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La charge de rupture de dalles en béton armé

Die Bruchlast von Eisenbetonplatten

The ultimate strength of reinforced concrete slabs

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It is evident that in the determination of the ultimate load the theory of elasticity is inapplicable. Already after the development of cracks and, more particularly, after yielding of the reinforcement has begun, the state is not elastic any longer. As the working load has to be fixed in proportion to the ultimate load, a theory of the yielding or plastic state of reinforced concrete slabs is desirable. An outline of the « theory of lines of fracture » will therefore be given in the following.

Let us consider a slab with uniform reinforcement in two directions at right angles to each other. When the reinforcement is evenly distributed, the yield value will be the same in all sections of the slab. The yielding will begin where the values will have maximum magnitude, and proceed along the lines of fracture. At the ultimate load the yielding has reached the edges, and along the lines of fracture the bending moment m per unit length is constant and equals the yield value corresponding to the reinforcement. This moment m is a maximum value in relation to the moments in all sections in the proximity of the lines of fracture. The lines of fracture divide the slab into several parts, and if now we assume the elastic deformations of these slab parts to be insignificant in comparison with the plastic deformations along the lines of fracture, the slab parts may be considered as plane. It then follows that the lines of fracture are straight lines. On the said assumption the deformation may be considered as angular rotations of the plane slab parts about the supports, and consequently the line of fracture between two slab parts must pass through the point of intersection of the axes of rotation of the two slab parts. Figure 1 shows some typical figures of fracture of slabs supported on four, three, and two sides, as well as on two sides and by one column, and on one side and by two columns, respectively. The axes of rotation lie in the supported sides and pass through the columns. The final determination of the figure of fracture and the breaking moment m is achieved with the conditions

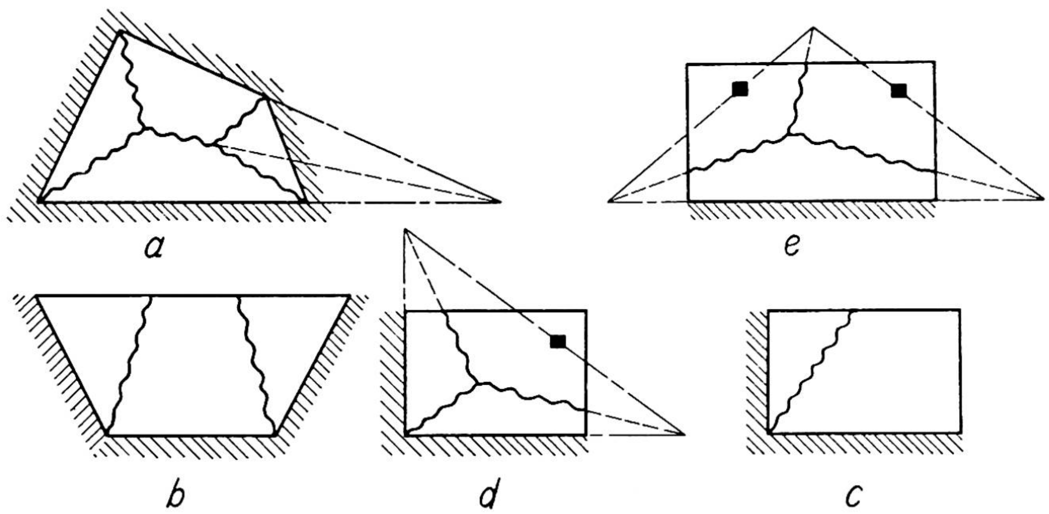

Fig. 1.

of equilibrium for the individual parts of the slab. For recording these conditions it is necessary to know the shearing forces along the lines of fracture. m being a maximum value in relation to the moments in sections in other directions through the same point, it is one of the principal moments, that is to say, the twisting moment is zero along the lines of fracture. We then find — as is also the case with a beam — that the shearing force is zero because m is maximum in relation to the moments in sections parallel to the section of fracture through adjacent points. Thus, only the bending moment m acts in the section of fracture, and the total moment may be represented by a vector equal to the line of fracture. The resulting moment for a part of the slab is found by vector addition.

Example 1. — A triangular slab with evenly distributed load is simply supported along its sides. According to the above, the figure of fracture will be as shown in figure 2. Let us consider the slab part at a. In the line of fracture OB acts the moment $m \cdot \overline{BO}$, and in the line of fracture OC the moment $m \cdot CO$. On the whole slab part acts $m\ (BO + \overline{OC}) = m \cdot BC = m \cdot a$. The moment about a gives the condition of equilibrium $m \cdot a = 1/6\ wa \cdot h^2_a$, or $m = 1/6\ wh^2_a$. Correspondingly, we get for the other slab parts $m = 1/6\ wh^2_b$ and $m = 1/6\ wh^2_c$, i.e., $h_a = h_b = h_c = r$, where r is the radius of the inscribed circle. The breaking moment is $m = 1/6\ wr^2$, where w is the ultimate load.

For all polygons circumscribed the circle with the radius r (fig. 3), it will be seen directly that also here we have $m = 1/6\ wr^2$; for the square having the side a, specially $m = 1/24\ wa^2$. (For rectangle see INGERSLEV, *Institution of Structural Engineers' Journal*, 1923.)

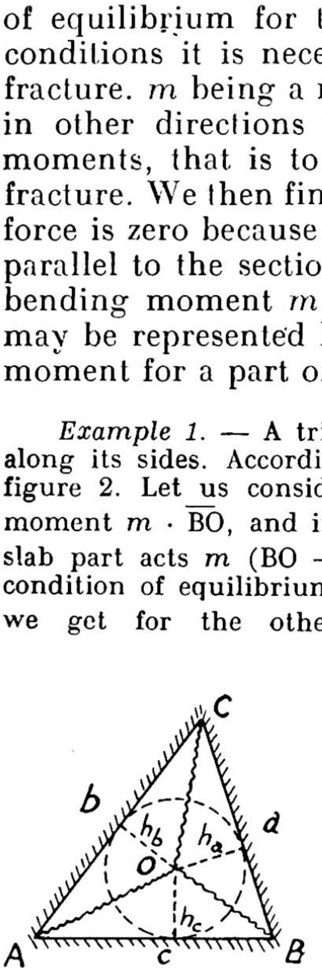

Fig. 2.

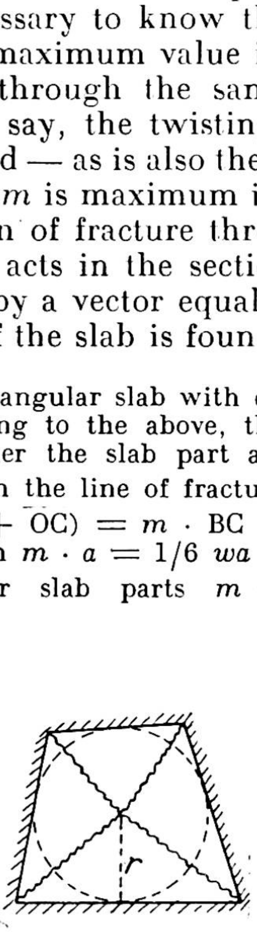

Fig. 3.

The ultimate load being n times the working load, and the breaking moment n times the working moment, where n is the coefficient of safety, we can also let m and w mean the permissible values.

With a free or simply supported edge, both the bending moment and the twisting moment should strictly speaking be zero. This involves that the lines of fracture should be at right angles to the edge. This is really

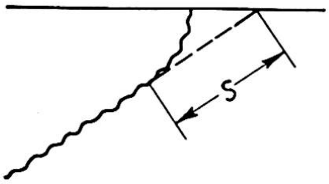

Fig. 4.

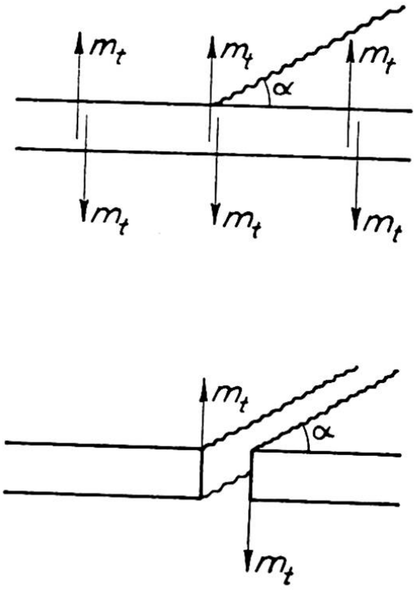

Fig. 5.

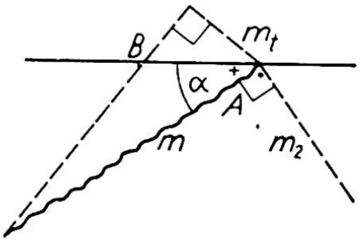

Fig. 6.

the case (fig. 4), but only quite close to the edge do the cracks suddenly turn so as to be at right angles to it.

As is known from the theory of elasticity for thin slabs, there are also here difficulties with the twisting moment at the edge. These difficulties are overcome by transforming the twisting moment into shear forces, as shown in figure 5 and first indicated by Kelvin and Tait. While the two single forces m_t neutralize each other when the slab is considered as a whole, they must be considered as acting each on its slab part when the equations of equilibrium for the individual slab parts are to be developed. The principal moments being m and m_2 (fig. 6), the bending moment along the edge will according to the above be

$$m \cos^2 \alpha + m_2 \sin^2 \alpha = 0$$

and the twisting moment

$$m_t = (m - m_2) \cos \alpha \sin \alpha = m \cot \alpha, \text{ as } m_2 = - m \cot^2 \alpha .$$

If we make the same transformation for plastic slabs, this will correspond to a rectilinear extension of the line of fracture to the edge (fig. 4). The single force $m \cot \alpha$ is then a static equivalent of the twisting moments and the shearing forces on the stretch s. Incidentally, this force the edge force, can also easily be deduced directly from the equation of equilibrium for the infinitesimal triangle AOB shown in figure 6. As m is a maximum value, the adjacent section OB has the same m, and as the bending moment is zero along AB $= ds$, the resultant for the whole triangle

$$m\,(\overline{AO} + \overline{OB}) = m \cdot \overline{AB} = m \cdot ds .$$

The moment about BO gives then, when magnitudes of a higher order are ignored,

$$m \cdot ds \cdot \cos \alpha = m_t \cdot ds \cdot \sin \alpha ; \qquad m_t = m \cot \alpha .$$

Example 2. — A rectangular slab with evenly distributed load is simply supported on two adjacent sides and free on the two others (fig. 7). The figure of fracture shown gives the edge force $m \cdot \frac{x}{a}$. The moment about a for the slab part A gives

$$ma = \frac{1}{6}\, wax^2 + m \cdot \frac{x}{a} \cdot x\,,$$

and the moment about b for the slab part B gives

$$m \cdot x + m\,\frac{x}{a} \cdot a = \frac{1}{2}\, wba - \frac{1}{3}\, wa^2x\,.$$

From these two equations of equilibrium are found :

$$\frac{a}{x} = \frac{a}{3\,b} + \sqrt{1 + \left(\frac{a}{3\,b}\right)^2}\,;$$

$$m = \frac{3}{4} \cdot \frac{wb^2}{1 + \sqrt{1 + 9\,\frac{b^2}{a^2}}} \qquad b \geqslant a\,.$$

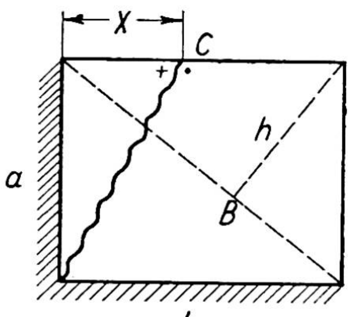

Fig. 7.

In the diagonal section shown is found the negative moment

$$m' = \frac{1}{6}\, wh^2 = \frac{1}{6}\, w\, \frac{a^2 b^2}{a^2 + b^2}\,.$$

It can be proved quite simply that the number of equations is always equal to the number of unknowns (*IABSE, Publications I*, 1932, p. 283). The equations are not linear, so that superposition cannot be applied, but it can be proved that it is safe to superpose loads acting jointly (that is, do not counteract each other).

Should the solution of the equations be too cumbersome, the following method can always be used in practice : By the principle of virtual work, m can be determined directly for an arbitrarily chosen figure of fracture (*loc. cit.*, p. 284). The real value of m being a maximum value, the proper figure of fracture will be the one making the corresponding m the maximum. As the variations in the proximity of a maximum are very small, a fair approximation for m can be obtained by estimating the figure of fracture. By the equations of equilibrium for the individual slab parts the estimated figure of fracture may be improved and a better approximation be attained. With a little experience it is possible to estimate immediately the figure of fracture so well that the corresponding m will differ only a few percent from the real value.

In the development of the equation of work advantage is taken of the fact that the work of the moment vector M in the rotation θ, which is likewise a vector, is the scalar product $(M\theta) = M\theta \cos(M\theta)$. If the vec-

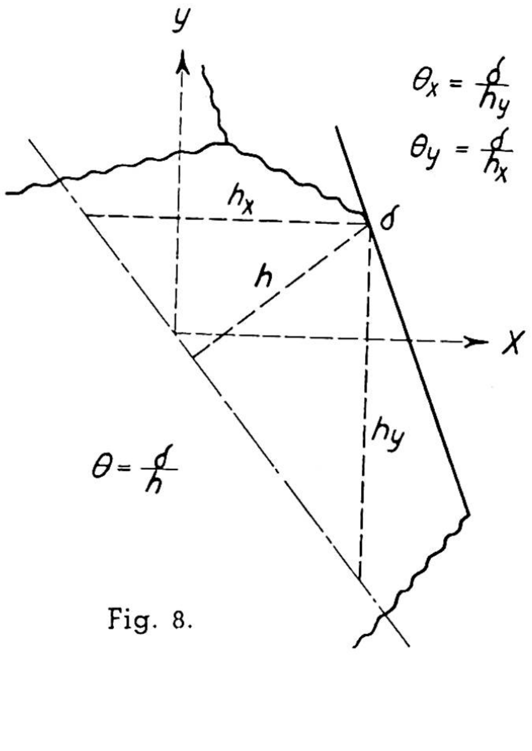

Fig. 8.

tors are resolved into components along two axes which are at right angles to each other, we get the expression $(M\theta) = M_x\theta_x + M_y\theta_y$. The rotation is determined, for instance, by the sinkings as shown in figure 8.

Example 3. — We will apply the equation of work to the preceding example. We lower C δ; thereby the slab part A will get the rotation $\theta_A = \delta \div x$; and the slab part B, $\theta_B = \delta \div a$, about the supports. The virtual work will be made $M_A\theta_A + M_B\theta_B = 0$, i.e.,

$$\left(ma - \frac{1}{6} wax^2\right)\frac{\delta}{x} + \left(m \cdot x - \frac{1}{2} wba^2 + \frac{1}{3} wa^2x^2\right)\frac{\delta}{a} = 0 \, ,$$

which gives :

$$m = \frac{1}{6} wab \frac{3 - \frac{x}{b}}{\frac{a}{x} + \frac{x}{a}} \, .$$

The shearing forces do not contribute, since the two slab parts do not move vertically in relation to each other. The real figure of fracture is now found by the condition $dm \div dx = 0$, which gives the result previously found

If we use as approximation $x = a$, we get $m = 1/12 \, wab\left(3 - \frac{a}{b}\right)$. For $b = a$ we then have $m = wab : 6$, exactly $wab : 5.55$. For $b = 2a$ we get $m = wab : 4.8$, exactly $wab : 4.72$. The error is 7.5 and 1.6 percent, respectively, which is of no practical consequence.

Example 4. — A square slab, simply supported on two adjacent sides and by one column in the opposite corner, is loaded with a single force P in the centre.

The figure of fracture will be as in figure 9. When the force P is lowered $\delta = 1$, the slab parts A get the rotations $1 : \frac{a}{2} = 2 : a$, while the rotation for the slab part B has the components $1 : a$, as $h_x = h_y = a$ (fig. 8) and the moment has the components $\frac{2}{3}\, ma$. Hence is obtained the equation of work :

$$P \cdot l = 2 \cdot ma \cdot \frac{2}{a} + 2 \cdot \frac{2}{3} ma \cdot \frac{1}{a} = \frac{16}{3} m \, ;$$

$$m = \frac{3}{16} P \, .$$

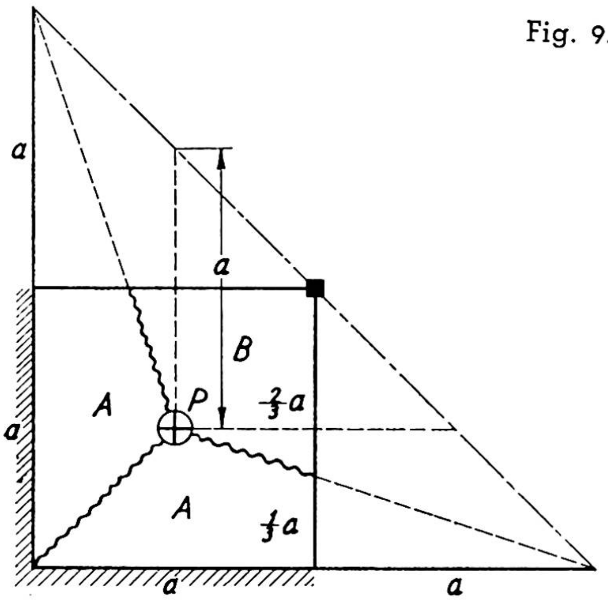

Fig. 9.

For fixed-end slabs and slabs which are continuous over supports are assumed lines of fracture over the supports with negative moments corresponding to the upper reinforcement of the slab.

When the reinforcement is uniform, but not equal in the two orthogonal directions, so that the corresponding yield values are m and μm, m is the same as in a slab with equal yield values m, and affines to the given slab in the proportion $1 : \sqrt{\mu}$ and with the same load per unit of area.

The theory is very well verified by the tests, both as regards the figures of fracture and the ultimate loads.

Résumé

Ce mémoire donne une description de la théorie des lignes de rupture de dalle en béton armé. On peut déterminer ces lignes de rupture géométriquement grâce au fait que les déformations élastiques sont faibles comparées aux déformations plastiques. Le moment fléchissant atteint sa valeur maximum le long de ces lignes, ce qui permet de déterminer les efforts transversaux et de torsion. Ils sont nuls sauf au bord libre. On en déduit les conditions d'équilibre pour les surfaces partielles limitées par les lignes de rupture connaissant ces lignes de rupture et le moment de rupture. Une méthode approchée peut être déduite du principe du travail virtuel. Cette méthode est illustrée par des exemples.

Zusammenfassung

Eine Beschreibung der Bruchlinientheorie für Eisenbetonplatten wird gegeben. Mittels geometrischer Bedingungen, welche aus der Tatsache folgen, dass die elastischen Deformationen unbedeutend sind gegenüber den plastischen, kann die Form der Bruchfigur bestimmt werden. Da das Biegungsmoment längs der Bruchlinien einen Grösstwert hat, können die Querkräfte und die Drillungsmomente bestimmt werden. Sie sind null, ausgenommen an einem freien Rand. Hierauf können die Gleichgewichtsbedingungen aufgestellt werden für die Teilflächen, in die die Platte durch die Bruchlinien geteilt wird, wobei die Bruchfigur und das Bruchmoment bekannt sind. Eine einfache Näherungsmethode kann durch die Anwendung des Prinzips der virtuellen Arbeit entwickelt werden. Die Theorie wird durch Beispiele illustriert

Summary

An outline of the theory of lines of fracture of reinforced concrete slabs is given. Through the geometric conditions which are a consequence of the fact that the elastic deformations are insignificant as compared with the plastic ones the character of the figure of fracture can be determined (fig. 1). The moment in the lines of fracture being a maximum value, the transversal force and the twisting moment can be determined. They become zero, except at a free edge. The equations of equilibrium for the individual parts into which the lines of fracture divide the slab can then be set up, whereby the figure of fracture and the breaking moment are determined. A simple method of approximation can be indicated by application of the principle of virtual work. The method is illustrated by examples.