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## VIII 4

### Analysis of Bars on Elastic Foundation.<sup>1</sup>

Berechnung von Balken auf elastischer Bettung.<sup>1</sup>

Etude analytique des poutres sur sol élastique.<sup>1</sup>

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Accepting *Winkler's* assumption, that the distributed reaction forces of the foundation are proportional at every point to the deflection of the bar at that point ( $p = Cy$ ), following this we shall propose two methods of analyzing the problem. The first, based on the differential equation of the elastic line, will be called exact analysis. The second one, where the elastic line is represented by trigonometric series, will be termed as approximative analysis.

#### I. Exact Analysis.

As it is well known, the solution of the differential equation of the elastic line

$$\frac{d^4y}{dx^4} + Ky = 0$$

takes a comparatively simple form when the bar is assumed to be infinitely long. Complications enter when we want to obtain solutions for bars of finite length. The straight forward mathematical determination of the constants of integration from the end conditions leads to complicated expressions as it is to be seen in other works. It will be shown here that a solution for a bar of finite length can be obtained from formulas derived for the infinitely long bar by using simply the principle of superposition.

The procedure will be that we load with the given loading an infinitely long bar, and then at the points where ends are required we apply some certain concentrated forces and moments which create at these points the conditions of ends. In this way instead of determining the constants of integration from the end conditions we *create* the required end conditions at the required points on the infinitely long bar. Consequently the forces and moments applied for this purpose will be called *end-conditioning forces*.

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<sup>1</sup> This is an abstract of the work done under the direction of Prof. S. Timoshenko at the Univ. of Mich., where the writer was sent under the Jeremiah Smith fellowship from Budapest, Hungary.

When a simple concentrated force,  $P$ , is acting at point 0 on the infinitely long bar (Fig. 1a) we obtain the following expressions for the side of the right of point 0:

$$y = \frac{Ps}{2bk} A_{sx}; \quad y' = -\frac{Ps^2}{bk} B_{sx}; \quad a)$$

$$M = \frac{P}{4s} C_{sx}; \quad Q = -\frac{P}{2} D_{sx}. \quad b)$$

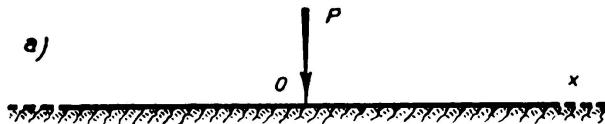


Fig. 1.

Where  $b$  is the width of the beam

$k$  is the modulus of the foundation  $\text{lbs/inch}^3$

$$s = \sqrt{\frac{bk}{4E}}$$

and where

$A_{sx} = e^{-sx} (\cos sx + \sin sx)$ ,  $B_{sx} = e^{-sx} \sin sx$ ,  $C_{sx} = e^{-sx} (\cos sx - \sin sx)$ , and  $D_{sx} = e^{-sx} \cos sx$ , are the so called Zimmermann characteristic functions. To the left of the point 0 the  $y'$  and  $Q$  will be taken with reversed sign. For a single concentrated moment,  $M_0$ , acting at 0 (Fig. 1b) we obtain the following expressions for the right side of the beam:

$$y = \frac{M_0 s^2}{bk} B_{sx}; \quad y' = \frac{M_0 s^3}{bk} C_{sx}; \quad M = \frac{M_0}{2} D_{sx}; \quad Q = -\frac{M_0 s}{2} A_{sx}.$$

Formulas for distributed loadings on the infinitely long bar can be derived from the above formulas by simple integration.

Let us assume now that an infinitely long bar is loaded with some given forces ( $P$  and  $q$ ), (Fig. 2a). We wish to obtain from this case the solution of a bar with finite length having free ends at A and B.

On the basis of what has been stated we shall apply at A and B, such end-conditioning forces,  $M_{OA}$ ,  $P_{OA}$ ,  $M_{OB}$  and  $P_{OB}$ , as will cancel the force functions  $M_A$ ,  $Q_A$ ,  $M_B$  and  $Q_B$ , arising at these points in the infinitely long bar as a result of the given loading, thus creating at A and B the conditions of free ends ( $M = 0$ ;  $Q = 0$ ).

The four unknown end-conditioning forces can be determined from the four end-conditions, but it is simpler to resolve the original loading (Fig. 2a) into symmetrical (Fig. 2b) and antisymmetrical (Fig. 2c) components of loading,

then to determine the end-conditioning forces in each case separately, and add them together. In this way we shall obtain for the symmetrical case (Fig. 2b) in the case of free ends, the following expressions:

$$P'_o = 4 F'_I \{ Q'_A (1 + D_{sl}) + s M'_A (1 - A_{sl}) \}$$

$$M'_o = -\frac{2}{s} F'_I \{ Q'_A (1 + C_{sl}) + 2 s M'_A (1 - D_{sl}) \}$$

and for the antisymmetrical case (Fig. 2c).

$$P''_o = 4 F''_I \{ Q''_A (1 - D_{sl}) + s M''_A (1 + A_{sl}) \}$$

$$M''_o = -\frac{2}{s} F''_I \{ Q''_A (1 - C_{sl}) + 2 s M''_A (1 + D_{sl}) \}$$

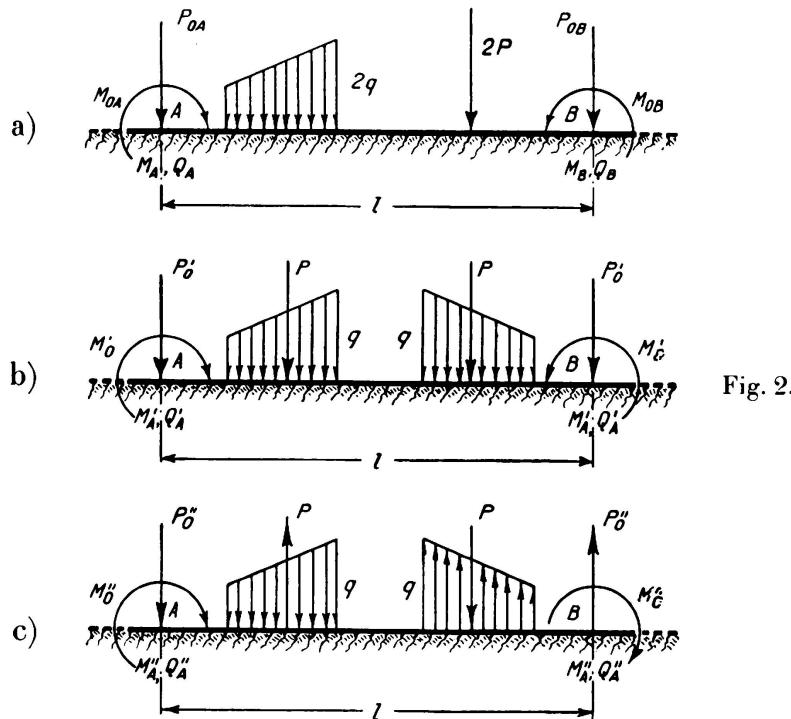


Fig. 2.

Where  $M'_A$ ,  $Q'_A$ , and  $M''_A$ , and  $Q''_A$ , are moments and shearing forces arising at A from the symmetrical and antisymmetrical loading components respectively, and the F symbols denote the functions shown in Fig. 3:

$$F'_I = \frac{1}{1 - e^{-2sl} + 2e^{-sl} \sin sl}$$

$$F''_I = \frac{1}{1 - e^{-2sl} - 2e^{-sl} \sin sl}$$

$$F'_{II} = \frac{1}{1 + e^{-2sl} + 2e^{-sl} \cos sl}$$

$$F''_{II} = \frac{1}{1 + e^{-2sl} - 2e^{-sl} \cos sl}$$

With the aid of these  $F$  functions, expressions for the end-conditioning forces can be derived for other cases than those with the free ends shown above. But these  $F$  functions have other applications besides the computation of the end-conditioning forces. For example, if a bar of length  $l$  is loaded by a concentrated force  $P$  at the end A (Fig. 4a), the displacements and the rotations of the ends can be expressed as

$$y_{A, B} = \frac{Ps}{bk} \left\{ \frac{F'_I}{F'_{II}} + \frac{F''_I}{F''_{II}} \right\}$$

$$\theta_{A, B} = - \frac{Ps^2}{bk} \left\{ \frac{F_1}{F''_1} \pm \frac{F''_1}{F'_1} \right\}$$

Under the action of a moment  $M$  at A (Fig. 4b) the displacements and rotations of the ends will be:

$$y_{A, B} = - \frac{Ms^2}{bk} \left\{ \frac{F'_I}{F''_I} + \frac{F''_I}{F'_I} \right\}$$

$$\theta_{A, B} = \frac{2Ms^3}{bk} \left\{ \frac{F'_I}{F''_{II}} + \frac{F''_I}{F'_{II}} \right\}$$

In case of simply supported ends (Fig. 4c) the rotations of the ends due to the moment  $M$  will be

$$\theta_{A, B} = \pm \frac{Ms^3}{bk} \left[ \frac{F'_{II}}{F'_{I}} + \frac{F''_{II}}{F''_{I}} \right].$$

This additional use of the F functions greatly simplifies the analysis of continuous beams and of beams where the cross-sections are, or the modulus of the foundation, varies by steps.

The same principle can be used when the bar, in addition to the transverse loading, is subjected to axial forces and twisting moments. The deflection and its higher derivatives will be always proportional to the transverse loading and so the superposition of the end-conditioning forces is applicable. This will be prese

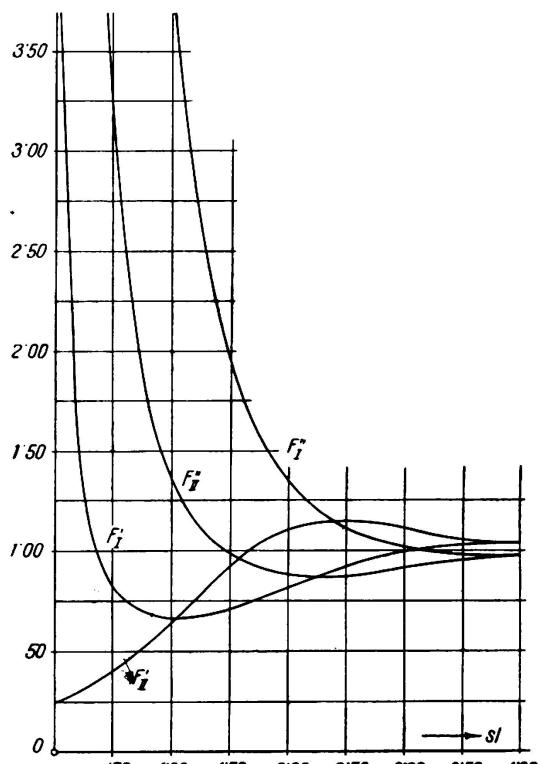


Fig. 3.

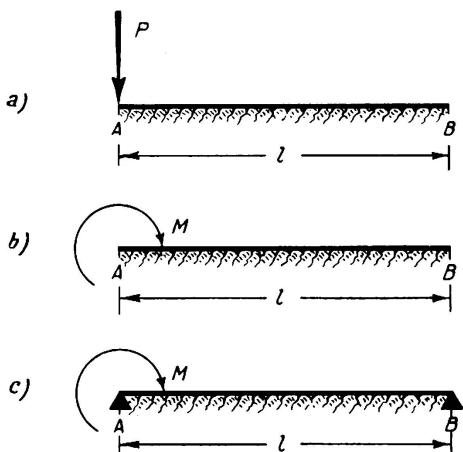


Fig. 4.

<sup>2</sup> Following the completion of this method the writer became acquainted with a Russian paper by *H. V. Klischewitch* and observed that he used the same principles with the exception that not having resolved the loading into symmetrical and antisymmetrical components, his formulas were more complicated and that he restricted himself to the case when the bar is subjected only to transverse loading.

## II. Approximative Analysis.

Taking a bar of finite length loaded with a concentrated force  $2P$  (Fig. 5a), the deflection of the bar can be assumed to consist of a constant vertical displacement  $y_0$ , of an angular rotation  $\alpha$ , and of the ordinates of the elastic line  $A'B'$ , which latter can be expressed in the form of a sine series. Hence the total deflection at any point at a distance  $x$  from the left end can be written in the form:

$$y = y_0 + \alpha \left[ \frac{1}{2} - x \right] + \sum_{n=1, 2, 3, \dots} a_n \sin \frac{n\pi x}{l}$$

If the original loading (Fig. 5a) is resolved into symmetrical (Fig. 5b) and anti-symmetrical (Fig. 5c) parts there will be evidently, in the first one, no rotation ( $\alpha = 0$ ), and in the second one there will be no constant vertical displacement

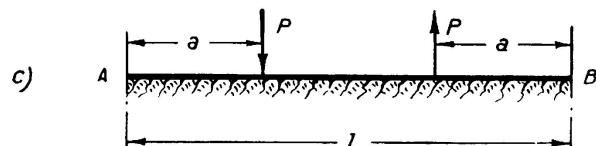
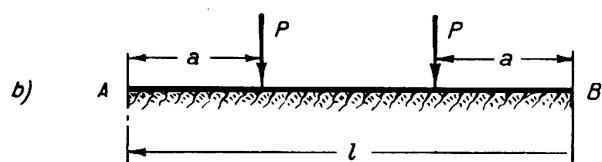
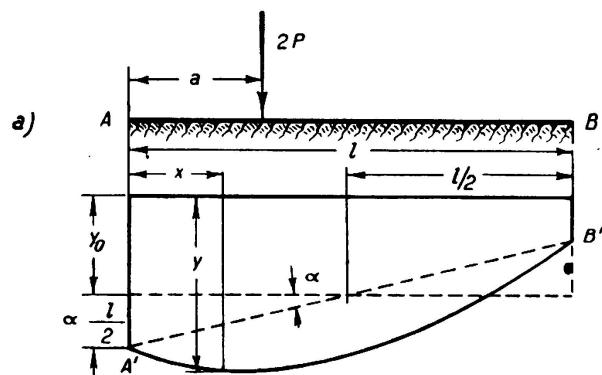


Fig. 5.

( $y_0 = 0$ ). Considering the conditions of equilibrium and equating the strain energy of the bar to the increase of potential energy in the foundation, we get,

For the symmetrical case (Fig. 5b), this expression for the elastic line:

$$y = \frac{2P}{bkl} - \sum_{n=1, 3, 5, \dots} a_n \left[ \frac{2}{n\pi} - \sin \frac{n\pi x}{l} \right]$$

Where any  $a_n$  term can be determined from the correlation

$$2P \left\{ \sin \frac{n\pi a}{l} - \frac{2}{n\pi} \right\} = a_n \left\{ n^4 \frac{\pi^4 EJ}{2l^3} + \frac{1}{2} lbk \right\} - \frac{4}{\pi^4} \cdot \frac{1}{n} \sum_{i=1, 3, 5, \dots} \frac{1}{i} a_i$$

For the antisymmetrical case the expression of the elastic line will take the form:

$$y = \frac{12 P (l - 2a)}{b k l^3} \left( \frac{l}{2} - x \right) - \left[ \frac{12}{l \pi} \left( \frac{l}{2} - x \right) - 1 \right] \sum_{n=2,4,6} a_n \sin \frac{n \pi x}{l}$$

Where the  $a_n$  terms can be determined from:

$$2 P \left\{ \sin \frac{n \pi a}{l} - \frac{6}{n \pi l} (l - 2a) \right\} = a_n \left( \frac{n^4 \pi^4 E J}{2 l^3} - \frac{l b k}{2} \right) - \frac{12 k l}{n \pi^2} \sum_{i=2,4,6} \frac{1}{i} a_i$$

Although this method is termed approximative, any degree of approximation can be obtained, depending on the number of the  $a$  terms taken into consideration.

But this method has a special advantage over the previous exact analysis. In this exact method the dimensions of the bar and the modulus of the foundation appear in exponents and trigonometric functions, thus making it practically impossible to find expressions for these dimensions as unknown quantities. On the contrary in this approximate method the quantities  $E$ ,  $I$ ,  $l$  and  $b$ ,  $k$  appear separately and are easily determined. This is especially true if only one term of the series is used, in which case the above expressions can be applied as useful design formulas.

If we deal with a bar of simply supported ends, the elastic line can be represented as a simple sine series of the form

$$y = \sum_{n=1,2,3} a_n \sin \frac{n \pi x}{l}$$

In the case of fixed-end bars the deflection line can be written in the approximate form of

$$y = \sum_{n=1,2,3} \frac{1}{2} a_n \left( 1 - \cos \frac{2 n \pi x}{l} \right)$$

In both cases the  $a$  terms can be determined from the condition of the minimum strain energy for the whole system.

This approximative method can be applied just as well when axial forces are acting on the bar in addition to the transverse loading. The formulas are especially applicable when the bar is comparatively stiff,  $sl \leq \pi$ . In such cases usually the first one or two terms of the series give a good accuracy for the elastic line. Having obtained this, the  $Q$  and  $M$  curves can be determined by integration, which will still increase the convergency of the results.

If the loading consists of several parts or the bar is more flexible, then it is advisable to use the exact analysis. However, when  $sl > 4$ , then the counter effect of the end-conditioning forces upon each other can be safely neglected ( $A_{sl} = B_{sl} = C_{sl} = D_{sl} = 0$ ), and if the bar is so long that the distance  $a$  between the last loading and the neighboring end of the bar gives a value  $sa > 4$ , then the simple formulas for the infinitely long bar can be applied.