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Equation différentielle pour le calcul des vibrations produites dans les constructions portantes par les charges mobiles

Differentialgleichung für die Schwingungsberechnung von Tragkonstruktionen infolge beweglicher Lasten

Differential equation for calculation of vibrations produced in load-bearing structures by moving loads

SVEN T. A. ÖDMAN, C. E.

Head, Technical Department
Swedish Cement and Concrete Research Institute at the Royal Institute of Technology
Stockholm, Sweden

This paper is a contribution to the theoretical study of the problem of forced vibrations in load-bearing structures of finite extent subjected to any arbitrary boundary conditions. The vibrations are assumed to be produced by one or several non-elastically applied loads, and possibly also transverse forces devoid of mass, which move on the structure with a constant velocity.

A usual treatment of similar problems consists in deducing a differential equation which represents the motion of the load-bearing structure and the load, and in finding a formal solution to this equation by means of a series expansion. This method has been applied by many authors to a beam which is hinged and freely supported at both ends, and is acted upon by a single moving load. Among these authors, the following deserve to be mentioned in this connection. Kryloff (1) neglects the influence exerted by the mass of the load on the natural vibration of the system, and thus obtains a simple formula for the deformation at any arbitrary subsequent instant. Inglis (2) expresses the deformation and the load by a Fourier series, in which he disregards all terms except the first. In other words, he imagines the concentrated load to be replaced by a load distri-

(1) Kryloff, A. N., *Mathematische Annalen*, Vol. 61, 1905. See also Timoshenko, S.; *Vibration Problems in Engineering*, U. S. A., 1928.

(2) Inglis, C. E., *A Mathematical Treatise on Vibrations in Railway Bridges*, Cambridge. 1934.

buted over the whole beam according to a sine function having a half wave length which is equal to the length of the beam and an amplitude which varies in accordance with the same sine function. Schallenkamp ([3]) deals in a similar manner with the vibration of the load in a vertical direction only. Looney ([4]) tackles the problem by means of the calculus of differences on the assumption that the natural vibration can be represented by a sine function, and disregards all harmonics except the fundamental. Contrary to Inglis, Rinkert ([5]) takes into account a finite number of terms in the series expressing the concentrated load.

A procedure that is commonly used by most of the investigators referred to in the above is to obtain the solution of the differential equation by means of sine series in which each term can be regarded as an analytical expression of one of the types of vibrations performed by a beam which is hinged and freely supported at both ends. If the beam is subjected to an impact caused by a force which is devoid of mass, vibrations will be produced, the frequencies of which are determined solely by the properties of material, dimensions, and boundary conditions of the beam, and each of these frequencies corresponds to a definite type of vibration. On the other hand, if the force acts in conjunction with a mass which takes part in the vibration, for instance, if the beam is submitted to a moving load, then different conditions will arise. In this case, a continuous change in the position of the mass on the beam gives rise to a continuous variation in each frequency and in the corresponding type of vibration. If we try to find a formal solution to the differential equation of vibration produced by a moving load by using series of functions, which are not variable with time, this implies an attempt to calculate a resulting vibration at any arbitrary gauge point by the aid of a limited number of terms. In reality, this motion is composed of several entirely separate vibrations, and an intricate analysis of frequencies will be required in order to segregate these vibrations.

The use of formal solutions entails insufficient accuracy in the calculation of stresses. Even if the series used for solving the differential equation is found to be convergent, it is not certain that the second derivative will approach a correct value, or will be convergent at all. This is due to the fact that a given initial substitution used in solving the differential equation will prove successful depending on the extent to which the first term of the series agrees with the actual, total deformation. Consequently, the fact that the boundary conditions are satisfied by all terms of the series alone is not sufficient. An analogous statement has been made by Courant ([6]) regarding variational and buckling problems. Therefore, as the type of vibration varies continuously in the case of moving loads, there must always be an uncertainty in the calculation of stresses.

The purpose of this paper is to demonstrate a method for a more general study of this problem under any arbitrary boundary conditions.

([3]) Schallenkamp, A., *Schwingungen von Trägern bei bewegten Lasten* (*Ingenieur-Archiv*, 1937).

([4]) Looney, Ch. T. G., *Impact on Railway Bridges* (*University of Illinois, Bulletin* No. 19, Vol. 42, 1944).

([5]) Rinkert, A., *Vibrations of a Beam with Hinged Ends under Action of a Load Moving with Constant Speed.* Examination Work at the Institution of Structural Engineering and Bridge-building at the Royal Institute of Technology, Stockholm 1945 (in Swedish).

([6]) Courant, R., *Variational Methods for the Solution of Problems of Equilibrium and Vibrations* (*Bulletin of the American Math. Soc.*, Vol. 49, No. 1, Jan. 1943).

In contradistinction from the earlier investigations, we shall take into account the variation in the type of natural vibration due to the change in the position of the load. Each term of the series used in solving the differential equation corresponds to the type of natural vibration performed at a given instant, and is therefore dependent on the position of the load, and hence on time. Accordingly, the state of vibratory motion is known at any instant, and the increments in moment and in stress due to each natural vibration can therefore also be calculated. Just as most other investigators who have dealt with this problem, we neglect the influence of damping, which has been discussed by Holzer (7) and Sezawa (8), among others. The deformations are assumed to be so small that the effect of rotatory inertia and of shear can be disregarded. These questions have been studied by Timoshenko (9), Goens (10), Pickett (11) and others.

Consider free harmonic vibration of a system which is compound in the range G and consists of a load-bearing structure in conjunction with one or several stationary masses. In this case, we can deduce a harmonic differential equation of the following well-known type, which is independent of the time factor

$$L[\varphi_n] + \rho \cdot \lambda_n \cdot \varphi_n = 0 \tag{1}$$

where φ_n is a characteristic function which represents the n-th type of deformation of the system within the range G, and λ_n is the corresponding characteristic value (12). $L[\varphi_n]$ is a linear differential expression formed with respect to the space coordinates x, y, z, and is defined within the same range, and ρ is a given dimensionless function which represents the relative mass density distribution of the system. Eq. (1) is often called Euler's differential equation.

If the section of the load-bearing structure is uniform, the caracteristic functions and the corresponding natural frequencies can be computed from the above differential equation by means of the methods given by Den Hartog (13) or Kármán and Biot (14), and others. For this purpose, a general solution of the differential equation is found for the ranges between the boundaries and the point of application of the load. By applying the boundary conditions, we obtain a frequency equation, and by means of this equation we can calculate a set of roots, each of which corresponds to a definite form of the characteristic function. The lowest root value corresponds to the fundamental frequency. Berg (15) has carried out this calculation for a hinged, freely supported beam, and has tabulated the values of the characteristic function as a function of the position of the

(7) Holzer, H., *Zeitschrift für angew. Math. und Mech.*, V. 8, p. 272, 1928.

(8) Sezawa, K., *Zeitschrift für angew. Math. und Mech.*, V. 12, p. 275, 1932.

(9) Timoshenko, S., *On the Correction for Shear of the Differential Equation for Transverse Vibration of Prismatic Bars* (*Philosophical Magazine*, Ser. 6, Vol. 41, p. 744 and Vol. 43, p. 125).

(10) Goens, E., *Ueber die Bestimmung des Elastizitätsmoduls von Stäben mit Hilfe von Biegungsschwingungen* (*Annalen der Physik*, 5. Ser., Vol. 11, p. 649, 1931).

(11) Pickett, G., *Equations for Computing Elastic Constants from Flexural and Torsional Resonant Frequencies of Vibration of Prisms and Cylinders* (*American Society for Testing Materials*, Vol. 45, 1945).

(12) Courant, R. and Hilbert, D., *Methoden der mathematischen Physik*, Band 1, Berlin, 1924, Kap. V.

(13) Den Hartog, J. P., *Mechanical Vibrations*, New York and London, 1940.

(14) v. Kármán, T. and Biot, M. A., *Mathematical Methods in Engineering*, New York and London, 1940.

(15) Berg, Owe, *Biegungsschwingungen eines in beiden Enden unterstützten punktförmig belasteten Balkens* (*Zeitschr. angew. Math. Mech.*, Bd 24, Nr 1, 1944).

load and definite given ratios between the masses of the load and the beam.

For the majority of practical purposes, some approximate method can usually be applied. For instance, Lord Rayleigh ([16]) assumes that the characteristic function approximates to the static deformation curve produced by the weight of the load, provided that the inertia forces are neglected, and calculates the frequency by means of the energy method. Ritz ([17]) puts the characteristic function equal to a series, and determines the coefficients of expansion and the frequencies by the aid of Hamilton's variational principle. Galerkin ([18]) applies a variational method by which, however, one arrives at exactly the same expression as is obtained by the method devised by Ritz. Gran Olsson ([19]) has found that with the aid of the principle of virtual displacements the same results may be obtained as by the methods of Ritz and Galerkin. Finally, Grammel ([20]) takes the integral equation of the system as a starting-point and determines the frequency from the relation between an assumed deformation and the nucleus.

All these methods give a sufficiently accurate upper limit of the natural frequency. The deviations of the approximate expression of the characteristic function from its exact form have scarcely any influence on the ultimate value, and the error is therefore usually inconsiderable. If greater importance is attached to the form of the characteristic function, as is often the case in the determination of stresses, or if the rigidity, section area and the mass density of the load-bearing structure are variable, it is convenient to use Vianello's ([21]) approximate method for plotting the characteristic function graphically by means of a funicular polygon. Great accuracy can be obtained by applying this procedure several times. This problem has also been dealt with by Inglis ([22]).

The characteristic functions $\varphi_1, \varphi_2, \varphi_3, \ldots$, etc., form a complete orthogonal system which satisfies the conditions for orthogonality, viz.,

$$\left.\begin{aligned} &\int_G \rho \cdot \varphi_r \cdot \varphi_n d\tau = 0 , && r \neq n \\ &\int_G \rho \cdot \varphi_n^2 d\tau = \text{constant} , && r = n \end{aligned}\right\} \tag{2}$$

where G denotes the limit range of the integral and τ is used to designate one or several coordinates in space. If the constant is put equal to unity, the characteristic functions are termed normalised.

Furthermore, the theorem of expansion of characteristic functions states that any stepwise finite arbitrary function f which statisfies the same

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([16]) Lord Rayleigh, *The Theory of Sound*, Vol. 1, 2nd Ed., pp. 111 and 287. *Phil. Mag.*, Vol., 47, p. 566, 1899 and Vol. 22, p. 255, 1911.

([17]) Ritz, W., *Ueber eine neue Methode zur Lösung gewissen Variationsprobleme der mathematischen Physik.* (*J. f. reine und angew. Math.*, Bd 135, pp. 1-61, 1909. — *Gesammelte Werke*, p. 265, Paris, 1911).

([18]) Galerkin, B. G., *Expansion in Series for Solving Some Equilibrium Problems for Plates and Beams* (*Wjestnik Ingenerow Petrograd*, 1915, Heft 19, in Russian).

([19]) Gran Olsson, R., *Die Anwendung des Prinzips der virtuellen Arbeit bei der Lösung von Knickproblemen* (*Det Kongelige Norske Videnskabers Selskab, Forhandlinger* Bd XVII, Nr 46).

Gran Olsson, R., *The Principle of Virtual Displacement Applied in Approximate Solutions of Eigenvalue Problems* (*Dixième Congrès des Mathématiciens Scandinaves*, Copenhague, 1946).

([20]) Grammel, R., *Ein neues Verfahren zur Lösung Technischer Eigenwertprobleme* (*Ing.-Arch.*, Bd 10/1939, pp. 35-46). See also : Lösch, Fr., *Berechnung der Eigenwerte linearer Integralgleichungen* (*Zeitschr. angew. Math. Mech.*, Bd 24, Nr 1. 1944).

([21]) Vianello, L., *Grafische Untersuchung der Knickfestigkeit gerader Stäbe* (*Z. V. D. I*, 1898, Juli-Dez., p. 1436).

([22]) Inglis, C. E., *Natural Frequencies and Modes of Vibration in Beams of Non-uniform Mass and Section* (*Trans. I. N. A.*, Vol. LXXI, p. 145, 1929).

boundary conditions as the characteristic functions φ_n and has a self-adjungated finite linear differential expression $L[\varphi_n]$ can be expressed by an absolutely and uniformly convergent series composed of these characteristic functions, viz.,

$$f = \sum_{n=1}^{\infty} c_n \cdot \varphi_n$$

where

$$c_n = \int_G \rho \cdot f \cdot \varphi_n d\tau \,. \tag{3}$$

For simplification, we shall deduce, in the first place, the differential equation for a single moving load having a mass which cannot be neglected.

It is convenient to imagine the masses of the load-bearing structure and of the load as a compound system whose natural vibration is unambiguously determined by the position of the load in a steady state of vibratory motion and varies with the position of the load, whereas the weight of the load is regarded as an external vertical force devoid of mass applied at the centre of gravity of the moving mass. In the treatment of the problem it makes no difference whether a force devoid of mass, e.g. a pulsating force, is added to the weight of the load, and the problem is thus reduced to the determination of the vibration produced by the resultant external force.

Since the system has an infinite number of degrees of freedom for every position of the moving mass, the deformation from the position of equilibrium at any arbitrary instant $t = t_j$ can be expressed by a series comprising all those degrees of freedom which come into play at that instant. Accordingly, we put

$$w = \sum_{n=1}^{\infty} q_n(t_j) \cdot \varphi_{nj}(t_j, s) \tag{4}$$

where w denotes the deformation from the position of equilibrium, and φ_{nj} corresponds to the n-th characteristic function determined in a steady state of vibratory motion with the moving mass in the j-th position on the structure. For instance, φ_{nj} can be imagined to be composed of trigonometric and hyperbolic functions in which the arguments are also variable with time. The quantity q_n is an unknown factor which varies with time only, and is termed a generalised coordinate.

The maximum kinetic and potential energies of the system, which are denoted by T and V respectively, are determined at the instant when the load is at the point s_j.

For the load-bearing structure alone, we obtain

$$T_b = \frac{m}{2} \int_G \left(\frac{\partial w}{\partial t_j}\right)^2 d\tau$$

where m denotes the mass per unit length τ of the structure.

The velocity of the moving mass at any arbitrary point is

$$\frac{\partial w}{\partial t_j} = \sum_{n=1}^{\infty} (\dot{q}_n \cdot \varphi_{nj} + q_n \cdot \dot{\varphi}_{nj})$$

where

$$\dot{q}_n = \frac{dq_n}{dt_j} \quad \text{and} \quad \dot{\varphi}_{nj} = \frac{\partial \varphi_{nj}}{\partial t_j} .$$

Therefore, we get

$$T_b = \frac{m}{2} \int_G \left[\sum_{n=1}^{\infty} (\dot{q}_n \cdot \varphi_{nj} + q_n \cdot \dot{\varphi}_{nj}) \right]^2 d\tau .$$

In calculating the corresponding increase in kinetic energy due to the moving mass having the weight P, we must take into account its curvilinear motion whose component in the direction w is determined by

$$w_P(s_j) = \sum_{n=1}^{\infty} q_n(t_j) \cdot \varphi_{nj}(t_j, s_j)$$

where s_j is also a function of time. Then the velocity of the load in the direction w can be written

$$\frac{\partial w_P}{\partial t_j} = \sum_{n=1}^{\infty} [\dot{q}_n \cdot \varphi_{nj}(s_j) + q_n \cdot \dot{\varphi}_{nj}(s_j)]$$

and the amount contributed by this velocity to the kinetic energy of the system is

$$T_P = \frac{P}{2g} \left[\sum_{n=1}^{\infty} [\dot{q}_n \cdot \varphi_{nj}(s_j) + q_n \cdot \dot{\varphi}_{nj}(s_j)] \right]^2 .$$

The potential energy of the system is equal to the sum of all inertia forces times the respective displacement of these forces. Since the inertia forces are proportional to the deformation, we consider their mean value, and the amount V_b contributed by the load-bearing structure to the potential energy of the system is determined by the expression

$$V_b = \frac{m}{2} \int_G \frac{\partial^2 w}{\partial t_j^2} \cdot \widehat{w} d\tau$$

$$= \frac{m}{2} \int_G \left(\sum_{n=1}^{\infty} q_n \cdot \omega_n^2 \check{}(s_j) \cdot \varphi_{nj} \right) \cdot \left(\sum_{n=1}^{\infty} q_n \cdot \varphi_{nj} \right) d\tau$$

where $\omega_n(s_j)$ is the frequency of the n-th natural vibration.

The corresponding increase in potential energy V_p due to the load is

$$V_p = \frac{P}{2g} \left(\sum_{n=1}^{\infty} q_n \cdot \omega_n^2(s_j) \cdot \varphi_{nj}(s_j) \right) \cdot \left(\sum_{n=1}^{\infty} \widehat{q_n} \cdot \check{\varphi}_{nj}(s_j) \right) .$$

Using the notation

$$\int_G \varphi^2_{nj} d\tau + \frac{P}{mg} \cdot \varphi^2_{nj}(s_j) = H_n(s_j)$$

and observing that the terms of the form

$$\int_G \varphi_{rj} \varphi_{nj} d\tau + \frac{P}{mg} \cdot \varphi_{rj}(s_j) \cdot \varphi_{nj}(s_j) ; \qquad r \neq n$$

are zero according to the conditions for orthogonality, see Eq. (2), we can write the total maximum kinetic and potential energies of the system

$$T = \frac{m}{2} \int_G \left[\sum_{n=1}^{\infty} (\dot{q}_n \cdot \varphi_{nj} + q_n \cdot \dot{\varphi}_{nj}) \right]^2 d\tau$$

$$+ \frac{P}{2g} \left[\sum_{n=1}^{\infty} [\dot{q}_n \cdot \varphi_{nj}(s_j) + q_n \cdot \dot{\varphi}_{nj}(s_j)] \right]^2$$

$$V = \frac{m}{2} \sum_{n=1}^{\infty} q_n^2 \cdot \omega_n^2 (s_j) \cdot H_n(s_j) \ . \tag{5}$$

These values are inserted in Lagrange's equation of motion

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial (T - V)}{\partial q_r} = Q_r$$

when Q_r denotes the force corresponding to q_r and is termed the generalised force. We then obtain

$$\ddot{q}_r \cdot H_r(s_j) + \int_G \varphi_{rj} \sum_{n=1}^{\infty} (2\, \dot{q}_n \cdot \dot{\varphi}_{nj} + q_n \cdot \ddot{\varphi}_{nj})\, d\tau$$

$$+ \frac{P}{mg} \cdot \varphi_{rj}(s_j) \cdot \sum_{n=1}^{\infty} [2\, \dot{q}_n \cdot \dot{\varphi}_{nj}(s_j) + q_n \cdot \ddot{\varphi}_{rj}(s_j)] + q_r \cdot \omega_r^2 (s_j) \cdot H_r(s_j) = \frac{Q_{rj}}{m} \ . \tag{6}$$

The generalised force Q_{rj} is determined so that the work done by the external force in the case of variation in the generalised coordinate q_r divided by this variation should be equal to Q_{rj}.

In this case, the work is equal to $P(s_j) \cdot \delta q_r \cdot \varphi_{rj}(s_j)$, where $P(s_j)$ denotes the weight of the load, possibly with the addition of an external force devoid of mass, which is applied at the same point. We then obtain

$$Q_{rj} = P(s_j) \cdot \varphi_{rj}(s_j) \ .$$

Eq. (6) gives a system of linear inhomogeneous differential equations of the second order having an infinite number of terms and variable coefficients. This equation can be regarded as the complete differential equation of forced vibration produced in a load-bearing structure by a moving non-elastic mass. It is very difficult to find an exact solution of this equation. In order to avoid this difficulty, we must resort to simplifications.

The influence of the curvilinear motion is so slight that it can be regarded as a correction, at least in normal structures met with in practice and subjected to ordinary permissible loads. In such cases it is obvious that the error will be negligible if the correction consists in disregarding the influence of all vibrations except the r-th. If the characteristic functions are normalised so that $\int_G \varphi_{rj}^2 \cdot d\tau = 1$, the variation in the form of the characteristic function must be so small that it could be neglected, with the result that both $\dot{\varphi}_{rj}$ and $\ddot{\varphi}_{rj}$ would become equal to zero. If the velocity

of the moving mass is assumed to be constant and equal to v, we get

$$\dot{\varphi}_{rj}(s_j) = v \cdot \overset{\mathrm{I}}{\varphi}_{rj}(s_j) \qquad \text{and} \qquad \ddot{\varphi}_{rj}(s_j) = v^2 \cdot \overset{\mathrm{II}}{\varphi}_{rj}(s_j)$$

where $$\overset{\mathrm{I}}{\varphi}_{rj}(s_j) = \frac{d\varphi_{rj}(s_j)}{ds_j} \qquad \text{and} \qquad \overset{\mathrm{II}}{\varphi}_{rj}(s_j) = \frac{d^2\varphi_{rj}(s_j)}{ds_j^2}$$

Consequently, the complete differential equation takes the simplified form

$$\ddot{q}_r \cdot \mathrm{H}_r(s_j) + \dot{q}_r \cdot \dot{\mathrm{H}}_r(s_j) + q_r \cdot [\omega_r^2(s_j) \cdot \check{\mathrm{H}}_r(s_j) + \mathrm{N}_r(s_j)] = \frac{\mathrm{P}(s_j) \cdot \varphi_{rj}(s_j)}{m} \tag{8}$$

where
$$\mathrm{H}_r(s_j) = 1 + \frac{\mathrm{P}}{mg} \cdot \varphi_{rj}^2(s_j)$$
$$\check{\dot{\mathrm{H}}}_r(s_j) = \frac{2v}{mg} \cdot \mathrm{P} \cdot \overset{\mathrm{I}}{\varphi}_{rj}(s_j) \cdot \varphi_{rj}(s_j)$$
$$\check{\mathrm{N}}_r(s_j) = \frac{v^2}{mg} \cdot \mathrm{P} \cdot \overset{\mathrm{II}}{\varphi}_{rj}(s_j) \cdot \varphi_{rj}(s_j)$$
$$s_j = v \cdot t_j$$

It makes no great difference whether one or several masses move with the same velocity on the load-bearing structure. The variation in mass distribution with time influences the form and the frequency of the characteristic function in a similar manner. It is convenient to keep the notations φ_n and ω_n unchanged, but they are used to denote the new characteristic function and the corresponding natural frequency. Accordingly, the j-th position of the load at any arbitrary instant $t = t_j$ signifies the location of k loads having the respective weights P_1, P_2, P_3, ..., P_k, and $\mathrm{P}_k(s_{kj})$ denotes the weight P_k, possibly with the addition of an external force devoid of mass, which is applied at the point s_{kj}. The new differential equation can be written in the form

$$\ddot{q}_r \cdot \mathrm{H}_r(s_j) + \dot{q}_r \cdot \dot{\mathrm{H}}_r(s_j) + q_r \cdot [\omega_r^2(s_j) \cdot \mathrm{H}_r(s_j) + \mathrm{N}_r(s_j)]$$
$$= \frac{1}{m} \sum_{n=1}^{k} \mathrm{P}_k(s_{kj}) \cdot \varphi_{rj}(s_{kj}) \tag{9}$$

where
$$\mathrm{H}_r(s_j) = 1 + \frac{1}{mg} \cdot \sum_{n=1}^{k} \mathrm{P}_k \cdot \check{\varphi}_{rj}^2(s_{kj})$$
$$\dot{\mathrm{H}}_r(s_j) = \frac{2v}{mg} \cdot \sum_{k=1}^{k} \mathrm{P}_k \cdot \overset{\mathrm{I}}{\varphi}_{rj}(s_{kj}) \cdot \varphi_{rj}(s_{kj})$$
$$\mathrm{N}_r(s_j) = \frac{v^2}{mg} \cdot \sum_{k=1}^{k} \mathrm{P}_k \cdot \overset{\mathrm{II}}{\varphi}_{rj}(s_{kj}) \cdot \varphi_{rj}(s_{kj})$$
$$s_j = v \cdot t_j$$

Eq. (9) is an ordinary linear inhomogeneous differential equation of the second order with variable coefficients.

If the fact that the characteristic function varies with time is completely disregarded, then Eq. (8) corresponds to a formal solution of the problem

under consideration. In that case, the functions φ_r and H_r are referred to the non-loaded structure, and $\omega_r^2(s_j) \cdot H_r(s_j)$ is constant and equal to the square of the natural frequency of this structure. For instance if we put

$$\varphi_r = \sqrt{\frac{2}{l}} \cdot \sin \frac{r\pi}{l} x ,$$

which is identical with the normalised characteristic function of the freely supported beam (l = length of beam, $0 \leqslant x \leqslant l$, and r = integers 1, 2, 3, ...), then we obtain for $r = 1$, the same differential equation as that deduced by Inglis. Therefore, it follows that his results can be regarded as a special case.

The tests made lately by A. Hillerborg ([23]) seem to indicate that the variation in the form of the characteristic function with the position of the load could be neglected in the case of the freely supported beam. It remains to be demonstrated whether this variation may also be disregarded in dealing with other structures or under different boundary conditions. This question can be examined theoretically on the basis of the present investigation.

An approximate solution of the differential equation (9) is briefly deduced in what follows.

By putting the right-hand member of Eq. (9) equal to zero, we obtain a homogeneous differential equation. After dividing by $H_r(s_j)$, this equation can be written

$$\ddot{q}_r + \frac{\dot{H}_r(s_j)}{H_r(s_j)} \cdot \dot{q}_r + \left[\omega_r^2(s_j) + \frac{N_r(s_j)}{H_r(s_j)}\right] \cdot q_r = 0 \tag{10}$$

and can be interpreted as a differential equation of free amplitude-modulated and frequency-modulated vibration.

To solve this equation, we put

$$q_r = e^{-\int \beta_r dt} (A \sin \int \varpi_r dt + B \cos \int \varpi_r dt) \tag{11}$$

where A and B are arbitrary constants, whereas s_r and ϖ_r are arbitrary functions of time which express the damping and the frequency of vibration respectively. If this expression is inserted in eq. (10) the two relations

$$\beta_r = \frac{1}{2}\left[\frac{\dot{\varpi}_r}{\varpi_r} + \frac{\dot{H}_r}{H_r}\right] \tag{12a}$$

$$\varpi_r^2 = \omega_r^2 + \frac{N_r}{H_r} - \dot{\beta}_r - \beta_r \cdot \frac{\dot{H}_r}{H_r} + \beta_r^2 \tag{12b}$$

must hold good in order that the above expression should satisfy the differential equation (10).

If we assume that the vibration sets in at the intsant $t = t_i$, the value of $\exp[-\int \beta_r dt]$ at any arbitrary subsequent instant $t = t_j$ can be calculated by integration. By using eq. (12a), we then obtain

$$exp\left[-\int_{t_i}^{t_j} \beta_r dt\right] = \sqrt{\frac{\varpi_r(s_i)}{\varpi_r(s_j)}} \cdot \frac{\omega_r(s_j)}{\omega_r(s_i)} . \tag{13}$$

([23]) HILLERBORG, A. L., *A Study of Dynamic Influences of Moving Loads on Girders (International Association for Bridge and Structural Engineering. Congress 1948 at Liège. Preliminary Publication).*

If the natural frequency is determined by Rayleigh's method, it can be demonstrated that

$$\sqrt{\frac{H_r(s_i)}{H_r(s_j)}} \approx \frac{\omega_r(s_j)}{\omega_r(s_i)} . \tag{14}$$

Since the modulated frequency ϖ_r does not differ from the natural frequency of the system to any notable extent, $\frac{\varpi_r(s_i)}{\varpi_r(s_j)}$ can be put approximately equal to $\frac{\omega_r(s_i)}{\omega_r(s_j)}$. Then

$$exp\left[-\int_{t_i}^{t_j} \beta_r dt\right] \approx \sqrt{\frac{\omega_r(s_j)}{\omega_r(s_i)}} \tag{15}$$

that is to say, the free vibration is damped approximately in proportion to the square root of the natural frequency.

The difference between the right-hand and left-hand members of eq. (14) is so small that derivation can be admitted without involving any considerable error. We get

$$\frac{\dot{\omega}_r}{\omega_r} \approx -\frac{1}{2}\left(\frac{\dot{H}_r}{H_r}\right) .$$

Noticing that

$$\frac{\dot{\varpi}_r}{\varpi_r} \approx \frac{\dot{\omega}_r}{\omega_r}$$

and inserting this expression in eq (12*a*) gives

$$\beta_r \approx \frac{1}{4}\left(\frac{\dot{H}_r}{H_r}\right) .$$

Using this relation it can be deduced that the frequency of vibration is approximately determined by

$$\varpi_r^2 = \omega_r^2 + \frac{1}{2} \cdot \frac{N_r}{H_r} - \frac{1}{16}\left(\frac{\dot{H}_r}{H_r}\right)^2 \cdot \frac{H_r + 1}{H_r - 1} . \tag{16}$$

Consequently

$$q_r = \sqrt{\frac{\varpi_r(s_i)}{\varpi_r(s_j)}} \cdot \frac{\omega_r(s_j)}{\omega_r(s_i)} \cdot \left(A \sin \int_{t_i}^{t_j} \varpi_r dt + B \cos \int_{t_i}^{t_j} \varpi_r dt\right) \tag{17}$$

can be regarded as a general solution of this equation (10). The arbitrary constants A and B are determined by the initial conditions at the instant $t = t_i$.

The solution of eq. (9) can now also be found by means of generally known methods. If we assume that the first of several consecutive loads travelling on the load-bearing structure passes over the first support at the instant $t = 0$, the solution can be written in the form

$$q_r = \frac{1}{m}\int_0^{t_j}\left[\frac{\sum_{k=1}^{k} P_k(s_{ki}) \cdot \varphi_{ri}(s_{ki})}{H_r(s_i) \cdot \varpi_r(s_i)} \cdot \frac{\omega_r(s_j)}{\omega_r(s_i)} \cdot \sqrt{\frac{\varpi_r(s_i)}{\varpi_r(s_j)}} \cdot \sin \int_{t_i}^{t_j} \varpi_r dt\right] dt_i . \tag{18}$$

This expression can also be simplified. If the static deformation of the load-bearing structure due to the loads in the i-th position is calculated with the help of series composed of the functions φ_{ri} and φ_{rj}, it can readily be demonstrated that

$$\frac{\varphi_{ri}(s_{ki})}{H_r(s_i)\cdot\omega_r^2(s_i)} \approx \frac{\varphi_{rj}(s_{ki})}{H_r(s_j)\cdot\omega_r^2(s_j)} \,. \tag{19}$$

By using this relation and putting $m\cdot H_r(s_j)\cdot\omega_r^2(s_j)=\lambda_r(s_j)$ the solution of the differential equation (9) can finally be written

$$q_r = \frac{1}{\lambda_r(s_j)}\cdot\frac{\omega_r(s_j)}{\sqrt{\varpi_r(s_j)}}\int_0^{t_j}\left[\frac{\omega_r(s_i)}{\sqrt{\varpi_r(s_i)}}\left(\sum_{k=1}^{k} P_k(s_{ki})\cdot\varphi_{rj}(s_{ki})\right)\cdot\sin\int_{t_i}^{t_j}\varpi_r\,dt\right]dt_i \tag{20}$$

and the problem under consideration can thus be regarded as theoretically solved.

The present work will be continued in a manner to allow a comparison between experimentally obtained results and theoretical values calculated with the aid of the developed theory.

Résumé

Le calcul théorique des déformations et des efforts causés dans les constructions portantes par les charges mobiles peut aussi être effectué dans le cas des constructions relativement compliquées. Dans la présente étude, ce problème est traité d'une manière plus générale que dans les travaux précédents, en faisant usage des fonctions caractéristiques, de sorte que l'équation différentielle établie dans le présent rapport est applicable aux conditions aux limites quelconques et à plusieurs charges mobiles inélastiques avec addition éventuelle d'autres forces dépourvues de masse. L'étude tient compte de la variation de la forme de la vibration naturelle avec la position de la charge. Une solution approximative de l'équation différentielle est présentée. L'auteur examine aussi d'autres méthodes applicables à l'étude de ce problème.

Zusammenfassung

Die Formänderungen und Spannungen, die in Tragkonstruktionen infolge einer oder mehrerer beweglichen Lasten entstehen, können auch bei verhältnismässig verwickelten Konstruktionen theoretisch berechnet werden. Zum Unterschied von früheren Untersuchungen, wird diese Frage in der vorliegenden Arbeit einer allgemeineren Behandlung unterzogen, und zwar mit Hilfe von Eigenfunktionen, so dass die aufgestellte Differentialgleichung für beliebige Randbedingungen sowie auch für mehrere bewegliche, nichtfedernde Lasten gegebenenfalls in Verbindung mit anderen massenlosen Kräften gültig ist. Die Änderung der Eigenschwingungsform mit der Lage der Last wird ebenfalls berücksichtigt. Eine angenäherte Lösung der Differentialgleichung ist gegeben. Auch andere Verfahren, die auf diese Frage anwendbar sind, werden besprochen.

Summary

The deformations and stresses produced in load-bearing structures by one or several moving loads can be calculated theoretically, even in the case of relatively complicated structures. In contradistinction from previous investigations, the present paper deals with this problem in a more general manner with the help of characteristic functions, so that the differential equation deduced in the paper holds good for any arbitrary boundary conditions and for several non-elastic moving loads, possibly acting in conjunction with other forces devoid of mass. The variation in the form of the natural vibration with the position of the load is also taken into account. An approximate solution of the differential equation is presented. Other methods for studying the problem are discussed.