[4]{\frac{\mu\omega^2}{\mathrm{EJ}}\left(1 - 2i\frac{\omega_b}{\omega}\right)} = \lambda\sqrt[4]{1 - 2i\frac{\omega_b}{\omega}}$$

avec

$$\Lambda = \lambda\sqrt{\frac{1}{2}\sqrt[4]{1 + 4\frac{\omega_b^2}{\omega^2}} + \sqrt{\frac{1}{8}\sqrt{1 + 4\frac{\omega_b^2}{\omega^2}} + \frac{1}{8}}} \tag{6}$$

$$\bar{\Lambda} = -\lambda\sqrt{\frac{1}{2}\sqrt[4]{1 + 4\frac{\omega_b^2}{\omega^2}} - \sqrt{\frac{1}{8}\sqrt{1 + 4\frac{\omega_b^2}{\omega^2}} + \frac{1}{8}}}$$

Si l'on ne considère que les membres réels on déduit de l'expression (3) que la composante de la vibration ayant l'amplitude $\bar{v}(x)$ précède la composante avec l'amplitude $v(x)$ de l'angle $\varphi = \frac{\pi}{2}$.

Si les extrémités de la barre subissent le mouvement harmonique (d'après la fig. 1) avec l'amplitude

$$\gamma_g + i\bar{\gamma}_g\,,\quad v_g + i\bar{v}_g\,,\quad \gamma_h + i\bar{\gamma}_h\quad \text{et}\quad v_h + i\bar{v}_h\,,$$

nous obtenons pour les constantes C quatre équations

$$\mathrm{C}_1 + \mathrm{C}_3 = v_g + i\bar{v}_g$$

$$\mathrm{C}_2 + \mathrm{C}_4 = \frac{l}{\Lambda + i\bar{\Lambda}}(\gamma_g + i\bar{\gamma}_g)$$

$$\mathrm{C}_1\cos h\,(\Lambda + i\bar{\Lambda}) + \mathrm{C}_2\sin h\,(\Lambda + i\bar{\Lambda}) + \mathrm{C}_3\cos(\Lambda + i\bar{\Lambda}) + \mathrm{C}_4\sin(\Lambda + i\bar{\Lambda})$$
$$= v_h + i\bar{v}_h$$

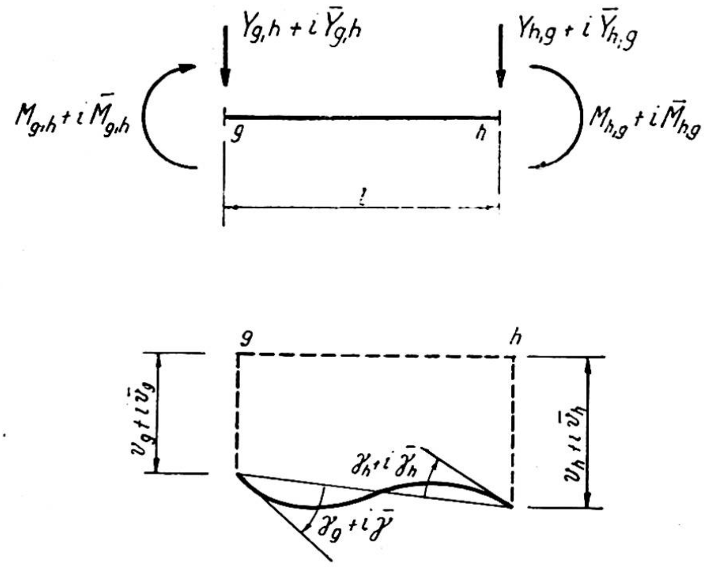

Fig. 1.

$$C_1 \sin h\,(\Lambda + i\bar{\Lambda}) + C_2 \cos h\,(\Lambda + i\bar{\Lambda}) - C_3 \sin(\Lambda + i\bar{\Lambda}) + C_4 \cos(\Lambda + i\bar{\Lambda}) = \frac{l}{\Lambda + i\bar{\Lambda}}(\gamma_h + i\bar{\gamma}_h)$$

d'où

$$C_1 = \frac{1}{2(\Lambda + i\bar{\Lambda})^2}\left[F_4(\Lambda + i\bar{\Lambda})\cdot(v_g + i\bar{v}_g) + F_3(\Lambda + i\bar{\Lambda})\cdot(v_h + i\bar{v}_h)\right.$$
$$\left. - l\cdot F_2(\Lambda + i\bar{\Lambda})\cdot(\gamma_g + i\bar{\gamma}_g) - l\cdot F_1(\Lambda + i\bar{\Lambda})\cdot(\gamma_h + i\bar{\gamma}_h)\right] + \frac{1}{2}(v_g + i\bar{v}_g)$$

$$C_2 = \frac{1}{2(\Lambda + i\bar{\Lambda})^3}\left[F_6(\Lambda + i\bar{\Lambda})\cdot(v_g + i\bar{v}_g) + F_5(\Lambda + i\bar{\Lambda})\cdot(v_h + i\bar{v}_h)\right. \qquad (7)$$
$$\left. - l\cdot F_4(\Lambda + i\bar{\Lambda})(\gamma_g + i\bar{\gamma}_g) + l\cdot F_3(\Lambda + i\bar{\Lambda})\cdot(\gamma_h + i\bar{\gamma}_h)\right] + \frac{l}{2(\Lambda + i\bar{\Lambda})}(\gamma_g + i\bar{\gamma}_g)$$

$$C_3 = -C_1 + v_g + i\bar{v}_g$$

$$C_4 = -C_2 + \frac{l}{\Lambda + i\bar{\Lambda}}(\gamma_g + i\bar{\gamma}_g)$$

avec

$$F_1(\Lambda + i\bar{\Lambda}) = -(\Lambda + i\bar{\Lambda})\,\frac{\sin h\,(\Lambda + i\bar{\Lambda}) - \sin(\Lambda + i\bar{\Lambda})}{\cos h\,(\Lambda + i\bar{\Lambda})\cos(\Lambda + i\bar{\Lambda}) - 1}$$

$$F_2(\Lambda + i\bar{\Lambda}) = -(\Lambda + i\bar{\Lambda})\,\frac{\cos h\,(\Lambda + i\bar{\Lambda})\sin(\Lambda + i\bar{\Lambda}) - \sin h\,(\Lambda + i\bar{\Lambda})\cos(\Lambda + i\bar{\Lambda})}{\cos h\,(\Lambda + i\bar{\Lambda})\cos(\Lambda + i\Lambda) - 1}$$

$$F_3(\Lambda + i\bar{\Lambda}) = -(\Lambda + i\bar{\Lambda})^2\,\frac{\cos h\,(\Lambda + i\bar{\Lambda}) - \cos(\Lambda + i\bar{\Lambda})}{\cos h\,(\Lambda + i\Lambda)\cos(\Lambda + i\bar{\Lambda}) - 1} \qquad (8)$$

$$F_4(\Lambda + i\bar{\Lambda}) = (\Lambda + i\bar{\Lambda})^2\,\frac{\sin h\,(\Lambda + i\bar{\Lambda})\sin(\Lambda + i\bar{\Lambda})}{\cos h\,(\Lambda + i\bar{\Lambda})\cos(\Lambda + i\bar{\Lambda}) - 1}$$

$$F_5(\Lambda + i\overline{\Lambda}) = (\Lambda + i\overline{\Lambda})^3 \frac{\sin h(\Lambda + i\overline{\Lambda}) + \sin(\Lambda + i\overline{\Lambda})}{\cos h(\Lambda + i\overline{\Lambda}) \cos(\Lambda + i\overline{\Lambda}) - 1}$$

$$F_6(\Lambda + i\overline{\Lambda}) = -(\Lambda + i\overline{\Lambda})^3 \frac{\cos h(\Lambda + i\overline{\Lambda}) \sin(\Lambda + i\overline{\Lambda}) + \sin h(\Lambda + i\overline{\Lambda}) \cos(\Lambda + i\overline{\Lambda})}{\cos h(\Lambda + i\overline{\Lambda}) \cos(\Lambda + i\overline{\Lambda}) - 1}$$

Pour le moment fléchissant et l'effort tranchant dans la section x et pour le temps t on a

$$M(x, t) = -EJ \frac{\partial^2 v(x, t)}{\partial x^2}, \qquad T(x, t) = -EJ \frac{\partial^3 v(x, t)}{\partial x^3} \tag{9}$$

et par conséquent pour les moments et les forces aux extrémités de la barre déformée on a les équations suivantes (écrites pour la barre horizontale)

$$M_{g,h} + i\overline{M}_{g,h} = M(0) + i\overline{M}(0) = -\frac{EJ}{l^2} F_4(\Lambda + i\overline{\Lambda}) \cdot (v_g + i\overline{v}_g) - \frac{EJ}{l^2}$$

$$F_3(\Lambda + i\overline{\Lambda}) \cdot (v_h + i\overline{v}_h) + \frac{EJ}{l} F_2(\Lambda + i\overline{\Lambda}) \cdot (\gamma_g + i\overline{\gamma}_g) + \frac{EJ}{l}$$

$$F_1(\Lambda + i\overline{\Lambda}) \cdot (\gamma_h + i\overline{\gamma}_h) \tag{10}$$

$$Y_{g,h} + i\overline{Y}_{g,h} = -T(0) - i\overline{T}(0) = \frac{EJ}{l^3} F_6(\Lambda + i\overline{\Lambda}) \cdot (v_g + i\overline{v}_g) + \frac{EJ}{l^3}$$

$$F_5(\Lambda + i\overline{\Lambda}) \cdot (v_h + i\overline{v}_h) - \frac{EJ}{l^2} F_4(\Lambda + i\overline{\Lambda}) \cdot (\gamma_g + i\overline{\gamma}_g) + \frac{EJ}{l^2}$$

$$F_3(\Lambda + i\overline{\Lambda}) \cdot (\gamma_h + i\overline{\gamma}_h) \tag{11}$$

En résolvant les systèmes en portique par la méthode de déformation on tire les déplacements et les rotations inconnus des équations qui ressortent des conditions d'équilibre dans les nœuds singuliers. Pour le nœud g quelconque il vient

$$\begin{aligned} \Sigma(M_{g,h} + i\overline{M}_{g,h}) - (M_g^e + i\overline{M}_g^e) &= 0 \\ \Sigma(X_{g,h} + i\overline{X}_{g,h}) - (X_g^e + i\overline{X}_g^e) &= 0 \\ \Sigma(Y_{g,h} + iY_{g,h}) - (Y_g^e + i\overline{Y}_g^e) &= 0 \,. \end{aligned} \tag{12}$$

$\Sigma(X_{g,h} + i\overline{X}_{g,h})$, $\Sigma(Y_{g,h} + i\overline{Y}_{g,h})$ étant les composantes horizontale et verticale des forces aux extrémités de toutes les barres, qui aboutissent au nœud g ;

$(X_g^e + i\overline{X}_g^e)$, $(Y_g^e + i\overline{Y}_g^e)$, $(M_g^e + i\overline{M}_g^e)$ étant les forces et le moment extérieurs agissant dans le nœud g.

Exemple numérique

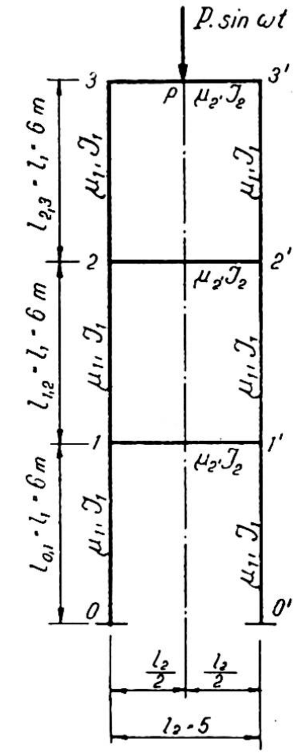

Fig. 2.

Considérons le portique étagé d'après la figure 2, chargé au milieu de la poutre $\overline{3,3'}$ par la force verticale harmoniquement variable $P \sin \omega t$.

Les moments d'inertie des poutres horizontales sont

$$J_2 = J_{1,1'} = J_{2,2'} = J_{3,3'} = 2 \cdot 10^{-4} \text{m}^4$$

et des piliers

$$J_1 = J_{0,1} = J_{1,2} = J_{2,3} = 1 \cdot 10^{-4} \text{m}^4.$$

La masse par l'unité de longueur est dans le cas des poutres horizontales

$$\mu_2 = \mu_{1,1'} = \mu_{2,2'} = \mu_{3,3'} = 0{,}5/9{,}81 = 0{,}0510 \quad \text{tonne.m}^{-2}.\text{sec}^2$$

et des piliers

$$\mu_1 = 0{,}2/9{,}81 = 0{,}0204 \quad \text{tonne.m}^{-2}.\text{sec}^2\,.$$

On calcule avec le module d'élasticité $E = 21 \cdot 10^6$ tonne/m^2 et avec le coefficient de l'amortissement $\omega_b = 2\pi \cdot 10 = 62{,}83$ sec^{-1} (on adopte pour ω_b une si grande valeur pour rendre l'exemple plus instructif).

Des conditions d'équilibre des moments [d'après (10) et (12)] dans les nœuds 1, 2 et 3 on obtient pour les amplitudes de déformations, en cas de la vibration forcée, amortie et symétrique, les trois équations du tableau I.

Le membre absolu de l'équation 3 du tableau I est déterminé comme le moment à l'extrémité de la barre parfaitement encastrée 3, p (d'après la fig. 3) de l'équation

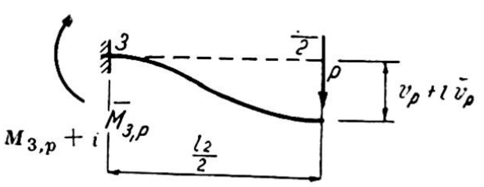

Fig. 3.

	$\gamma_1 + i\bar{\gamma}_1$	$\gamma_2 + i\bar{\gamma}_2$	$\gamma_3 + i\bar{\gamma}_2$	
1	$\frac{2EJ_1}{l_1} F_2(\Lambda_1 + i\bar{\Lambda}_1) + \frac{EJ_2}{l_2} [F_2(\Lambda_2 + i\bar{\Lambda}_2) - F_1(\Lambda_2 + i\bar{\Lambda}_2)]$	$\frac{EJ_1}{l_1} F_1(\Lambda_1 + i\bar{\Lambda}_1)$		$= 0$
2	$\frac{EJ_1}{l_1} F_1(\Lambda_1 + i\bar{\Lambda}_1)$	$\frac{2EJ_1}{l_1} F_2(\Lambda_1 + i\bar{\Lambda}_1) + \frac{EJ_2}{l_2} [F_2(\Lambda_2 + i\bar{\Lambda}_2) - F_1(\Lambda_2 + i\bar{\Lambda}_2)]$	$\frac{EJ_1}{l_1} F_1(\Lambda_1 + i\bar{\Lambda}_1)$	$= 0$
3		$\frac{EJ_1}{l_1} F_1(\Lambda_1 + i\bar{\Lambda}_1)$	$\frac{EJ_1}{l_1} F_2(\Lambda_1 + i\bar{\Lambda}_1) + \frac{EJ_2}{l_2} [F_2(\Lambda_2 + i\bar{\Lambda}_2) - F_1(\Lambda_2 + i\bar{\Lambda}_2)]$	$= \frac{Pl_2}{4} \frac{F_3\left(\frac{\Lambda_2 + i\bar{\Lambda}_2}{2}\right)}{F_6\left(\frac{\Lambda_2 + i\bar{\Lambda}_2}{2}\right)}$

TABLEAU I

$$M_{3,p} + i\bar{M}_{3,p} = -\frac{EJ_2}{\left(\frac{l_2}{2}\right)^2} F_3\left(\frac{\Lambda_2 + i\bar{\Lambda}_2}{2}\right) \cdot (v_p + i\bar{v}_p) \tag{13}$$

où

$$v_p + i\bar{v}_p = \frac{P}{2} \frac{\left(\frac{l_2}{2}\right)^3}{EJ_2} \frac{1}{F_6\left(\frac{\Lambda_2 + i\bar{\Lambda}_2}{2}\right)} \cdot \tag{14}$$

Si nous supposons que la fréquence angulaire de la force harmonique $P \sin \omega t$ est $\omega = 103,7 \text{ sec}^{-1}$ (ce qui est dans le cas donné égal à la première fréquence angulaire de la vibration symétrique propre non amortie), nous obtenons

$$\lambda_1 = l_1 \sqrt[4]{\frac{\mu_1 \omega^2}{EJ_1}} = 6 \sqrt[4]{\frac{0,0204 \cdot 103,7^2}{21 \cdot 10^6 \cdot 1 \cdot 10^{-4}}} = 3,410$$

et d'après (6)

$$\Lambda_1 = 3,410 \sqrt{\frac{1}{2} \sqrt[4]{1 + \frac{4.62,83^2}{103,7^2}} + \sqrt{\frac{1}{8} \sqrt{1 + \frac{4.62,83^2}{103,7^2}} + \frac{1}{8}}}$$

$$= 3,410 \sqrt{0,6267 + 0,5669} = 3,726$$

$$\bar{\Lambda}_1 = -3,410 \sqrt{0,6267 - 0,5669} = -0,834.$$

Il s'ensuit

$$\cos h(\Lambda_1+i\overline{\Lambda}_1)=\cos h\Lambda_1 \cos \overline{\Lambda}_1+i \sin h\Lambda_1 \sin \overline{\Lambda}_1=13{,}96-15{,}37\ i$$
$$\sin h(\Lambda_1+i\overline{\Lambda}_1)=13{,}94-15{,}39\ i$$
$$\cos(\Lambda_1+i\overline{\Lambda}_1)=-1{,}141-0{,}516\ i$$
$$\sin(\Lambda_1+i\overline{\Lambda}_1)=-0{,}755+0{,}779\ i$$

et d'après (8)

$$\frac{EJ_1}{l_1} F_1(\Lambda_1+i\overline{\Lambda}_1)=858-664\ i$$
$$\frac{EJ_1}{l_1} F_2(\Lambda_1+i\overline{\Lambda}_1)=1\,132+810\ i$$

Par analogie on obtient pour $\lambda_2=3{,}005$

$$\Lambda_2+i\overline{\Lambda}_2=3{,}283-0{,}735\ i$$

et

$$\frac{EJ_2}{l_2}\left[F_2(\Lambda_2+i\overline{\Lambda}_2)-F_1(\Lambda_2+i\overline{\Lambda}_2)\right]=748+1\,851\ i\ .$$

Le membre absolu a la valeur

$$\frac{Pl_2}{4}\,\frac{F_3\left(\frac{\Lambda_2+i\overline{\Lambda}_2}{2}\right)}{F_6\left(\frac{\Lambda_2+i\overline{\Lambda}_2}{2}\right)}=P\,(0{,}718-0{,}1887\,i)\ .$$

Si nous substituons les valeurs déterminées dans les équations du tableau I nous obtenons les équations du tableau II.

	$\gamma_1+i\overline{\gamma_1}$	$\gamma_2+i\overline{\gamma_2}$	$\gamma_3+i\overline{\gamma_3}$	
1	3 012 + 3 471 i	858 — 664 i		= O
2	858 — 664 i	3 012 + 3 471 i	858 — 664 i	= O
3		858 — 664 i	1 880 + 2 661 i	= P (0,718 — 0,1887 i)

TABLEAU II

Il résulte des équations de ce tableau

$$\gamma_1=-0{,}050\cdot 10^{-4}\ P \qquad \overline{\gamma_1}=0{,}100\cdot 10^{-4}\ P$$
$$\gamma_2=0{,}435\cdot 10^{-4}\ P \qquad \overline{\gamma_2}=0{,}188\cdot 10^{-4}\ P$$
$$\gamma_3=0{,}742\cdot 10^{-4}\ P \qquad \overline{\gamma_3}=-1{,}986\cdot 10^{-4}\ P$$

ainsi que les rotations des nœuds sont

$$\gamma_1(t)=\gamma_1 \sin \omega t+\overline{\gamma_1}\cos \omega t=(-0{,}050 \sin 103{,}7\ t+0{,}100 \cos 103{,}7\ t)\cdot 10^{-4}\ P$$
$$\gamma_2(t)=\gamma_2 \sin \omega t+\overline{\gamma_2}\cos \omega t=(0{,}435 \sin 103{,}7\ t+0{,}188 \cos 103{,}7\ t)\cdot 10^{-4}\ P$$
$$\gamma_3(t)=\gamma_3 \sin \omega t+\overline{\gamma_3}\cos \omega t=(0{,}742 \sin 103{,}7\ t-1{,}986 \cos 103{,}7\ t)\cdot 10^{-4}\ P$$

Il vient des résultats précédents que le décalage de phase

$$\varphi = \operatorname{arctg} \frac{\bar{\gamma}}{\gamma}$$

est différent dans chaque nœud envisagé.

L'auteur a contrôlé les résultats obtenus par une autre méthode de calcul qui consiste à décomposer la vibration dans ses formes propres ([3]).

Nous ne reproduisons ici que les résultats, par suite de manque de place. En ne considérant que les trois membres de la série des vibrations propres, il vient

$$\gamma_1 = (0 - 0{,}328 + 0{,}306) \cdot 10^{-4}\,P = -0{,}022 \cdot 10^{-4}\,P$$

$$\bar{\gamma}_1 = (-0{,}435 + 0{,}825 - 0{,}298) \cdot 10^{-4}\,P = 0{,}092 \cdot 10^{-4}\,P$$

$$\gamma_2 = (0 + 0{,}037 + 0{,}464) \cdot 10^{-4}\,P = 0{,}501 \cdot 10^{-4}\,P$$

$$\bar{\gamma}_2 = (0{,}717 - 0{,}092 - 0{,}451) \cdot 10^{-4}\,P = 0{,}174 \cdot 10^{-4}\,P$$

$$\gamma_3 = (0 + 0{,}324 + 0{,}395) \cdot 10^{-4}\,P = 0{,}719 \cdot 10^{-4}\,P$$

$$\bar{\gamma}_3 = (-0{,}745 - 0{,}814 - 0{,}384) \cdot 10^{-4}\,P = -1{,}943 \cdot 10^{-4}\,P$$

où les termes entre parenthèses expriment la participation des formes particulières des vibrations propres pour les résultats finaux.

Résumé

Pour le calcul de la vibration forcée amortie des portiques on peut appliquer la méthode des déformations. La résolution est analogue à celle des vibrations non amorties, grâce à l'emploi dans le calcul de fonctions complexes.

Zusammenfassung

Für die gedämpfte erzwungene Schwingung von Rahmenträgern kann die Deformationsmethode angewendet werden. Die Lösung gleicht derjenigen der ungedämpften Schwingungen, wenn man in der Berechnung komplexe Funktionen einführt.

Summary

The deformation method can be used for forced suppressed oscillation of portal frames. The solution is similar to that of unsuppressed oscillations if complex functions are used in the calculation.

([3]) Voir dans la référence ([2]), chap. II, 4.

Vb5

L'influence des sollicitations dynamiques sur les constructions

Einfluss dynamischer Beanspruchung auf die Bauwerke

Effect of dynamic forces on structures

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If a load is applied to an elastic structure very slowly so that the velocity u, imparted to the mass elements of the loaded member may be neglected in comparison to the velocity of sound v, in the material, the load is usually said to be static, otherwise it is called dynamic. The definition is one of practical convenience to be understood in the same sense as the practical definition of the elastic limit. In this paper we shall be mainly concerned with ratios $\frac{u}{v} > 10^{-4}$, in general corresponding to impulses generated by high explosives or impact of missiles.

Shock waves

The velocity of a wave in a compressible medium depends on the density of the medium in the way that increased density corresponds to increased velocity. It is accordingly possible to describe the generation of a shock wave as occurring in successive steps, where each subsequent part of the wave moves in an increasingly dense medium at a greater speed than its forerunners which will be successively overtaken. The steepness of the wave front will consequently increase and would finally become discontinuous, if such a physically instable state were not prevented by heavy energy losses. For the practical treatment of such a wave, however, the wave front may be described as discontinuous.

The shock wave emerging from a detonating explosive may for practical purposes be considered to consist of two parts, extending over different ranges from the center of explosion. In the first range the particle velocity is nearly equal to the phase velocity corresponding to the pseudo-discontinuous wave front and the wave is associated with a considerable transportation of mass. In the second range the particle velocity lags behind the phase velocity and the shock wave is successively transformed into an ordinary sound wave, in which the particle movements may be considered as infinitesimal. The zone of transition between the first and the second range is fairly well defined, as can be seen in fig. 1, showing a photograph taken immediately after that the wave front has left the expanding luminous detonation gases. The steep front of the shock wave may be seen as an oblate halfsphere which after short progress will take the form of a perfect halfsphere.

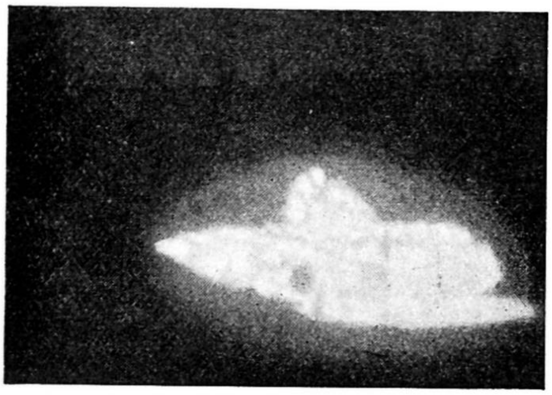

Fig. 1. Detonation of TNT in air. The visible wave front has just left behind the expanding luminous detonation gases.

For the present purpose it may suffice to state that the form of the waves and their impulses can be measured experimentally as a function of time, distance and the amount and type of explosive, and that impulses imparted to structures may be calculated from these data. For most practical purposes it is sufficient to know the total wave-impulse imparted to a plane rigid surface of unit area, while the wave form is unimportant. The last statement is justified by the fact that the time, during which energy is transferred from the wave to the structure, is usually very short in comparison to the transverse vibration period of the most common structures. We shall discuss this question in some detail later together with the characteristics of various types of structures. It should be mentioned, however, that the effect of the blast from an atomic bomb is expected to be different as a consequence of the very large total impulse and its long duration.

Impact

When a missile hits an object, the duration of the impact is determined by the elastic and plastic properties of the two bodies, their geometrical form and extension and the phenomena of rupture that may occur in and around the zone of contact. The impact may under certain conditions develop in a special way that is characterized by a secondary ejection of material from the hit structure. The elastic contact between the two bodies generates a set of compression waves which may be reflected at a free surface of the structure. Upon reflection it is turned into a dilatational wave progressing in opposite direction to the primary wave. A

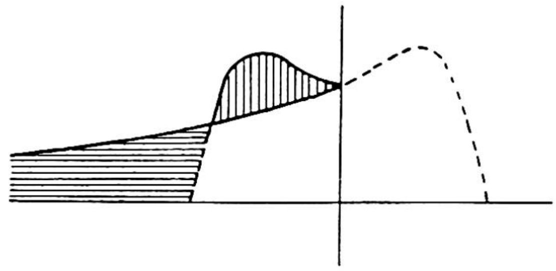

Fig. 2. Schematic representation of the reflection of a shock wave at a solid-air interface.

═ Compression in the primary wave.
||||| Tension in the reflected wave.

simplified representation of the actually very complicated phenomenon is given in fig. 2. We see that a maximum of tensile stress is to be expected when the reflected wave has penetrated to a certain depth of the material. If the tensile stress exceeds the critical value for the material, rupture will occur and the part set free will leave the structure as a missile, often with a considerable velocity. It is also seen that a smooth wave with small variations in intensity cannot be expected to produce the ejection effect. The effect will generally lead to changes of frequency and boundary conditions of the structural member involved. Similar effects may also be obtained by detonation of explosives in contact with the material.

Behaviour of different building materials

For a heterogeneous and anisotropic material, the modulus of elasticity may be defined by means of the statistical mean value of the velocity of sound waves in the material. For steel this averaging effect may take place over volumes of material that are very small in comparison to the volumes of material used in structural engineering. Steel may consequently be considered as quasi-isotropic and the dispersion of a wave progressing through the material may be neglected. Such materials as brickwork and concrete, however, will require larger volumes for the averaging effect to take place and the dispersion of a wave passing through the material is considerable. This is particularly important to observe when dealing with shock waves, because the wave front will soon loose its steepness and the length of the wave group will be increased after a relatively short passage. By these materials, the elastic constants derived from static or slow loading tests differ considerably from those obtained by dynamic methods, because the plastic deformation which takes place in the former case is substantially eliminated when the test is dynamically conducted. The rate of loading, however, also influences the magnitude of the elastic limit and the rupture strength of the material. The same effect is also observed for steel, although to a smaller extent.

The resistance against local damage from impact is increased both for steel and concrete, if the modulus of elasticity is increased. It is well known, however, that the brittleness of steel increases with the hardness, as well as the risk of secondary ejection of material. A good armour-plate, for instance, must consequently possess good ductility on the rear side. As to concrete it has been empirically observed that the resistance against local damage by impact increases with increasing modulus of elasticity, i. e. with high contents of stone aggregates and increased density of the mortar. As the dimensions of the concrete structure increase, the dispersion of the primary waves will substantially reduce the risk of secondary ejection, which will be easily understood with reference to fig. 1.

The dynamic behaviour of various structural elements

Our discussion may be limited to three types of structural elements, viz. columns, beams and slabs, as the characteristic properties of most structures can be referred to these elements. If such a structural element is subjected to an impulse, a vibrational state is generated that may be considered to be composed of superimposed characteristic vibrations, each corresponding to a discrete energy level and a definite shape of deformation. The characteristic vibration frequencies and the deformation types are determined by the shape and density distribution of the structural element, the elastic and plastic properties of the material and the boundary conditions. A decrease of mass density, increase of the rigidity or reduced degree of freedom will generally increase the frequency and vice versa.

The characteristic functions, or eigenfunctions, are solutions of the general amplitude equation

$$\Delta\Delta\varphi - \lambda\varphi = 0$$

and satisfy the relation of orthogonality

$$\int \varphi_i \, \varphi_k \, d\tau = 0 \quad (i \neq k)$$

The physical significance of the orthogonality relation is that the vibrational states corresponding to the separate characteristic functions may exist simultaneously without mutually disturbing each other, i. e. they are linearly independent.

The characteristic functions may be obtained as mathematically exact solutions of the amplitude equation or as approximate solutions to the variational problem. In the case of concrete, the modulus of elasticity determined by dynamical means must be used and the material may be treated as homogeneous and isotropic without consideration of reinforcement and microscopic cracks, provided that the interaction between the concrete and the reinforcement is intact.

A study of the characteristic functions of slabs subjected to various boundary conditions is being made at this institute by Mr. Ödman and it is expected that his work will facilitate the practical treatment of the problem.

Columns and beams

A column is usually designed for carrying an axial load, with due consideration to the question of stability against buckling. In practice, the actual load is either excentric or combined with a bending moment that will produce an initial lateral deformation of the column, whose carrying capacity is consequently determined by the stresses in the external fibres. When a vibrational state is set up in such a column by a lateral impulse, the superimposed stresses may eventually reach the critical value for the material and cause a collapse. Disregarding the practical impossibility of applying a centric load, the excentricity of the external load is always assured for a column of concrete as a consequence of the heterogeneity of the material and its capacity of plastic deformation. The amount of mass per unit length in relation to the surface exposed to the impulse will be greater for columns made of concrete compared to columns made of steel and the response to lateral impulses will be reduced. A design to reduce the risk of buckling will as a rule keep the initial lateral deformations and the secondary additional moments small.

Generally, the impulses corresponding to the second range of the detonation wave have no dangerous influence on the ordinary column, because it is designed for high buckling stability and exposes a small area to the relatively weak impinging wave. The above mentioned effect may, however, occur in the first wave range. For common explosives and ordinary conditions, the duration of the impulse is short enough to make the energy absorption of the column practically independent of its lowest characteristic frequency and we find from the impulse equation

$$mv = \int_{T} p\,dt$$

that more favourable conditions will be produced by increasing the mass of the column which will lead to a decrease of the initial velocity and the maximum amplitude. If the mass increment, however, is associated with decreased characteristic frequency, while the buckling risk at normal load remains unchanged, the favourable effect is counteracted. The mass increment should in other words be combined with increased rigidity. For designing purposes it is usually not necessary to calculate the reactions at the supports, unless the system is very rigid and exceptionally susceptible to shearing forces.

Partial destruction through impact will, as a rule, cause complete collapse of loaded steel columns. In the case of reinforced concrete columns, in which the plastic deformation has caused a transfer of load from the concrete to the reinforcememnt, even a superficial damage to the concrete surrounding the reinforcement may be sufficient to produce partial buckling of the reinforcement bars, which under unfavourable conditions may lead to a sudden collapse of the column. Usually, however, the central part of the concrete column has to be damaged, before its carrying capacity is appreciably reduced. The effect of a lateral impulse located at the base of a column will be discussed later in connection with various structural arrangements.

It is obvious that the behaviour of beams is in principle similar to that of columns, except that the absence of axial load and the presence of a lateral dead load will diminish the probability of damaging effects of lateral shock waves to a considerable degree. It is generally to be observed that damages to beams, due to blast, occur through secondary influence from surrounding structural elements.

Slabs

When a slab is subjected to a shock wave, the excited vibrational state is extremely sensitive to the loading and boundary conditions. It is consequently practically impossible to predict which mode of vibration will predominate, especially as the energy levels sometimes are very close and a sort of degeneracy occurs. Empirically, however, it has been observed that the real behaviour of slabs designed by use of one of the fundamental states is reasonably in accord with theory, provided that the impulse is close to the critical value for rupture.

As for beams it is usually not necessary to consider the reactions at the supports if these extend continuously along the edges. Discrete supporting arrangements necessitate a detailed analysis with regard to shearing effects.

If the slab is subjected to the detonation wave of an explosive in contact with the slab or the impact of a missile, the longitudinal compression wave will produce a local damage around the contact zone, eventually accompanied by a secondary ejection of material from the opposite side of the slab. It has been empirically observed that such local effects have very small influence on the characteristic frequency of the slab and that the damage must be extensive in order to produce an appreciable change. This implies that the structure of which the slab is an integrating part retains its normal function.

Types of structures

As representing the various possible combinations of the aforementioned structural elements three types of structures will be considered, namely the framework, the mushroom structure and the cell structure. All these represent structures which are highly statically indeterminate. Their main dynamical characteristics are fairly well known from the study of earth-quake effects on buildings and we shall limit our discussion to some questions that may be of interest for the planning and designing of factories or other constructions where explosions may take place.

For the framework and also for the mushroom structure a much discussed question concerns the advantages gained by the use of walls consisting of light-weight materials in order to reduce and limit the effects of blast on the carrying structure.

From theoretical considerations it might be expected that impulses transmitted by a shock wave, emerging from the center of a closed room where the distances to all walls are equal, will be absorbed in the same degree, independently of the resistance and the mass of the walls, provided that the fundamental frequencies are low enough to permit the impulses to be considered as momentary. In other words, if one of the walls should be removed without change of the boundary conditions for the remaining walls, the latter would be affected by the impulse in quite the same manner. This has, as a matter of fact, been verified experimentally. Should the intensity of the wave suffice to produce rupture, this could accordingly not have been prevented by making one of the walls less resistant. If, however, the residual static pressure of the explosion gases within the closed space is high enough to produce rupture, the effects will be abated, if one of the walls is easily destroyed. This advantage is, however, only apparent or fortuitous.

In a sufficiently limited space for the static pressure to produce rupture, the impulse by high explosives will be amply sufficient for producing it and the static pressure will be of less consequence as it will only complete the destruction. The explanation of the favourable effect sometimes observed as arising from the premature destruction of one wall may be sought in an inadequate design of the structure supporting the walls horizontally, or the roof vertically. For instance, the tensile stresses set up in a reentrant corner usually start the destruction at impulse intensities far below the intensity producing rupture in the central part of the slab, and a rapid decrease of the static pressure will consequently be favourable. With properly designed walls of uniform and equal strength, no damage at all would have occurred. The guiding principle for the design should be to assure a satisfactory resistance to the shock wave.

In factories and similar buildings, where the amount of explosives contained in a limited space can be controlled, it is technically and economically possible to give the room suitable volume with regard to the permissible static gas pressure. The walls should be designed to resist the shock wave so as to prevent damage to adjoining rooms.

With reference to the discussed questions it will thus be seen that the advantages which are claimed to be gained by framework or mushroom structures are much overrated by this type of load. In our opinion, the most adequate construction is provided by the cellular system as composed of elastically clamped slabs, limiting the damaging effects to the closed cell. Constructional systems of this kind must be considered as the most effective for avoiding total damage by locally occurring explosions and they should be used more frequently for factories and other constructions where risks of explosion are involved at operation.

From a general point of view, and especially with regard to shock waves transmitted through the subsoil, all the structures discussed are particularly well adapted to withstand dynamic action. Even if one or several of the carrying parts are destroyed, the statically indeterminate system will continue to function, causing a redistribution of loads and stresses but preventing the structure from collapse.

Damage to foundations may be caused by shock waves generated in well graded moraine soils. Such shock waves generally occur as longitudinal waves and are easily dispersed by applying a filling of stone around the structural element. If the detonation takes place below a certain depth in layers of plastic clay, however, more dangerous effects may be produced. Besides a primary longitudinal wave, a transverse wave of great amplitude and low frequency is generated, the propagation of which is confined to the surface of the layer. This latter wave, from which damage may arise, resembles the Rayleigh wave, with accelerations comparable to those occurring in earth-quakes. The range of propagation and the energy content, however, are rather limited and depend on the properties of the clay layers and their boundary conditions. The absorption of the wave energy by an ordinary, heavy structure, founded on clay with the load concentrated on pile groups or distributed over a continuous slab, is in general so complete that the wave is extinguished by the obstacle. On

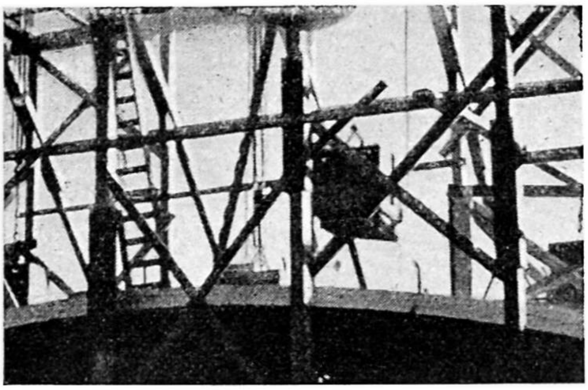

Fig. 3. Impact load on a 9.5 cm reinforced shell roof with a clear span of 14.2 meters. A concrete block weighing 1 000 kgs is released from a height of 3 meters. The picture was taken immediately before the impact.

account of the relatively small energy content of the wave, the effects will only be local and the risks of damage will as a rule be restricted to the pile groups. Rupture would, however, only occur in the immediate neighbourhood of the center of explosion and the risk of damage to the structure as a whole would thus be greatly reduced or entirely eliminated, if the possibilities of redistribution of load from the damaged group of piles to the surrounding ones were assured by an adequate design. Even in this case the cellular structure is less sensitive to local damage of the foundation. Although the framework or the mushroom structure may represent good solutions of the structural problem, the cellular system should be preferred if other circumstances allow it.

In such cases, where a structure is designed for the sole purpose of protecting people or machinery from heavy falling objects, as for instance linings in rock tunnels, the construction should permit a high deformation in order to reduce the risks of damage by piercing.

Figures 3 and 4 show an experiment which was carried out in order to verify the theoretical treatment of local impact on a thin concrete shell roof with a clear span of 14.2 meters and a height at its centre of 1.25 meters. The shell was 9.5 cm thick and was reinforced with 5 mm bars, spaced at a distance of 65 mm. A concrete block, weighing 1 000 kilos, with an effective impact area of 47.5 $\times$ 47.5 cm was released from a height of 3 meters. The two pictures show the undeformed shell, respectively its maximum deformation. The test was repeated with a sharp rock replacing the concrete block and in both cases the missile was arrested by the shell, although a certain amount of local penetration occurred, as is shown in fig. 5.

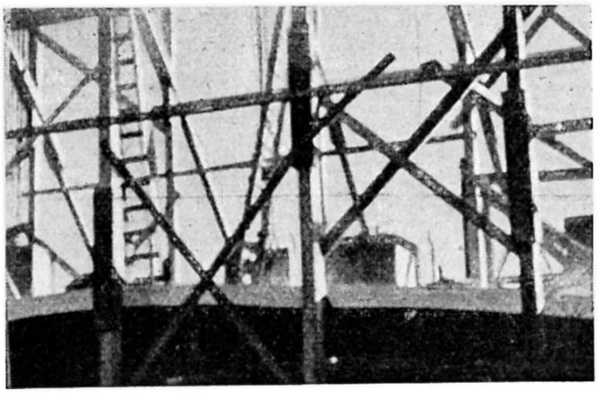

Fig. 4. Conditions as in fig. 3. The deflection of the shell under impact is clearly shown in the picture.

Fig. 5. Local penetration of sharp rock weighing 2 000 kg and falling from a height of 3 meters on the shell roof shown in figure 3.

In another test with a 2 000 kilos missile, the observed vertical deformation, without serious damage to the shell, was 25 cm.

Résumé

La nature des sollicitations dynamiques produites par explosion ou choc est discutée; les propriétés caractéristiques et le comportement de quelques matériaux de construction sous l'effet de sollicitations dynamiques sont étudiés; l'auteur étudie également les déformations subies par trois éléments de construction (colonnes, poutres et dalles), ainsi que le comportement de ces éléments dans diverses constructions.

Zusammenfassung

Die Art der dynamischen Beanspruchung bei Explosion oder Stoss wird besprochen; die charakteristischen Eigenschaften und das Verhalten einiger Baustoffe des Hochbaues unter der Einwirkung von dynamischer Beanspruchung werden in aller Kürze behandelt, ebenso wie die charakteristischen Formänderungseigenschaften von drei typischen Konstruktionselementen, nämlich Stützen, Balken und Platten. Einige Erfahrungen über die Wirkungsweise der besprochenen Konstruktionselemente in verschiedenen Bauwerken werden erörtert.

Summary

The nature of dynamic load as produced by explosives and impact is discussed; the properties and behaviour of some building materials under the action of dynamic load are briefly related and the characteristic deformation properties of three typical structural elements, viz. columns, beams and slabs are discussed, as well as some questions with regard to their mode of function in various structural systems.