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Calcul des ponts suspendus à grande portée

Berechnung der weitgespannten Hängebrücke

Analysis of the long span suspension bridge

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In deriving the governing equation of the stiffened suspension bridge, the following simplifying assumptions are generally made. That the hanger pull is constant along the cable, and the hangers are inextensible, that the cable displacements in each span are purely vertical, and the cable slides over rigid towers, and that the stiffening truss is of uniform rigidity.

Following the lead of Timoshenko and Priester, in recent years the exponential form of the equation has been replaced by a solution in the form of a trigonometric series, which takes account of variable hanger pull. By applying Southwell's Relaxation technique to the series solution, the author presents a method of analysis which does not embody the remaining assumptions. In the following pages the method is developed, and applied to the calculations for a bridge of 3 300 ft span. It is then shewn that for a structure of this magnitude the usual simplifications are legitimate, but that the effects of non-uniform stiffening truss rigidity, and of hanger extension are not negligible.

Derivation of the increment in the horizontal cable tension

The increment H_L in the horizontal component of the cable tension resulting from application of live load, and temperature change, may be derived from considerations of energy, or from the kinematics of the distorted cable. The energy solution assumes that the cable displacements are purely vertical, or if horizontal movements take place, that they are of no significance. It is not intuitively apparent that this is so, and by a kinematic approach the effect of the horizontal movements of the cable can be investigated. One result of these horizontal displacements and the consequential inclination of the hangers and translation of the tower tops, is

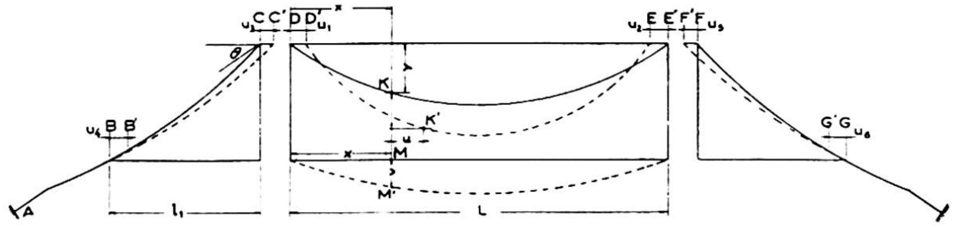

Fig. 1.

that the horizontal component of the cable tension is not constant across the bridge. The unbalance between centre and side spans arising from the stiffness of the towers may be corrected for without difficulty. On page 448 (post) the effect of hanger inclination on the moments in the truss is assessed, and it is found to be small. Neglect is generally on the side of safety. In what follows, the horizontal component of the cable tension is assumed to be constant across the bridge.

In fig. 1, which represents a bridge over three spans BC, DE and FG, as a result of temperature change and the application of live load, point K on the cable, distant x, y, from D, moves to K′, at $(x + u)$, $(y + v)$. At the towers there will be horizontal displacements U_1 and U_2, at each end of the centre span, and corresponding displacements U_3 and U_5 in the side spans. There will also, in general, be movements indicated by BB′ and GG′, due to the extension of the cable and the anchorage steelwork beyond the side spans.

An element ds of the cable at K will undergo a change in length of

$$(dx^2 + dy^2 + dv^2 + 2\,du \cdot dx + 2\,dv \cdot dy)^{\frac{1}{2}} - (dx^2 + dy^2)^{\frac{1}{2}},$$

or to the first order of small quantities,

$$dx\,(1 + y'^2)^{\frac{1}{2}} \left(1 + \frac{u' + y'v' + \dfrac{v'^2}{2}}{1 + y'^2}\right) - dx\,(1 + y'^2)^{\frac{1}{2}}$$

$$= dx \left(\frac{u' + y'v' + \dfrac{v'^2}{2}}{s'}\right) = \Delta ds \text{ (say)},$$

where « dashes » denote differentiation with respect to x, and $s' = (1 + y'^2)^{\frac{1}{2}}$.

Δds can also be expressed in terms of the elastic extension of the cable, and the effect of temperature change as

$$\Delta ds = \frac{H + H_L}{EA} \cdot s'^2 \left(1 + \frac{2\,u' + 2\,y'v' + v'^2}{s'^2}\right) dx - \frac{H}{EA} \cdot s'^2\, dx$$
$$+ \omega t \cdot s' \left(1 + \frac{u' + y'v}{s}\right) dx$$

where H denotes dead-load horizontal component of cable tension ;
H_L denotes live-load horizontal component of cable tension ;
ω denotes coefficient of expansion of the cable ;
E denotes modulus of elasticity of the cable ;
A denotes cross-sectional area of the cable.

Now H/EA, the dead load extension of the cable is small, H_L/EA is in general much smaller, so noting thats s' will not exceed about 1.1, and neglecting very small quantities,

$$\frac{H_L}{EA}\cdot s'^2\,dx + \omega t\cdot s'\,dx = \left(\frac{u' + y'v' + \frac{v'^2}{2}}{s'}\right)\left(1 - \frac{2\,H}{EA}\right)dx$$

and
$$\frac{H_L}{EA}\cdot s'^3\,dx + \omega t\cdot s'^2 dx = \left(\frac{u' + y'v' + \frac{v'^2}{2}}{s'^2}\right)\left(1 - \frac{2\,H}{EA}\right)dx\,.$$

Integrating between 0 and L,

$$\left(1 - \frac{2\,H}{EA}\right)(U_2 - U_1) = \int_0^L \left\{\frac{H_L}{EA}\cdot s'^3 + \omega t\cdot s'^2\right\}dx + \left(1 - \frac{2\,H}{EA}\right)$$
$$\int_0^L \left(vy'' - \frac{v'^2}{2}\right)dx \qquad (1)$$

Integrating between 0 and x, and neglecting small quantities,

$$u = U_1 + \int_0^x \left\{\frac{H_L}{EA}\cdot s'^3 + \omega t\cdot s'^2\right\}dx - \int_0^x y'v'\,dx \qquad (2)$$

For the side spans, writing w, and z, in place of v, the centre span deflection, the corresponding equations are

$$\left(1 - \frac{2\,H}{EA}\right)(U_4 - U_3) = \int_0^{l_1} \left\{\frac{H_L}{EA}\cdot s'^3 + \omega t\cdot s'^2\right\}dx + \left(1 - \frac{2\,H}{EA}\right)$$
$$\int_0^{l_1} \left(wy'' - \frac{w'^2}{2}\right)dx \qquad (3)$$

$$u = U_3 + \int_0^x \left\{\frac{H_L}{EA}\cdot s'^3 + \omega t\cdot s'^2\right\}dx - \int_0^x y'w'dx \qquad (4)$$

and similar equations in U_5, U_6, and z, for the second side span.

Then expressing the extension of the cable and anchorage steelwork between A and B as $\frac{H_L}{EA}\cdot l_2 + \omega t l_3$, and the extension between C and D over the tower as $\frac{H_L}{EA}\cdot l_4 + \omega t l_5$, equation (3) can be written

$$\left(1 - \frac{2\,H}{EA}\right)U_1 = \frac{H_L}{EA}\left\{\int_0^{l_1} s'^2\,dx + l_2 + l_4\right\} + \omega t\left\{\int_0^{l_1} s'^2\,dx + l_3 + l_5\right\}$$
$$+ \left(1 - \frac{2\,H}{EA}\right)\int_0^{l_1}\left(\omega y'' - \frac{\omega'^2}{2}\right)dx\,. \qquad (5)$$

Eliminating U_1 and U_2 from equations (1) and (5)

$$\frac{H_L}{EA}\left[\int_0^L s'^3\,dx + 2\left\{\int_0^{l_1} s'^3\,dx + l_2 + l_4\right\}\right]$$
$$+ \omega t\left[\int_0^L s^{2\prime}\,dx + 2\left\{\int_0^{l_1} s^{2\prime}\,dx + l_3 + l_5\right\}\right] =$$
$$= \left(1 - \frac{2\,H}{EA}\right)\left[\int_0^L \left(\frac{v'^2}{2} - vy''\right)dx\right.$$
$$\left. + \int_0^{l_1}\left(\frac{\omega'^2}{2} - \omega y'' + \frac{z'^2}{2} - zy''\right)dx\right] \qquad (6)$$

It is apparent from (6) that the value of H_L, although derived from consideration of the « u » displacements, is independent of them.

The shape of the erected cable is intermediate between catenary and parabola, but closer to the latter. Treatment is simpler with the parabolic form.

Then for the centre span,

$$y = \frac{4\,fx}{L} - \frac{4\,fx^2}{L^2} = \frac{m}{2}\left(x - \frac{x^2}{L}\right) \quad \text{(say)},$$

For the side spans,

$$y = \frac{4\,f_1 x}{l_1} - \frac{4\,f_1 x^2}{l_1^2} + x\tan\theta = \frac{m_1}{2}\left(x - \frac{x^2}{l_1}\right) + x\tan\theta \quad \text{(say)}$$

Expressing the deflections of centre and side spans by the sine series,

$$\left.\begin{aligned} v &= \Sigma_n V_n \sin n\pi \frac{x}{L} \qquad n = 1,\ 2,\ 3 \\ w &= \Sigma_n W_n \sin n\pi \frac{x}{l_1} \\ z &= \Sigma_n Z_n \sin n\pi \frac{x}{l_1} \end{aligned}\right\} \tag{7}$$

and substituting in (6),

$$\frac{H_L}{EA}\cdot L_s + \omega t \cdot L_t = \left(1 - \frac{2\,H}{EA}\right)\Sigma_n \left[K_n V_n + k_n\,(W_n + Z_n) + \frac{n^2\,V_n^2}{4\,L} + n^2\,\frac{(W_n^2 + Z_n^2)}{4\,l_1}\right] \tag{8}$$

where

$$L_s = \int_0^L s'^3 dx + 2\left\{\int_0^{l_1} s'^3 dx + l_2 + l_4\right\}$$

$$L_t = \int_0^L s'^2 dx + 2\left\{\int_0^{l_1} s'^2 dx + l_3 + l_5\right\}$$

$$K_n = m\,\frac{(1 - \cos n\pi)}{n\pi} \quad \text{and} \quad k_n = m_1\,\frac{(1 - \cos n\pi)}{n\pi}.$$

The term $2\,H/EA$ is for long span bridges of the order of $1/200$. It will however be found that omission of this term from equation (8) changes the value of H_L by very much less than 0.5 %.

Effect of « u » displacements on the partition of load between stiffening-truss and cable

Referring to fig. 1, the load taken by the cable on a length dx at K is originally

$$-H \cdot y''.$$

Point K on the cable now moves to K′, and point M on the stiffening-truss to M′, the load taken by the cable becoming

$$-(\mathrm{H}+\mathrm{H_L})(y''+v''), \text{ on a length } dx\left(1+\frac{du}{dx}\right).$$

The change in the cable load is to the first order of small quantities,

$$-\mathrm{H_L}\cdot y''-(\mathrm{H}+\mathrm{H_L})(v''+u'y'')=w_c \qquad \text{(say)},$$

If M_c is the change in the bending moment sustained by the cable, by integrating w_c twice it is found that

$$\mathrm{M_C}=\mathrm{H_L}\cdot y+(\mathrm{H}+\mathrm{H_L})\left(v+y''\int_0^x u\,dx-\frac{xy''}{\mathrm{L}}\int_0^{\mathrm{L}} u\,dx\right. \tag{9}$$

since for the parabola, y'' is constant, and $y'''=0$.

The moment sustained by the stiffening-truss, of flexural rigidity B is

$$-\mathrm{B}\cdot v''=\mathrm{M_G} \qquad \text{(say)}. \tag{10}$$

Then if M_L denote the bending moment from the applied live load and temperature change,

$$\mathrm{M_L}=\mathrm{M_C}+\mathrm{M_G}=\mathrm{H_L}\cdot y-\mathrm{B}\cdot v''+(\mathrm{H}+\mathrm{H_L})\,v$$
$$+(\mathrm{H}+\mathrm{H_L})\,y''\left[\int_0^x u\,dx-\frac{x}{\mathrm{L}}\int_0^{\mathrm{L}} u\,dx\right] \tag{11}$$

Substituting for u and v from equations (2) and (7),

$$\mathrm{M_C}=\mathrm{H_L}\cdot y+(\mathrm{H}+\mathrm{H_L})\,\Sigma_n\mathrm{V}_n\sin\frac{n\pi x}{\mathrm{L}}+(\mathrm{H}+\mathrm{H_L})\left(\frac{\mathrm{H_L}}{\mathrm{EA}}+\omega t\right)y$$

$$+(\mathrm{H}+\mathrm{H_L})\,m^2\,\frac{\Sigma_n\mathrm{V}_n}{n\pi}\left[\frac{1-\cos\frac{n\pi x}{\mathrm{L}}}{2}\right.$$

$$\left.+\frac{x}{\mathrm{L}}\cos\frac{n\pi x}{\mathrm{L}}-\frac{2\sin\frac{n\pi x}{\mathrm{L}}}{n\pi}-\frac{x}{\mathrm{L}}\left(\frac{1+\cos n\pi}{2}\right)\right| \tag{12}$$

Equation (12) can be expressed as the Fourier series,

$$\mathrm{M_C}=\Sigma_\lambda\,(\mathrm{M_C})_\lambda\sin\frac{\lambda\pi x}{\mathrm{L}}$$

where

$$(\mathrm{M_C})_\lambda=\frac{2}{\mathrm{L}}\int_0^{\mathrm{L}}\mathrm{M_C}\sin\left(\frac{\lambda\pi x}{\mathrm{L}}\right)dx\,.$$

Then $\mathrm{H_L}\cdot y$ gives

$$\frac{2\,\mathrm{H_L}}{\mathrm{L}}\int_0^{\mathrm{L}}\left(\frac{4\,fx}{\mathrm{L}}-\frac{4\,fx^2}{\mathrm{L}^2}\right)\sin\left(\frac{\lambda\pi x}{\mathrm{L}}\right)dx=\frac{16\,f}{\lambda^3\pi^3}(1-\cos\lambda\pi)\,\mathrm{H_L}$$
$$=\mathrm{H_L}\,(\mathrm{G})_\lambda\ \text{(say)} \tag{13}$$

$$(\mathrm{H}+\mathrm{H_L})\,\Sigma_n\mathrm{V}_n\sin\frac{n\pi x}{\mathrm{L}}, \quad \text{gives}$$

$$\frac{2(H+H_L)}{L}\int_0^L \sin\left(\frac{\lambda\pi x}{L}\right)\Sigma_n V_n \sin\left(\frac{n\pi x}{L}\right)dx = (H+H_L)V_\lambda, \text{ when } \lambda = n.$$

When $\lambda \neq n$, the integral is zero (14)

$$(H+H_L)\left(\frac{H_L}{EA}+\omega t\right)y \quad \text{gives} \quad H\left(\frac{H_L}{EA}+\omega t\right)(G)_\lambda, \tag{15}$$

as $H_L \cdot y\left(\frac{H_L}{EA}+\omega t\right)$ can be neglected in comparison with $H_L \cdot y$. The last line of equation (12) gives

$$(H+H_L)\Sigma_n A_{\lambda,n} V_n$$

where, when

$$\lambda = n, \quad \Sigma_n A_{\lambda,n} V_n = -\frac{m^2 V_\lambda}{2\lambda^2\pi^2}.$$

and, when $\lambda \neq n$,

$$\Sigma_n A_{\lambda,n} V_n = \Sigma_n V_n \frac{m^2 n \{1+\cos(\lambda-n)\pi\}}{\lambda\pi^2(n^2-\lambda^2)} \tag{16}$$

Expressing M_G also in its harmonic components,

$$(M_G)_\lambda = \frac{2}{L}\Sigma_n V_n \frac{n^2\pi^2}{L^2}\int_0^L B\sin\left(\frac{\lambda\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx.$$

When the flexural rigidity B of the stiffening truss is uniform throughout its length,

$$(M_G)_\lambda = \frac{\lambda^2\pi^2 B}{L^2}V_\lambda. \tag{17}$$

When B is not uniform, the variation of B along the span can always be expressed as

$$B = B_0 + \Sigma_m B_m \sin\frac{m\pi x}{L} \tag{18}$$

where in any practical design, at the most three odd values of B_m will represent the variation of B along the span.

Then $(M_G)_\lambda = \Sigma_n R_{\lambda,n} V_n$, where,

$$\Sigma_n R_{\lambda,n} V_n = \frac{\lambda^2\pi^2 B_0}{L^2}V_\lambda - \Sigma_{m,n} V_n \cdot \frac{4\pi m n^3 \lambda \{1-\cos(\lambda-n+m)\pi\} B_m}{L^2\{(\lambda^2-n^2)^2-2m^2(\lambda^2+n^2)+m^4\}} \tag{19}$$

$(\lambda - n)$ is either even or zero.

Now since $M_L = M_C + M_G$ the Fourier series for M_L, and $M_C + M_G$, must be equal term by term, and

$$(M_L)_\lambda - H_L\left(1+\frac{H}{EA}\right)(G)_\lambda - H\omega t\,(G)_\lambda$$

$$= (H+H_L)(\Sigma_n A_{\lambda,n} V_n + V_\lambda) + \Sigma_n R_{\lambda,n} V_n \tag{20}$$

For the side spans, corresponding to equation (20),

$$(M_t)_\lambda - \left\{ H_L \left(1 + \frac{H}{EA} \sec^3 \theta\right) + H\omega t \sec^2 \theta \right\} (g)\, \lambda$$

$$= (H + H_L)(\Sigma_n a_{\lambda,n} W_n + W_\lambda) + \frac{\lambda^2 \pi^2 B_1}{l_1^2} W_\lambda \qquad (21)$$

where

$$(g)_\lambda = \frac{16\, f_1 (1 - \cos \lambda\pi)}{\lambda^3 \pi^3} \qquad (22)$$

B_1 is the flexural rigidity of the side span trusses. (It will rarely be necessary to take account of variable stiffness in the side spans), and

$$\Sigma_n a_{\lambda,n} W_n = -\frac{m_1^2 W_\lambda}{2 \lambda^2 \pi^2}, \quad \text{when } \lambda = n,$$

when $\lambda \neq n$,

$$\Sigma_n a_{\lambda,n} = \Sigma_n W_n \frac{n [m_1^2 \{1 + \cos (\lambda - n) \pi\} + 2 m_1 \tan \theta \{1 - \cos (\lambda - n) \pi\}]}{\lambda \pi^2 (n^2 - \lambda^2)}$$

$$(23)$$

The terms in $\frac{H}{EA}$, and $H\omega t$ in equations (20) and (21), included for completeness, can usually be neglected.

For a uniform load extending from $x = a$, to $x = b$,

$$(M_L)_\lambda = \frac{2\, p l^2}{\lambda^3 \pi^3} \left(\cos \frac{\lambda \pi a}{L} - \cos \frac{\lambda \pi b}{L}\right) \qquad (24)$$

For a concentrated load P at $x = a$,

$$(M_L)_\lambda = \frac{2\, PL}{\lambda^2 \pi^2} \sin \frac{\lambda \pi a}{L} \qquad (25)$$

It will be appreciated that H_L is an unknown in (20) and (21). A direct solution may be obtained by substituting for H_L from equation (8), and liquidating the deflection coefficients V_n, W_n, Z_n, by Southwell's Relaxation method. Numerical work is much simplified if this substitution is not made, but instead an approximate value of H_L is introduced in equations (20) and (21), and subsequently corrected for. Now very closely equation (8) can be written

$$\frac{H_L}{EA} \cdot L_s + \omega t L_t = \Sigma_n K_n V_n + \Sigma_n k_n (W_n + Z_n)$$

and then approximately

$$\frac{H_L}{EA} \cdot L_s + \omega t L_t = \frac{2\, m V_1}{\pi} + \frac{2\, m_1}{\pi} (W_1 + Z_1) \qquad (26)$$

Writing

$$\frac{H_L}{H} = \beta, \quad \frac{HL^2}{\pi^2 B} = \alpha, \quad \frac{H l_1^2}{\pi^2 B_1} = \alpha_1,$$

$$\beta^2\left[\frac{HL_s}{EA}(\alpha+2\,\alpha\alpha_1+\alpha_1)+\omega tL_t\alpha\alpha_1+\frac{2\,\alpha\alpha_1}{\pi}(mG_1+2\,m_1g_1)\right]$$
$$+\beta\left[\frac{HL_s}{EA}(1+\alpha)(1+\alpha_1)+\omega tL_t(\alpha+2\,\alpha\alpha_1+a_1)\right.$$
$$\left.+\frac{2}{\pi}\left\{m\alpha(1+\alpha_1)\,G_1+2\,m_1\,\alpha_1(1+\alpha)\,g_1\right\}-\frac{2\,m}{\pi}\,\frac{(M_L)_1}{H}\,\alpha\alpha_1\right]$$
$$+\omega tL_t(1+\alpha)(1+\alpha_1)-\frac{2\,m}{\pi}\,\frac{(M_L)_1}{H}\,\alpha\,(1+\alpha_1)=0\,. \qquad (27)$$

For live load on a side span, and the terms,

$$-\beta\frac{2\,m_1}{\pi}\,\frac{(M_l)_1}{H}\,\alpha\alpha_1,\ \text{and}\ -\frac{2\,m_1}{\pi}\,\frac{(M_l)_1}{H}\,\alpha_1\,(1+\alpha)\,.$$

The value of H_L given by equation (27) is substituted in (20) and (21). The « given » moments M_L and M_l are liquidated, and by interpolation corrected values of H_L and V_n are derived.

The moment M_G taken by the stiffening truss is given by

$$M_G=\Sigma_{\lambda,n}R_{\lambda,n}V_n\sin\lambda\pi\frac{x}{L}$$

This expression may be found to converge too slowly.

Now $M_L=\Sigma_\lambda\,(M_L)_\lambda\sin\frac{\lambda\pi x}{L}$, and from equation (13),

$$H_L\cdot y=\Sigma_\lambda H_L\,(G)_\lambda\sin\left(\frac{\lambda\pi x}{L}\right);$$

therefore

$$M_G=M_L-H_L\cdot y-\Sigma_\lambda\left[(M_L)_\lambda-H_L\,(G)_\lambda-\Sigma_nR_{\lambda,n}V_n\right] \qquad (28)$$

The term in square brackets is rapidly convergent, since, from (20), $(M_L)_\lambda-H_L\cdot(G)_\lambda$ approximates to $\Sigma_nR_{\lambda,n}V_n$ as λ increases.

The effect of the « u » displacements as represented by the parameters $A_{\lambda,n}$, $a_{\lambda,n}$ is extremely small on long span bridges, and equation (28) becomes $M_G=M_L-H_L\cdot y-(H+H_L)\,\Sigma_\lambda V_\lambda\sin\left(\frac{\lambda\pi x}{L}\right)$ (29)

Numerical application (Bridge of 3.300 ft span)

$L=3\,280$ ft $f=326$ ft $\frac{f}{L}=0.09939$ $m=0.7951$ $L_s=6\,366$ ft

$l_1=1\,000$ ft $f_1=30.30$ ft $\frac{f_1}{l_1}=\frac{1}{33}$ $m_1=0.2424$ $\tan\theta=0.3708$

$A=980$ sq.in $EA=27.44\times10^9$ $H=58.5\times10^6$ lb.

$p=6\,100$ lb.ft from $a=\frac{3}{16}L$ to $b=\frac{5}{16}L$ on centre span.

$B=28.51\times10^{12}$ (average value) $B_1=28.51\times10^{12}$ lb.ft²

$\frac{\pi^2B}{L^2}=2.615\times10^6$ lb (average value) $\frac{\pi^2\,B_1}{l^2}=28\cdot11\times10^6$ lb .

The effective area of the centre span stiffening truss, averaging 260 sq. in, varies from 200 sq. in at the ends, to 290 sq. in at the quarter points, and 240 sq. in at centre span. This variation is closely represented by

$$200.58 + 80 \sin \frac{\pi x}{L} + 40 \sin 3 \frac{\pi x}{L}$$

giving the values of $R_{\lambda,n}$ in table no. I.

In table no. IV, the case of a load of 6 100 lb from $a = 3/16$ L to $b = 5/16$ L is analysed. This is the loading found to give maximum positive moment at the quarter point. « U » effects are not considered in the analysis.

Solving equation (27) for $(M_L)_1 = 1\,168.1 \times 10^6$ lb ft, an approximate value of $H_L = 3.15 \times 10^6$ lb gives $(M_L)_\lambda - H_L(G)$ in line 4. The Relaxation process begins in line no. 5, where the whole of $(M_L)_1$ is liquidated by a value of $V_1 = 108.2/64.28$, which contributes $-R_{31}V_1 = -0.2$ to $(M)_3$, etc. In lines 19 to 25 the residual moments are liquidated by small contributions to $V_1 - V_9$. Account is kept of $V_n K_n + \dfrac{n^2 V_n^{\,2}}{4\,L}$, in columns 3 to 4 as relaxation proceeds. Lines 27-28 give the contribution to H_L from the side spans, and a value of H_L $2\,690 \times 10^6$ is obtained. It is now apparent that the starting value of H_L was too high, so in lines 32-36 relaxation is continued for an increment in H_L of -0.04×10^6. Finally H_L is interpolated as $3\,114 \times 10^6$ lb. In lines 38-41, the V_λs are corrected for the change in H_L, giving a bending moment in the stiffening truss of 147×10^6 lb. ft. On the assumption of uniform truss rigidity, this moment is found to be 139×10^6 lb. ft, or 5.5 % less. If account is taken in the above analysis of the « u » effect as expressed in the parameters $A_{\lambda,n}$ and $a_{\lambda,n}$, the moment at the quarter point is reduced by only 0.2 %.

Effect of hanger extension

As a result of hanger extension, the stiffening truss is subject to additional moments and shears. The increase is not negligible near the ends of the truss.

Denoting by Δv the hanger extension at any section x, of the centre span, corresponding to a stiffening truss deflection v, and neglecting horizontal displacements,

$$\Delta M_G + M_G = M_L - (H + H_L)(v - \Delta v + \Delta v_0) \qquad (30)$$

where Δv_0 denotes the hanger extension at each end of the span, and ΔM_G the additional moment transferred to the truss by the hanger extension.

Therefore

$$\Delta M_G = (H + H_L)(\Delta v - \Delta v_0). \qquad (31)$$

The hanger pull per unit length is given by

$$q = -\frac{d^2 M_C}{dx^2} = -(H + H_L)\frac{d^2 v}{dx^2} - H_L \cdot y'' \qquad (32)$$

But $M_G = -\dfrac{Bd^2 v}{dx^2}$ therefore

$n \backslash \lambda$	1	2	3	4	5	6	7	8	9	10	11
1	2.63		0.13		— 0.10		— 0.02		— 0.01		
2		11.04		0.104		— 0.49		— 0.118		— 0.051	
3	1.14		23.93		0.04		— 1.17		— 0.305		— 0.139
4		0.415		42.2		— 0.055		— 2.155		— 0.582	
5	— 2.52		0.13		65.7		— 0.20		— 3.44		— 0.95
6		— 4.39		— 0.12		94.4		— 0.36		— 5.00	
7	— 1.03		— 6.39		— 0.37		128.4		— 0.58		— 6.86
8		— 1.38		— 8.62		— 0 67		167.66		— 0.82	
9	— 0.688		— 2.74		— 11.1		— 1.0		212.1		— 1.10
10		— 1.28		— 3.64		— 13.9		— 1.3		261	
11	— 0.527		— 1.87		— 4.6		— 16.9		— 1.66		316
12		— 0.99		— 2.45		— 5.62		— 20.3		— 2.07	
13	0.428		— 1.43		— 3.06		— 6.74		— 23.9		— 2.56
14		— 0.815		— 1.88		— 3.70		— 7.92		— 27.7	
15	— 0.364		— 1.18		— 2.34		— 4.37		— 9.20		— 31.99
16		— 0.693		— 1.51		— 2.80		— 5.09		— 10.5	

$$R_{\lambda,n} = 2.018 \times 10^6 \lambda^2 - \Sigma_n n^2 \lambda \left[\frac{2.050}{(\lambda^2 - n^2)^2 - 2(\lambda^2 + n^2) + 1} + \frac{3.075}{(\lambda^2 - n^2)^2 - 18(\lambda^2 + n^2) + 81}\right] \times 10^6$$

$\lambda = n$	1	2	3	4	5	6	7	8	9	10
$G\lambda$	336.45									
K_n	0.5062		0.1687		0.1009		0.0723		0.0562	
$\lambda^2 \pi^2 B/L^2 \times 10^{-6}$	2.6152	10.46	23.51	41.84	65.38	94.1	128.1	167.4	211.8	262
$H \times 10^{-6}$	58.5									
$H(1 + A\lambda\lambda)\, 10^{-6}$	56.61	58.03	58.29	58.38	58 42	58.5				
$\lambda^2 \pi^2 B/L^2 + H \times 10^{-6}$	61.12	68.96	82.04	100.34	123.88	152.6	186.6	225.9	270.3	320
$\lambda^2 \pi^2 B^2/L^2 + H(1 + A\lambda\lambda) \times 10^{-6}$	59.23	68.49	81.83	100.22	123.80	152.6	186.6	225.9	270.3	320

$\lambda = n$	1	2	3	4	5	6	7	8	9	
$g\lambda$	31.27									
k_n	0.1513		0.0514		0.0309		0.0220		0.0172	
$\lambda^2 \pi^2 B_1/l^2 \times 10^{-6}$	28.707	114.8	258.4	459	718	1 033	1 407	1 837	2 325	
$H \times 10^{-6}$	58.50		58.5		58.5		58.5			
$H(1 + A\lambda\lambda) \times 10^{-6}$	58.32	58.46								
$\lambda^2 \pi B_1/l_1^2 + H \times 10^{-6}$	87.21	173.3	316.9	517	776.5	1 091	1 465	1 895	2 384	
$\lambda^2 \pi^2 B_1^2/l_1 + H(1\, A\lambda\lambda) \times 10^{-6}$	87.03									

Table I

VALUES OF

$R_{\lambda, n} \times 10^{-6}$

FOR CENTRE SPAN

(3 300 ft)

12	13	14	15	16
− 0.027		− 0.016		− 0.011
	− 0.077		− 0.047	
− 0.272		− 0.154		− 0.097
	− 0.452		− 0.259	
− 1.41		− 0.679		− 0.394
	− 1.95		− 0.951	
− 9.02		− 2.59		− 1.27
	− 11.5		− 3.31	
− 1.45		− 14.2		− 4.12
	− 1.8		− 17.3	
376		− 2.19		− 20.5
	442		− 2.55	
− 2.98		513		− 3.12
	− 3.4		587	
− 36.4		− 4.0		669

Table II A

PARAMETERS

FOR CENTER SPAN

(3 300 ft)

11	12	13	14	15	16
0.0460		0.0389		0.0337	
316	377	442	513	588	669
374	435	500	571	646	727
374	435	500	571	646	727

Table II B

PARAMETERS

FOR SIDE SPANS

(1 000 ft)

n \ λ	1	2	3	4	5	6	7	8	9	10	11	12
1	— 0.0322		— 0.0054		— 0.0011		— 0.00038		— 0.00018		— 0.00011	
2		— 0.0080		— 0.0054		— 0.0013		— 0.00054		— 0.00027		
3	0.0485		— 0.0036		— 0.0048		— 0.0014		— 0.0006			
4		0.0215		— 0.0020		— 0.0013		— 0.0013				
5	0.0268		0.0135		— 0.0013		— 0.0038		— 0.0013			
6		0.0121		0.0097		— 0.0009		— 0.0035				
7	0.0188		0.0075		0.0075		— 0.0066		— 0.0031			
8		0.0086										
9	0.0145		0.0054									
10		0.0067										
11	0.012											
12												
13	0.010											
14												
15	0.009											

n \ λ	1	2	3	4
1	— 0.0030	— 0.006	— 0.0005	
2	0.024	— 0.00075	— 0.005	— 0.0005
3	0.0041	0.011	— 0.00033	— 0.004
4	0.0096	0.002	0.007	

TABLE III. — VALUES OF $A_{\lambda, n}$

Above : for 3 300 ft centre span
Left : for 1 000 ft side spans

$$q = - H_L \cdot y'' + \frac{H + H_L}{B} \cdot M_G \tag{33}$$

and the hanger extensions are given by

$$\Delta v = \frac{F_C + f - y}{E_h A_h} \left\{ - H_L \cdot y'' \frac{(H + H_L)}{B} M_G \right\}$$

$$\Delta v_0 = \frac{F_C + f}{E_h A_h} (- H_L \cdot y'') \tag{34}$$

where F_c denotes length of centre hanger, E_h denotes hanger modulus of elasticity, and A_H denote area of hangers per unit length.

Then equation (31) can be written

$$\Delta M_G = \frac{H + H_L}{E_h A_h} \left[(F_c + f - y) \frac{(H + H_L)}{B} M_G + y y'' \cdot H_L \right] \tag{35}$$

and the additional shear ΔS_g is

$$\Delta S_G = \frac{H + H_L}{E_h A_h} \left[\frac{H + H_L}{B} \left\} (F_c + f - y) S_G - y' M_G \right\{ + y' y'' H_L \right] \tag{36}$$

BRIDGE $H = 58.5 \times 10^6$ lb
$A = 980$ sq in
$E = 28 \times 10^6$ lb/in²
$\frac{2H}{EA} = 0.0043$
$\frac{EA}{LS} = 4.3104 \times 10^6$

LOADED LENGTHS
3/16 L — 5/16 L
Centre Span

CENTRE SPAN $\pi^2 B/L^2 = 2.615 \times 10^6$ lb
$L = 3.280$ ft
$f/L = 0.09939$
$f = 326$ ft
LIVE LOAD PER LIN : FOOT 6 100 lb/ft

SIDE SPANS $B_1 = 28.71 \times 10^6$ lb
$l_1 = 1\,000$ ft
$f_1/l_1 = 0.0303$
$\tan\theta = 0.3708$
TEMPERATURE CONDITION NORMAL

Col. N°	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Line N°	ΔV_n	V_n	$\Delta V_n K_n + n^2 V_n^2/4L$	$V_n K_n + n^2 V_n^2/4L$	MOMENTS $\times 10^6$															
					$(M)_1$	$(M)_2$	$(M)_3$	$(M)_4$	$(M)_5$	$(M)_6$	$(M)_7$	$(M)_8$	$(M)_9$	$(M)_{10}$	$(M)_{11}$	$(M)_{12}$	$(M)_{13}$	$(M)_{14}$	$(M)_{15}$	$(M)_{16}$
1	$(H + H_L) + R\lambda\lambda$ for $H_L = 3.15 \times 10^6$				64.28	72.69	85.58	103.8	127.4	156.1	190	229	274	324	378	438	503	574	649	731
2	$(M_L)\lambda$				1168.1	405.0	123.2	—	− 39.9	− 36.3	− 17.2	—	8.05	7.85	3.76	—	− 1.52	− 1.19	− 0.35	—
3		$-H_L(G)\lambda = (G)\lambda \times 3.15 \times 10^6$			− 1059.9		− 39.3		− 8.5		− 3.1		− 1.46		− .80		− .50		− .32	
4					108.2	405.0	− 83.9	—	− 48.4	− 36.3	− 20.3	—	6.6	7.8	3.0	—	− 2.0	− 1.2	− 0.7	—
5	$V_1 = 1.6840$				− 108.2		− 0.2		0.1											
6	$V_2 = 5.576$					− 405.0		− 0.6		2.7		0.7		0.3		0.2		0.1		0.1
7	$V_3 = 0.978$				− 1.1		− 83.7				1.2		0.3		0.1		0.1			
8	$V_4 = -0.0056$	$V_4 = -0.0056$						0.6												
9	$V_5 = -0.379$				− 1.0				48.3		− 0.1		− 1.3		− 0.4		− 0.2		− 0.1	
10	$V_6 = -0.215$					− 0.9		—		33.6		− 0.1		− 1.1		− 0.3		− 0.2		− 0.1
11	$V_7 = -0.101$				− 0.1		− 0.0		—		19.2		− 0.1		− 0.7		− 0.2		− 0.1	
12	$V_8 = 0.0026$	$V_8 = 0.0026$										− 0.6								
13	$V_9 = 0.0201$								0.2		—		− 5.5				0.2		0.1	
14	$V_{10} = 0.0217$	$V_{10} = 0.0217$						0.1		0.3				− 7.0				0.3		0.1
15	$V_{11} = 0.0055$	$V_{11} = 0.0055$	0.0003	0.0003							0.1				− 2.0				0.1	
16	$V_{13} = -0.0042$	$V_{13} = -0.0042$	− 0.0002	0.0001									− 0.1				2.1			
17	$V_{14} = -0.0017$	$V_{14} = -0.0017$																1.0		
18	$V_{15} = -0.0011$	$V_{15} = -0.0011$																	0.7	
19	$V_1 = -0.0312$	$V_1 = 1.6498$	0.8348	0.8349	2.2															
20	$V_2 = -0.0124$	$V_2 = 5.564$	0.0094	0.8443		0.9														
21	$V_3 = -0.007$	$V_3 = 0.970$	0.1638	1.0081			0.6													
22	$V_5 = 0.0016$	$V_5 = -0.377$	− 0.0380	0.9701					− 0.2											
23	$V_6 = 0.0019$	$V_6 = -0.213$								− 0.3										
24	$V_7 = 0.0005$	$V_7 = -0.101$	− 0.0073	0.9628							− 0.1									
25	$V_9 = -0.0004$	$V_9 = 0.0197$	0.0011	0.9639									0.1							
26		Residuals			—	—	—	0.1	—	—	—			—	—	− 0.1	—	—	—	0.1
27	SIDE SPANS																			
28	$H + H_L + \lambda^2 \pi^2 \frac{B_1}{l_1^2}$ for $H_L = 3.15 \times 10^6$				90.36		320													
29			$-3.15 \times (g)_\lambda \times 10^6$		− 98.4		− 3.6													
30	$W_1 = -1.089$		− 0.1680	− 0.1680	98.4															
31	$W_3 = -0.0113$		− 0.0006	− 0.1686			3.6													
				− 0.3372																
				0.9639																
				0.6267	− 0.0027	0.6240	$H_L = 2.690 \times 10^6$													
32	$\Delta H_L = -0.04$																			
33	$W_1 = 0.0139$		0.0021	0.0042	1.25															
	MAIN SPAN				13.5		0.5		0.1											
34	$V_1 = 0.210$		0.1061	0.1061	− 13.5															
35	$V_3 = 0.0059$		0.0010	0.0010			− 0.5													
36	$V_5 = 0.0008$		0.0001	0.0001					− 0.1											
				0.7381	− 0.0032	0.7349	$H_L = 3.168 \times 10^6$		3.11											
							2.690		3.15											
							.478		0.04											
							$K = \frac{0.46}{0.518} = 0.89$													
							$H_L = 3.114 \times 10^6$													
37	Interpolated values of V_1, etc																			
38	$V_1 = 0.1870$	1.8368																		
39	$V_3 = 0.0052$	0.9752																		
40	$V_5 = 0.0007$	− 0.3763																		
41	Final values of V_λ																			

V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}	V_{13}	V_{14}	V_{15}
1.8368	5.564	0.9752	− 0.0056	− 0.3763	− 0.213	− 0.101	0.0026	0.0197	0.0217	0.0055	− 0.0042	− 0.0017	− 0.0011

42 $\Sigma V\lambda \sin \lambda\pi/4 = 8.1478$

Moment at ¼ point $= 1409.9 \times 10^6 - H_L y - (H + H_L) v$
$= (1409.9 - 761.0 - 502.0) \times 10^6$
$= 147 \times 10^6$ lb ft

TABLE IV. — CALCULATION OF STIFFENING TRUSS MOMENTS

For the structure under consideration the maximum end shear is increased by 4 %, and the moment at the 1/16 point by about the same amount. The effect falls off rapidly towards the centre of the span.

There are also increases in the moments and shears at the ends of the truss beyond what are given by (35) and (36) arising from the rigid support afforded to the ends of the truss. In the derivation of equations (35) and (36) it was assumed that the ends of the truss were on hangers.

If Δv_0 is the end hanger extension given by (34), and K is the average hanger reaction for unit extension per unit length, then the additional end shear taken by the truss is

$$\Sigma_n \frac{8\,\mathrm{K}\Delta\mathrm{V}_0}{n^2\pi^2 + \frac{\mathrm{KL}^2}{\mathrm{H}} + \frac{\mathrm{KL}^4}{n^2\pi^2\mathrm{B}}} \qquad n = 1, 3, 5, \ldots \tag{37}$$

The increase in end shear from this cause is only about 1 %.

The effect of hanger inclination

The importance of the inclination of the hangers discussed on p. 2, in changing the horizontal component of the cable tension must now be considered.

At any section x of the centre span, the hanger inclination is given by $\frac{u}{h_x}$, where h_x is the length of the hanger at x. The increment in horizontal cable tension is $\frac{u \cdot q}{h_x} dx$, where u is given by equation (2), and q by equation (33).

The total increment of horizontal cable tension at x is

$$\int_0^x \frac{uq}{h_x} dx\,. \tag{38}$$

Performing this integration graphically, it is found that at the quarter point the effect of hanger inclination is to increase $\mathrm{H_L}$ by 1 %, with a corresponding decrease in $\mathrm{M_G}$ of only 0.3 %.

Conclusions

A careful study undertaken for a bridge of 3 300 ft span, indicates that the usual assumptions in the orthodox deflection theory are legitimate when analyzing a structure of this magnitude. On long span bridges, the principal contribution to moments and shears comes from the higher harmonics of the deflection. These harmonics are but little affected by quite large changes in $\mathrm{H_L}$ arising from uncertainties as to the exact behaviour of the structure. On the other hand, the primary harmonic $\mathrm{V_1}$, which is highly geared to variation in $\mathrm{H_L}$, contributes only a small amount to the moments and shears taken by the stiffening truss. On the structure discussed in this paper, a 1 % change in $\mathrm{H_L}$, causes the bending moment to change by only 0.2 % on the avérage. Making the assumption of uniform truss rigidity, it is thus possible to simplify the computation very considerably. Equation (8) is replaced by

$$\frac{H_L}{EA} \cdot L_s + \omega t L_t = \Sigma_n K_n + \Sigma_n k_n (W_n + Z_n) . \tag{31}$$

and equations (20) and (28) by

$$M_G = M_L - H_L \cdot y \frac{- \Sigma_\lambda \{ (M_L)_\lambda - H_L (G)_\lambda \}}{1 + \dfrac{\lambda^2 \pi^2 \dfrac{B}{L^2}}{H + H_L}} \cdot \sin \frac{\lambda \pi x}{L} , \tag{40}$$

It will be found that on long span bridges, maximum moments and shears are given by short loaded lengths of the order of 1/8 to 1/4 of the span. It will also be found that the maximum moment at any section from a given loading occurs when the loading extends equally on either side of the section. Work can be systematized and simplified by tabulating multiples sines and cosines of $\frac{\lambda\pi}{32}$, and constructing tables of cos $n\pi a/L$-cos $n\pi b/L$ for several loaded lengths. The author has made up tables of this nature which can be applied without modification to a bridge of any span. By this means, and the use of equations (39) and (40), moments and shears at any section for a given loading can be obtained very rapidly. The effect of changes in make-up giving revised dead load and truss rigidity can be assessed without difficulty.

Résumé

La méthode par approximations successives de Southwell s'applique à la résolution de ponts suspendus à l'aide de solutions trigonométriques. Ce procédé de calcul ainsi obtenu ne nécessite pas la simplification habituelle de suspentes inextensibles à traction constante, un déplacement vertical des câbles, et un coefficient de rigidité constant. Ce procédé est amélioré et étendu à un pont d'une portée de 1 000 mètres. L'influence de ce que l'on néglige les corrections habituelles est étudiée et l'auteur montre que seuls les deux facteurs suivants influencent pour des ponts suspendus de grande portée : rigidité variable des poutres et longueur variable des suspentes. Le mémoire se termine par une représentation simplifiée de la résolution par série suffisante pour la plupart des ouvrages courants.

Zusammenfassung

Die Iterationsmethode von Southwell wird zur Lösung des Hängebrückenproblems mit Hilfe von trigonometrischen Reihen angemeldet. Das sich dabei ergebende Rechnungsverfahren verzichtet auf die üblichen vereinfachenden Annahmen über konstanten Hängezug unelastischer Hänger, vertikale Kabelverschiebungen und konstante Steifigkeit der Versteifungsträger. Das Verfahren wird entwickelt und auf die Berechnung einer Brücke von 1 000 m Spannweite angewendet. Die Einflüsse der üblichen Vernachlässigungen werden untersucht und es wird gezeigt, dass nur die Auswirkungen einer veränderlichen Trägersteifigkeit und der Hänger-

dehnungen bei Brücken dieser Grössenordnung von Bedeutung sind. Die Arbeit schliesst mit einer vereinfachten Darstellung der Lösung mit Reihen, welche für die meisten Tragwerke genügend ist.

Summary

Southwell's Relaxation technique is applied to the trigonometric series solution of the problem of the stiffened suspension bridge. The resulting method of analysis is free from the usual simplifying assumptions of constant hanger pull, inextensible hangers, vertical cable displacements and uniform truss rigidity. The method is developed and applied to the calculations for a bridge of 3 300 ft span. The results of neglecting the corrections to the orthodox theory are then assessed, and it is shown that only the effects of non-uniform truss rigidity and hanger extension are significant on structures of this magnitude. The paper terminates with a simplified presentation of the series solution, which is sufficiently accurate for most structures.