I 5

Critical observations on the theory of plasticity.

Kritische Betrachtungen zur Plastizitätstheorie.

Considérations critiques sur la théorie de la plasticité.

Oberbaurat Dr. G. v. K a z i n c z y,
Budapest.

In 1914 the present author expressed the opinion, in a Hungarian journal,[1] that the permanent deformation of the steel ought to be taken into account in deciding the true carrying capacity of statically indeterminate structures. Since this true carrying capacity is greater than the capacity worked out in accordance with the elasticity theory, it seems reasonable to allow for permanent deformation even in the practical design of structures. In the meantime the problem has been discussed, illuminated and checked by experiment in all its aspects, and it is proposed to give here a critical survey of the whole field.

The new method of calculation is known by various names. The term *"plasticity theory"* is understood to mean a method of calculation in which account is taken of permanent strains, by contrast with the elasticity principle which takes account only of the elastic strains. The term *"carrying capacity method"* [theory of plastic equilibrium] is used also, but this lacks definition, because carrying capacity is often identified with maximum supportable load (as for instance by *Stüssi*), while on the other hand *Bleich, Maier-Leibnitz* and the author in his earlier publications understand this term to mean the "practically" supportable load. The position, as regards this problem, is governed by reference to certain main principles. *What is the purpose of structural calculations?* It is to ensure that structures shall remain serviceable whilst in service; inaccuracies of calculation and in determining the properties of materials, as well as the loading, make it necessary to adopt certain guarantees against structures becoming unserviceable. At the Vienna Congress[2] the author argued that the factor of safety is a question of economics. It behoves on the one hand to build as cheaply as possible and on the other hand to build in such a way that any damage likely to be sustained — due account being taken of the propability of its occurrence — will not outweigh the saving achieved through adopting smaller dimensions for the structural member. The factor of safety must, therefore, be increased in proportion to the magnitude of the damage apprehended. These considerations serve to explain why we are content with a factor of safety of between 1.6 and 1.8 in cases where unserviceability will not cause an unacceptable amount of deflection, but insist on perhaps three times this amount in cases where over-stressing of the structural members might result in immediate collapse (for instance by buckling) without warning. In the case of

members which are rendered useless by any considerable change of shape our endeavours are directed to ensuring adequate safety against excessive deformation, rather than against failure by rupture. To lay down a general rule for the amount of permissible bending, it might perhaps be stated that the limiting load (critical load or practically supportable load) should be taken as the load which, when gradually increased, causes a rapid increase in the deflection. In the experiments by *F. Stüssi* and *C. F. Kollbrunner*[3] (Fig. 1) the present author would take 1.71 tonnes and not 2.35 tonnes as being the limiting load in the case


Fig. 1.

of the beam on two supports; from this point of view the upshot of the experiments reads rather differently: namely, that even in extreme cases the limiting load (not the true carrying load) has two values* if the beams are built in at the ends, except where the fixation is too flexible with the result that the elastic deflection at the middle increases so rapidly, once the yield point is reached, that it becomes excessive even before the yield point is reached over the inner support. Fig. 2 shows how the lines of bending would appear in an ideally plastic beam under uniformly distributed loading with different amounts of end fixation. It will be seen that in certain cases deformation must be taken into account.

There are two methods of introducing the desired factor of safety when dimensioning a structure: either the calculation may be based on the actual load multiplied by the factor of safety, or the limiting stress divided by the factor of safety may be regarded as the permissible stress. The latter method is the usual
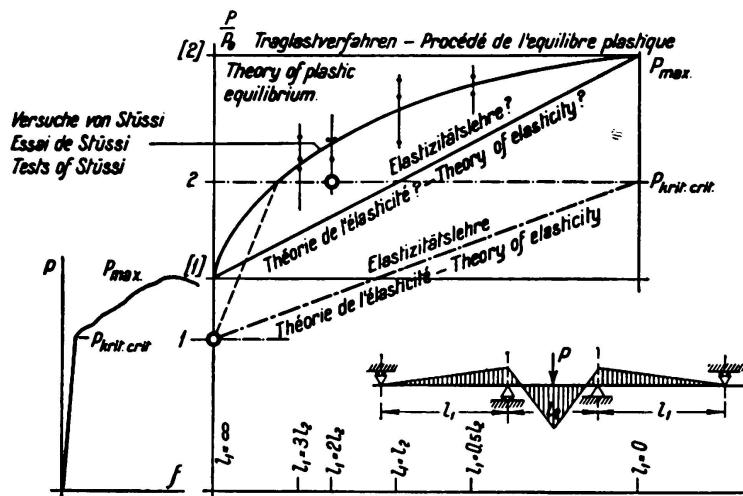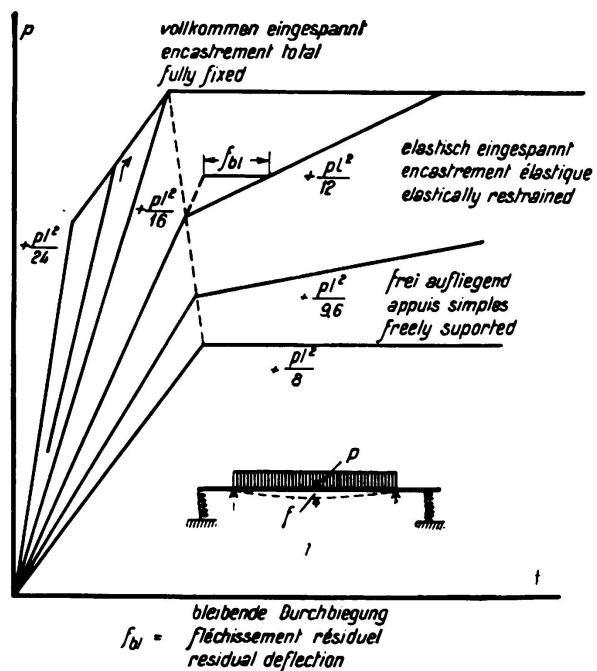

Fig. 2.

---

* Beam simply supported on two points $P_{Tl} = 1{,}71$ tonnes; $P_v = 2{,}35$ tonnes. $\dfrac{3{,}46}{1{,}71} = 2{,}02$.

Continuous beam $P_{Tl} = 3{,}46$ tonnes; $P_v = 3{,}82$ tonnes; $\dfrac{3{,}82}{2{,}35} = 1{,}60$.

one, the ratio between the permissible stress and the limiting stress representing
the factor of safety. This would be correct if the stresses followed a linear law of
growth up to the limiting load, but in the case of statically indeterminate struc-
tures that condition does not, as a rule, obtain, owing to equalisation of stress. If
calculations were based on the working stresses multiplied by the factor of safety
it would be easier to account for the equalisation of stresses, seeing that this
operates only when the working stress is exceeded and is, therefore, a criterion
of safety rather than of actual stress.

In order to determine the limiting loads of statically indeterminate girders
theoretically, a material has been assumed which possesses ideal properties, that
is to say which gives an idealised stress-strain diagram. It has further been
assumed that cross sections remain plane when deformed, and that the yield
increases gradually from the extreme fibres towards the inside of the beam.
According to this theory the possibility of a cross section subject to bending
stresses continuing to deform without the moment being further increased
depends on its having become plastic as far as the neutral axis. The plastic hinge
effect therefore implies an unlimited degree of bending, and in mild steel
this condition cannot be fulfilled on account of the cold working effect. Hence
a number of investigators have recently turned closer attention to the pheno-
menon of plastic deformation with special reference to cases where the field of
stress is not uniform and the yield phenomena do not progress uniformly, but
where the less heavily stressed portions hinder the deformation of the plasticised
elements (works by $W. Kuntze$[4],
$W. Prager$[5]; also $J. Fritsche$[6]:
The New Theory of Plasticity).
Observation however, has not
confirmed this theory, the yield
figures not being affected to
such an extent that the beams
yield all at once as far as the
neutral axis. The same fact may
also be seen in the illustrations
to Nadai's work, „Der Bildsame
Zustand der Werkstoffe", p. 127,
Fig. 130 showing how the signs
of yield spread gradually in-
wards. On the other hand it
may be observed in the case of
I beams that the yield figures

Theorie u Beobachtung
Théorie et observation
Theory and observation

in der praktischen Berechnung
zu verwenden.
à employer dans le calcul pratique.
to be used in praxis
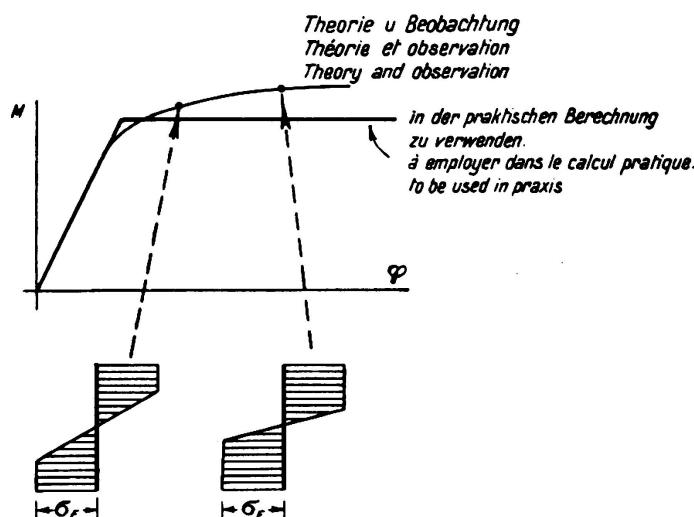
Fig. 3.

appear simultaneously throughout one half flange. $Rinagl$[7], however, states that
the delay in yielding which this view of the matter implies is a mistake, and
that the effect is attributable to the existence of an upper yield point which always
operates in bending but which appears only indistinctly in a tensile test. The
present author does not agree with Professor $Rinagl$ about this, as he has
himself been able to observe the delay to the yield in a non-uniform field of
stress in frame members, and an account of these observations will be given below.

An attempt to allow for the true properties of the material by reference

to any of these theories leads to complicated calculations. Since, however, our objective is the dimensioning of structures and not the theoretical interpretation of experimental results, we must devise a simple method of calculation. This can be done if we assume a sharp transition from the elastic to the plastic condition even under bending stress. Maier-Leibnitz[8] has shown the possibilities of solving
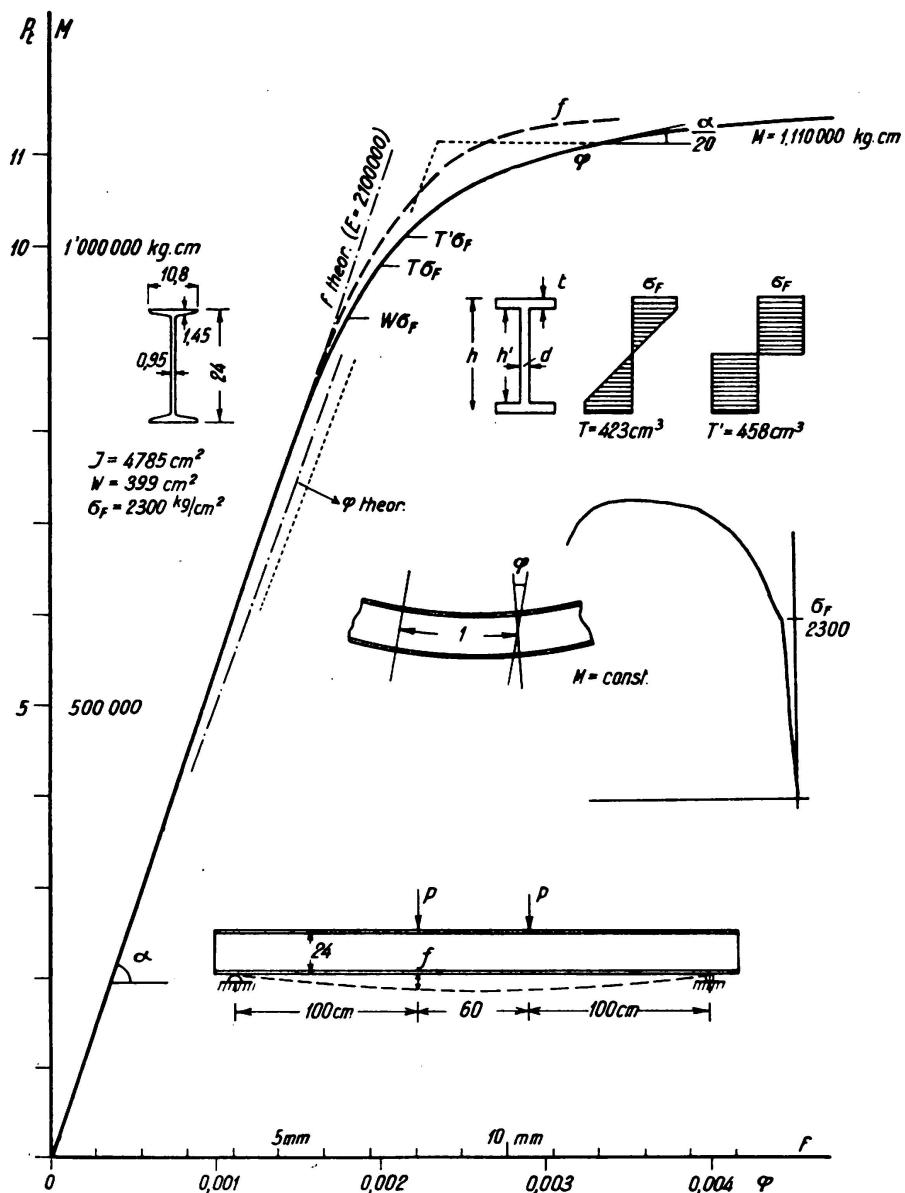
Fig. 4.

simple problems by reference to the true theory of moments and deformation, but a practical method can be built up only on a simplified interpretation (Fig. 3). Maier-Leibnitz suggests that the moment to be regarded as a limiting moment should be that corresponding to the sharpest curvature of the residual strain-moment curve. The author suggests, however, that the value to be taken as a limiting moment should be the one at which permanent bending is twenty times as great as elastic bending.

To throw further light on this question the author has carried out experiments

on an I beam MP 24 (W = 399 cm³) coated with lacquer so that the appearance
of yield could be more readily detected. Up to σ = 2250 kg/cm² the bending
line was practically straight (Fig. 4). On the tensile flange the yield figures appe-
ared at 2500 kg/cm² whereas on the compression flange they appeared as
early as $\sigma = \dfrac{M}{W} = 2120$ kg/cm², but obviously this was due to a local une-
vennes. At 2800 kg/cm² that amount of deformation was obtained which
the author has adopted as the criterion of the limiting moment. The beam was
then removed from the bending machine and was carefully examined and photo-
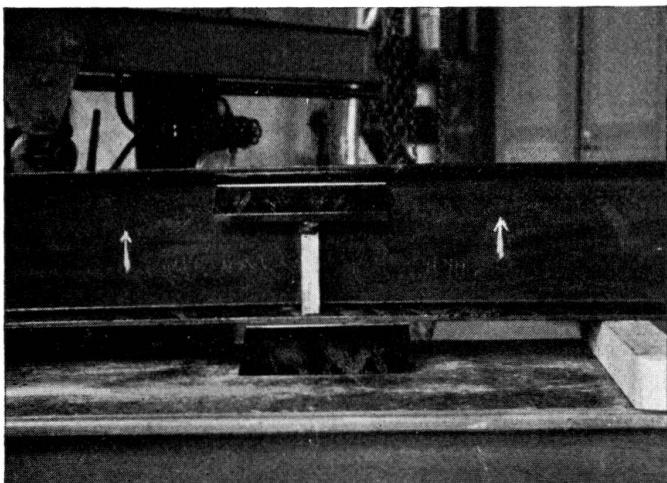
Fig. 5.

graphed (Fig. 5). The condition
of yield had been reached
only over approximately one
half of the length containing
the places most heavily stressed
under constant maximum mo-
ment. Contrary to theoretical
indications, the yield figures ex-
tended to the neighbourhood of
the neutral axis. A tensile test
bar cut from the unloaded end
of the beam after the experi-
ment, was found to have a yield
point of 2300 kg/cm² in a very
small length of yield. The con-
clusion which has to be drawn
from this experiment is that the limiting moment must be determined on empiri-
cal, not theoretical, grounds. It appears that the limiting moment and the yield
point stand in no simple relationship to one another as the matter is affected by
the shape of cross section and by the properties of the material. If these limiting
moments are determined empirically* for certain cross sections and types
of steel no obstacle stands in the way of adopting the new method of inter-
pretation. If we have decided to calculate from the idealised bending line
(M-φ-diagram) we may design our structures in accordance with the following
rules.

1) Statically determinate structures subject to bending.

The limit of carrying capacity is reached not when the extreme fibre attains
to the yield point but when the "beam" as a whole is yielding. The carrying
moment is not M = W · σ$_F$ but M = T · σ$_F$ where T is approximately 6 to 20 %
greater than W and must be determined by experiment beforehand.

2) Statically determinate trussed girders (lattice girders).

The calculation is made as hitherto. Secondary stresses resulting from the
rigidity of the intersections may be left out of account, but as regards com-

---

* Suggestions for calculating the limiting moment have been put forward by *von Kazinczy* (9),
Kist (10), *Fritsche* (11) and *Kuntze* (4), but these all give lower values of carrying moment
than the present author's experiments.

pressive forces the buckling length is to be taken as equal to the whole theoretical length of the bar, even in the plane of the truss. Compression members should be dimensioned with a higher factor of safety than tension members because if the buckling load is exceeded the result may be a collapse of the structure.

3) Calculation of the connecting rivets.

This is calculated as hitherto, that is to say the total load in the member is uniformly divided among all the connecting rivets. Here practice and experience have fully confirmed the correctness of the plasticity theory. The connecting rivets or weld seams should be calculated not from the tabulated but from the maximum permissible load in the member, so that in the case of an excess it is the member and not the connections which will yield. Owing to the redistribution of the secondary stresses amongst the members it is desirable that the connections should be made rigid against bending.

4) The calculation of continuous beams.

In the case of beams consisting of a single rolled section (of constant cross section) the M_o moment should be determined in each opening as for a simple beam, and the closing line should be so drawn that the negative and positive moments are of equal magnitude; the beam should then be designed to suit the maximum moment so calculated.

In the case of beams wherein the cross section has been made to vary with the moment by adding flange plates, there is not much point in designing by reference to the plasticity theory, but if, with a view to greater economy, it is still desired to make use of the new method, the closing line should be drawn in arbitrarily in such a way as to minimise the cost of construction. The rule to remember is that the negative moment can be minimised as desired, whereas a yield taking place in the middle of the beam is always associated with large deflections.

Under moving loads the limiting moments are to be determined according to the elasticity theory, the closing line being moved about at will in the direction of equalising the moment.[12, 13]

One of the most important results of the plasticity theory is that permanent settlements of the supports need no longer be taken into account, while on the other hand, the effect of elastic settlement calls for examination.

Rolling and shrinking stresses need not be taken into account but stresses resulting from irregularity of temperature in service should be considered.[13]

If a greater amount of equalisation of moment is counted upon — especially by yielding at the middle of the beam — it is desirable to make the compression flange heavier so that the yield may take place only in the tension flange.

5) Structures in which members are restrained against bending.

Several authors have indicated that in a framed structure which is statically indeterminate to the n-th degree, the yield point may be reached at n places without the structure thereby being rendered useless. The problem may be represented by assuming hinges to exist at these places and to be subject to moments of constant magnitude. Previously the author has himself put forward this view[14] but he now feels compelled to change it slightly. In order that a struc-

ture of this kind may be rendered unstable sufficient hinges must be introduced to form a kinematic chain. When movement takes place the hinges turn in a given direction. The yield hinge, however, turns only in one direction, for in the other direction it acts as a completely elastic member of the girder. Those yield hinges which turn in a direction opposite to that which should exist in a kinematic chain should not, therefore, be considered as hinges. Thus it may be possible in a structure which is statically indeterminate to the n-th degree for the yield point to be exceeded at more than n places before the structure becomes unstable.

Hence a framed structure is safe to carry a given load as long as it is possible to draw a moment line satisfying the conditions of equilibrium with the external stresses and not exceeding the value $M = T \cdot \sigma_{zul}$ at any point. A more exact


Nach der Elastizitätslehre
D'après la théorie de l'élasticité
With theory of elasticity

Ausgleich durchgeführt
Après l'egalisation
Adjustment made

M_A ⟶ M_A

$M_B + M_C = M_A$

Ausgleich. Zusatzmomente
Egalisation. Moments additionnels.
Adjustement. Additional moments.

Fig. 6.

method may be developed similarly to that of Prof. *Cross*: first of all the moment is determined in accordance with the elasticity theory, then, at the places where the moments are to be reduced, the structure is assumed to be cut in two and additional moments are assumed to be introduced in order to relieve the load. At those places where the reduction has already been effected hinges are introduced, but only where an increase in the moments is to be expected (fig. 6). The principal advantage of the plasticity theory lies in the fact that it enables the moment to be regulated, and in this way the more dangerous places to be protected from over-stressing. Gene-

rally speaking the verticals are the more important members in a frame. It is possible, therefore, by making the beams weaker at the connection, to relieve the verticals when the beam reaches its limiting moment, since no further moment can be transferred to the verticals once the limiting moment at the connection has been reached. The danger of buckling of the verticals can thus be avoided by making the beam yield at the connection, which is in no way dangerous.

6) Lattice girders.

Lattice girders which outwardly are statically indeterminate can be designed in the same way as beams and framed girders. The yield phenomena occur in one part of a bar, but for the purpose of equalisation use should be made only of tension bars, because the resistance of a compression bar after it has buckled decreases immediately to a low value, as the author has already indicated at Liége. More recently *E. Chwalla*[15] has worked out this problem, and has also confirmed by experiment that the compression resistance falls away very rapidly.

In the case of lattice girders which are internally statically indeterminate, it happens not seldom, according to the elasticity theory, that full use cannot be made of all the bars — as for instance in the lattice girder shown in Fig. 7 where, according to the elasticity theory, the parts marked B cannot be fully utilised. In such a case the plasticity theory offers an economic advantage by
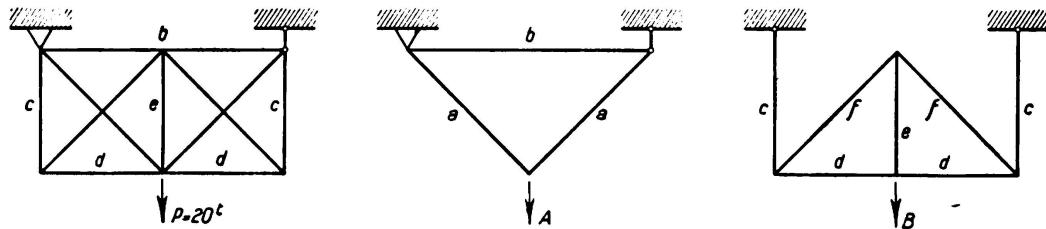

Fig. 7.

enabling all the bars to be fully utilised. The design of such a structure is usually very simple: the supernumerary tension bars are left out, and in their place the known forces $F \cdot \sigma_{zul}$ are inserted; the most heavily stressed tension bars which first reach their yield point are thus eliminated, and when the decision cannot be made by simple means the elasticity theory is used. The cross section should always be so adjusted that only tension members are brought to the yield point and no compression member is exposed to buckling.
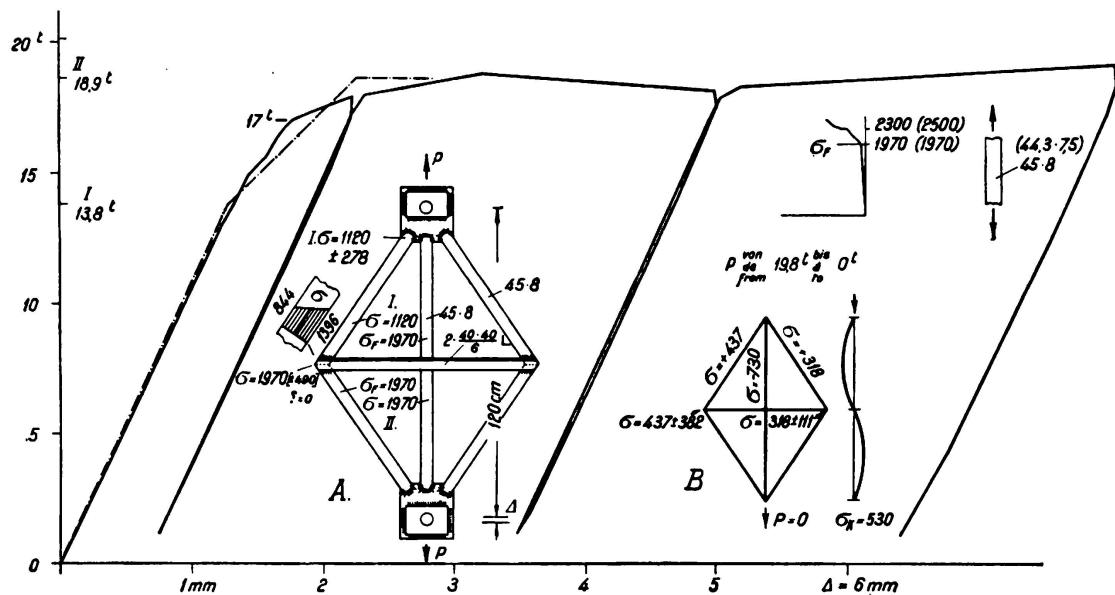
Fig. 8.

For moveable loads special methods must be adopted as, for instance, that of E. Melan[16] — but always with the condition that no plastic deformations are to be permitted in compression bars.

The author has carried out some experiments to confirm theoretical considerations on the plasticity theory and a brief account of these will now be given. He has examined two kinds of internally statically indeterminate frames

— welded and riveted — whereas *G. Grüning* and *E. Kohl*[17] carried out their experiments on externally statically indeterminate frames in which the most heavily stressed tension bars were made in the form of eye-bars, so that no conclusion as to the usual form of connection can be drawn from these. The form of girder adopted and the dimensions and results may be seen in Fig. 8. The behaviour, assuming an ideally plastic material, is represented in Fig. 9. The frame may be regarded as built up from two fundamental system A and B. The resistances offered by the separate systems A and B are plotted along ordinates as functions of the resulting changes in length. P_I is designed as the "first limiting load" and P_{II} as the "second limiting load" (or limit of carrying capacity). After the removal of load the separate systems will not remain free from stress, the residual forces being shown in Fig. 9.

Fig. 9.

An investigation of the mechanical properties of the material chosen showed that the strip steel was very soft, and that as the stress increased it showed a very large region of yield. The yield point was first observed in the vertical tension bar (first limiting load). On the load being further increased the stresses in the vertical bar remained constant, any notable increase being limited to the other bars until the yield point had been obtained in these also (second limiting load). According to the present day view of yield, the theoretical secondary stresses (also included in Fig. 8) disappear. On being relieved of load the structure behaved in an entirely elastic manner, and the residual stresses are shown in Fig. 8. The vertical bar did not however withstand residual stresses of 740 kg/cm² because it consisted of strip steel and was bound to buckle even at 530 kg/cm². This buckling could also be observed in the experimental test piece.
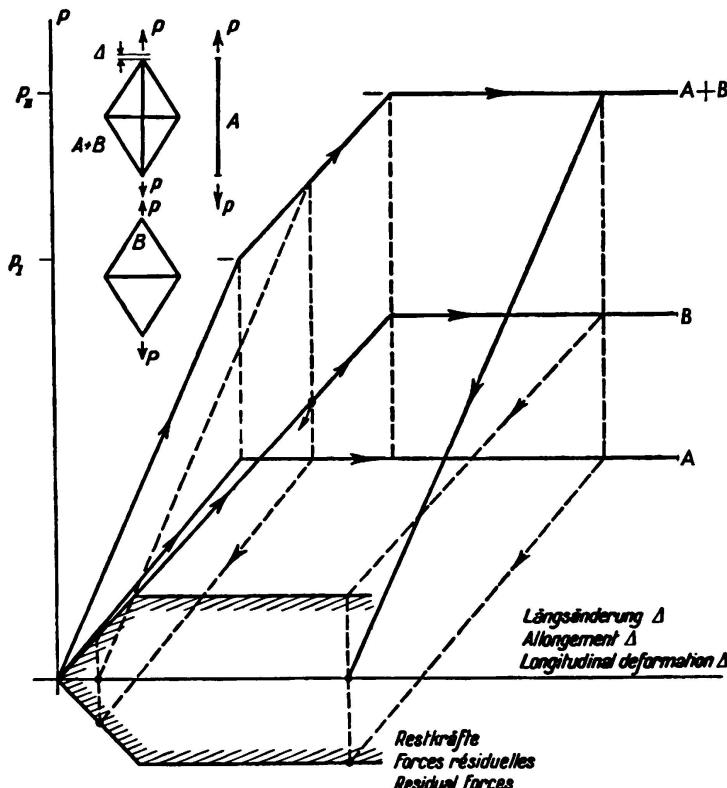
The first lines of yield were noted in the neighbourhood of the middle of the vertical bar with P = 14 tonnes but the actual yield figure in the vertical bar not until 17 tonnes. The specimen had undergone considerable change of shape but nevertheless only quite small portions of the bar yielded (Fig. 10). The plastic alteration in length is limited to certain places and at these it attained a constant percentage. The changes in length of a mild steel bar must be visualised as in Fig. 11, wherein k_I and k_{II} are different alterations in length. The lines marked e represent elastic and the lines marked p represent plastic elongations. The limits

of carrying capacity (second limiting load) agree well with the theoretical values, which shows that the shrinking stresses due to welding have no effect on the carrying capacity, but only on the beginning of the equalisation of stress.

In order to determine the amount of these shrinkage stresses the author had further experimental specimens constructed on which observations were made of elongations at a number of points during the welding and cooling processes: the result of this was to indicate shrinkage stresses of 900 kg/cm². No delay of the yield phenomena — that is to say no upper yield point — could be observed. In the inclined bars with heavy secondary stresses no yield occurred when the average stresses had already reached the yield point. These
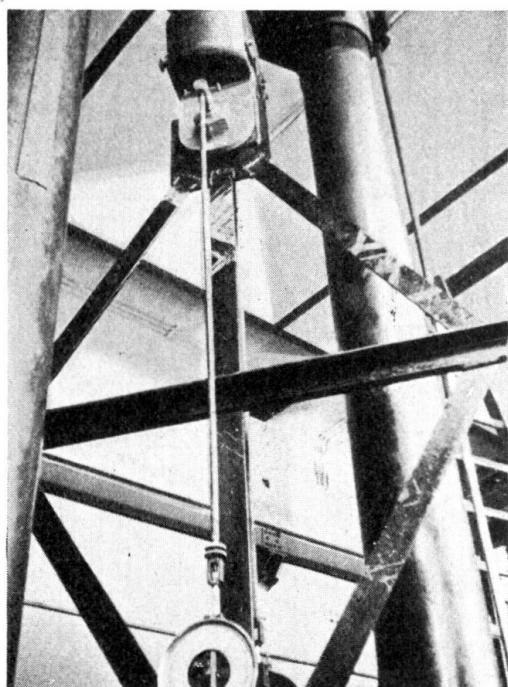

Fig. 10.
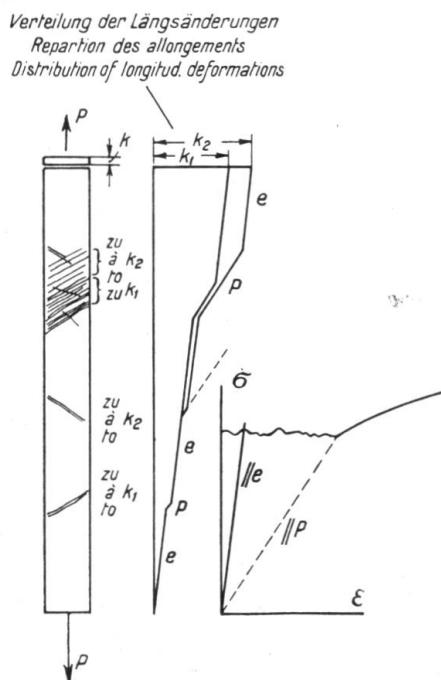
Fig. 11.

experiments appear, therefore, to support the new plasticity theory, and in none of the experiments carried out was it possible to observe the old yield conditions. The author proposes to treat these experiments more fully in the technical press.[19]

A similar framework was made in riveted construction. As a result of the somewhat higher elastic limit of the steel used, the maximum load obtained was higher than in the case of the welded girder (20,4 as against 19,1 tonnes). When first loaded a movement of the rivets was observed, but under further loading the performance was elastic. Despite the presence of the rivet holes the yield point could be attained over the full cross section also.

From these experiments the following conclusions may be drawn: In welded statically indeterminate lattice girders the shrinkage stresses have an influence only over the beginning of the process of equilisation of stress, and not on the magnitude of the critical load. It is to be noted that the shrinkage stresses had the effect of increasing the primary stresses in the tension bars, and of decreasing those in the compression bars. (Note the choice of working method.)

In riveted statically indeterminate lattice girders the plastic elongation begins at the connection and the necessary amount of force is somewhat increased by friction, but the increased yield point around the hole or the toughening of the steel through the use of riveting tools may have had a similar effect. With short lengths of bar only a limited amount of ductility of the connection is necessary for the equalisation of stress, but the connections should always be strong enough to ensure that the full bar reaches its yield point before the connections are affected. The limiting load of a riveted framed structure is approximately the same as if the bars were not weakened by rivet holes according to the plasticity
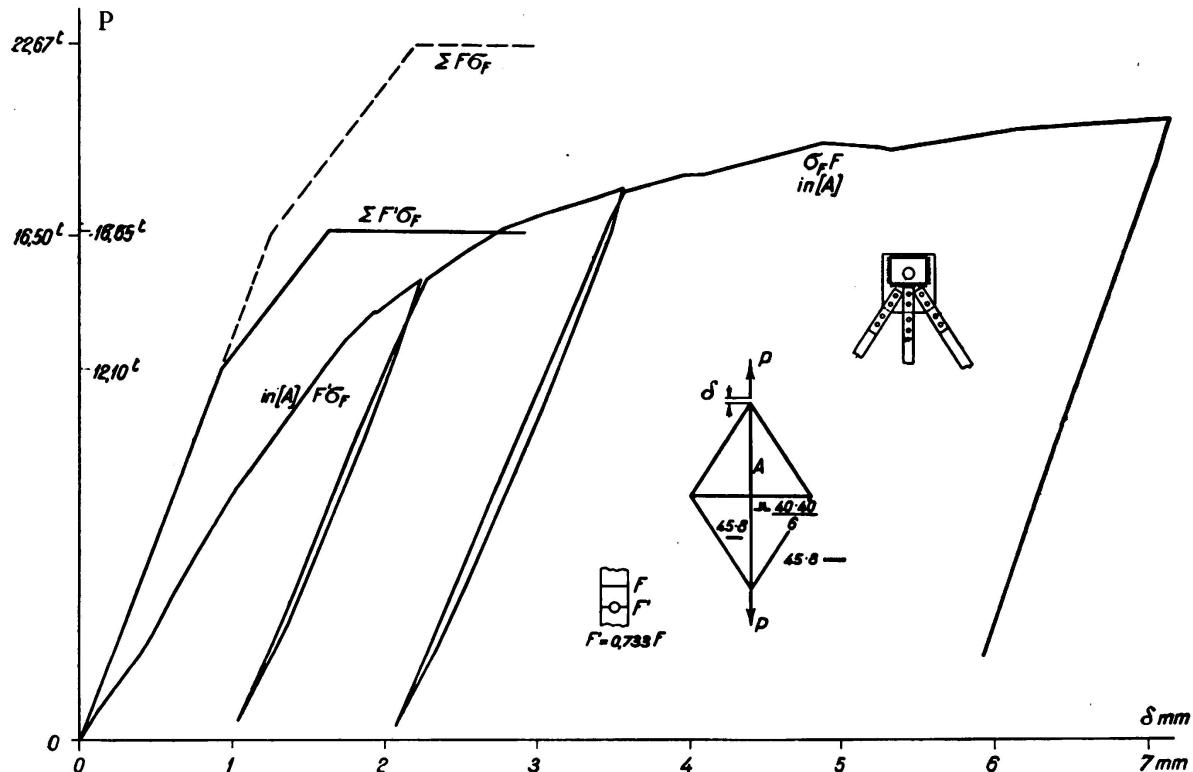

Fig. 12.

theory, always provided that no compression bar reaches a condition of buckling. To take account of the large permanent deformation we may obtain the practical limiting load by deducting the rivet holes and allowing for equalisation of stress, and the degree of safety will then always be greater than in welded structures which have been calculated as for the whole cross section.

Besides the lattice girders, experiments were also made on riveted plate web girders, resting on two supports and loaded at the third points. Measurements were made of the angle of bending of the middle portion under a constant moment, and the results of the experiment are shown in Fig. 13. In calculating the rivet holes were not deducted. The measured deflection is somewhat greater than the calculated amount with $E = 2{,}100$ tonnes per sq. cm but the amount of bending agrees fairly well with the recovery on removing the load (showing the elastic action). After a period of two days the yield point increased by 6 % and the girder then behaved in a purely elastic manner. Under the proposed assumption that $\Delta\sigma/\Delta E = 1/20\,E$ the critical load was determined to be

14 tonnes. Fig. 13 shows the result of comparing this experimental value with different assumptions, the minimum yield point of the angle being taken as 2,500 kg per sq. cm. In this way the maximum extreme fibre stress of the flange plates would be 2780 kg per sq. cm. For determining the carrying moment $T \cdot \sigma_{zul}$ the condition was assumed that the flange plate had reached its yield point. Since in this experiment a second unknown enters, namely, the value to be attributed to the rivet holes, a comparative experiment was carried out as between riveted and welded girders of the same section and material, and the results of this are shown in table I.

Table I.

Welded I-beam				Riveted I-beam $d = 16$ mm	
σ_F kg/cm²	Section mm			Section mm	σ_F kg/cm²
2680	152.6 · 13	compression flange		152 · 12.8	2680
2620	155 · 7.7	tension flange		154 · 7.7	2590
2750	60 · 60 · 6.1	4 L		60 · 60 · 6.1	2780
4280	182 · 8.2	Web		183 · 8.6	4060
1 513 000		critical moment according to tests kgcm		**1 266 000**	
tension 1 180 000	compression 1 420 000	W_{σ_F} (σ_F = flange)		tension 1 170 000	compression 1 400 000
		W_{σ_F} deducting rivet		**965 000**	**1 135 000**
		W_{σ_F} deducting rivet holes in web also		906 000	1 087 000
1 644 000		T_{σ_F} Full section		1 632 000	
		T_{σ_F} deducting rivet holes		**1 387 000**	
		T_{σ_F} deducting rivet holes in web also		1 266 400	
1 513 000		T_{σ_F} of flange plates and angles $+ W_{\sigma_F}$ of web		deducting rivet holes **1 259 000**	

A continuous riveted girder over three supports was also examined (Fig. 15). The deflections are greater than calculated even when the girder is unloaded.
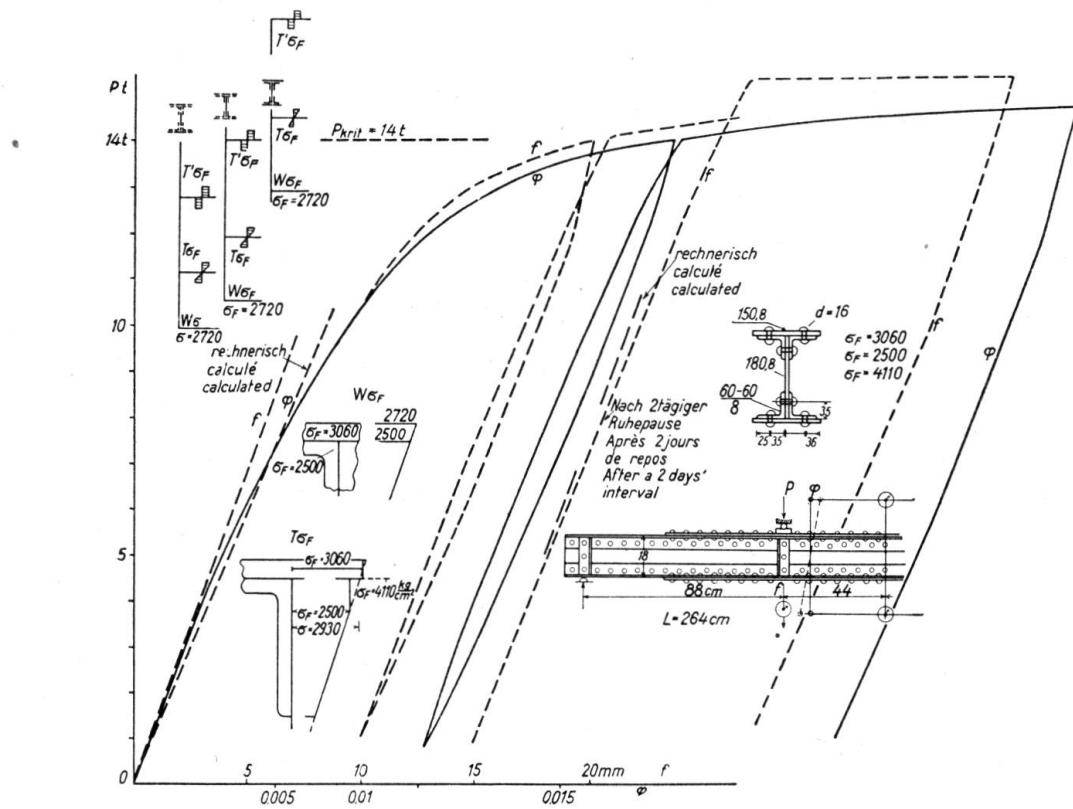
Fig. 13.

From the central support to the point of load the web attained its yield point through shear force, whereby *Stüssi's* argument[18] from theory that the shear stresses must considerably increase when the yield spreads to certain points below
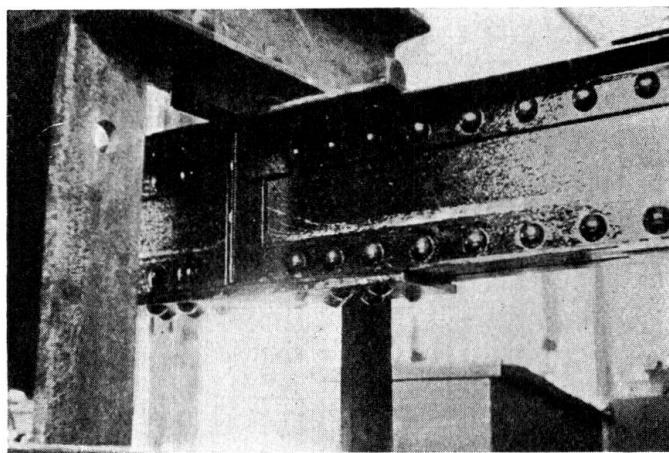
Fig. 14.

the edge of the beam was experimentally confirmed, though to a smaller extent than anticipated by him. This is perhaps attributable to the fact that, as a result of the rapid removal of the moment, only a short portion of the girder is

affected, which moreover is hindered in its movement by the neighbouring portion. The final result found by the author was that the maximum load was represented by T · σ_F assuming a complete equalisation of the moment.
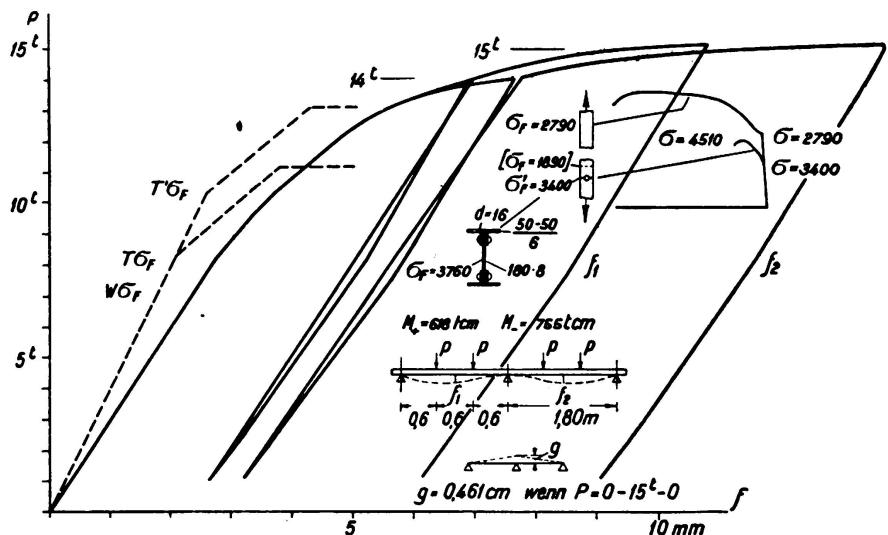
Fig. 15.

Bibliography.

[1] von Kazinczy: Versuche mit eingespannten Trägern. Betonszemle 1914, Nos. 4, 5, 6.

[2] von Kazinczy: Report of the 2nd International Congress for Bridge and Structural Engineering. Vienna 1928, p. 249.

[3] F. Stüssi and C. F. Kollbrunner: Beitrag zum Traglastverfahren. Bautechnik 1935, N° 21. p. 264.

[4] W. Kuntze: Ermittlung des Einflusses ungleichförmiger Spannungen und Querschnitte auf die Streckgrenze. Stahlbau 1933, N° 7, p. 19.

[5] W. Prager: Die Fließgrenze bei behinderter Formänderung. Forschungen auf dem Gebiete des Ingenieurwesens, 1933.

[6] J. Fritsche: Grundsätzliches zur Plastizitätstheorie. Stahlbau 1936, N° 9.

[7] F. Rinagl: Yield Limits and Characteristic Deflection Lines. Preliminary Publication, p. 1561.

[8] H. Maier-Leibnitz: Test Results, their Interpretation and Application. Preliminary Publication, p. 97.

[9] G. von Kazinczy: International Congress on Steel, Liége 1930.

[10] N. C. Kist: International Congress on Steel, Liége 1930.

[11] J. Fritsche: Die Tragfähigkeit von Balken aus Stahl mit Berücksichtigung des plastischen Verformungsvermögens. Der Bauingenieur 1930, Nos. 49—51.

[12] G. von Kazinczy: Die Weiterentwicklung der Plastizitätslehre. Technika 1931.

[13] H. Bleich: Über die Bemessung statisch unbestimmter Stahltragwerke unter Berücksichtigung des elastisch-plastischen Verhaltens des Baustoffes. Der Bauingenieur 1932, Nos. 19, 20, p. 261.

[14] G. von Kazinczy: Statisch unbestimmte Tragwerke unter Berücksichtigung der Plastizität. Der Stahlbau 1931, p. 58.

[15] E. Chwalla: Three Contributions on the Loading Question of Statically Indeterminate Steel Trusses. I.A.B.S.E. Publications, Vol. 2, p. 96.

[16] E. Melan, Theory of Statically Indeterminate Systems. Preliminary Publication, p. 43.

[17] G. Grüning and E. Kohl: Tragfähigkeitsversuche an einem durchlaufenden Fachwerkbalken aus Stahl. Der Bauingenieur 1933, N° 5/6, p. 67.

[18] F. Stüssi: Über den Verlauf der Schubspannungen in auf Biegung beanspruchten Balken aus Stahl. Schweizerische Bauzeitung 1931, Vol. 98, N° 1, p. 2.

[19] von Kazinczy: Versuche mit innerlich statisch unbestimmten Fachwerken. Der Bauingenieur 1938, N° 15/16, p. 236.