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# IVc5

## Considérations concernant le calcul et le projet des voûtes cylindriques

## Kritische Betrachtungen zur Berechnung und zum Entwurf von Zylinderschalen

## Critical notes on the calculation and design of cylindrical shells

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The forces and bending moments in cylindrical shells are usually determined by a differential equation of the 8th order. The mathematical work is very complicated, especially the determination of the arbitrary constants at the boundary conditions. The numerical computations must be worked out to six decimals, when the final results are required to have three. The forces and moments thus obtained determine the dimensions and reinforcement of the shell. How this is done is never mentioned in the literature, and, I think, with good reason. The fact is that it is done in a very irrational manner. The tensile stresses are summed up into a tensile force, and the area of reinforcement is determined by dividing this tensile force by the working stress, ignoring the fact that the deformation and the working stress do not correspond, as they are not constant over the entire region of tension. With this serious discrepancy between stress and strain the basis of all the fine mathematical work and all the complicated computations is gone, the basis being the theory of elasticity, where stress and strain correspond. We may put it this way : First we calculate the forces and moments according to the theory of elasticity, and then we design according to assumptions contrary to the same theory. Being very sensitive to the distribution of the shear forces, the transversal bending moments may be multiplied when cracks develop and the bars yield, and we must therefore calculate the shell in a state of rupture at the ultimate load, the more so as the working load must be fixed according to the ultimate load.

For long shells the theory of rupture is very simple. The shell may be considered as a reinforced concrete beam, and the tension of the concrete in the tension zone is not taken into account. The neutral axis is determined

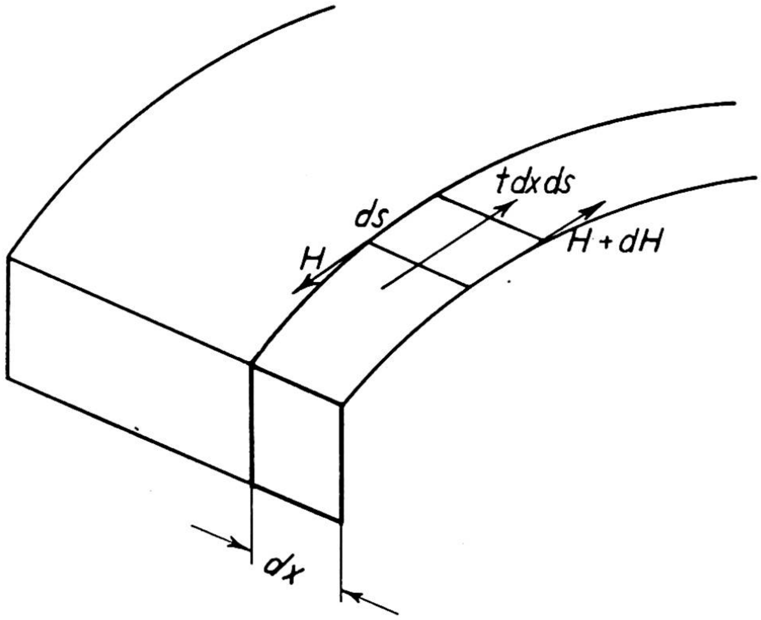


Fig. 1.

as corresponding to the stresses in the concrete and the reinforcement. Then also the moment arm $h_t$ is determined, whereby the tensile force T, equal to the compressive force C, can be calculated, as $M = T \cdot h_t$. On a strip of the shell act the shear forces — H and H + $d$H, calculated according to the elementary theory for reinforced concrete beams. The resultant of — H and H + $d$H acts as a tangential force $tdx$ on the shell. The forces $t = \frac{dH}{dx}$ and the load $p$ then determine the transverse moments $m$ in the shell. Between the reinforcement and the neutral axis, H, and consequently also $t$, are constant. For the sake of simplicity the compressive stress is assumed constant, as in the modern methods for calculation of reinforced concrete beams. H will then vary in the compression zone proportionally to the arc length of the normal section.

When calculating the transverse moment $m$, the normal force $n$, and

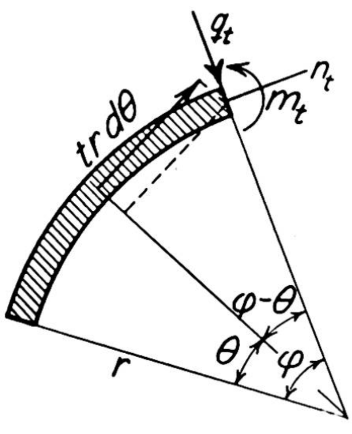


Fig. 2.

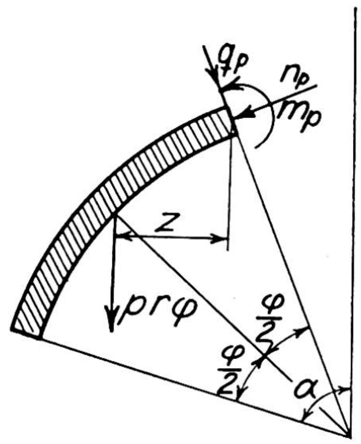


Fig. 3.

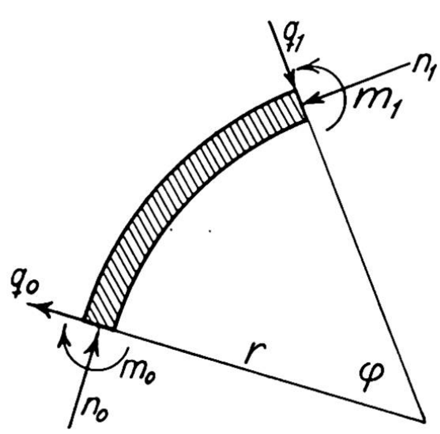


Fig. 4.

the transverse force $q$ we have the following formulae for circular shells

$$\left.\begin{aligned} m_t &= \int_0^{\varphi} r^2 t\,[1 - \cos(\varphi - \theta)]\, d\theta\,; \quad n_t = \int_0^{\varphi} rt\cos(\varphi - \theta)\, d\theta\,; \\ q_t &= \int_0^{\varphi} rt\sin(\varphi - \theta)\, d\theta\,. \end{aligned}\right\} \quad (1)$$

For $t =$ constant are obtained

$$m_t = tr^2(\varphi - \sin\varphi)\,; \qquad n_t = tr\sin\varphi\,; \qquad q_t = tr(1 - \cos\varphi)\,. \qquad (2)$$

For $t = k \cdot \theta$, i.e. proportional to the arc length, are obtained

$$\left.\begin{aligned} m_t &= kr^2\left(\frac{\varphi^2}{2} - 1 + \cos\varphi\right) - \frac{kr^2\varphi^4}{24}\,; \qquad n_t = kr(1 - \cos\varphi)\,; \\ q_t &= kr(\varphi - \sin\varphi)\,. \end{aligned}\right\} \quad (3)$$

From constant vertical load we get

$$\left.\begin{aligned} m_p &= -pr^2\begin{pmatrix}\cos\alpha - \cos(\alpha - \varphi) \\ +\varphi\sin(\alpha - \varphi)\end{pmatrix}; \qquad n_p = -pr\,\varphi\sin(\alpha - \varphi)\,; \\ q_p &= -pr\,\varphi\cos(\alpha - \varphi)\,. \end{aligned}\right\} \quad (4)$$

Finally, from support, or from the adjacent part, we obtain

$$\left.\begin{aligned} m_1 &= m_0 + n_0 r(1 - \cos\varphi) + q_0 r\sin\varphi\,; \quad n_1 = n_0\cos\varphi - q_0\sin\varphi\,; \\ q_1 &= q_0\cos\varphi + M_0\sin\varphi\,. \end{aligned}\right\} \quad (5)$$

The calculation can be made by means of a combination of these formulae. Some of the expressions are not suitable for practical calculations, but can be developed in series as shown, or tables of these expressions may be calculated once and for all.

With a skew load, or unsymmetrical cross-section, the influence of the twisting may become so great that two compression zones may be required. This will complicate the calculations somewhat, but they remain elementary all the same. It is, however, best — and frequently also feasible — to design the shell and the supports in such a way that this case does not occur. As an example, let us consider the well-known saw-tooth roof. The resultant P of the load acts here at a distance $a$ from the resultant — P of the shear forces by mere bending. We then have at twisting moment $P \cdot d$ per unit length of the shell. The best way of getting rid of this is to design the top-lights so that they can transfer a force $F = \frac{Pd}{l}$ per unit length from shell to shell. In a thesis for the doctorate, now in the press, H. Lundgren has offered a further contribution on this point — also as far as short shells are concerned.

The stresses being proportional to the load, it is possible to divide by the coefficient of safety, whereby the permissible stresses and the permissible loads will be included in the calculation instead of the rupture values.

*Example.* — A circular shell of the cross-section shown in the figure is calculated for symmetrical loading. Its length is 30.0 m, and the load is :

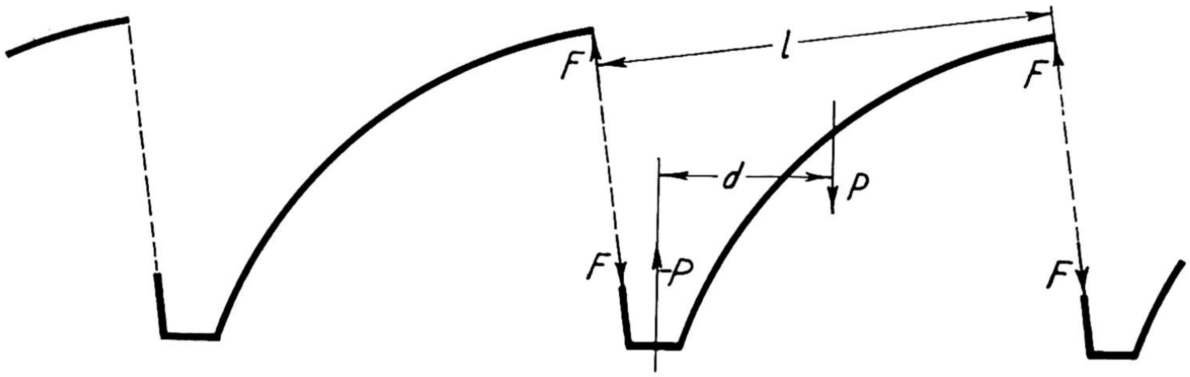


Fig. 5.

Circular Part :

| | | | |
|---|---|---|---|
| Self-weight 0.08 · 2 400 = | 192 kg/sq.m | | For the whole arc |
| Insulation, etc. | 53 » | | $2 \cdot 0.6 \cdot 8.85 \cdot 320 = 3\,400$ kg/m |
| Snow | 75 » | | |
| In total | 320 kg/sq.m | | |

Vertical Part :

| | | |
|---|---|---|
| Reinforced concrete 0.15 · 2 400 = | 360 kg/sq.m | For the whole edge girder |
| Insulation, etc. | 40 » | $2 \cdot 0.9 \cdot 400 = 720$ kg/m |
| In total | 400 kg/sq.m | |

For the whole cross-section $P = 4\,120$ kg/m. $M = 1/8 \cdot 4\,120 \cdot 30^2 = 464\,000$ kg/m. The neutral axis is assumed to correspond to $\beta = 0.3$, i.e. 0.40 m under the top. The centre of pressure is then $8.85\left(1 - \frac{\sin\beta}{\beta}\right) - \frac{1}{6} \cdot 8.85 \cdot \beta^2 = 0{\cdot}13$ m under the top, so that the moment arm $h_t$ is $2.30 - 0.13 = 2.17$ m. The tensile force and the compressive force will then be $464\,000 : 2.17 = 214\,000$ kg. The concrete stress will be

$$\sigma_b = \frac{214\,000}{2 \cdot 0.3 \cdot 885 \cdot 8} = 50.5 \text{ kg/sq. cm} .$$

With the reinforcement rod tension $= 1\,800$ kg/sq.cm, $\frac{214\,000}{2 \cdot 1\,800} = 59.4$ sq.cm be required for each edge girder, for instance 19 reinforcement rods of 20 mm dia.

Owing to the symmetry, $H = \frac{Q}{2 \cdot h_t}$, where Q is the transverse force in the shell considered as beam with 30 m span. We then have $t = \frac{1}{2\,h_t} \cdot \frac{dQ}{dx} = \frac{P}{2\,h_t}$, where P is the load per unit length, i.e. $t = \frac{4\,120}{2 \cdot 2.17} = 950$ kg/sq.m between the reinforcement and the neutral axis. From the latter to the top it decreases to zero proportionnally to $\theta$. The factor of proportionality $k$ is $950 : 0.3 = 3\,170$ kg/m. In the edge girder we get no moment, and the resultant of $t$ and the load is $R = 950 \cdot 0.75 - 400 \cdot 0.9 = 353$ kg/m. The following transverse moment is then found

$$m = R\,(5.00 - 8.85 \sin\theta) + m_p + m_t,$$

where $m_p$ is calculated from formula (4), $m_t$ from (2), which, for points over the neutral axis, is corrected by means of formula (3). $n$ and $q$ are calculated correspondingly. As a check serves that $q = 0$ at the top, i.e. $q_t = 1/2\ 4\,120 - 0.75 \cdot 950 = 1\,345$ kg/m $= tr\,(1 - \cos\alpha) - \frac{tr}{3}\,(\beta - \sin\beta) = 950 \cdot 8.85 \cdot 0.16$, which tallies. $tr^2$ being $= 950 \cdot 8.85^2 = 74\,400$ kg, $1/24\ kr^2 = 103\,000$ kg, and $pr^2 = 320 \cdot 8.85^2 = 25\,100$ kg, we get in kg

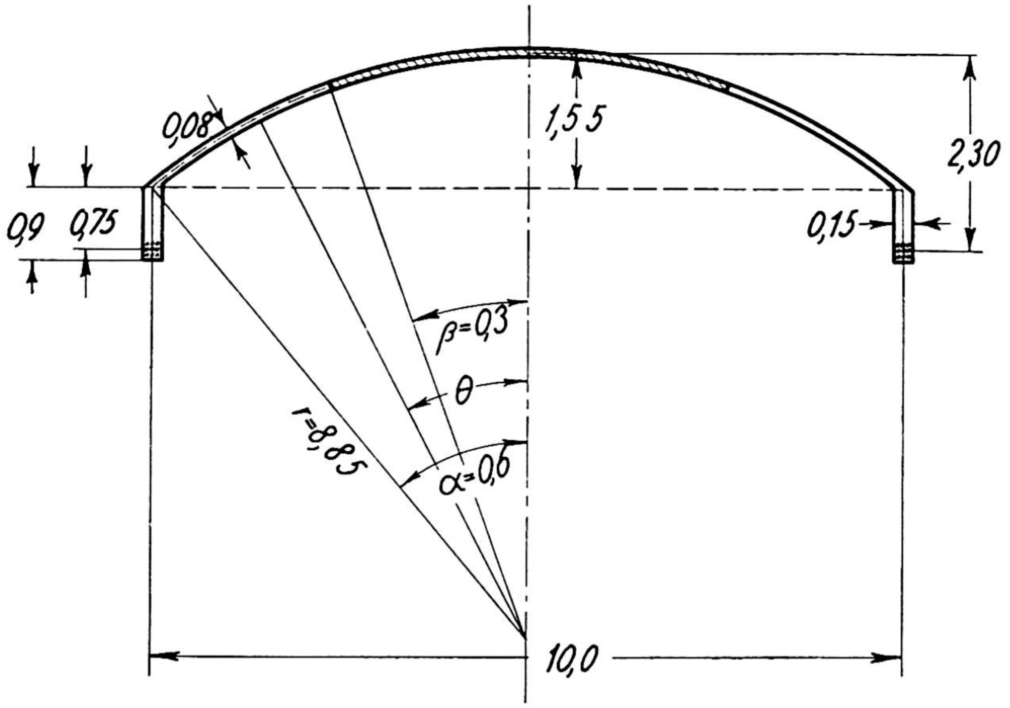


Fig. 6.

| $\theta$ | 0.60 | 0.55 | 0.50 | 0.45 | 0.40 | 0.35 | 0.30 | 0.25 | 0.20 | 0.15 | 0.10 | 0.05 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_R$ | 0 | 131 | 266 | 405 | 547 | 691 | 840 | 991 | 1 145 | 1 290 | 1 450 | 1 610 | 1 765 |
| $m_p$ | 0 | — 28 | — 107 | — 246 | — 448 | — 712 | —1 036 | —1 430 | —1 888 | —2 411 | —3 006 | —3 662 | —4 386 |
| $m_t$ | 0 | 2 | 13 | 42 | 99 | 193 | 332 | 528 | 785 | 1 120 | 1 525 | 2 030 | 2 620 |
| corr. | 0 | — | — | — | — | — | — | 0 | — 1 | — 5 | — 16 | — 40 | — 83 |
| $m$ | 0 | 105 | 172 | 201 | 198 | 172 | 136 | 89 | 41 | — 6 | — 47 | — 62 | — 84 |

It will be seen that the location of the neutral axis can have little infuence on $m$, the correction terms in the compression zone being rather small.

## Résumé

On calcule en général les voûtes cylindriques en se basant sur la théorie d'élasticité, mais les dimensions choisies ne sont pas en accord avec cette théorie. C'est contraire à la logique. Comme la charge de rupture est à la base de la détermination de la charge utile, l'auteur indique une méthode de calcul se basant sur l'état de rupture. Il l'applique à un exemple numérique.

## Zusammenfassung

Zylinderschalen werden gewöhnlich nach der Elastizitätstheorie berechnet, aber unter Annahmen dimensioniert, die mit dieser Theorie nicht übereinstimmen. Dies ist nicht logisch. Da die Bruchlast als Grundlage für die Bestimmung der Tragkraft genommen werden muss, wird gezeigt, wie eine einfache Berechnungsmethode, die vom Bruchstadium ausgeht, entwickelt werden kann. Sie wird angewendet auf ein Beispiel.

**Summary**

Cylindrical reinforced concrete shells are usually calculated on the theory of elasticity, but are dimensioned under assumptions at variance with this theory. This is irrational. As the ultimate load must be the starting point for the estimation of the carrying capacity, it is shown how a simple calculation-method based on the rupture stage can be developed. It is illustrated by an example.