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Considérations sur la flexion de poutres droites à section variable sous l'influence de charges extérieures

Betrachtungen über die Biegung von geraden Balken mit veränderlichem Querschnitt unter äusseren Lasten

Considerations on bending straight beams of variable cross section under action of external loads

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1. The method

The method which we will use in this case has been proposed by S. Timoshenko (1). It consists in the expansion of the function representing the deflection curve of a simply supported beam (fig. 1) into a trigonometric series. To all reputed advantages of this method we intend to add a new one, obtained when not only the deflection curve, but also when all functions representing the external loading and variable stiffness in bending of the beam are resolved into similar series.

We will endeavour to resolve the problem of the general case of a simply supported beam of variable cross section supported and loaded in the plane of a principal axe of its cross section by all possible forms of external loading, i.e. by concentrated forces and couples, as well as distributed loads which might have normal as well as axial components, so that the resulting compressive axial force might vary from section to section either continuously or discontinuously.

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(1) S. Timoshenko, *Application of generalized coordinates to solution of problems on bending of bars and plates* (*Bull. Polyt. Inst.*, Kiev, 1909) (Russian).
S. Timoshenko, *Bull. Soc. Eng. Techn.*, St. Petersburg, 1913.
S. Timoshenko, *Theory of Elastic Stability*, New York and London, 1936 (see page 23); *Strength of Materials*, Vol. 1, 1930 (see page 306).

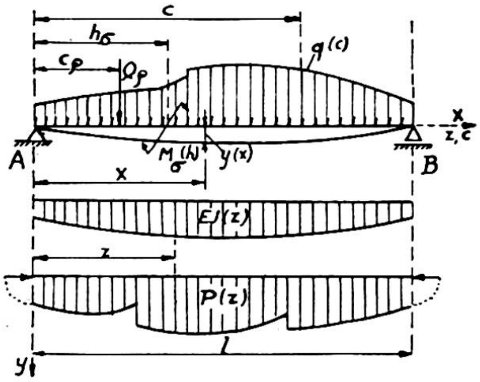

Fig. 1.

Let the variable stiffness in bending $E \cdot J = F(x)$ of our beam be represented by either of two types of convergent Fourier series :

$$E \cdot J = F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos n \frac{2\pi x}{l} + \sum_{n=1}^{\infty} B_n \cdot \sin n \frac{2\pi x}{l}, \tag{1}$$

or:

$$E \cdot J = A_0 + \sum_{n=1}^{\infty} A_{\frac{n}{2}} \cdot \cos\left(\frac{n}{2} \cdot \frac{2\pi x}{l}\right) \equiv \sum_{n=0}^{\infty} A_{\frac{n}{2}} \cdot \cos \frac{n \cdot \pi \cdot x}{l}. \tag{1'}$$

We suppose further, that the external load consists of:

1. Lateral components of concentrated loads Q_ρ, where $\rho = 1, 2, \ldots, \varkappa$, acting at distances c_ρ from left support A of the beam;

2. Distributed lateral loads $q = q(c)$, which are functions of the distance c from the left support A;

3. Lateral couples, whose moments M_σ, $(\sigma = 1, 2, \ldots, \tau)$ acting in the plane of fig. 1, are applied to the beam sections at distances from A equal to h_σ;

4. Compressive (when positive) axial forces $P = P(x)$, which may continuously or discontinuously vary from section to section, because of axial components of external concentrated or distributed loads.

We will further consider the expansion of the functions $q(c)$ and $P(x)$ in following necessarily convergent trigonometrical series:

$$q = q(c) = \sum_{n=1}^{\infty} \beta_{\frac{n}{2}} \cdot \sin\left(\frac{n}{2} \cdot \frac{2\pi c}{l}\right) \equiv \sum_{n=1}^{\infty} \beta_{\frac{n}{2}} \cdot \sin \frac{n\pi c}{l}; \tag{2}$$

$$P = P(z) = \sum_{n=0}^{\infty} p_n \cdot \cos n \frac{2\pi z}{l} + \sum_{n=1}^{\infty} \cdot \nu_n \cdot \sin n \frac{2\pi z}{l}; \tag{3}$$

or:

$$P = \sum_{n=0}^{\infty} p_{\frac{n}{2}} \cdot \cos\left(\frac{n}{2} \cdot \frac{2\pi z}{l}\right) \equiv \sum_{n=0}^{\infty} p_{\frac{n}{2}} \cdot \cos \frac{n\pi z}{l}. \tag{3'}$$

Finally, we will accept the possibility of developing the ordinates of the deflection curve into a similar "half-wave" series:

$$y = \sum_{n=1}^{\infty} b_{\frac{n}{2}} \cdot \sin\left(\frac{n}{2} \cdot \frac{2\pi x}{l}\right) \equiv \sum_{n=1}^{\infty} b_{\frac{n}{2}} \cdot \sin\frac{n \cdot \pi \cdot x}{l}, \tag{4}$$

which is always possible, as Dirichlet's conditions for expanding into a Fourier series are always fulfilled for a deflexion curve of a continuous beam.

The series (4) satisfies all the end conditions of a beam, freely supported at its ends.

We have introduced here four different terms x, z, c, h for the same abscissae, for the sake of facilitating some later integrations.

By differentiating once and twice the series (4) we obtain expressions for the approximate values of the slope ϑ and curvature $\frac{1}{\rho}$ of the deflexion curve:

$$\vartheta \cong y' = \frac{\pi}{l} \cdot \sum_{n=1}^{\infty} n \cdot b_{\frac{n}{2}} \cdot \cos\frac{n \cdot \pi \cdot x}{l}; \tag{4'}$$

$$\frac{1}{\rho} \cong y'' = -\frac{\pi^2}{l^2} \cdot \sum_{n=1}^{\infty} \cdot n^2 \cdot b_{\frac{n}{2}} \cdot \sin\frac{n \cdot \pi \cdot x}{l}. \tag{4''}$$

2. Bending of the beam

We propose now to derive the equations of the deflexion curve of our beam, subject to the external loads: Q_ζ, M_σ, $q(c)$, $P(z)$, by using the general principle of the *strain energy* in bending. To obtain the expressions for the Fourier series coefficients $b_{\frac{n}{2}}$ $(n = 1, 2, \ldots, \infty)$, we must derive the expressions for the change of strain energy δV, when the coefficients $b_{\frac{n}{2}}$ are variated independently of each other by small increases $\delta b_{\frac{n}{2}}$. We will further equate this change to the work δT of external forces during this additional deflexion.

1. *The strain energy* of a beam [2] may be given in this case by the equation:

$$V = \frac{1}{2} \int_0^l E \cdot J \cdot \left(\frac{d^2 y}{dx^2}\right)^2 \cdot dx \tag{5}$$

Suppose at first, that the stiffness in bending $E \cdot J$ of our beam is constant on an extent from $x = 0$ to $x = z$, then for this part of the beam the partial strain energy V' will be:

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[2] M. T. Huber, *On critical loads of axially compressed bars of discontinuously variable cross sections* (*Publications of the Tech. Inst. for Aeronautics*, Warsaw, 1930) (Polish).

$$V' = \frac{1}{2} E \cdot J \int_0^z \left(\frac{d^2 y}{dx^2}\right)^2 \cdot dx \tag{6}$$

$$= \frac{\pi^4 \cdot E \cdot J}{2\, l^4} \cdot \int_0^z dx \left[\sum_{n=1}^{\infty} n^4 \cdot b^2_{\frac{n}{2}} \cdot \sin^2 n \frac{\pi x}{l} \right.$$

$$\left. + 2 \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n^2 \cdot k^2 \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \cdot \sin \frac{n \pi x}{l} \cdot \sin \frac{k \pi x}{l} \right]$$

$$= \frac{\pi^3 \cdot EJ}{2 \cdot l^3} \cdot \sum_{n=1}^{\infty} n^3 \cdot b^2_{\frac{n}{2}} \cdot \int_0^z \sin^2 \frac{n \cdot \pi \cdot x}{l} \cdot d\left(\frac{n \pi x}{l}\right)$$

$$+ \frac{\pi^3 \cdot E \cdot J}{l^3} \cdot \sum_{\substack{n,k=1 \\ n \neq k}}^{\infty} n^2 \cdot k^2 \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \int_0^z \sin \frac{n \pi x}{l} \cdot \sin \frac{k \pi x}{l} \cdot d\left(\frac{\pi x}{l}\right)$$

$$= \frac{\pi^3 \cdot E \cdot J}{2\, l^3} \cdot \left\{ \sum_{n=1}^{\infty} n^3 \cdot b^2_{\frac{n}{2}} \left[n \frac{\pi \cdot z}{2\, l} - \frac{1}{4} \cdot \sin n \frac{2\, \pi z}{l} \right] \right.$$

$$+ \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n^2 \cdot k^2 \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \cdot \left[\frac{1}{k-n} \cdot \sin (k-n) \frac{\pi z}{l} \right.$$

$$\left.\left. - \frac{1}{k+n} \cdot \sin (k+n) \frac{\pi \cdot z}{l} \right] \right\}$$

$$= \frac{\pi^3 \cdot E \cdot J}{2\, l^3} \cdot \left\{ \frac{\pi z}{2\, l} \cdot \sum_{n=1}^{\infty} n^4 \cdot b^2_{\frac{n}{2}} - \frac{1}{4} \sum_{n=1}^{\infty} n^3 \cdot b^2_{\frac{n}{2}} \cdot \sin\left(n \frac{2\, \pi \cdot z}{l}\right) \right.$$

$$+ \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n^2 \cdot k^2 \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \cdot \left[\frac{1}{k-n} \cdot \sin (k-n) \frac{\pi \cdot z}{l} \right.$$

$$\left.\left. - \frac{1}{k+n} \cdot \sin (k+n) \frac{\pi z}{l} \right] \right\}.$$

Having thus obtained the strain energy V' for $0 \leqslant x \leqslant z$, under the condition that there was $E \cdot J = \text{const}$, we may now suppose that $E \cdot J$ remains constant only in the interval of $z < x < z + dz$, and afterwards integrate the obtained differential for the full length of the beam. We obtain in this manner the full strain energy V of the beam with variable stiffness:

$$V = \frac{\pi^3}{2\, l^3} \left[\frac{\pi}{2\, l} \cdot \sum_{n=1}^{\infty} n^4 \cdot b^2_{\frac{n}{2}} \cdot \int_0^l EJ \cdot dz - \frac{1}{4} \sum_{n=1}^{\infty} n^3 \cdot b^2_{\frac{n}{2}} \int_0^l E \cdot J \cdot d\left(\sin n \frac{2\, \pi z}{l}\right) \right.$$

$$+ \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n^2 \cdot k^2 \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \left\{ \frac{1}{k-n} \int_0^l EJ \cdot d \left[\sin (k-n) \frac{\pi z}{l} \right] \right.$$

$$\left. - \frac{1}{k+n} \cdot \int_0^l EJ \cdot d \left[\sin (k+n) \frac{\pi z}{l} \right] \right\}$$

$$= \frac{\pi^4}{4\, l^4} \cdot \left\{ \sum_{n=1}^{\infty} n^4 \cdot b^2_{\frac{n}{2}} \int_0^l \mathrm{EJ} \cdot dz - \sum_{n=1}^{\infty} n^4 \cdot b^2_{\frac{n}{2}} \int_0^l \mathrm{EJ} \cdot \cos n \frac{2\pi z}{l} \cdot dz \right.$$

$$+ 2 \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n^2 \cdot k^2 \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \left[\int_0^l \mathrm{EJ} \cdot \cos (k - n) \frac{\pi z}{l} \cdot dz \right.$$

$$\left. \left. - \int_0^l \mathrm{EJ} \cdot \cos (k + n) \frac{\pi z}{l} \cdot dz \right] \right\}$$

$$= \frac{\pi^4}{4\, l^3} \cdot \left[\mathrm{A}_0 \sum_{n=1}^{\infty} n^4 b^2_{\frac{n}{2}} - \frac{1}{2} \sum_{n=1}^{\infty} \mathrm{A}_n \cdot n^4 \cdot b^2_{\frac{n}{2}} \right.$$

$$\left. + \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n^2 \cdot k^2 \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \left(\mathrm{A}_{\frac{k-n}{2}} - \mathrm{A}_{\frac{k+n}{2}} \right) \right] \tag{7}$$

Here A_i are the coefficients of the series (1), whereas the "half-wave" coefficients $\mathrm{A}_{\frac{n \pm k}{2}}$, when $(n \pm k)$ are uneven numbers, belong to series (1′):

$$\mathrm{A}_0 = \frac{1}{l} \int_0^l \mathrm{E} \cdot \mathrm{J} \cdot dz\,,$$

$$\mathrm{A}_n = \frac{2}{l} \int_0^l \mathrm{E} \cdot \mathrm{J} \cdot \cos n \frac{2\pi z}{l} \cdot dz\,,$$

$$\mathrm{A}_{\frac{n \pm k}{2}} = \frac{2}{l} \int_0^l \mathrm{E} \cdot \mathrm{J} \cdot \cos \left[\frac{n \pm k}{2} \cdot \frac{2\pi z}{l} \right] \cdot dz\,. \tag{8}$$

What concerns the integral (8), in the case when $(n \pm k)$ is an uneven number, we way interprete the development (1′) as an expansion of a symmetrical or antisymmetrical curve, consisting of two curves $f(x)$ symmetrically or antisymmetrically disposed to each other into a normal Fourier series (1) as shown in fig. 2.

We will calculate now the variation $\delta \mathrm{V}$ of the strain energy, when the deflexion is variated, while every coefficient b obtains a small increase $\delta b_{\frac{n}{2}}$. From the form of the series it may be deduced at once, that variations $\delta b_{\frac{n}{2}}$ may be considered as wholely independent from each other. Hence

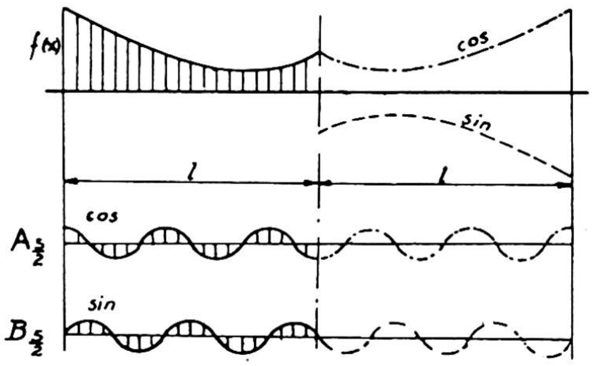

Fig. 2.

we may regard δV as consisting of a whole series of independent "partial" variations δV_n, obtained from each variation $\delta b_{\frac{n}{2}}$ separately. In fact, as $\delta b_{\frac{n}{2}}$ are independent from each other, we may take only one $\delta b_{\frac{n}{2}} \neq 0$, and all other $\delta b_{\frac{k}{2}} = 0$ (for $n \neq k$). From the type of equation (7) it is evident, that

$$\delta V = \sum_{n=1}^{\infty} \delta V_n.$$

Such a "partial" variation of V will be:

$$\delta V_n = \frac{\partial V}{\partial b_{\frac{n}{2}}} \cdot \delta b_{\frac{n}{2}}$$

$$= \delta b_{\frac{n}{2}} \cdot \frac{\pi^4}{2\, l^3} \cdot \left\{ \left(A_0 - \frac{1}{2} A_n \right) \cdot n^4 \cdot b_{\frac{n}{2}} + \frac{1}{2} n^2 \cdot \sum_{\substack{k=1 \\ (k \neq n)}}^{\infty} k^2 \cdot b_{\frac{k}{2}} \cdot \left(A_{\frac{k-n}{2}} - A_{\frac{k+n}{2}} \right) \right\}. \tag{9}$$

We must still draw attention of the reader to the fact, that since the factor 2 has been taken in front of the double sum signs, these double sums may not include both permutations of every pair of values n and k, but only one. It may be clearly seen from the second line of eq. (6), from which had been deduced all the double sums of this Report.

2. *The work T of external loads* consists of two principal parts, namely, the work of lateral and axial loads:

$$T = T_L + T_P .$$

T_L is the sum of the works of all types of lateral loading: Q_ρ, $q(c)$, M^σ, and we may separately calculate the work of every type of such load:

a) The work of concentrated lateral loads Q_ρ, is the product of their values and respective transversal displacements y_ρ:

$$T_Q = \sum_{\rho=1}^{\chi} Q_\rho \cdot y_\rho = \sum_{\rho=1}^{\chi} Q_\rho \cdot \sum_{n=1}^{\infty} b_{\frac{n}{2}} \cdot \sin \frac{n\pi c_\rho}{l};$$

b) The work of the continuous lateral load $q(c)$ will be:

$$T_q = \sum_{n=1}^{\infty} b_{\frac{n}{2}} \int_0^l q \cdot \sin n \frac{\pi c}{l} \cdot dc = \frac{l}{2} \sum_{n=1}^{\infty} \cdot b_{\frac{n}{2}} \cdot \beta_{\frac{n}{2}} ;$$

c) The work of concentrated couples, whose moments M_σ are acting in the plane of fig. 1, will be:

$$T_m = \sum_{\sigma=1}^{\tau} M_\sigma \cdot y_\sigma' = \frac{\pi}{l} \sum_{\sigma=1}^{\tau} M_\sigma \cdot \sum_{n=1}^{\infty} n \cdot b_{\frac{n}{2}} \cos n \frac{\pi h_\sigma}{l} .$$

Hence the general expression for the work of lateral loads of every possible kind will be:

$$T_L = T_Q + T_q + T_m$$

$$= \sum_{n=1}^{\infty} b_{\frac{n}{2}} \left(\frac{l}{2} \cdot \beta_{\frac{n}{2}} + \sum_{\rho=1}^{x} Q_\rho \cdot \sin \frac{n \cdot \pi}{l} c_\rho + \frac{\pi}{l} \sum_{\sigma=1}^{\tau} M_\sigma \cdot n \cdot \cos \frac{n\pi}{l} h_\sigma \right). \quad (12)$$

It remains to calculate the work of the variable axial loads along the length of the beam. This work will be represented by an integral, taken along the length of the beam, of the axial force P over elementary axial displacements due to bending only. Such a displacement is equal to the difference between the length of an element of the deflection curve and the corresponding element of a straight line, parallel to the axe of abscissae. Denothing this displacement by $d\lambda$, we obtain its value:

$$d\lambda = ds - dx \cong \frac{1}{2} \left(\frac{dy}{dx} \right)^2 \cdot dx .$$

Thus the elementary work of the compressive force will be:

$$dT_P' = P \cdot d\lambda \cong \frac{1}{2} \cdot P \cdot \left(\frac{dy}{dx} \right)^2 dx$$

so that finally:

$$dT_P' = \frac{1}{2} P \frac{\pi^2}{l^2} \left(\sum_{n=1}^{\infty} n^2 \cdot b^2_{\frac{n}{2}} \cdot \cos^2 \frac{n\pi x}{l} + 2 \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n \cdot k \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \cos \frac{n\pi x}{l} \cos \frac{k\pi x}{l} \right) dx .$$

We will assume at first, that the compressive force P is constant along a certain part of the beam, within the limits: $0 < x < z$. Then the work T_P' will be:

$$T_P' = \frac{\pi^2 \cdot P}{2 l^2} \cdot \left[\sum_{n=1}^{\infty} n^2 \cdot b^2_{\frac{n}{2}} \cdot \cos^2 \frac{n\pi x}{l} + 2 \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n \cdot k \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \int_0^z \cos \frac{n\pi x}{l} \cdot \cos \frac{k\pi x}{l} \cdot dx \right]$$

$$= \frac{\pi \cdot P}{2 l} \cdot \sum_{n=1}^{\infty} n \cdot b^2_{\frac{n}{2}} \left(\frac{n \cdot \pi z}{2 l} + \frac{1}{4} \sin \cdot n \frac{2 \pi z}{l} \right)$$

$$+ \frac{\pi P}{2 l} \cdot \sum_{\substack{n,k=1 \\ (n \neq k)}}^{\infty} n \cdot k \cdot b_{\frac{n}{2}} \cdot b_{\frac{k}{2}} \left[\frac{1}{k-n} \cdot \sin (k-n) \frac{\pi z}{l} + \frac{1}{k+n} \cdot \sin (k+n) \frac{\pi z}{l} \right]$$

$$= \frac{\pi^2 \cdot P \cdot z}{4 l^2} \cdot \sum_{n=1}^{\infty} n^2 \cdot b^2_{\frac{n}{2}} + \frac{\pi \cdot P}{8 l} \sum_{n=1}^{\infty} n \cdot b^2_{\frac{n}{2}} \cdot \sin n \frac{2 \pi z}{l}$$

$$+\frac{\pi\cdot P}{2\,l}\cdot\sum_{\substack{n,k=1\\(n\neq k)}}^{\infty} n\cdot k\cdot b_{\frac{n}{2}}\cdot b_{\frac{k}{2}}\left|\frac{1}{k-n}\cdot\sin(k-n)\frac{\pi z}{l}\right.$$

$$\left.+\frac{1}{k+n}\cdot\sin(k+n)\frac{\pi z}{l}\right]. \quad (13)$$

But in reality the axial load P may be considered as constant only along an elementary length, say in the limits:

$$z < x < (z+dz)\,.$$

Hence, to obtain the true work T we must integrate the equation (13) for a variable $P = P(x)$ along the full length of our beam, i.e. within the limits $0 < z < l$, in the following way:

$$T_P = \frac{\pi^2}{4\,l^2}\sum_{n=1}^{\infty} n^2\cdot b^2_{\frac{n}{2}}\int_0^l P\cdot dz + \frac{\pi}{8\,l}\cdot\sum_{n=1}^{\infty} n\cdot b^2_{\frac{n}{2}}\int_0^l P\cdot d\left(\sin n\,\frac{2\,\pi z}{l}\right)$$

$$+\frac{\pi^2}{2\,l}\cdot\sum_{\substack{n,k=1\\(n\neq k)}}^{\infty} n\cdot k\cdot b_{\frac{n}{2}}\cdot b_{\frac{k}{2}}\left\{\int_0^l \frac{P}{k-n}\cdot d\left[\sin(k-n)\frac{\pi z}{l}\right]\right.$$

$$\left.+\int_0^l \frac{P}{k+n}\cdot d\left[\sin(k+n)\frac{\pi z}{l}\right]\right\}$$

$$=\frac{\pi^2}{4\,l^2}\cdot\sum_{n=1}^{\infty} n^2\cdot b^2_{\frac{n}{2}}\int_0^l P\cdot dz + \frac{\pi^2}{4\,l^2}\cdot\sum_{n=1}^{\infty} n^2\cdot b^2_{\frac{n}{2}}\int_0^l P\cdot\cos n\,\frac{2\,\pi z}{l}\cdot dz$$

$$+\frac{\pi^2}{2\,l}\cdot\sum_{\substack{k,n=1\\(n\neq k)}}^{\infty} n\cdot k\cdot b_{\frac{n}{2}}\cdot b_{\frac{k}{2}}\cdot\left\{\int_0^l P\cdot\cos(k-n)\frac{\pi z}{l}\cdot dz\right.$$

$$\left.+\int_0^l P\cdot\cos(k+n)\frac{\pi z}{l}\cdot dz\right\}$$

$$=\frac{\pi^2}{4\,l}\cdot\left[\sum_{n=1}^{\infty} n^2\cdot b^2_{\frac{n}{2}}\left(p_0+\frac{1}{2}p_n\right)\right.$$

$$\left.+\sum_{\substack{n,k=1\\(n\neq k)}}^{\infty} n\cdot k\cdot b_{\frac{n}{2}}\cdot b_{\frac{k}{2}}\left(p_{\frac{k-n}{2}}+p_{\frac{k+n}{2}}\right)\right]. \quad (14)$$

Now we may write the complete expression of the work T in full length:

$$T = T_L + T_P$$

$$=\sum_{n=1}^{\infty} b_{\frac{n}{2}}\left(\frac{l}{2}\cdot\beta_{\frac{n}{2}}+\sum_{\rho=1}^{x} Q_\rho\cdot\sin\frac{n\cdot\pi}{l}\cdot c_\rho+\sum_{\sigma=1}^{z} M_\sigma\cdot\frac{n\cdot\pi}{l}\cos n\,\frac{\pi h\sigma}{l}\right)$$

$$+\frac{\pi^2}{4\,l}\left[\sum_{n=1}^{\infty} n^2\cdot b^2_{\frac{n}{2}}\left(p_0+\frac{1}{2}p_n\right)+\sum_{\substack{n,k=1\\(n\neq k)}}^{\infty} n\cdot k\cdot b_{\frac{n}{2}}\cdot b_{\frac{k}{2}}\left(p_{\frac{k-n}{2}}+p_{\frac{k+n}{2}}\right)\right]. \quad (15)$$

We must now calculate the additional work δT produced by the external forces because of a small arbitrary variation of the deflexion curve, due to small changes $\delta b_{\frac{n}{2}}$ of the coefficients $b_{\frac{n}{2}}$. The partial displacement due to one change $\delta b_{\frac{n}{2}}$ only, may be denoted δT_n. These partial displacements are all independent from each other, hence we may study the whole process on the example of one variation $\delta b_{\frac{k}{2}}$, while the others $\delta b_{\frac{k}{2}}$, $(k \neq n)$ are zero. The sum of all such partial values will be the full virtual work:

$$\delta T = \sum_{n=1}^{\infty} \delta T_n .$$

Now we obtain:

$$\delta T_n = \frac{\partial T}{\partial b_{\frac{n}{2}}} \delta b_{\frac{n}{2}} . \tag{16}$$

3. As every component of δT_n depends only on the small arbitrary increase $\delta b_{\frac{n}{2}}$ and, as we saw before, the same concerns the components of δV_n, instead of writing

$$\delta V = \delta T ,$$

we may write:

$$\delta V_n = \delta T_n \qquad (n = 1, 2, \ldots, \infty) . \tag{17}$$

Thus we obtain as many equations, as there are unknown coefficients $b_{\frac{n}{2}}$. By putting values (16) and (9) into eq. (17), we obtain:

$$\frac{l}{2} \cdot \beta_{\frac{n}{2}} + \sum_{\rho=1}^{x} Q_\rho \cdot \sin\left(\frac{\pi n}{l} \cdot c_\rho\right) + \frac{n \cdot \pi}{l} \sum_{\sigma=1}^{z} M_\sigma \cdot \cos\left(\frac{\pi \cdot n}{l} \cdot h_\sigma\right)$$

$$= \frac{\pi^4 \cdot n^2}{2\, l^3} \cdot \left[n^2 \cdot \left(A_0 - \frac{1}{2} A_n\right) - \left(\frac{l^2}{\pi^2} p_0 + \frac{1}{2} p_n\right)\right] \cdot b_{\frac{n}{2}}$$

$$+ \frac{\pi^4 \cdot n}{4\, l^3} \cdot \sum_{\substack{k=1 \\ (k \neq n)}}^{\infty} k \cdot b_{\frac{k}{2}} \left[n \cdot k \cdot \left(A_{\frac{k-n}{2}} - A_{\frac{k+n}{2}}\right) - \frac{l^2}{\pi^2} \left(p_{\frac{k-n}{2}} + p_{\frac{k+n}{2}}\right)\right] . \tag{18}$$

These equations are the *exact solution* of the general problem of plane bending a straight beam on two simple supports, because they allow to calculate all coefficients $b_{\frac{n}{2}}$ one by one; first of all, we must compute the coefficients A_n, p_n, $A_{\frac{k \pm n}{2}}$, $p_{\frac{k \pm n}{2}}$, $\beta_{\frac{n}{2}}$; then, as first approximation, we compute one by one the coefficients $b_{\frac{n}{2}}$ from the transformed equation:

$$b_{\frac{n}{2}} = \frac{1}{\pi^4 \cdot n^2 \cdot \left[n^2 \cdot \left(A_0 - \frac{1}{2} A_n\right) - \frac{l^2}{\pi^2}\left(p_0 + \frac{1}{2} p_n\right)\right]} \cdot \Bigg\{ 2\, l^3 \left[\frac{l}{2} \beta_{\frac{n}{2}} + \sum_{\rho=1}^{x} Q_\rho \cdot \sin\left(\frac{n\pi}{l} \cdot c_\rho\right) + \frac{n \cdot \pi}{l} \cdot \sum_{\sigma=1}^{\tau} M_\sigma \cdot \cos\left(\frac{n\pi}{l} \cdot h_\sigma\right)\right] - \frac{\pi^4 \cdot n}{2} \sum_{\substack{k=1 \\ (k \neq n)}}^{\infty} k \cdot b_{\frac{k}{2}} \cdot \left[n \cdot k \left(A_{\frac{k-n}{2}} - A_{\frac{k+n}{2}}\right) - \frac{l^2}{\pi^2}\left(p_{\frac{k-n}{2}} + p_{\frac{k+n}{2}}\right)\right] \Bigg\} \tag{19}$$

neglecting the infinite series in the last term, as it is always much smaller than the first expression in square brackets, because of the factor $2\, l^3$ in front of it.

After having calculated in this manner the first approximation of the values of $b_{\frac{n}{2}}$, we put them into the last factor of the right side of eq. (19), and then repeat the evaluation of $b_{\frac{n}{2}}$ as a second approximation, etc.

The series (4) is in most cases sharply convergent, so that there is no need in practice to compute more than two approximations. In general we proceed the calculus so long, until the difference between two consecutive approximate values of $b_{\frac{n}{2}}$,

$$\Delta b^{(i)}_{\frac{n}{2}} = b^{(i+1)}_{\frac{n}{2}} - b^{(i)}_{\frac{n}{2}}$$

becomes less than the allowable error.

If $q = Q_\rho = M_\sigma = 0$, the formula (19) will give the values $b_{\frac{n}{2}} = 0$, as it should be.

There may occur some simplifications in the calculus, if the distribution of the stiffness $E \cdot J$ is such as to give above a certain value of n:

$$A_{n+1} = A_{n+2} = \ldots = 0 .$$

3. Elastic stability

An analysis of the equation (19) will prove, that every coefficient b will become infinite (or eventually indefinite) as soon as the denominator at the right side of the chosen n-th equation of the system (19) becomes zero. There occurs then a case of instability, and the bar will have the tendency to buckle in the shape of n half-waves:

$$p_0 + \frac{1}{2} p_n = \frac{\pi^2 \cdot n^2}{l^2}\left(A_0 - \frac{1}{2} A_n\right) = \frac{\pi^2}{\left(\frac{l}{n}\right)^2} \cdot \left(A_0 - \frac{1}{2} A_n\right) \tag{20}$$

Naturally in most cases of engineering practice the buckling will occur in the form of one half-wave, because of the dominating factor n^2 at the right side of eq. (20). But there may certainly occur special cases, when a form with n half-waves might also become dangerous; but we may accept as a rule, that the most dangerous is the form buckling corresponding to the value $n = 1$:

$$p_0 + \frac{1}{2} p_1 = \frac{\pi^2}{l^2} \cdot \left(A_0 - \frac{1}{2} A_1 \right) \tag{21}$$

When $A_1 = p_1 = 0$, the eq. (21) reduces to the common Euler formula for $E \cdot J = \text{const}$ and $P = \text{const}$ along the length of the bar. The more general case (20) for $n > 1$ may be of interest to the designer in some exceptional cases only, when the so obtained critical mean compressive load p_0 might become smaller, than some higher component p_n $(n > 1)$. This would be possible only if the value of A_n would be near to A_0, what is very improbable in practice, especially for higher values of n. In any case, we must remember that the very simple conditions (20) and (21) are exactly derived from the laws of Mechanics, and include neither simplifications, nor approximations of any sort, although they are valable for the most general case of load and stiffness distributions.

On the contrary, the eq. (19) involving infinite series, bear an approximate character; the conditions of stability, although deduced from these equations, are absolutely exact, and throw some new light upon the instability effects in beams.

The condition (20) shows that for the buckling of a beam in the form of n half-waves, only the n-th harmonics of the axial load and stiffness distributions are decisive, and other harmonics have no influence on this effect at all. This bears a deep analogy, to the effect of simple forced harmonic vibrations, which has been pointed out before in some more special case by M. T. Huber (3).

Therefore, we dare say, that the expansion into a trigonometric series is not only a convenient method of mathematical computation of bending problems of compressed beams, but this method divulges some new aspects of the effect of instability itself, and has a deeper meaning in itself, as it allowed to discover a new general law of static instability of inhomogeneous compressed beams.

4. Appendix

For practical use of equations (19), (20), (21) we need a method of expanding functions $E \cdot J$, q, P in trigonometric series. This may be done by means of several methods, as for example: by Fischer-Hinnen (4), or by F. M. Lewis (5) and, especially for lower values of n, by P. V. Melentiev (6).

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(3) M. T. Huber, *On an analogy of some effects of stability of slightly curved elastic bars with a simple case of forced vibrations* (*Polish Academy of Sciences*, P. A. U., Kraków, Poland, 1934) (Polish).

(4) Fischer-Hinnen, *Elektrotechnische Ztg.*, Vol. 22 (1901), pages 396-398 (German).

A. Hussmann, *Rechnerische Varfahren zur harnonischen Analyse und Synthese*, Berlin, 1938 (German).

(5) F. M. Lewis, *A method of harmonic analysis* (*J. Appl. Mech.*, Vol. 2 [1935], Nr. 4, pages 137-140).

(6) P. V. Melentiev, *Some New Methods in Approximate Calculus*, Leningrad and Moscow, 1937 (see pages 139-147) (Russian).

The harmonic synthesis of the deflection curve, after obtaining the values of $b_{\frac{n}{2}}$, may be done by means of Howard's circles, or by simple computation from goniometric tables.

Résumé

L'auteur examine dans ce mémoire des poutres dont les centres des diverses sections droites constituent une ligne droite et les axes principaux de ces mêmes sections deux plans orthogonaux entre eux; toutes les charges et réactions extérieures se trouvent dans l'un de ceux-ci.

Ces hypothèses admises, la rigidité à la flexion peut être variable, même avec solution de continuité. Les charges axiales peuvent également varier. En exprimant tous les paramètres sous la forme de séries trigonométriques, variables le long de la portée, on obtient la résolution de la ligne élastique sous la forme d'une autre série trigonométrique. On obtient ainsi un critère très simple pour la stabilité élastique [Eq. (20)].

Zusammenfassung

Bei den Balken, die in diesem Beitrag betrachtet werden, liegen die Schwerpunkte aller Querschnitte auf einer Geraden und die Hauptachsen bilden zwei aufeinander senkrecht stehende Ebenen, deren eine auch die äusseren Lasten und die Reaktionen enthält. Unter diesen Voraussetzungen kann die Biegesteifigkeit noch beliebig kontinuierlich oder diskontinuierlich veränderlich sein. Auch die Axiallasten können sich beliebig ändern. Indem alle über die Spannweite veränderlichen Parameter des Problems in trigonometrische Reihen entwickelt werden, erhält man die Lösung der Gleichung der elastischen Linie auch in der Form von trigonometrischen Reihen. Zugleich folgt aus dieser Entwicklung noch ein sehr einfaches Kriterium für die elastische Stabilität [Gl. (20)].

Summary

Beams taken under consideration in this paper have all cross-section centers lying on one straight line, all principal axes of cross-sections are contained in two perpendicular planes, one of which contains all external loads and reactions acting on the beams.

Under assumption of these restrictions, the stiffness in bending may be arbitrarily continuously or discontinuously variable. The axial loads may also vary arbitrarily. By means of expansion into trigonometric series of all parameters of this problem, variable along the span, a solution for the line of elastic deflection is obtained, in the shape of a trigonometric series. By the way, a very simple criterion for the elastic stability [Eq. (20)] is obtained.