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Rubrik: I. Importance of the ductility of steel for calculating and dimensioning steel structural work, especially when statically indeterminate

I

Importance of the ductility of steel for calculating and dimensioning steel structural work, especially when statically indeterminate.

Die Bedeutung der Zähigkeit des Stahles für die Berechnung und Bemessung von Stahlbauwerken, insbesondere von statisch unbestimmten Konstruktionen.

La ductilité de l'acier. Sa définition. Manière d'en tenir compte dans la conception et le calcul des ouvrages, notamment des ouvrages hyperstatiques.

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I

General Report.

Generalreferat.

Rapport Général.

Dr. Ing. L. Karner †,
Generalsekretär der I.V.B.H., Prof. an der Eidg. Techn. Hochschule, Zürich.

Economic construction demands that the available margin of safety should be fully utilised. It was formerly the practice to base the calculations of structures on the hypothesis of permissible stresses determined from the factor of safety which in turn was governed by the breaking stress, but this practice gave a completely false picture. In the course of the last ten years the yield point strength in the stress-strain diagram obtained by the tensile test has become of ever increasing importance in the evaluation of material and, therefore, in the determination of the degree of safety.

Experience on completed works has shown that under certain conditions stresses may be present which considerably exceed the permissible values, and which may even exceed the yield point strength, without the structure being thereby endangered. As examples one may cite the secondary stresses present in trussed lattice girders, the conditions of stress in riveted connections, the effects in continuous girders, attributable to settlement of the supports, etc.

It is well known that in hyperstatic systems stresses are allowed to exist under certain conditions which exceed the yield point stress without prejudice to the factor of safety assumed.

These observations call for a fundamental revision of the concept of the factor of safety in steel structures, for the yield point stress no longer constitutes an adequate criterion of the material to the designing engineer.

When a bar subjected to pure tension reaches the yield point stress its resistance is destroyed, except in so far as the elongation may be hindered through the material forming part of an internally hyperstatic system and the interplay of forces within this system being altered. In beams exposed to bending, or to bending combined with a normal force, it is possible for the load to increase, even if the deformation is not hindered, after the yield point stress is reached in the extreme fibres, because in such a case the stresses are of a hyperstatic nature.

Les us consider the case of a rectangular beam subjected to bending. According to the old hypothesis of plasticity the "permissible moment" is to be explained on the assumption of a regular yield which takes place in layers, until a completely plastic condition has been reached or until a plastic joint has been formed. In accordance with the new theory of plasticity, however, the matter is to be explained on the assumption of an increase in the yield point stress followed by abrupt and complete plastification over the whole section, or by the abrupt formation of a plastic joint.

From the point of view of the designer and engineer neither theory is more than an interesting explanation of the nature of a process of development devoid of importance so far as the final result is concerned — the latter being the formation of a plastic joint, and, therefore, a modification in the statical composition of the structure.

The plastic behaviour depends largely on the type of section, and particularly on the system of multi-axial stress. The phenomena which occur are very difficult to explain mathematically on the basis of our present knowledge.

In none of the large number of researches hitherto carried out in reference to the theory of plasticity has account been taken of the influence of time. It is likely enough, however, that a better explanation of the conditions which exist will be reached by considering the problem as being properly a dynamic one, that is by introducing a new variable "t".

It is a matter of common knowledge that the "yield point stress" as determined by the ordinary tensile test does not depend only on the shape of the section of the specimen, but also on the manner of execution of the test in relation to time.

The fact that the properties of ductility possessed by steel allow of its being dimensioned with greater economy in hyperstatic structures has exerted an influence on the German regulations, and on other regulations relating to structural steelwork. In the design of continuous girders it is, in fact, necessary to allow for the circumstance that an equalisation of the moments occurs over the supports and in the spans (though the allowance may perhaps not correspond exactly with the true conditions). *Kazinczy* in Hungary and *Kist* in Holland have suggested that the design of hyperstatic systems should be based on a new definition of the factor of safety.

A more exact understanding of the problem has only been rendered possible by the very full and interesting works carried out by *Grüning,* who was the first to suggest an analytical conception of the relationship with which we are concerned. *Grüning* confined his investigations at first to the action of a permanent super-load on the supporting system, but *Hans Bleich* has taken account also of differences in the arrangement of the load, and has introduced the idea of self-stress lines to serve as a basis for calculation based on plastic equilibrium ["carrying capacity method"].

In the papers before the I.A.B.S.E. great importance has been attached to the principle that the questions outlined above should be discussed only by persons fully competent to do so, in order that as adequate as possible a review of the problem might be obtained. The works by *Fritsche, Freudenthal* and *Rinagl* are concerned mainly with questions relating to the technical aspects of materials, while another group of writers, including *Melan, Kohl* and *Lévi,* have treated the problem from a theoretical point of view supported by an idealised stress-strain diagram.

It is clear that test results play a part of paramount importance in explaining questions of plasticity. *Maier-Leibnitz* in his paper refers to this subject, and bases his arguments regarding methods of calculation on tests. In a recent issue of the journal "Stahlbau", *Maier-Leibnitz* has described a series of more complete tests for determining the effective resistance of continuous girders and has arrived at simplified hypotheses for their interpretation which are likely to be very useful. Finally, there is a paper by *Bleich* in which the statical design of continuous girders and frames by reference to the theory of plasticity is explained.

The influence of plasticity on the dimensioning of structural steelwork is one which appears to be of great importance, in view of the urge towards greater economy in such structures without prejudice to their safety. This aim is capable of attainment, under certain limitations, by having recourse to the principle of plastic equilibrium [carrying capacity method] when dimonsioning hyperstatic systems. The limitations in question have reference, for instance, to frames and continuous girders, which according to the theory of elasticity are already fully utilised over the wohle of their sections, and which consequently possess a very small reserve of strength due to the transformation of the system which results from a plastic joint being introduced. The same is true of lattice girders, at any rate so far as these are at present understood, and in this instance account has also to be taken of the instability of elements stressed in compression.

So far we have tacitly confined our discussions to the subject of plasticity in girders and rigid structures, but the plastic behaviour of the material also plays a very important part in problems of unstable equilibrium, and from this point of view it is necessary to pay careful attention to what is already known. The theory of plasticity enables us to simplify the study of the stability of a bar, taking due account of the shape of its cross section and also of widely varying conditions of support. It will very soon be possible to establish practical methods of calculation for studying problems of the stability of bars and slabs on a greatly simplified basis.

Hitherto the methods followed in dimensioning the members of hyperstatic systems with regard to the plastic properties of the steel have usually taken no account of the contingency of breakage by fatigue effects, and neither the tests nor the experience at present available are sufficient to show how far, when alternating stresses have once exceeded the yield point stress and plastic joints have been formed, it is permissible to treat any subsequent excess of purely elastic stress in the same way as in structures where no local equalisation of moments occurs, from the point of view of fatigue.

Even if the design of hyperstatic systems be performed without reference to the theory of plasticity it remains true to say that the knowledge obtained from the solution of these problems will play a very important part in determining the form of structures. For instance, in the light of this knowledge the misgivings that have been entertained regarding the construction of continuous girders on account of the possibility of settlement of the supports would not appear to be justified. Economical structures of statically indeterminate nature are now possible where previously only statically determined structures could have been adopted in view of the risk of settlement of piers or the elasticity of supports.

Hitherto it has been customary to attribute too great an importance to secondary stresses, although *Engesser* showed more than forty years ago that such stresses are in fact diminished by the ductility of the steel. Examples in relation both to the resistance and the stability of metal structures might be multiplied indefinitely. We are, in fact, now in the middle of a revolution of our ideas as to the local critical stresses affecting structures, and we are demanding that the technique of testing materials shall yield better criteria, enabling the designer to dimension his structures with due regard to the degree of safety which is actually necessary. Our knowledge of these matters is continually increasing, and at many points its scope extends beyond the concepts hitherto entertained, but neither the new methods of calculation nor the new constructional ideas can do more than elucidate the details and peculiarities of any one element of structure.

A job as a whole is made up of a mass of details which act reciprocally upon one another, so that the complex problem which results is one which calls for all the attention of the engineer to resolve it. It behoves the designer worthy of his task to draw upon the new knowledge now available of the properties of materials, and upon the new methods of calculation based on the tests of the ductility of steel, in order to achieve steel structures which shall be at once reliable and economical.

I 1

Contribution to Discussion on Theory on Plasticity.

Diskussion über die Plastizität.

Discussion relative à la plasticité.

L. Baes,
Professeur à l'Université de Bruxelles.

I. *On the general theory of plasticity, the significance of yield lines, and the boundary of the elastic and plastic regions.*

1) Definitions of plasticity in general.

To avoid any risk of misinterpretation it may be well to recall that plastic strain is said to exist in parts of a body, or the material in question is said to be plastic, when the strain which there exists is not entirely of an elastic nature, and at the same time the cohesion of the material is not entirely destroyed, even though some change in the structural lattice may have taken place.

This definition is a general one, and the expressions "plastic strain" and "permanent deformation without destruction of cohesion" are, therefore, synonymous, the latter being in contradistinction to cases where permanent deformation is attended by cracking and an implication that the cohesion has been partially destroyed.

2) Definition of the phenomenon of plastic flow and of creep lines in mild steel.

One reason for the special interest which attaches to the study of plasticity is that it covers the existence of a very important property in mild steel: this metal, when tested in simple tension or compression, exhibits a very special kind of plastic creep. The phenomenon in question is such that when a particular intensity of stress is reached the longitudinal extension suddenly increases in an unmistakable manner. In an idealised form the phenomenon is represented by the horizontal interlude (in French, *palier,* "stair landing") in the stress-strain curve for tension or compression. It does not extend to a very large range of deformation, but its technical consequence is to bring about an appreciable amount of plastic strain, which follows upon a phase in which the elastic strains are very small.

This phenomenon and its results have, indeed, been the starting point of modern researches into plasticity, from which it is sought to profit by economising in the design of mild steel structures, the plastic effect serving as a valuable buffer against local increases of stress. It is also partly this phenomenon, idealised by the break in the stress-strain curve, which has given rise to the notion of perfect plasticity — a condition under which the deformation is assumed to increase while the stress remains constant. This conception implies the existence of single, duo-axial or tri-axial state of stress, and of the condition of perfect plasticity. Hence the justification, as well as the practical need, to develop a suitable hypothesis to account for this condition.

3) Significance of the creep lines which appear on the surface of mild steel pieces when plastically strained.

The author is inclined to accept the opinion of Messrs. Ititaro Takaba and Katumi Okuda as quoted in Paper N° 1 of this group: "The appearance of creep lines, and the sudden break in the stress-strain curve, are the result of one and the same phenomenon, that is to say, of the deformation in groups of a large number of crystalline grains".

There would appear, then, to exist a true discontinuity of the state of strain, which occurs in zones, and which appears to affect a whole region of the material instead of being localised. It appears plausible to suppose that this abrupt occurrence is due to a condition of molecular instability, comparable with the phenomenon of delayed deformation.

This would imply that at the moment when the change occurs there is also an alteration in the lattice structure, an alteration which is manifested in an appreciable amount of irreversible slip — in other words of plastic slip — which is attended by an increase in hardness through blocking of the slipped surfaces (see *Moser*).

It would appear manifest from the foregoing that the true boundary line between that portion of a body which has remained in the elastic condition and that portion which has changed to the plastic condition need not necessarily coincide with the lines of creep, for the latter appear to be related to the phenomenon of delayed action and to affect the whole of a zone. Where the portion of the body in question is subjected to a condition of simple stress the creep lines may be very wide apart, but where this condition does not obtain they are, on the contrary, often very close together.

4) Some characteristics of the creep lines.

Dr. *Freudenthal* states that the most important property of the creep lines appearing on the surface is that they coincide with the direction of maximum shear. The present writer would observe, however, that this is incorrect except in so far as the creep lines constitute a network made up of two sets of lines at right angles, and that other cases occur where these lines are merely one set of "slip crazings". Such cases are common enough but appear to have been overlooked, even though they are clearly indicated in the early descriptive paper by *Hartmann,* as well as in those by *Frémont,* and are easy to reproduce. In such cases the craze lines are evidently not associated with the direction of maximum shear, but on the contrary with that of one of the two principal stresses (isostatic lines). It may be observed, moreover, that such a creep line may disappear in the middle of its course, or if the stress increases it may spread while at the same time others are originated. Clearly, then, a creep line is not necessarily something which is originated once and for all, but it frequently happens that such a line may develop by successive steps as the load is increased.

All the evidence tends, then, to suggest that creep lines do not as a rule constitute boundary lines between the elastic and plastic regions. This is a conclusion which seems to follow from the fact that a line which does not make its appearance at one definite juncture, but which propagates itself as the stress increases, cannot be a boundary line: for if it were the latter it would, presumably, be a closed curve conforming in part to the shape of the material.

The conclusion reached by Dr. *Freudenthal* would seem, therefore, to be untenable: but to say this is very far from asserting that creep lines (particularly) where they very well defined) are devoid of mathematical interest.

5. The plastic condition.

In the case of metals which are capable of plastic deformation with or without a definite creep limit, the main hypotheses which have been put forward to account for the conditions that obtain at the boundary separating the regions of plastic and elastic strain are as follows[1]:

Hypothesis of Saint-Venant, Maurice Lévy, and Guest:

$$\tau_{max} \quad \text{or} \quad \frac{\sigma_I - \sigma_{III}}{2} = k = \frac{R_e}{2}$$

(R_e being the limit of elasticity for pure tension).

Hypothesis of Beltrami and Haigh: Here the criterion is the amount of specific energy involved in the elastic strain, and the condition to be satisfied may be written as follows:

$$(\sigma^2_I + \sigma^2_{II} + \sigma^2_{III}) - \frac{2}{m}(\sigma_{II} \cdot \sigma_{III} + \sigma_{III} \cdot \sigma_I + \sigma_I \cdot \sigma_{II}) = R_e^2$$

This may be represented in space as an ellipsoid, or in the case of duo-axial stress as an ellipse.

Hypothesis of von Mises and Hencky[2]: Here the criterion adopted is the value of the specific energy of change of shape by slip, and may be expressed as follows:

$$(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 = 2\,R_e^2 = 8\,k^2$$

or, in terms of the maximum tangential strains:

$$\tau^2_{I.\,II} + \tau^2_{II.\,III} + \tau^2_{III.\,I} = \frac{1}{2} R_e^2 = 2\,k^2$$

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[1] See *L. Baes:* Résistance des matériaux et éléments de la théorie de l'élasticité et de la plasticité des corps solides. Vol. 1, Chapter XI: «Le problème des critères de la résistance des matériaux». Brussels 1930—34.

[2] The French translation of *Dr. Freudenthal's* paper refers to: «3° Hypothèse de travail constant de déformation suivant la relation . . .». This is a dangerous way of putting the matter, for it should be made clear that only part of the strain energy is involved — that part implied by the expression "energy of change of shape due to slip". The qualification is all the more necessary because the expression is not well known in French.

This is represented in space by a cylinder of revolution, or in the case of duo-axial stress by an ellipse.

Hypothesis of von Mises and Hencky as modified by Huber: Here the criterion is the value of the specific energy of change of shape due to slip in so far as the cubical expansion, or the average stress

$$\frac{\sigma_{I} + \sigma_{II} + \sigma_{III}}{3}$$

are negative. If, on the contrary these quantities are positive, the criterion of Beltrami should be used instead. (It may be as well to draw attention here to the criterion of *Huber,* which is not identical with that of *von Mises* and *Hencky,* being a great deal more than the latter. This is often overlooked.)

Beltrami's criterion may be represented graphically as in Fig. 1 or in the case duo-axial stress it corresponds to a figure made up of two ellipses and differs little from that of *von Mises* and *Hencky.*

It is to be observed that the experiments now on record, notably those carried out by *Roš* and *Eichinger,* have shown that *Huber's* hypothesis is very satisfactory for mild steel or similar materials.

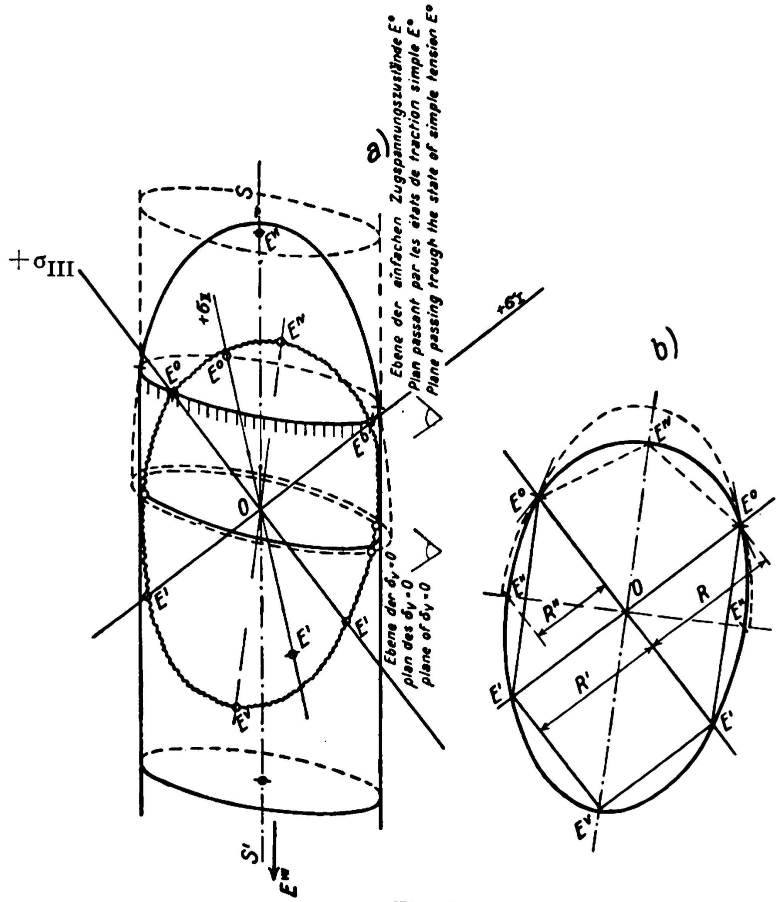


Fig. 1.

Huber's theory of limiting elastic equilibrium. (Graphical form).

a) Tri-axial stress conditions.

b) Duo-axial stress conditions.

It will now be expedient to consider two special cases which are of frequent occurrence:

Special case of deformation in a single plane, with perfect plasticity.

Here the plastic strain occurs in parallel planes, and if it be assumed that these are the planes of the principal stresses σ_{I} and σ_{III}, having perpendicular to them the plane of σ_{II}, then for perfect plasticity we shall have at all points

$$\sigma_{II} = \frac{\sigma_{I} + \sigma_{III}}{2}$$

whence

$$\sigma_{I} < \sigma_{II} < \sigma_{III}.$$

The conditions of plasticity between stresses, according to *St. Venant* and *Maurice Levy,* will then be expressible as follows:

$$\sigma_I - \sigma_{III} = 2\,k = R_e$$

The condition of plasticity according to *von Mises* and *Hencky* will be

$$(\sigma_I - \sigma_{III}) = \frac{4}{\sqrt{3}}\,k = \frac{2}{\sqrt{3}}\,R_e.$$

It will be noticed that these two conditions agree in a coefficient $\frac{2}{\sqrt{3}}$ and correspond to a particular value of τ_{max}. It follows from the fact that σ_{II} is intermediate between σ_I and σ_{III} that the facets at which plastic slip occurs will be those perpendicular to the plane I, III.

The slip surfaces are cylinders of which the axes are normal to this plane; the slip occurs parallel to this plane and is marked in the latter by two conjugate families of slip lines forming an orthogonal network, bisecting that of the isostatic network. Along these lines, as and when they occur, the tangential stress reaches its critical value.

Particular case of stress in one plane, or of duo-axial stress with perfect plasticity.

This case is of very frequent occurrence on the surface of pieces. One of the principal stresses is zero, say $\sigma_{II} = 0$, and if σ_I and σ_{III} are of contrary sign the condition of plasticity according to *St. Venant* and *Maurice Levy* is

$$\tau_{max} = \frac{\sigma_I - \sigma_{III}}{2} = \pm\,k = \pm\frac{R_e}{2}.$$

If, however σ_I and σ_{III} are of the same sign, the condition becomes σ_I or $\sigma_{III} = R_e$.

The condition of plasticity according to *von Mises* and *Hencky* is then

$$\sigma_I^2 - \sigma_I \cdot \sigma_{III} + \sigma_{III}^2 = 4\,k^2 = R_e^2$$

and in Cartesian co-ordinates σ_I, σ_{III} is represented by an ellipse.

It will be seen that where the principal stresses σ_I and σ_{II} are of different signs there is scarcely any numerical difference between the conditions of *St. Venant* and *von Mises.* The two conditions are not, however, proportional, as was the case for plane deformation. In the case where the two principal stresses σ_I and σ_{II} are of contrary sign, as in Fig. 2, there is formed a network of slip lines. According to the hypothesis of *St. Venant* this network bisects the network of isostatic lines, and at any point in such a line the corresponding τ_{max} reaches its critical value at the moment when slip occurs.

According to the hypothesis of von Mises, also, there is a network formed by two families of lines, but along the direction of slips in this network τ_{max} reaches no definite value, for it is the critical value that is obtained, and its amount is not based on a definite value of τ_{max}.

Where the two principal stresses σ_I and σ_{III} are of the same sign, as in Fig. 4, there no longer occurs any formation of a network of two families of lines at right angles, the slip faces not being normal to the free surface according to either of the hypothesis, and there is merely formed a single family of slip crazings common to the two groups of slip faces (Fig. 5 and 6).

According to the hypothesis of *St. Venant*, every element in these crazings coincides, at the moment of its formation, with the element of the isostatic line

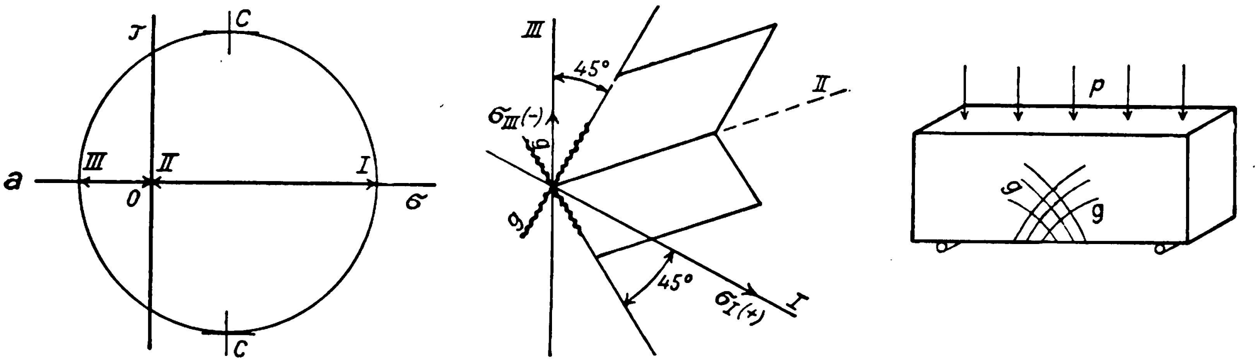


Fig. 2.

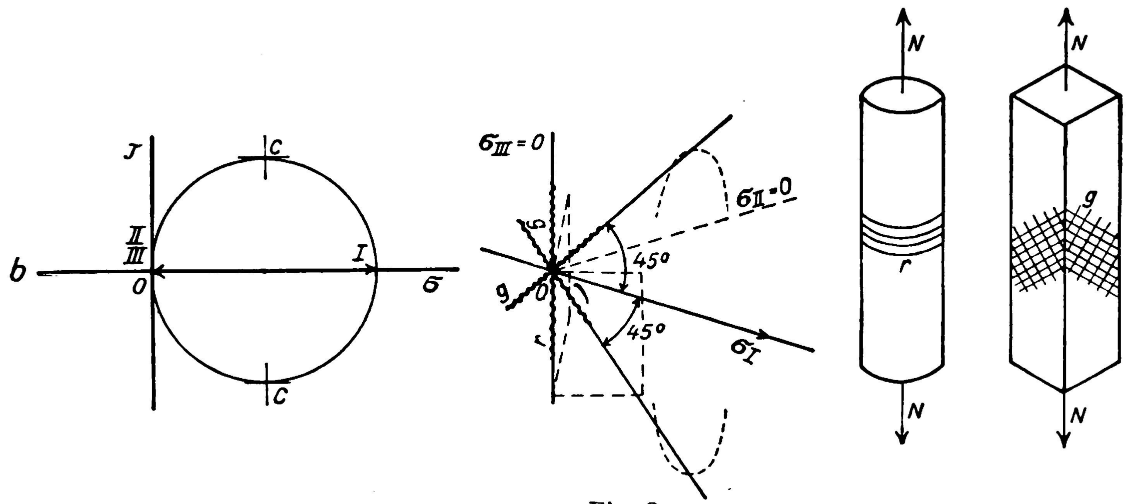


Fig. 3.

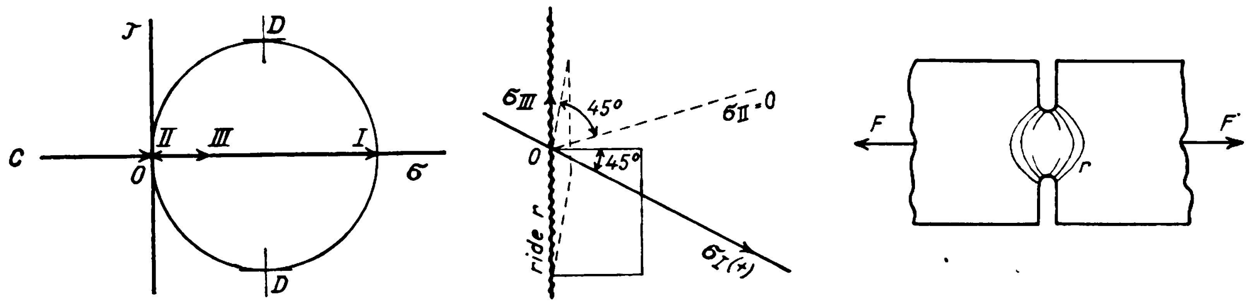


Fig. 4.

Figs. 2—4. Appearance of slip networks g or lines r on the outer faces of parts subjected to uni-planar stress.

Case a: $\sigma_{II} = 0$; σ_I and σ_{III} of opposite sign,

Case b: $\sigma_{II} = \sigma_{III} = 0$,

Case c: $\sigma_{II} = 0$; σ_I and σ_{III} of same sign.

which corresponds to the smallest absolute value of principal stress at that moment and in that place, and the maximum principal stress attains double the value of the critical tangential stress.

In the old papers by *Hartmann,* the distinction between the crazings and the slip surfaces is clearly apparent — see Fig. 7 — though *Hartmann* has not put forward an explanation of this difference.

According to the hypothesis of *von Mises* the stress attained at a given point in a craze line at the moment of its formation corresponds to the critical condition, but this is not altogether simple to understand; indeed the circumstances in which the crazing occurs would appear to have been overlooked, though the phenomenon is one of fairly frequent occurrence and the problem to which it gives rise is then altogether different from that of the formation of the network. It is a condition which often arises in flat pieces with lateral notches. Fig. 6.

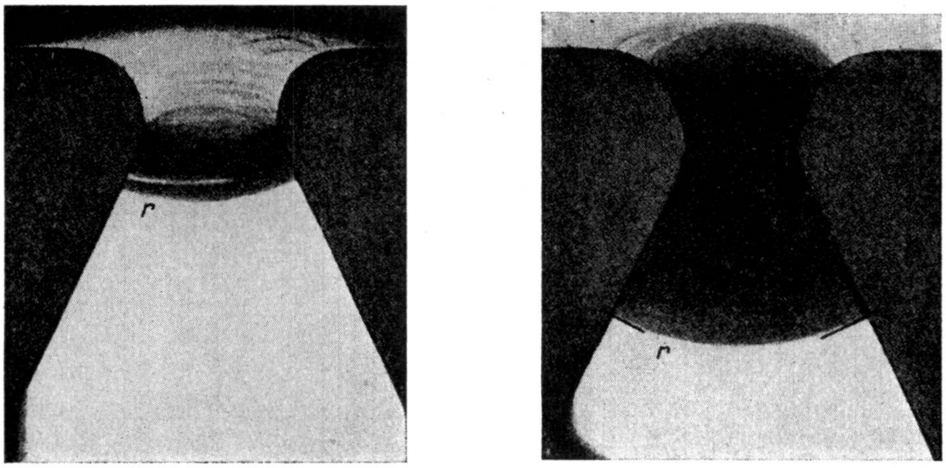


Fig. 5.

Sample of plate subjected to tension, showing the gradual development of slip lines r. (From "Mesure de la limite élastique des métaux", Ch. Frémont, 1903.)

Where only one of the principal stresses differs from zero, there is a theoretical possibility of the formation of the network of slip lines g or of the family of craze lines r (Fig. 3). In practice it is probably the network of slip lines g which will be formed, and so far as this is concerned the two stress hypotheses of *St. Venant* and *von Mises* are evidently identical.

Where a solid mass is in question, however, the hypothesis of *St. Venant* leads to the assumption of slip surfaces for all those elements which at the moment of slip have τ_{max} equal to the critical value; on the other hand by the hypothesis of *von Mises* no simple connection exists between the critical conditions and the maximum tangential stress.

6) Boundary between the region which has remained elastic and the region which has become plastic.

Dr. Freudenthal writes as follows:

"The acceptance of the slip lines as the boundaries of the plastic regions, and the development of solutions depending on properties of the slip lines themselves, will always lead to results which are not in agreement with reality."

The second part of this may be accepted with reserve, but the truth of the first part is evident in a general way. Generally speaking, inaccuracy must result from confusing the boundary of the plastic region with a slip surface, for it is evident that along the slip surfaces *within* the plastic zone the condition of plasticity is satisfied as it is on the actual boundary; actually at the boundary, however, it has to be reconciled with an elastic condition.

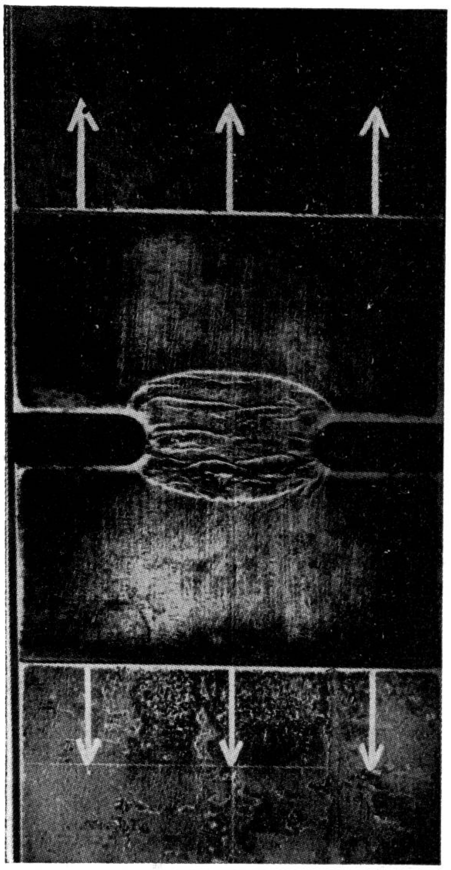


Fig. 6.
Slip bands in a notched piece of mild steel.

Generally speaking the boundary surface is not formed by any one slip surface, but by points on a number of slip surfaces. It is, therefore, inaccurate to assert generally "as is clearly indicated by all the observations, the shapes of these curves has nothing in common with the creep lines themselves, *but the curves correspond equally well with conditions of plastic or of elastic strain*".

The boundary surface must evidently be defined as a surface in the elastic state, wherein the function taken as a criterion is constant. In the case of a plane surface which is stressed in that plane, the bounding line corresponds with an isochromatic line in photo-elastic experiments, and this is true whether the criterion of *St. Venant* or that of *Von Mises* be applied. Where the same piece is stressed in another plane the boundary surface is not isochromatic except in accordance with the hypothesis of *St. Venant,* when the tensions σ_{I} and σ_{III} are of contrary sign, or when one of them is not equal to zero.

If the same piece, is subjected to two plane stresses of the same sign, then, in accordance with the hypothesis of *St. Venant,* the boundary line is a curve of equal value for the maximum principal stress, and is not an isochromatic curve as obtained in photo-elastic experiments.

According to the hypothesis of *Von Mises* in this condition of stress the boundary line is not an isochromatic curve. *The importance of distinguishing between the case of duo-axial stress and plane deformation will thus be apparent,* and this is the upshot of the author's present remarks.

In order to show that the boundary line is not, generally speaking, a slip line, it is merely necessary to mention two simple cases which are well known. The first of these is a thick cylindrical envelope subjected to a large difference of pressure, where on account of the axial symmetry the boundary surface between the plastic and the elastic regions is a cylinder concentric with the tube itself, whereas in each cross section the trace of the surfaces of slips are logarithmic spirals, there is nothing in common between these forms.

The second example is that of a plane slab which is stressed on its edge by the application of a load which is nearly concentrated. Here the lines of slip on the side of the slab are logarithmic spirals. The boundary line is an isochromatic line, that is to say a circumference which has its centre on the line of load and which is tangential to the boundary line of the piece. Yet another typical case is that of a circular disc loaded by two diametrically opposed loads.

It is manifest, then, that the boundary line between the plastic and elastic regions is not generally coincident with a slip line. The present writer is of an opinion that it may be useful to bring this point out more clearly and more simply than has been done by the author of the paper in question. But that is well established.

There are numerous and important questions which remain to be elucidated in the field of plasticity, for the present existing theories are no more than a simple outline needing to be filled in. It is apparent also from the point of view specially treated in Paper N° I 2 by *Dr. J. Fritsche,* that the condition of plastic creep is a function not of the local state of stress, but of the *stress prevailing over a whole region.* This new kind of creep is a very interesting one, and involves new features which are supported by undeniable experimental facts. Indeed the present writer has had an intuition of these ever since he undertook the tests to destruction on joists encased in concrete.

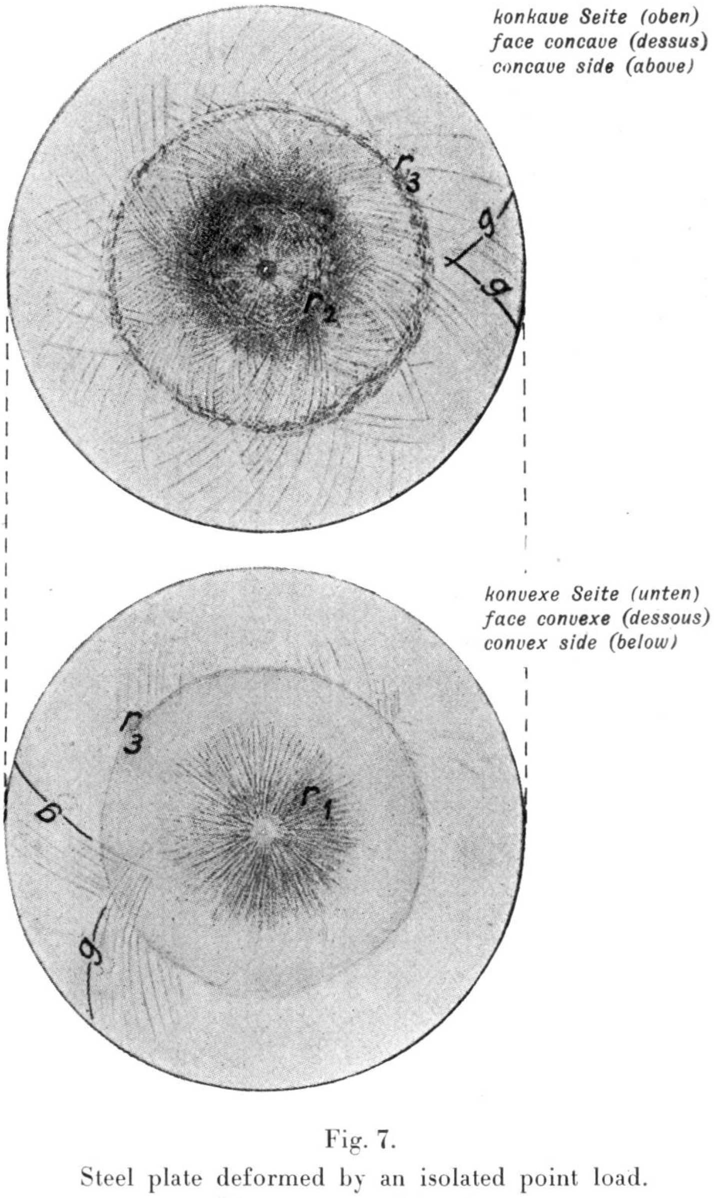


Fig. 7.
Steel plate deformed by an isolated point load.
Slip lines g and bands r.
(From „Distribution des déformations dans les métaux soumis à des efforts“ L. Hartmann, 1896.)

II. Application to the design of steel structures.

The writer wishes to record his complete agreement with *Dr. F. Bleich,* who draws attention to the necessity for the exercise of great prudence in applying these principles to structures at the present time, and has rightly urged that the new method of calculation must not be applied to systems in which account has to be taken of the fatigue resistance of the material. For the present it should be applied only to simple systems which are hyperstatic to no more than a very limited degree, consisting of members subject to bending, where the part in compression cannot become uncased; it is applicable only to those girder constructions or building frameworks which are not subject to repeated loading or to vibration by machinery.

It would further be a wise precaution, when designing structures on the hypothesis of plastic equilibrium, to work with stresses such that the creep stress will not actually be reached, so that the adjustability due to plasticity will be held in reserve.

I 2

Combined Bending and Shear Beyond the Range of Purely Elastic Deformation.

Biegung mit Querkraft, außerhalb des Gebietes der rein elastischen Formänderung.

Flexion et effort tranchant en dehors de la zone de déformation purement élastique.

A. Eichinger,

Dipl.-Ingenieur, Wissenschaftlicher Mitarbeiter der E.M.P.A. Zürich.

Given the stress strain diagram for simple tension or compression it is possible, assuming that cross sections remain plane, to determine the distribution of stress over the cross section of a beam which is subjected to a shear force simultaneously with bending. An example of this will now be given.

Introduction.

It is known that the total deformation is capable of subdivision into two parts.

a) An elastic strain made up of components which satisfy the following equations of elasticity:

$$e_1 = \frac{1}{E} \cdot \left[\sigma_1 - \frac{1}{m}(\sigma_2 + \sigma_3)\right]; \quad e_2 = \text{etc., and}$$

b) A plastic strain made up of components which satisfy the equation of plasticity:

$$\delta_1 = \frac{1}{D} \cdot \left[\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right]; \quad \delta_2 = \text{etc.}$$

wherein E represents Young's modulus and D the modulus of plasticity. In the case of plastic deformations the coefficient of transverse strain m has the value 2.

Hitherto it has been customary to base all statical calculations on the assumption that the behaviour of the structure is purely elastic, but recently attempts have been made to take account also of the influence of plastic deformation on: — 1) The distribution of stress in a beam or over its cross section, and 2) The flow of forces in the structure as a whole (statically indeterminate quantities M, Q and N).

[1] *M. Roš* and *A. Eichinger:* Versuche zur Klärung der Frage der Bruchgefahr. Reports on Discussions, Swiss Federal Testing Station, Zürich. N°. 14 of Sept 1926; N°. 34 of Feb. 1929; N°. 87 of Apr. 1934.

Principles of the theory of plasticity.

It may be recalled[2] that in the case of simultaneous action of a normal and a shear stress the following equations apply:

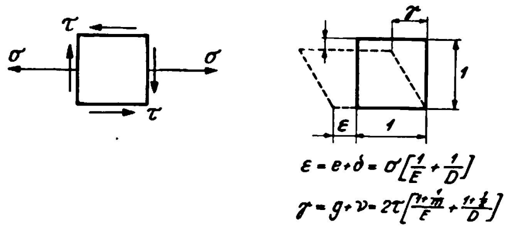


Fig. 1.
State of stress and strain in an element of a body.

for elastic deformation (strain of comparison)

$$e_g = \sqrt{e^2 \cdot \left(1 + \frac{1}{m}\right)^2 + \frac{3}{4} \cdot g^2} = \frac{\sigma_g}{E} \cdot \left(1 + \frac{1}{m}\right);$$

for plastic deformation

$$\delta_g = \sqrt{\delta^2 \cdot \left(1 + \frac{1}{2}\right)^2 + \frac{3}{4} \nu^2} = \frac{\sigma_g}{D} \cdot \left(1 + \frac{1}{2}\right)$$

wherein $\sigma_g = \sqrt{\sigma^2 + 3 \cdot \tau^2}$
denotes the stress of comparison. In such a case the elongation is expressed by

$$\text{elastic} \quad e = \frac{\sigma}{E}$$

$$\text{plastic} \quad \delta = \frac{\sigma}{D}$$

and the specific value of the slip is given by

$$\text{elastic} \quad g = \frac{\tau}{E} \cdot 2 \cdot \left(1 + \frac{1}{m}\right)$$

$$\text{plastic} \quad \nu = \frac{\tau}{D} \cdot 2 \cdot \left(1 + \frac{1}{2}\right).$$

The total deformation is equal to the sum of elastic and plastic deformation, namely,

deformation (strain of comparison) . $\varepsilon_g = e_g + \delta_g$
elongation . . . . $\varepsilon = e + \delta$ and
slip . . . . $\gamma = g + \nu$

If, then, the stress-strain diagram for a particular material subject to ordinary tension or compression is known, it becomes possible by the above formulae to arrive at the fundamental relationship between the stress of comparison and

[2] Discussion by *M. Roš* and *A. Eichinger:* The buckling of rectangular slabs compressed in excess of the elastic limit symmetrically along both axes. Final Report of the First Congress, I.A.B.S.E.

the deformation (strain of comparison) depending on this alone (changes in volume being always of an elastic character), see Fig. 2.

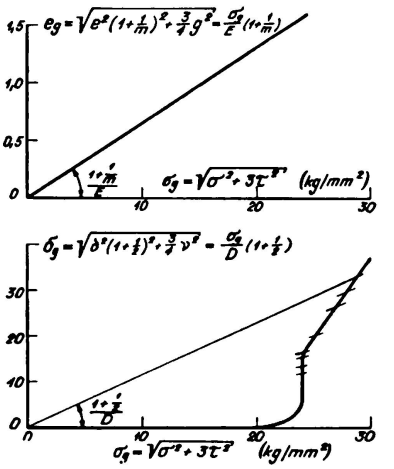


Fig. 2.
Relation of elastic and plastic strain of comparison to the stress of comparison.

Distribution of stress over the cross section of the beam.

Before attempting to estimate the influence of the statically indeterminate quantities M, Q and N upon the flow of forces within the structure it is necessary to determine the effect of plastic deformation on the distribution of stresses over the cross sectional area.

Assuming that the cross section remains plane throughout (or in more accurate language, assuming that the elongation follows a linear law) the elongation undergone by a fibre at any given distance y from the neutral axis is expressed by

$$\varepsilon = \varepsilon_r \cdot \frac{y}{h/2}$$

where ε_r denotes the elongation of the extreme fibre (Fig. 3). Since, moreover, the shear stress and therefore the amount of slip that occurs at the extreme fibre must be zero, it becomes possible to determine the extreme fibre stress σ_r by reference to Fig. 2.

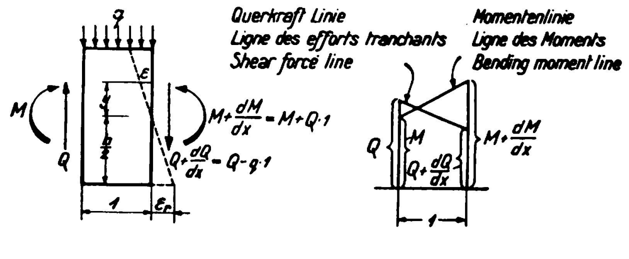


Fig. 3.
Element of a beam of unit length 1 under the influence of M, Q and q.

The distribution of normal stress over the cross sectional area is usally assumed to occur from $\sigma = 0$ to $\sigma = \sigma_r$ by analogy with a portion of the $\sigma - \varepsilon$ diagram. This assumption, however, is only justified if $\tau = 0$ throughout. If this condition is not satisfied the distribution of the normal stresses over the cross section may be very different, for the total elongation ε would be produced by a smaller normal stress σ according to the magnitude of the shear stress τ acting on the same element of material.

For the present, instead of attempting to determine the distribution of σ and τ for a given bending moment and shear force, let us be content with the σ distribution as in Fig. 4. On this assumption τ can be determined for every point in the cross section, since

$$\varepsilon = \varepsilon_r \cdot \frac{y}{h/2} = \sigma \cdot \left[\frac{1}{E} + \frac{1}{D}\right]$$

whence it follows that:

$$\frac{1}{D} = \frac{\varepsilon}{\sigma} - \frac{1}{E}.$$

In other words it is possible to determine the modulus of plasticity D in respect of every value of y and by drawing a line at an angle of $\frac{1+\frac{1}{2}}{D}$ through the origin of the coordinates as far as the σ_g—δ_g curve in Fig. 2 we can find the stress of comparison σ_g and hence determine the required shear stress from the formula

$$\tau = \sqrt{\frac{\sigma_g^2 - \sigma^2}{3}}.$$

Since the relation

$$\frac{\partial \tau}{\partial y} = \frac{\partial \sigma}{\partial x}$$

applies in the case of sectional elements of constant width the distribution of stresses σ' and τ' in a neighbouring section is also known, as in Fig. 4. It follows from this that the distribution of normal stresses σ over a cross section does not

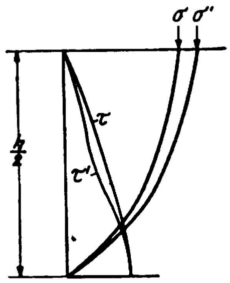


Fig. 4.
Distribution of stress in the upper half of cross section.

here depend entirely on the moment M which exists in this section, as is true for elastic deformation, but is influenced also by the shear force Q and by the distributed load, $q = \frac{dQ}{dx}$.

Strictly speaking, these considerations are valid only on a single occasion of stressing in excess of the limit of purely elastic behaviour. They are, therefore, of little practical importance. Whereas the first time that the elastic limit is exceeded a large amount of plastic deformation takes place at once, the material may fracture through fatigue even without any visible sign of permanent deformation.

It should also be noticed that in spite of the implied change in the upper limit of load or stress in the most heavily stressed member, which is attributable to plastic deformation, no change occurs in the amplitude of the range of loading (B—A) even at the critical points considered (where B represents the upper and A the lower limits of load). Since, however, in most forms of construction the fatigue strength depends mainly on the amplitude of the alternating stresses, and only slightly on the magnitude of the basic stress $\frac{A+B}{2}$, the advantage actually gained is much smaller than might have been expected from the reduction in the upper limit of stress as indicated in Fig. 5. For these

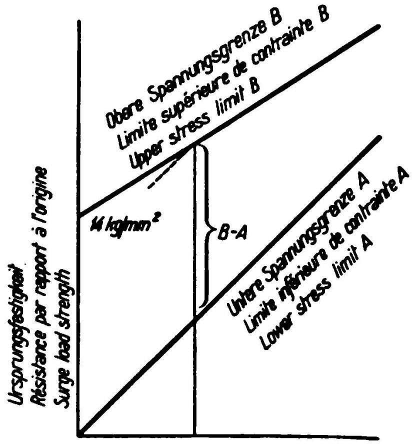


Fig. 5.
Tensile fatigue tests for butt welds. (Swiss Federal Testing Station, Zürich).

reasons the plastic deformation should be kept in reserve against accidental overloading in all cases where fatigue effects are likely to be present, the only exception being where it is due to buckling. Calculations for estimating the factor of safety against fatigue should continue to be based on the principles of elasticity, and exceptions to this rule should be admitted only where they are supported by endurance and fatigue tests as distinct from short period tests.

$\frac{y}{h/2}$	σ	ε	$\frac{\varepsilon}{\sigma}$	$\frac{1}{D}\cdot 10^3$	σ_g	τ	$\frac{\partial\tau}{\partial y}$	σ'	ε'	$\frac{\varepsilon'}{\sigma'}$	$\frac{1}{D}\cdot 10^3$	σ'_g	τ'
	kg/mm²	‰	·10³	mm²/kg	kg/mm²	kg/mm²	kg/mm³	kg/mm²	‰	10³	mm²/kg	kg/mm²	kg/mm²
1.0	25.0	14.58	0.584	0.534	25.0	0	0.40	29.0	23.5	0.810	0.760	29.0	0
0.8	23.2	11.66	0.503	0.453	24.2	4.0	0.35	26.7	18.8	0.705	0.655	27.1	2.7
0.6	20.8	8.75	0.421	0.371	24.0	6.9	0.29	23.7	14.1	0.595	0.545	25.1	4.8
0.4	17.0	5.84	0.343	0.293	24.0	9.8	0.28	19.8	9.4	0.475	0.425	24.1	7.9
0.2	10.5	2.92	0.278	0.228	24.0	12.5	0.20	12.5	4.7	0.376	0.326	24.0	11.8
0	0	0	—	—	—	13.8	0	0	0	—	—	—	13.8

Note: The section σ'—τ' is at a distance $\frac{h}{10}$ from the section σ—τ. See Fig. 4.

I 3

Observations on Ductility.

Betrachtungen über die Zähigkeit.

Considérations sur la ductilité.

Professor Dr. Ing. W. Kuntze,
Staatliches Materialprüfungsamt Berlin-Dahlem.

In the field of mechanical engineering, endeavours are now being made to construct heavily stressed shafts (such as crankshafts) which are subject to heavy stresses in cast iron instead of in high-ductility special steels, as the former is cheaper in first cost and gives almost equal performance. This analogy with a field of construction adjoining our own serves to illustrate the fact that we now attach a different meaning to the conception of ductility in steels than we formerly did, the crux of the matter being not the *gross* amount of plasticity but the *capacity to resist non-uniform conditions of stress.*

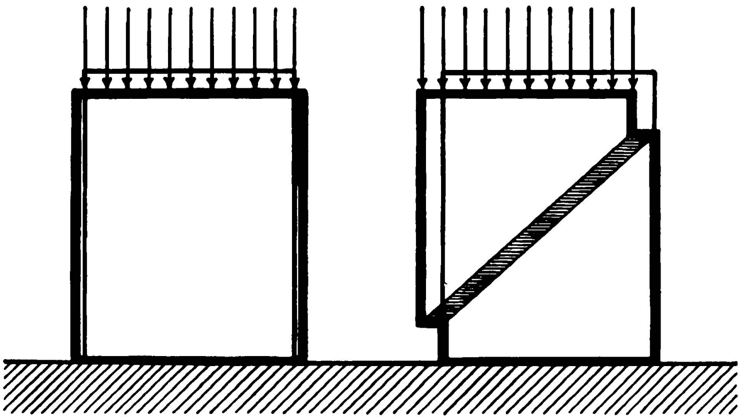


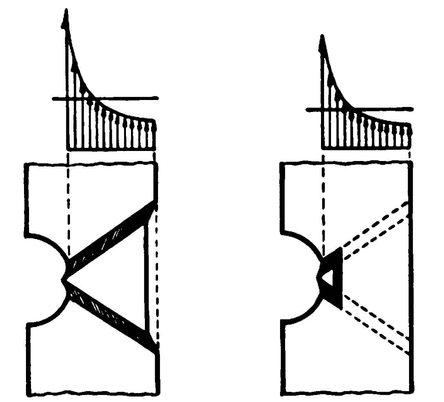
Fig. 1.
Elastic and plastic deformation.

Provided that the structural cohesion is high this capacity is attainable even though the plasticity of structure may be relatively small and only just sufficient. The mechanism of plastic deformation differs from that of elastic deformation as indicated in Fig. 1 through the spontaneous occurrence of layers of yield and rotation. By reason of their kinetic origin these yield layers are fundamentally insensitive, in the statical sense, to variations of stress.[1] They depend for their occurrence on an additive force, which may be determined from the conditions of equilibrium of vectors and is known as the "resisting medium"

[1] *W. Kuntze:* Einfluß ungleichförmig verteilter Spannungen auf die Festigkeit von Werkstoffen. Maschinenelemente-Tagung Aachen. Berlin, V.D.I.-Verlag, 1936.

(Widerstandsmittel).[2] On this basis *Fritsche* has made successful calculations for beams, and has carried the idea further in order to arrive at the carrying capacity of eccentrically loaded columns.[3] [4]

Our structural steels do not, however, actually behave in the ideal way represented in the diagram. It is true that the formation of a slip plane implies a change from purely elastic behaviour, and therefore a danger of brittle fracture, but in our commercial steels it is impossible entirely to eliminate the possibility of *internal microscopical breakdown.* Local microscopic cracks are a phenomenon which, according to the quality of the material, is apt in any case to accompany



sate against notching action   sensitive against notching action

Fig. 2.
Sensibility against notching action for alternating stresses.

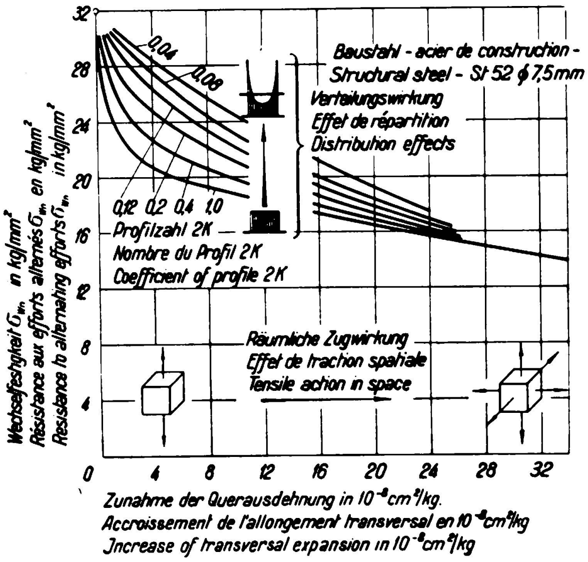


Fig. 3.
Relation between the resistance to alternating stresses and the stress distribution and multidimensional stressing.[5]

plastic deformation and particularly so in the presence of non-uniform stress. Such cracks cause a shortening of the yield paths corresponding to Fig. 2, and this is associated with a lower carrying capacity which is designated in practice as "notch sensitiveness". The ideal condition of insensitivity to notch effect, shown in the left-hand figure, is characterised on the contrary by continuous yield surfaces. Consequently the test results obtained in experiments made under non-uniform conditions of stress usually work out lower than those which would correspond to the carrying capacities calculated on the basis of an ideal resisting medium.

The tendency of materials to internal fracture must, therefore, limit the accuracy of calculations made by reference to the "resisting medium". What,

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[2] *W. Kuntze:* Ermittlung des Einflusses ungleichförmiger Spannungen und Querschnitte auf die Streckgrenze. „Der Stahlbau", Vol. 6 (1933), pp. 49/52.

[3] *J. Fritsche:* Grundsätzliches zur Plastizitätstheorie. „Der Stahlbau", Vol. 9 (1936), pp. 65/68.

[4] *J. Fritsche:* Der Einfluß der Querschnittsform auf die Tragfähigkeit außermittig gedrückter Stahlstützen. „Der Stahlbau", Vol. 9 (1936), pp. 90/96.

[5] *W. Kuntze:* Einfluß des durch die Gestalt erzeugten Spannungszustandes auf die Biegewechselfestigkeit. Arch. Eisenhüttenwesen 10 (1936/37) S. 369/73; Ber. Nr. 367 Werkstoffaussch. Ver. dtsch. Eisenhüttenl.

from this point of view, are the special cases liable to arise, and what is the nature of the forces tending to promote premature brittleness?

On relating the results of changes in the notch effects to the three-dimensional conditions of tensile stress, and to those of the distribution of stress as in Fig. 3,

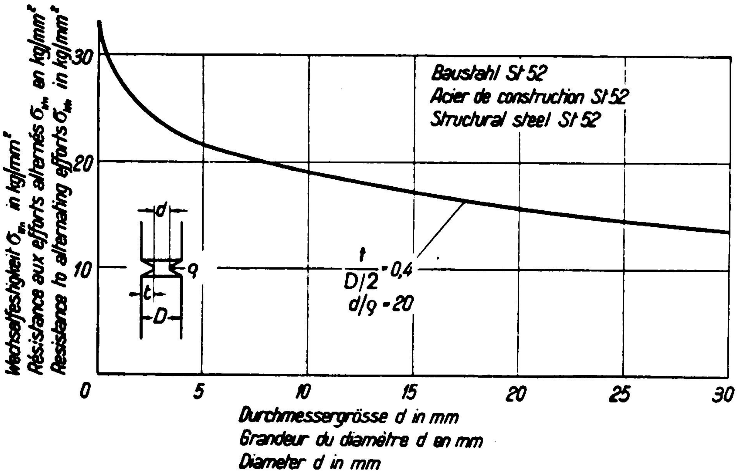

Fig. 4.
Resistance to alternating stresses in relation to size (diameter).

it will be seen that *it is not primarily the concentrations of stress but the multi-dimensional condition of stress which causes a reduction in the alternating fatigue strengths.* Indeed in the present series of experiments those samples which showed high concentrations of stress gave a higher notch fatigue resistance than the specimens under the same average three-dimensional conditions in which the

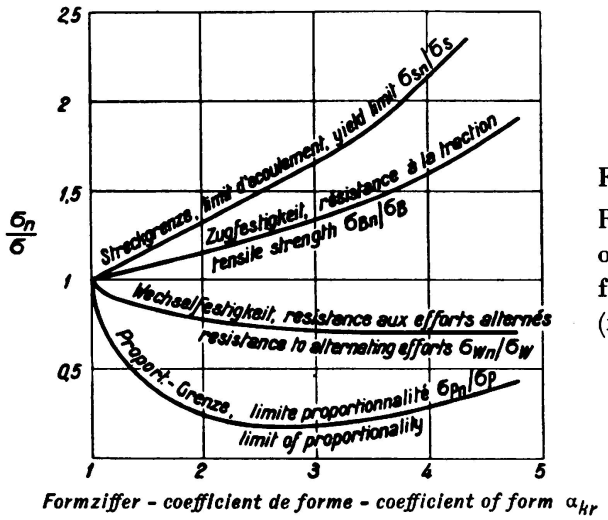

Fig. 5.
Relative increase or decrease of different limits of strength for increasing peak stress (form coefficient α_{kr}).

distribution was more uniform. These are actual experimental results, which cannot be set aside.

The reducing effect, on the alternating fatigue strength, exerted by the multi-dimensional condition of stress depends also on the absolute dimensions of the constructional member in question. Fig. 4 shows how if the depth of notch $\frac{t}{D/2}$ and its sharpness $\frac{d}{\rho}$ are maintained proportional then *the fatigue strength drops off steadily as the diameter of the specimen is increased.*

The *limit of proportionality* is another factor which greatly influences the structural cohesion,[1] its effect in the case of non-uniform stress being similar to that exerted by the alternating fatigue strength. But as regards the *yield point* (resistance to slipping) the existence of multi-dimensional stress has the opposite effect, this causing an increase whereas a concentration of stress results in a decrease, especially if the dimensions are large. Fig. 5.

A clue to these partly contradictory effects may be sought in the fact that frequently contradictory measurements are reported.

Multi-dimensional stresses occur in the construction (1) as the result of its external arrangement, and (2) through shrinkage in welded connections. Under what conditions do these two factors operate detrimentally?

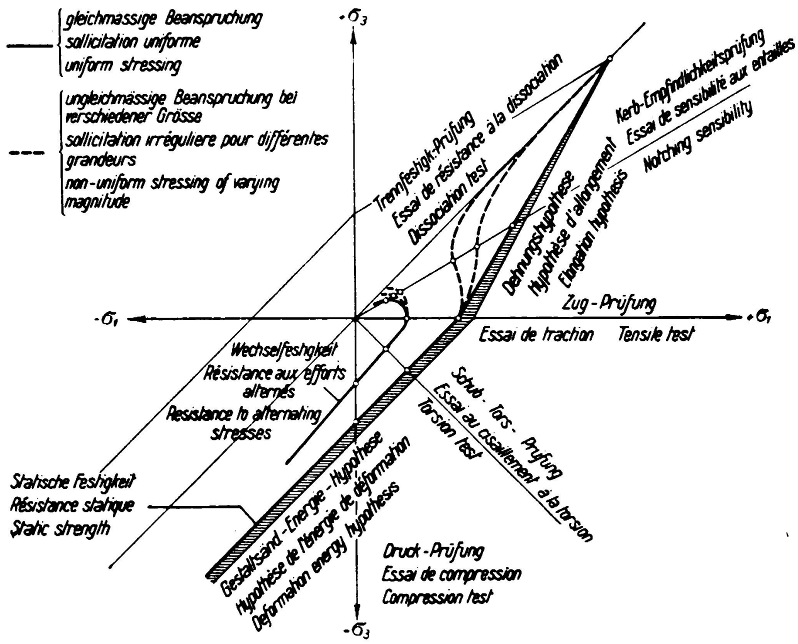

Fig. 6.

Arrangement for multi-axial resistance test.

If, for instance, at some point a multi-dimensional condition of stress is present as the result of shrinkage, compressive stresses will be present in the immediate neighbourhood since within the constructional member equilibrium must exist. By applying the law of the "resisting medium", which implies an average effect, it must be anticipated that in such a case the yield point will not be appreciably affected. *But if the dimensions are large the statical strength will be somewhat reduced as a result of the concentration of stress, and the alternating fatigue strength will be considerably reduced in consequence of the multi-dimensional condition of tensile stress.*

The effect represented here varies in magnitude according to the material, and this is due to the *mode of development of the testing of materials.* The classical tests for compressive, shear and tensile strength under statical and alternating stress are represented in the four quadrants of Fig. 6 showing the maximum tensile principal stress and the maximum compressive principal stress.

The new tests for "splitting strength" *(Trennfestigkeit)* and notch sensitivity, under statical or alternating loads, appear in the purely tensile quadrants,[6] and with the aid of these the material in question may be assessed as regards its behaviour under multi-dimensional tensile stresses in different sizes of member. The results of the test, then, provide an approximate measure as to how far, in the case of any given material, the calculated values will fail to be attained on account of the operation of the "resisting medium". In this way it becomes possible to introduce the factor of proportionality, suggested by *Klöppel*[7] as a means of correction.

These results of recent research may serve as directives for calculation and design, but they do not touch upon the question of how far in bridge work — especially in the case of statically indeterminate systems — the changes in the loads imposed operate as true alternating fatigue stresses in the sense that the term is used in the testing of materials. This remains a special problem of bridge engineering which must always be borne in mind.

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[6] *W. Kuntze:* Kohäsionsfestigkeit. Berlin 1932, J. Springer. Also Spezial N°. XX of Mitt. deutscher Materialprüfungsanstalten. (Methods of testing cohesion have meanwhile been further developed.)

[7] *K. Klöppel:* Gemeinschaftsversuche zur Bestimmung der Schwellfestigkeit voller, gelochter und genieteter Stäbe aus St. 37 und St. 52. „Der Stahlbau", Vol. 9 (1936), pp. 97/119.

I 4

The Ductility of Steel; the Effect of Rapidly Imposed and Repeated Loading.

Die Zähigkeit des Stahles, die Wirkung der raschen und der wiederholten Beanspruchungen.

La ductilité de l'acier, l'action des efforts rapides et des efforts répétés.

R. L'Hermite,

Directeur adjoint des Laboratoires du Bâtiment et des Travaux Publics, Paris.

Most of the works which have dealt with the ductility of steel and with the study of its consequences on the safety of structures have failed to throw any clear light on the influence of the time factor. There can be no doubt that this factor plays a predominant part, especially where the external effects operate rapidly; that is to say when the rate of the application of load and the velocity of deformation are high. This is true of dynamic loads, for which no equalisation of stresses takes place as a rule.

The relative deformation of two points in a solid body subjected to the action of a force F is the sum of a deformation which disappears more or less rapidly with F (known as the elastic viscous strain) and of a second kind of deformation which is permanent (known as plastic strain). This circumstance has led to the conception of "inherited action", introduced into physics by Volterra. In this special case it may be said that the application of an elementary force d F does not immediately produce the full amplitude of deformation, both elastic and plastic; there is a delay or reactivity in the occurrence of the deformation both when the load is applied and when it is removed, and the consequence of this reactivity is a residual deformation which is multiplied by a "heredity factor" Φ, the latter being a function of such a kind that its value decreases indefinitely with time. Under these conditions the expression for elastic viscous deformation is as follows:

$$x(t) = \int_0^t M[(t-z), F]\, N(F) \cdot \frac{dF}{dz}\, dz$$

The expression for plastic deformation is as follows:

$$x'(t) = \int_0^t \mathfrak{M}[(t-z), F]\, \mathfrak{N}(F) \cdot \frac{dF}{dz}\, dz$$

The first of these expressions applies to all cases of the imposition and removal of load, the second only to the case where $\frac{dF}{dz}$ is positive. In the case, for instance, of repeated loading, the plastic strain arises during the first imposition of load; for a first approximation it does not enter into account except as an initial constant.

The first approximate calculation gives for M (t) the expression $M = 1 - e^{-\lambda t}$, and for N a function which depends on the nature of the solid in question. In the same way we have

$$\mathfrak{M} = \alpha - \beta\, e^{-\mu t}.$$

We may thus recognise a number of expressions which are in current use:

Plastic flow under constant load:

$$x'(t) = [\alpha t + \beta (1 - e^{-\mu t})]\, \sigma(F).$$

(This formula agrees exactly with that obtained in experiment by Prof. *Roš*.)

Elastic strain under a load which increases in accordance with a definite law:

$$x(t) = \frac{F(t)}{E} - \frac{1}{E}\int_0^t e^{-\lambda(t-z)}\, \frac{dF}{dz} \cdot dz$$

In the case of a linear load we have:

$$x(t) = \frac{p}{E}\left(t - \frac{1 - e^{-\lambda t}}{\lambda}\right)$$

The first term represents the total elastic strain and the second represents the delay or elastic hysteresis.

The strain under a sinusoidal load is as follows:

$$x(t) = \frac{p}{E} \cdot \sin \varkappa\eta t - \frac{\varkappa\eta p}{E} \cdot \frac{\lambda \cos \varkappa\eta t + \varkappa\eta \sin \varkappa\eta t}{\lambda^2 + \varkappa^2\eta^2}$$

wherein the second term represents the diminution in amplitude of the strain as a function of the frequency. The coefficient λ may be calculated by comparing this with the experimental results of repeated bending. In a carbon steel which has an ultimate strength of 68 kg per sq. mm we have found $\lambda = 5.25 \times 10^3$.

The total deformation under an increasing load is given by the formula:

$$X(t) = \frac{F(t)}{E} - \frac{1}{E}\int_0^t e^{-\lambda(t-z)}\, \frac{dF}{dz}\, dz$$

$$+ \alpha \int_0^t \mathfrak{N}(F)\,(t - z)\, \frac{dF}{dz}\, dz + \beta \int_0^t \mathfrak{N}(F)\,(1 - e^{-\mu(t-z)})\, \frac{dF}{dz}\, dz$$

A detailed examination of this function shows that for any given total load the plastic strain falls off as the rate of application of load increases. The case of a rapidly increasing load is of very frequent occurrence in engineering work, and it is clear, therefore, that in such work it is not correct to assume the same mode of adaptation and the same laws of plasticity under impact as under a slowly-applied super-load.

The experimental study of these questions has shown, also, that in the case of repeatedly applied loads which obey a harmonic law the modulus of elasticity is variable in time; moreover it has been observed that this variation depended on the amplitude of the load. For a small load the coefficient λ diminishes and tends towards a value λ; in other words the solid adapts itself to the forces imposed upon it. On the other hand when the amplitude and the force exceed a certain value, which is perfectly definite, the coefficient λ tends to be increased. The value of this boundary between the two phenomena is practically the same as the fatigue limit measured independently for the same solid, and this fact provides the missing experimental link between strain and failure under repeated loading. Moreover it agrees with the measurements made of damping capacity, according to which the logarithmic decrements of the oscillations set up by the successive impulses diminish when a point in excess of the fatigue limit is reached, and increase when a point below this limit is attained.

Another series of questions which may be examined on the basis of this theory is that which relates to the propagation of vibrations in solids. It may be sufficient to say that under the high frequency and low amplitude of (for instance) acoustical vibration, plastic phenomena play a subordinate role compared with the conditions of propagation, and only elastic hysteresis may be of some importance. The general equation for the propagation of a vibratory movement, derived from our first equations is as follows:

$$\delta \frac{d^2u}{dt^2} = E\frac{d^2u}{dn^2} + \int_0^t e^{-\lambda(t-r)} \frac{d^3u}{dn^2 dz}\, dz$$

or

$$\delta \frac{d^2u}{dt^2} = E\frac{d^2u}{dn^2} + \frac{E}{\lambda}\cdot\frac{d^3u}{dn^2dt} - \frac{E}{\lambda^2}\frac{d^4u}{dn^2dt^2} + \dots + (-1)^{n+1}\frac{E}{\lambda^n}\frac{d^{n+2}u}{dn^2dt^2} + \dots$$

Since λ has a high value the equation just quoted may be limited to the first two terms of the second member. It is then exactly similar to the equation for propagation already well known in relation to viscous media, wherein $\frac{E}{\lambda}$ is the coefficient of viscosity.

I 5

Critical observations on the theory of plasticity.

Kritische Betrachtungen zur Plastizitätstheorie.

Considérations critiques sur la théorie de la plasticité.

Oberbaurat Dr. G. v. Kazinczy,
Budapest.

In 1914 the present author expressed the opinion, in a Hungarian journal,[1] that the permanent deformation of the steel ought to be taken into account in deciding the true carrying capacity of statically indeterminate structures. Since this true carrying capacity is greater than the capacity worked out in accordance with the elasticity theory, it seems reasonable to allow for permanent deformation even in the practical design of structures. In the meantime the problem has been discussed, illuminated and checked by experiment in all its aspects, and it is proposed to give here a critical survey of the whole field.

The new method of calculation is known by various names. The term *"plasticity theory"* is understood to mean a method of calculation in which account is taken of permanent strains, by contrast with the elasticity principle which takes account only of the elastic strains. The term *"carrying capacity method"* [theory of plastic equilibrium] is used also, but this lacks definition, because carrying capacity is often identified with maximum supportable load (as for instance by *Stüssi*), while on the other hand *Bleich, Maier-Leibnitz* and the author in his earlier publications understand this term to mean the "practically" supportable load. The position, as regards this problem, is governed by reference to certain main principles. *What is the purpose of structural calculations?* It is to ensure that structures shall remain serviceable whilst in service; inaccuracies of calculation and in determining the properties of materials, as well as the loading, make it necessary to adopt certain guarantees against structures becoming unserviceable. At the Vienna Congress[2] the author argued that the factor of safety is a question of economics. It behoves on the one hand to build as cheaply as possible and on the other hand to build in such a way that any damage likely to be sustained — due account being taken of the propability of its occurrence — will not outweigh the saving achieved through adopting smaller dimensions for the structural member. The factor of safety must, therefore, be increased in proportion to the magnitude of the damage apprehended. These considerations serve to explain why we are content with a factor of safety of between 1.6 and 1.8 in cases where unserviceability will not cause an unacceptable amount of deflection, but insist on perhaps three times this amount in cases where over-stressing of the structural members might result in immediate collapse (for instance by buckling) without warning. In the case of members which are rendered useless by any considerable change of shape our endeavours are directed to ensuring adequate safety against excessive deformation, rather than against failure by rupture. To lay down a general rule for the amount of permissible bending, it might perhaps be stated that the limiting load (critical load or practically supportable load) should be taken as the load which, when gradually increased, causes a rapid increase in the deflection. In the experiments by *F. Stüssi* and *C. F. Kollbrunner*[3] (Fig. 1) the present author would take 1.71 tonnes and not 2.35 tonnes as being the limiting load in the case of the beam on two supports; from this point of view the upshot of the experiments reads rather differently: namely, that even in extreme cases the limiting load (not the true carrying load) has two values* if the beams are built in at the ends, except where the fixation is too flexible with the result that the elastic deflection at the middle increases so rapidly, once the yield point is reached, that it becomes excessive even before the yield point is reached over the inner support. Fig. 2 shows how the lines of bending would appear in an ideally plastic beam under uniformly distributed loading with different amounts of end fixation. It will be seen that in certain cases deformation must be taken into account.

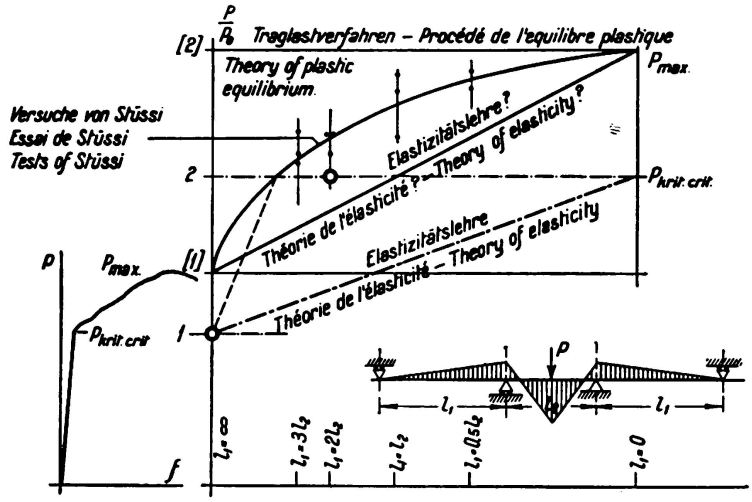

Fig. 1.

There are two methods of introducing the desired factor of safety when dimensioning a structure: either the calculation may be based on the actual load multiplied by the factor of safety, or the limiting stress divided by the factor of safety may be regarded as the permissible stress. The latter method is the usual

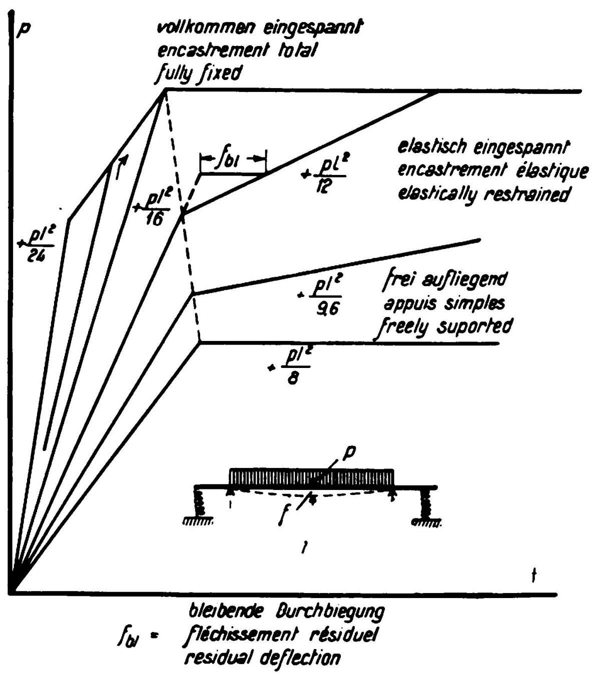

Fig. 2.

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* Beam simply supported on two points $P_{Tl} = 1{,}71$ tonnes; $P_v = 2{,}35$ tonnes. $\frac{3{,}46}{1{,}71} = 2{,}02$.

Continuous beam $P_{Tl} = 3{,}46$ tonnes; $P_v = 3{,}82$ tonnes; $\frac{3{,}82}{2{,}35} = 1{,}60$.

one, the ratio between the permissible stress and the limiting stress representing the factor of safety. This would be correct if the stresses followed a linear law of growth up to the limiting load, but in the case of statically indeterminate structures that condition does not, as a rule, obtain, owing to equalisation of stress. If calculations were based on the working stresses multiplied by the factor of safety it would be easier to account for the equalisation of stresses, seeing that this operates only when the working stress is exceeded and is, therefore, a criterion of safety rather than of actual stress.

In order to determine the limiting loads of statically indeterminate girders theoretically, a material has been assumed which possesses ideal properties, that is to say which gives an idealised stress-strain diagram. It has further been assumed that cross sections remain plane when deformed, and that the yield increases gradually from the extreme fibres towards the inside of the beam. According to this theory the possibility of a cross section subject to bending stresses continuing to deform without the moment being further increased depends on its having become plastic as far as the neutral axis. The plastic hinge effect therefore implies an unlimited degree of bending, and in mild steel this condition cannot be fulfilled on account of the cold working effect. Hence a number of investigators have recently turned closer attention to the phenomenon of plastic deformation with special reference to cases where the field of stress is not uniform and the yield phenomena do not progress uniformly, but where the less heavily stressed portions hinder the deformation of the plasticised elements (works by *W. Kuntze*[4], *W. Prager*[5]; also *J. Fritsche*[6]: The New Theory of Plasticity). Observation however, has not confirmed this theory, the yield figures not being affected to such an extent that the beams yield all at once as far as the neutral axis. The same fact may also be seen in the illustrations to Nadai's work, „Der Bildsame Zustand der Werkstoffe", p. 127, Fig. 130 showing how the signs of yield spread gradually inwards. On the other hand it may be observed in the case of I beams that the yield figures appear simultaneously throughout one half flange. *Rinagl*[7], however, states that the delay in yielding which this view of the matter implies is a mistake, and that the effect is attributable to the existence of an upper yield point which always operates in bending but which appears only indistinctly in a tensile test. The present author does not agree with Professor *Rinagl* about this, as he has himself been able to observe the delay to the yield in a non-uniform field of stress in frame members, and an account of these observations will be given below.

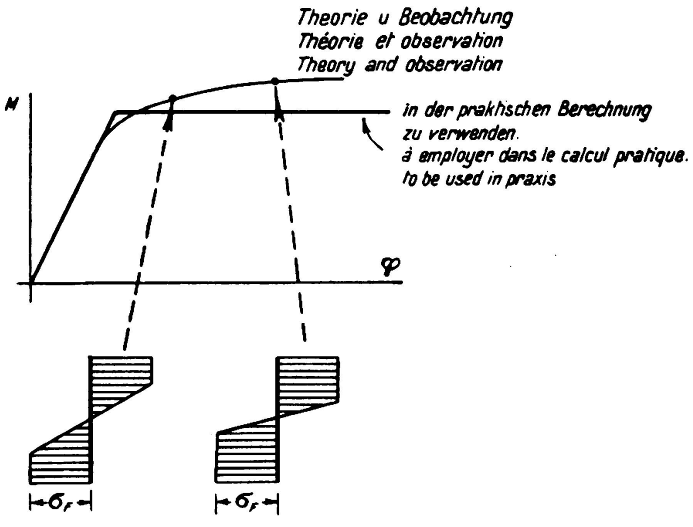

Fig. 3.

An attempt to allow for the true properties of the material by reference to any of these theories leads to complicated calculations. Since, however, our objective is the dimensioning of structures and not the theoretical interpretation of experimental results, we must devise a simple method of calculation. This can be done if we assume a sharp transition from the elastic to the plastic condition even under bending stress. *Maier-Leibnitz*[8] has shown the possibilities of solving

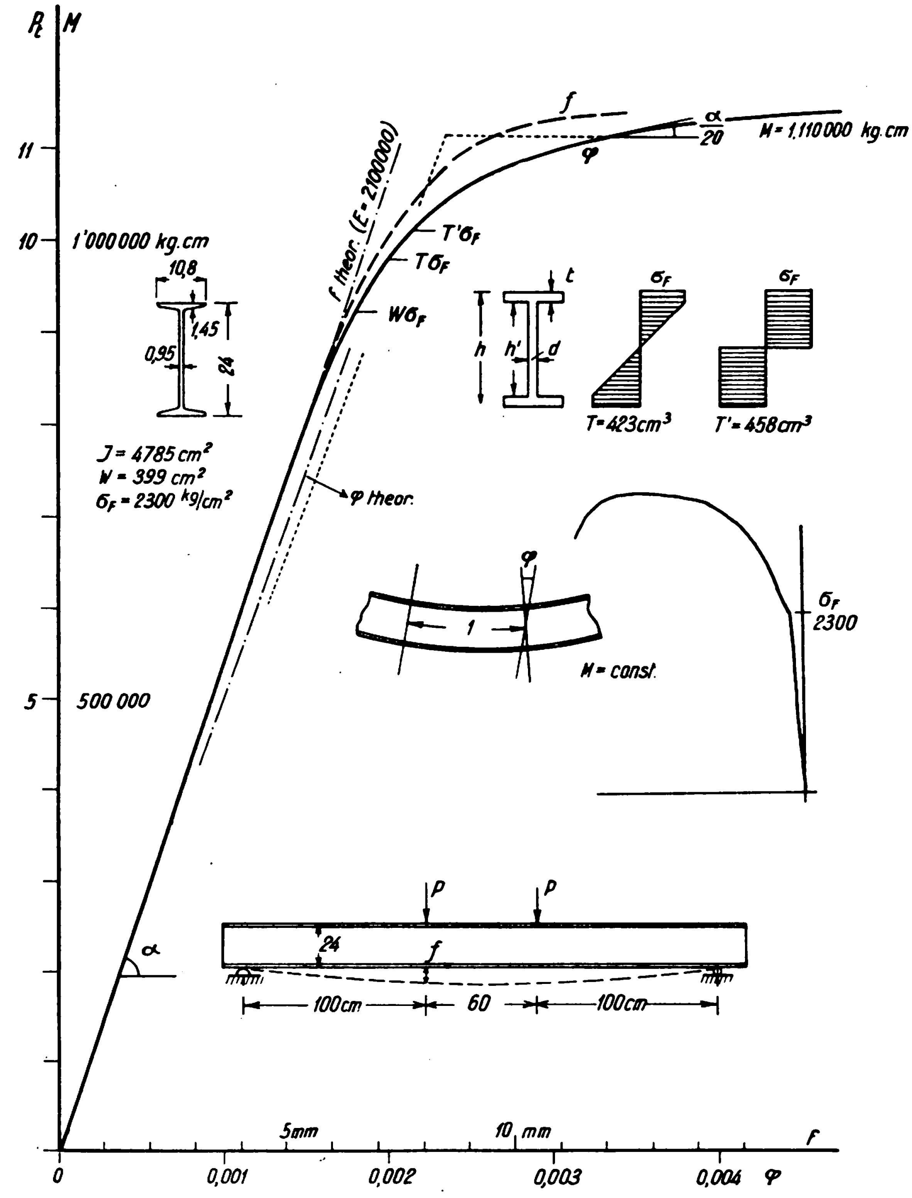

Fig. 4.

simple problems by reference to the true theory of moments and deformation, but a practical method can be built up only on a simplified interpretation (Fig. 3). *Maier-Leibnitz* suggests that the moment to be regarded as a limiting moment should be that corresponding to the sharpest curvature of the residual strain-moment curve. The author suggests, however, that the value to be taken as a limiting moment should be the one at which permanent bending is twenty times as great as elastic bending.

To throw further light on this question the author has carried out experiments on an I beam MP 24 ($W = 399$ cm^3) coated with lacquer so that the appearance of yield could be more readily detected. Up to $\sigma = 2250$ kg/cm^2 the bending line was practically straight (Fig. 4). On the tensile flange the yield figures appeared at 2500 kg/cm^2 whereas on the compression flange they appeared as early as $\sigma = \frac{M}{W} = 2120$ kg/cm^2, but obviously this was due to a local unevennes. At 2800 kg/cm^2 that amount of deformation was obtained which the author has adopted as the criterion of the limiting moment. The beam was then removed from the bending machine and was carefully examined and photographed (Fig. 5). The condition of yield had been reached only over approximately one half of the length containing the places most heavily stressed under constant maximum moment. Contrary to theoretical indications, the yield figures extended to the neighbourhood of the neutral axis. A tensile test bar cut from the unloaded end of the beam after the experiment, was found to have a yield point of 2300 kg/cm^2 in a very small length of yield. The conclusion which has to be drawn from this experiment is that the limiting moment must be determined on empirical, not theoretical, grounds. It appears that the limiting moment and the yield point stand in no simple relationship to one another as the matter is affected by the shape of cross section and by the properties of the material. If these limiting moments are determined empirically* for certain cross sections and types of steel no obstacle stands in the way of adopting the new method of interpretation. If we have decided to calculate from the idealised bending line (M-φ-diagram) we may design our structures in accordance with the following rules.

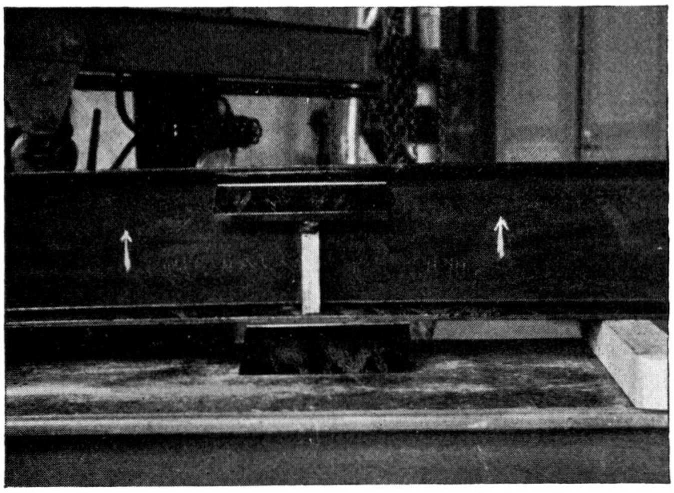

Fig. 5.

1) Statically determinate structures subject to bending.

The limit of carrying capacity is reached not when the extreme fibre attains to the yield point but when the "beam" as a whole is yielding. The carrying moment is not $M = W \cdot \sigma_F$ but $M = T \cdot \sigma_F$ where T is approximately 6 to 20 % greater than W and must be determined by experiment beforehand.

2) Statically determinate trussed girders (lattice girders).

The calculation is made as hitherto. Secondary stresses resulting from the rigidity of the intersections may be left out of account, but as regards com-

* Suggestions for calculating the limiting moment have been put forward by *von Kazinczy* (9), *Kist* (10), *Fritsche* (11) and *Kuntze* (4), but these all give lower values of carrying moment than the present author's experiments.

pressive forces the buckling length is to be taken as equal to the whole theoretical length of the bar, even in the plane of the truss. Compression members should be dimensioned with a higher factor of safety than tension members because if the buckling load is exceeded the result may be a collapse of the structure.

3) Calculation of the connecting rivets.

This is calculated as hitherto, that is to say the total load in the member is uniformly divided among all the connecting rivets. Here practice and experience have fully confirmed the correctness of the plasticity theory. The connecting rivets or weld seams should be calculated not from the tabulated but from the maximum permissible load in the member, so that in the case of an excess it is the member and not the connections which will yield. Owing to the redistribution of the secondary stresses amongst the members it is desirable that the connections should be made rigid against bending.

4) The calculation of continuous beams.

In the case of beams consisting of a single rolled section (of constant cross section) the M_0 moment should be determined in each opening as for a simple beam, and the closing line should be so drawn that the negative and positive moments are of equal magnitude; the beam should then be designed to suit the maximum moment so calculated.

In the case of beams wherein the cross section has been made to vary with the moment by adding flange plates, there is not much point in designing by reference to the plasticity theory, but if, with a view to greater economy, it is still desired to make use of the new method, the closing line should be drawn in arbitrarily in such a way as to minimise the cost of construction. The rule to remember is that the negative moment can be minimised as desired, whereas a yield taking place in the middle of the beam is always associated with large deflections.

Under moving loads the limiting moments are to be determined according to the elasticity theory, the closing line being moved about at will in the direction of equalising the moment.[12, 13]

One of the most important results of the plasticity theory is that permanent settlements of the supports need no longer be taken into account, while on the other hand, the effect of elastic settlement calls for examination.

Rolling and shrinking stresses need not be taken into account but stresses resulting from irregularity of temperature in service should be considered.[13]

If a greater amount of equalisation of moment is counted upon — especially by yielding at the middle of the beam — it is desirable to make the compression flange heavier so that the yield may take place only in the tension flange.

5) Structures in which members are restrained against bending.

Several authors have indicated that in a framed structure which is statically indeterminate to the n-th degree, the yield point may be reached at n places without the structure thereby being rendered useless. The problem may be represented by assuming hinges to exist at these places and to be subject to moments of constant magnitude. Previously the author has himself put forward this view[14] but he now feels compelled to change it slightly. In order that a structure of this kind may be rendered unstable sufficient hinges must be introduced to form a kinematic chain. When movement takes place the hinges turn in a given direction. The yield hinge, however, turns only in one direction, for in the other direction it acts as a completely elastic member of the girder. Those yield hinges which turn in a direction opposite to that which should exist in a kinematic chain should not, therefore, be considered as hinges. Thus it may be possible in a structure which is statically indeterminate to the n-th degree for the yield point to be exceeded at more than n places before the structure becomes unstable.

Hence a framed structure is safe to carry a given load as long as it is possible to draw a moment line satisfying the conditions of equilibrium with the external stresses and not exceeding the value $M = T \cdot \sigma_{zul}$ at any point. A more exact method may be developed similarly to that of Prof. *Cross*: first of all the moment is determined in accordance with the elasticity theory, then, at the places where the moments are to be reduced, the structure is assumed to be cut in two and additional moments are assumed to be introduced in order to relieve the load. At those places where the reduction has already been effected hinges are introduced, but only where an increase in the moments is to be expected (fig. 6). The principal advantage of the plasticity theory lies in the fact that it enables the moment to be regulated, and in this way the more dangerous places to be protected from over-stressing. Generally speaking the verticals are the more important members in a frame. It is possible, therefore, by making the beams weaker at the connection, to relieve the verticals when the beam reaches its limiting moment, since no further moment can be transferred to the verticals once the limiting moment at the connection has been reached. The danger of buckling of the verticals can thus be avoided by making the beam yield at the connection, which is in no way dangerous.

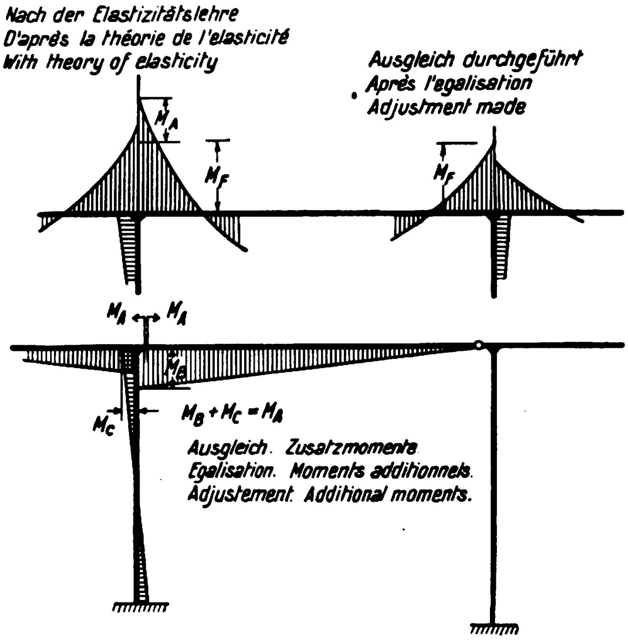

Fig. 6.

6) *Lattice girders.*

Lattice girders which outwardly are statically indeterminate can be designed in the same way as beams and framed girders. The yield phenomena occur in one part of a bar, but for the purpose of equalisation use should be made only of tension bars, because the resistance of a compression bar after it has buckled decreases immediately to a low value, as the author has already indicated at Liége. More recently *E. Chwalla*[15] has worked out this problem, and has also confirmed by experiment that the compression resistance falls away very rapidly.

In the case of lattice girders which are internally statically indeterminate, it happens not seldom, according to the elasticity theory, that full use cannot be made of all the bars — as for instance in the lattice girder shown in Fig. 7 where, according to the elasticity theory, the parts marked B cannot be fully utilised. In such a case the plasticity theory offers an economic advantage by

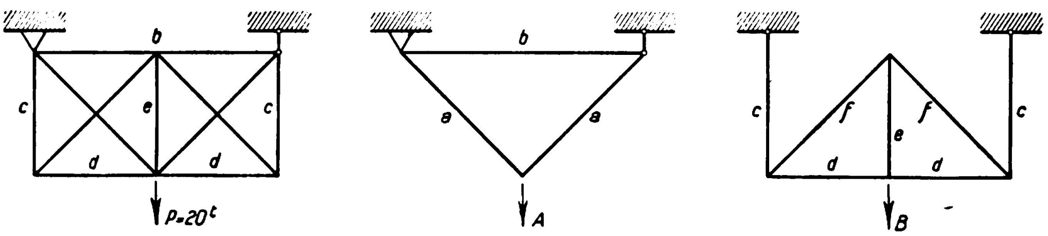

Fig. 7.

enabling all the bars to be fully utilised. The design of such a structure is usually very simple: the supernumerary tension bars are left out, and in their place the known forces $F \cdot \sigma_{zul}$ are inserted; the most heavily stressed tension bars which first reach their yield point are thus eliminated, and when the decision cannot be made by simple means the elasticity theory is used. The cross section should always be so adjusted that only tension members are brought to the yield point and no compression member is exposed to buckling.

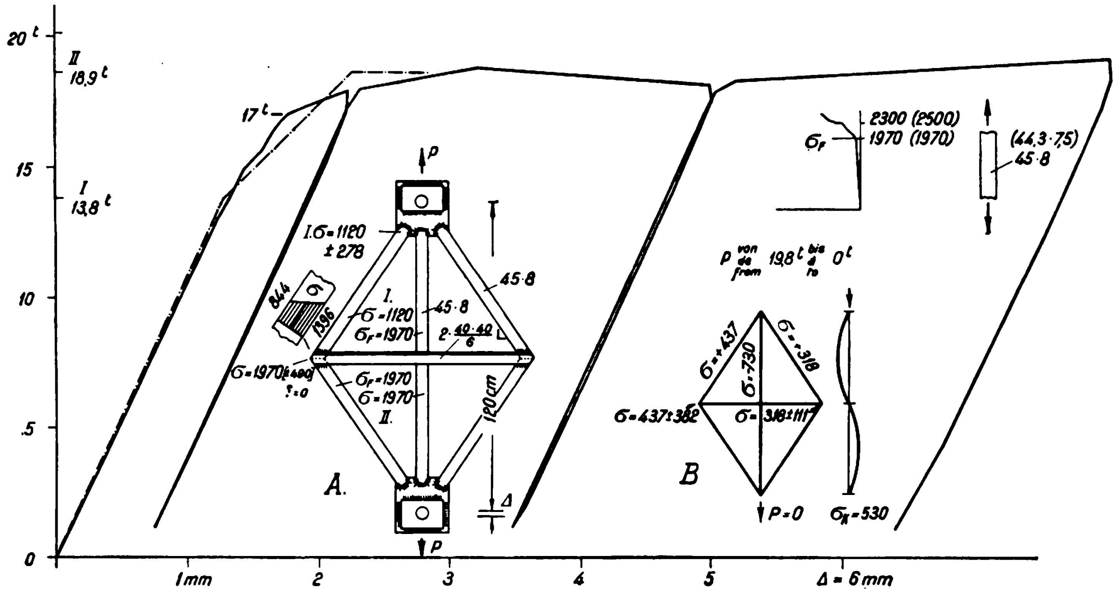

Fig. 8.

For moveable loads special methods must be adopted as, for instance, that of *E. Melan*[16] — but always with the condition that no plastic deformations are to be permitted in compression bars.

The author has carried out some experiments to confirm theoretical considerations on the plasticity theory and a brief account of these will now be given. He has examined two kinds of internally statically indeterminate frames

— welded and riveted — whereas *G. Grüning* and *E. Kohl*[17] carried out their experiments on externally statically indeterminate frames in which the most heavily stressed tension bars were made in the form of eye-bars, so that no conclusion as to the usual form of connection can be drawn from these. The form of girder adopted and the dimensions and results may be seen in Fig. 8. The behaviour, assuming an ideally plastic material, is represented in Fig. 9. The frame may be regarded as built up from two fundamental system A and B. The resistances offered by the separate systems A and B are plotted along ordinates as functions of the resulting changes in length. P_I is designed as the "first limiting load" and P_{II} as the "second limiting load" (or limit of carrying capacity). After the removal of load the separate systems will not remain free from stress, the residual forces being shown in Fig. 9.

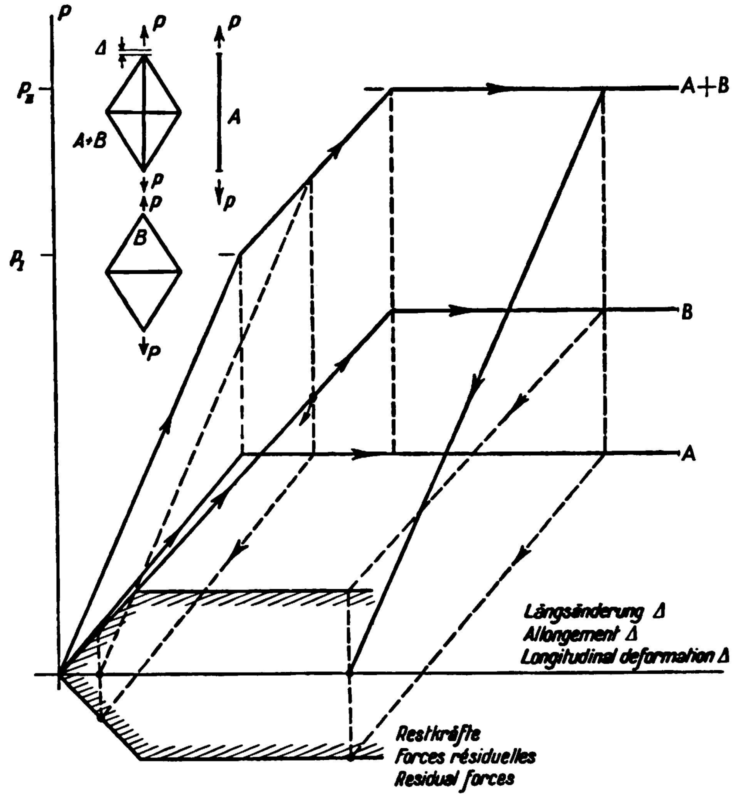

Fig. 9.

An investigation of the mechanical properties of the material chosen showed that the strip steel was very soft, and that as the stress increased it showed a very large region of yield. The yield point was first observed in the vertical tension bar (first limiting load). On the load being further increased the stresses in the vertical bar remained constant, any notable increase being limited to the other bars until the yield point had been obtained in these also (second limiting load). According to the present day view of yield, the theoretical secondary stresses (also included in Fig. 8) disappear. On being relieved of load the structure behaved in an entirely elastic manner, and the residual stresses are shown in Fig. 8. The vertical bar did not however withstand residual stresses of 740 kg/cm² because it consisted of strip steel and was bound to buckle even at 530 kg/cm². This buckling could also be observed in the experimental test piece.

The first lines of yield were noted in the neighbourhood of the middle of the vertical bar with P = 14 tonnes but the actual yield figure in the vertical bar not until 17 tonnes. The specimen had undergone considerable change of shape but nevertheless only quite small portions of the bar yielded (Fig. 10). The plastic alteration in length is limited to certain places and at these it attained a constant percentage. The changes in length of a mild steel bar must be visualised as in Fig. 11, wherein k_I and k_{II} are different alterations in length. The lines marked e represent elastic and the lines marked p represent plastic elongations. The limits of carrying capacity (second limiting load) agree well with the theoretical values, which shows that the shrinking stresses due to welding have no effect on the carrying capacity, but only on the beginning of the equalisation of stress.

In order to determine the amount of these shrinkage stresses the author had further experimental specimens constructed on which observations were made of elongations at a number of points during the welding and cooling processes: the result of this was to indicate shrinkage stresses of 900 kg/cm². No delay of the yield phenomena — that is to say no upper yield point — could be observed. In the inclined bars with heavy secondary stresses no yield occurred when the average stresses had already reached the yield point. These

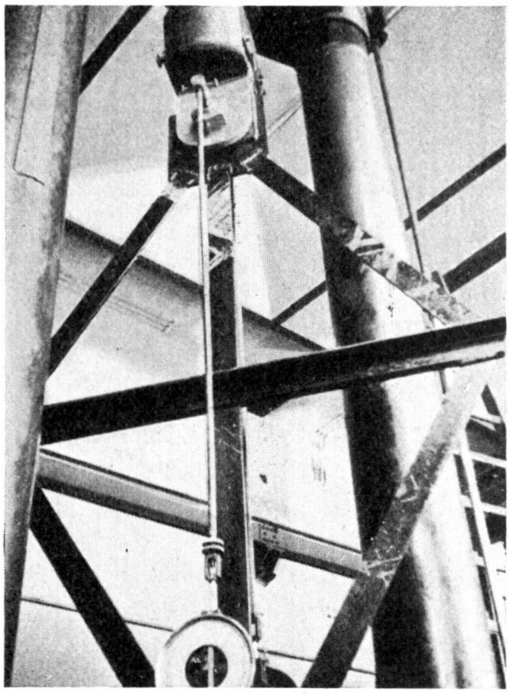

Fig. 10.

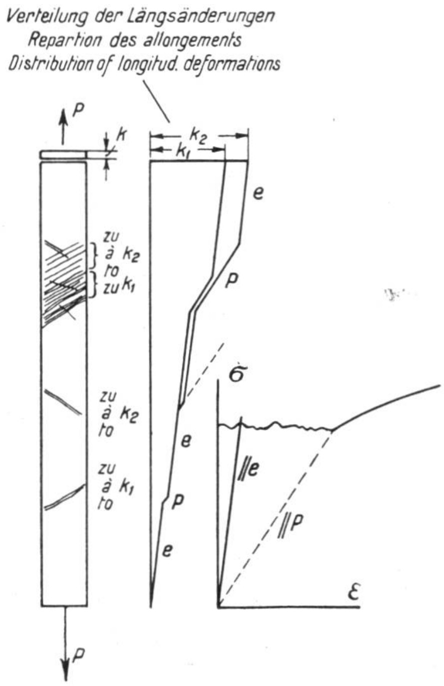

Fig. 11.

experiments appear, therefore, to support the new plasticity theory, and in none of the experiments carried out was it possible to observe the old yield conditions. The author proposes to treat these experiments more fully in the technical press.[19]

A similar framework was made in riveted construction. As a result of the somewhat higher elastic limit of the steel used, the maximum load obtained was higher than in the case of the welded girder (20,4 as against 19,1 tonnes). When first loaded a movement of the rivets was observed, but under further loading the performance was elastic. Despite the presence of the rivet holes the yield point could be attained over the full cross section also.

From these experiments the following conclusions may be drawn: In welded statically indeterminate lattice girders the shrinkage stresses have an influence only over the beginning of the process of equilisation of stress, and not on the magnitude of the critical load. It is to be noted that the shrinkage stresses had the effect of increasing the primary stresses in the tension bars, and of decreasing those in the compression bars. (Note the choice of working method.)

In riveted statically indeterminate lattice girders the plastic elongation begins at the connection and the necessary amount of force is somewhat increased by friction, but the increased yield point around the hole or the toughening of the steel through the use of riveting tools may have had a similar effect. With short lengths of bar only a limited amount of ductility of the connection is necessary for the equalisation of stress, but the connections should always be strong enough to ensure that the full bar reaches its yield point before the connections are affected. The limiting load of a riveted framed structure is approximately the same as if the bars were not weakened by rivet holes according to the plasticity

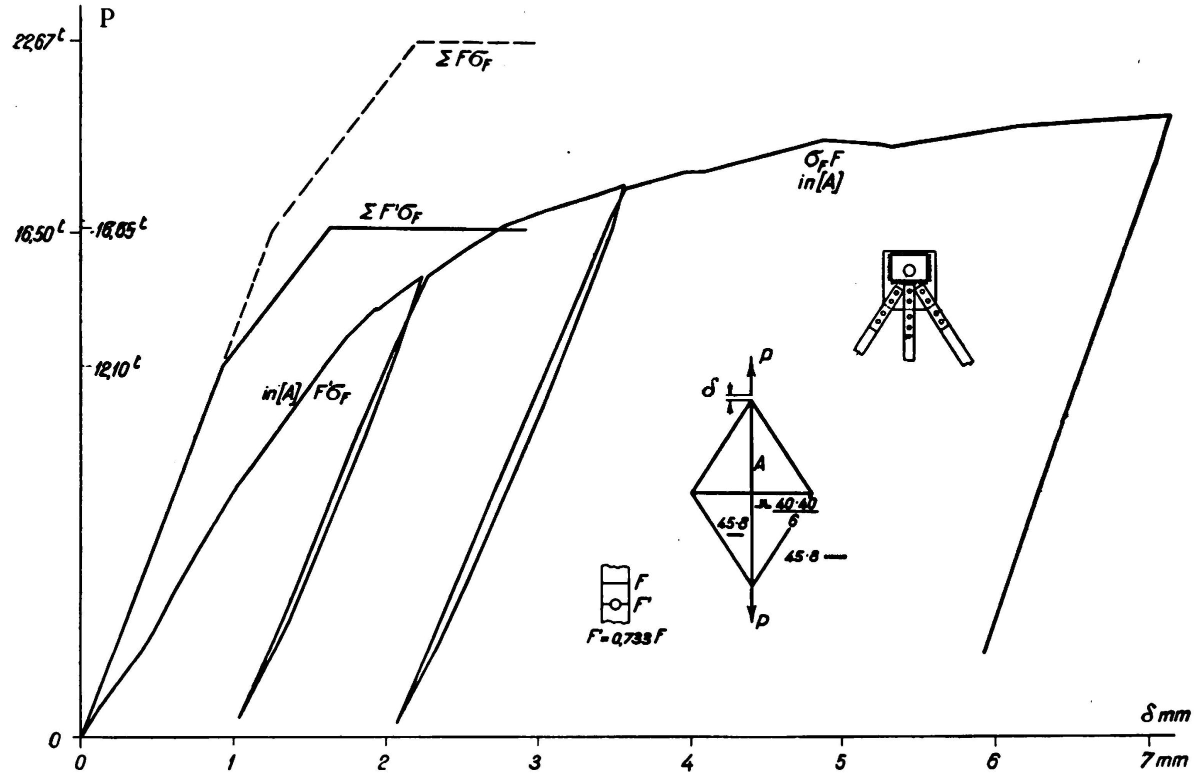

Fig. 12.

theory, always provided that no compression bar reaches a condition of buckling. To take account of the large permanent deformation we may obtain the practical limiting load by deducting the rivet holes and allowing for equalisation of stress, and the degree of safety will then always be greater than in welded structures which have been calculated as for the whole cross section.

Besides the lattice girders, experiments were also made on riveted plate web girders, resting on two supports and loaded at the third points. Measurements were made of the angle of bending of the middle portion under a constant moment, and the results of the experiment are shown in Fig. 13. In calculating the rivet holes were not deducted. The measured deflection is somewhat greater than the calculated amount with $E = 2{,}100$ tonnes per sq. cm but the amount of bending agrees fairly well with the recovery on removing the load (showing the elastic action). After a period of two days the yield point increased by 6 % and the girder then behaved in a purely elastic manner. Under the proposed assumption that $\Delta\sigma/\Delta E = 1/20\,E$ the critical load was determined to be

14 tonnes. Fig. 13 shows the result of comparing this experimental value with different assumptions, the minimum yield point of the angle being taken as 2,500 kg per sq. cm. In this way the maximum extreme fibre stress of the flange plates would be 2780 kg per sq. cm. For determining the carrying moment $T \cdot \sigma_{zul}$ the condition was assumed that the flange plate had reached its yield point. Since in this experiment a second unknown enters, namely, the value to be attributed to the rivet holes, a comparative experiment was carried out as between riveted and welded girders of the same section and material, and the results of this are shown in table I.

Table I.

Welded I-beam			Riveted I-beam d = 16 mm	
σ_F kg/cm²	Section mm		Section mm	σ_F kg/cm²
2680	152.6 · 13	compression flange	152 · 12.8	2680
2620	155 · 7.7	tension flange	154 · 7.7	2590
2750	60 · 60 · 6.1	4 L	60 · 60 · 6.1	2780
4280	182 · 8.2	Web	183 · 8.6	4060
1 513 000		critical moment according to tests kgcm	**1 266 000**	
tension 1 180 000	compression 1 420 000	$W\sigma_F$ (σ_F = flange)	tension 1 170 000	compression 1 400 000
		$W\sigma_F$ deducting rivet	**965 000**	**1 135 000**
		$W\sigma_F$ deducting rivet holes in web also	906 000	1 087 000
1 644 000		$T\sigma_F$ Full section	1 632 000	
		$T\sigma_F$ deducting rivet holes	**1 387 000**	
		$T\sigma_F$ deducting rivet holes in web also	1 266 400	
1 513 000		$T\sigma_F$ of flange plates and angles $+ W\sigma_F$ of web	deducting rivet holes **1 259 000**	

A continuous riveted girder over three supports was also examined (Fig. 15). The deflections are greater than calculated even when the girder is unloaded.

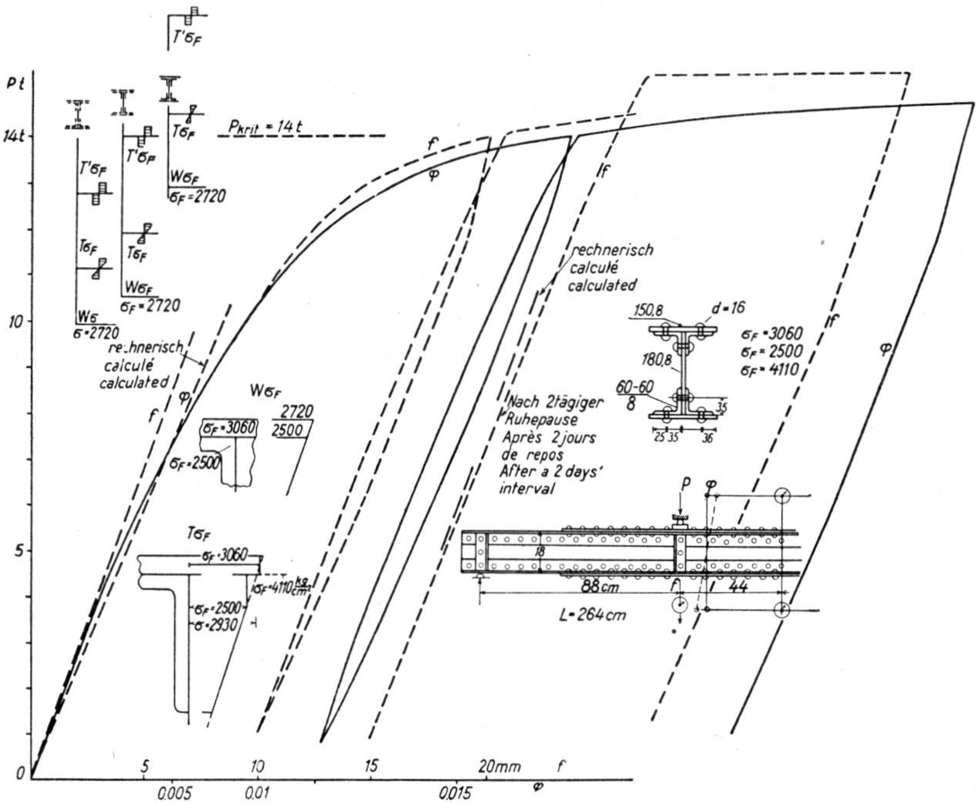

Fig. 13.

From the central support to the point of load the web attained its yield point through shear force, whereby *Stüssi's* argument[18] from theory that the shear stresses must considerably increase when the yield spreads to certain points below

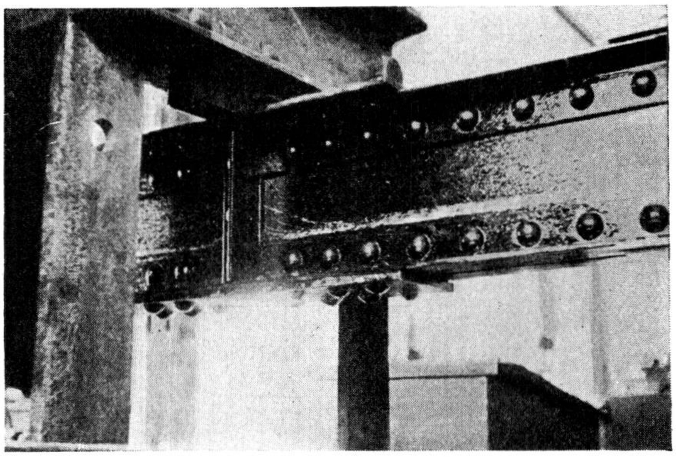

Fig. 14.

the edge of the beam was experimentally confirmed, though to a smaller extent than anticipated by him. This is perhaps attributable to the fact that, as a result of the rapid removal of the moment, only a short portion of the girder is affected, which moreover is hindered in its movement by the neighbouring portion. The final result found by the author was that the maximum load was represented by $T \cdot \sigma_F$ assuming a complete equalisation of the moment.

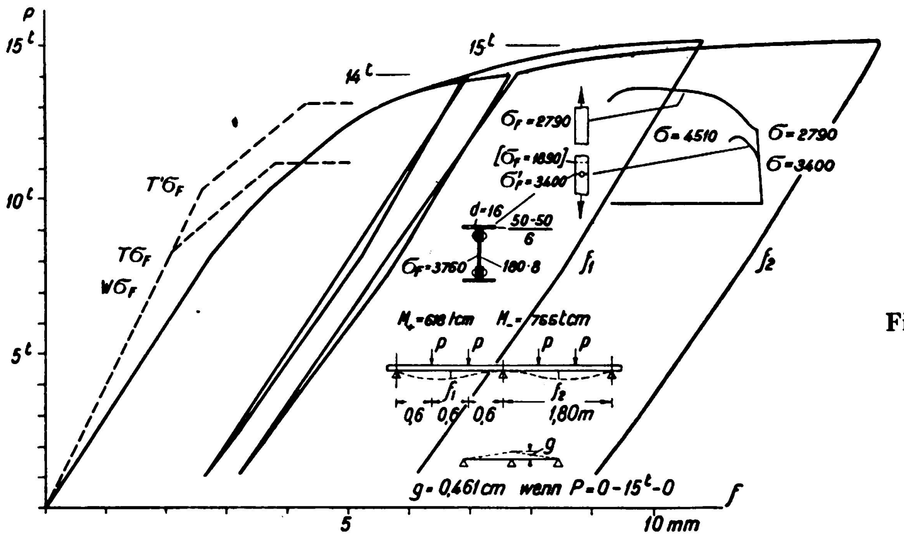

Fig. 15.

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[17] *G. Grüning* and *E. Kohl:* Tragfähigkeitsversuche an einem durchlaufenden Fachwerkbalken aus Stahl. Der Bauingenieur 1933, N° 5/6, p. 67.

[18] *F. Stüssi:* Über den Verlauf der Schubspannungen in auf Biegung beanspruchten Balken aus Stahl. Schweizerische Bauzeitung 1931, Vol. 98, N° 1, p. 2.

[19] *von Kazinczy:* Versuche mit innerlich statisch unbestimmten Fachwerken. Der Bauingenieur 1938, N° 15/16, p. 236.

I 6

The Relations M_{st} (P) and M_F (P) in Girders Continuous over Three Spans Carrying a Load P in the Central Span

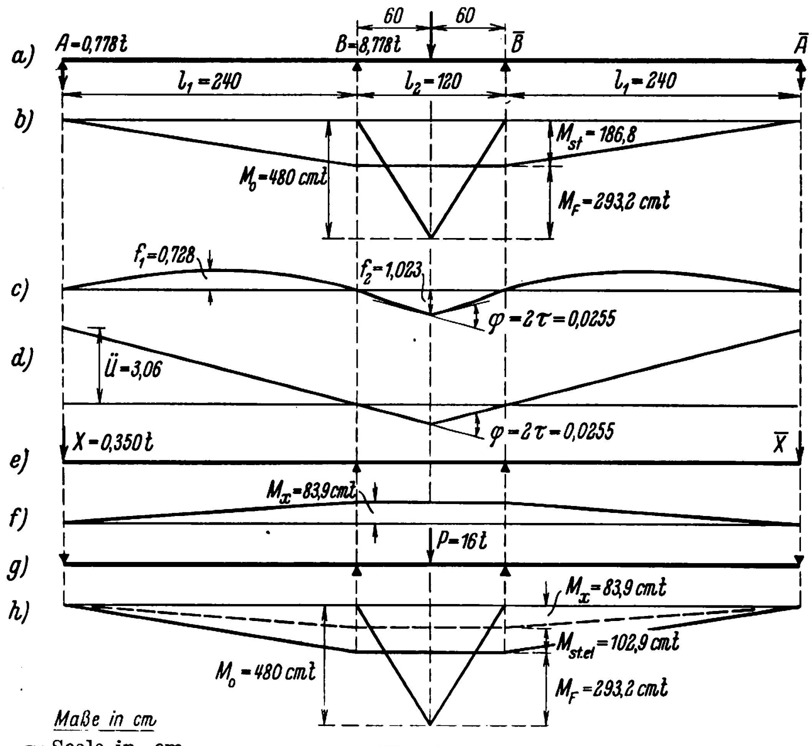
(see Preliminary Publication, pages 121—126).[1])

Die Beziehungen M_{st} (P) und M_F (P) beim durchlaufenden Balken mit drei Öffnungen, belastet durch P im Mittelfeld (siehe Vorbericht Seite 126—128).[1])

Les expressions M_{st} (P) et M_F (P) dans la poutre continue à trois ouvertures, soumise à une charge P agissant dans la travée médiane (voir la Publication Préliminaire, pages 121—126).[1])

Dr. Ing. H. Maier-Leibnitz,
Professor an der Technischen Hochschule, Stuttgart.

If the girder shown in Fig. 1 be subjected to a load increasing from 0 to P the first result is a bending moment diagram calculable from purely elastic



= Scale in cm

Fig. 1.

Behaviour of girder under a load P = 16 tonnes.

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1 See also the journal „Der Stahlbau", 1936. No. 20, pp. 153 foll.

principles and characterised by M_o, M_{st}, M_F. When $P = P_s$ ($= \sim 11$ tonnes) the yield stress σ_s is attained at the extreme fibre in the middle of the span.

As soon as $P > P_s$ (as, for instance, when $P = 16$ tonnes), M_F cannot increase appreciably above $M_s = W \cdot \sigma_s$. With the aid of *Mohr's* theorem an expression for the angle φ of the line of bending at the middle of the girder can be derived by considering the line of moments (Fig. 1b) from A to B and from B to the point where the line of bending has a kink. In the present case

$$EI\varphi = 6600\,P - 280\,M_F.$$

If the load is removed the shape of the beam in Fig. 1d shows at the ends an amount ü, and before the load is re-imposed ü has to be eliminated by the two

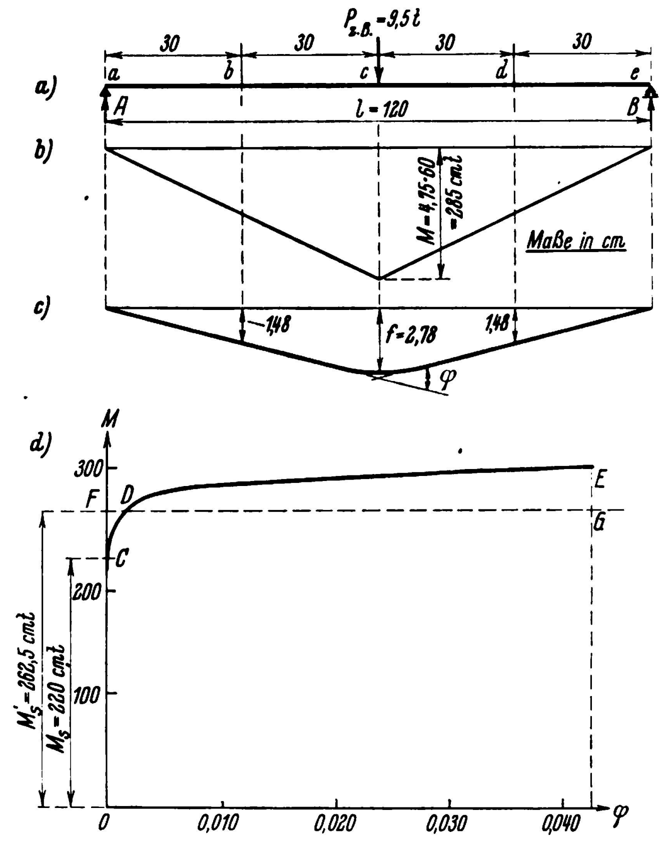


Fig. 2.
Relations of φ (M) in the simple beam.

forces X and $\overline{X}$ corresponding to the moments at the supports M_x. On reloading with $P = 16$ tonnes, M_x is augmented by the purely elastic support-moment M_{stel} (line of moments corresponding to Fig. 1h).

How great is M_F and, therefore, the support moment $M_{st} = M_o - M_F$ and the values of φ, ü and X?

In a simple beam, for instance in the beam of comparison $l = 120$ cm $= l_2$ the relation φ (M) may be obtained by purely experimental means as shown in Fig. 2, where M is the moment at the middle of the beam. In order to indicate the actual behaviour of the continuous beam this relationship may be transferred to the continuous beam, that is to say $M_{(\varphi)} = M_F\,(\varphi)$. On the other hand we may write the relationship stated above:

$$EI\varphi = 6600\,P - 280\,M_F$$

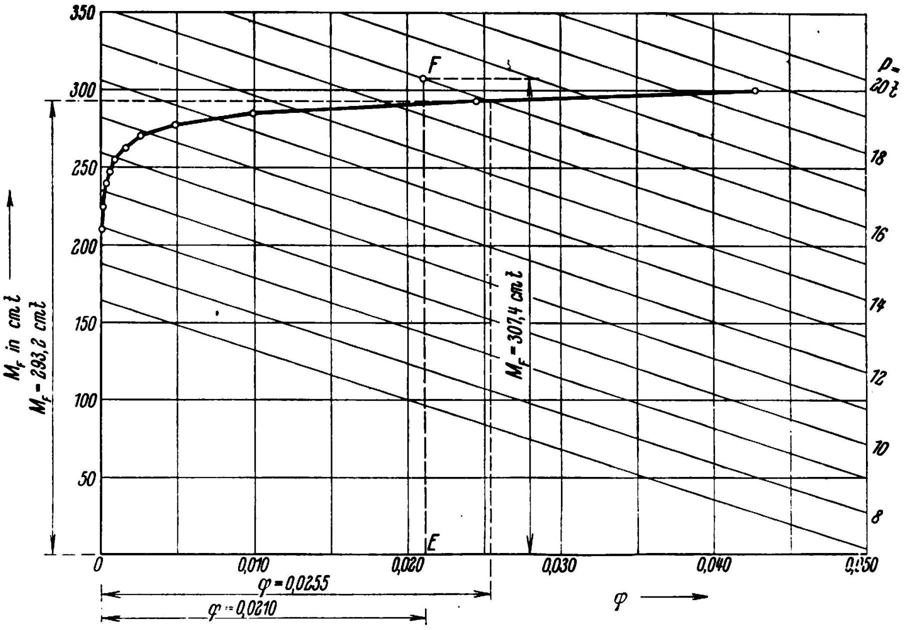
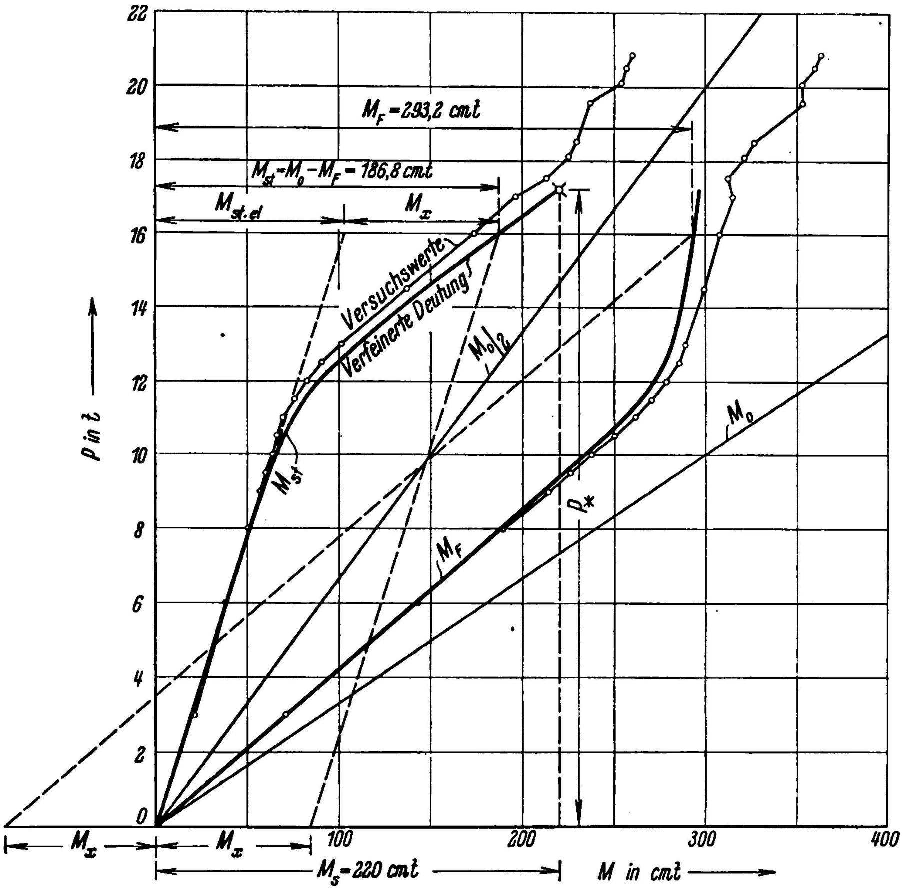


Fig. 3.

Determination of M_F for $l_1 = 240$ cm and $l_2 = 120$ cm, with the aid of $EI\varphi = 6600\ P - 280\ M_F$ (particularly with P = 16 tonnes).



Versuchswerte = Experimental values

Verfeinerte Deutung = More meticulous interpretation

Fig. 4.

Relations M_{st} (P) and M_F (P) for $l_1 = 240$ cm.

Fig. 3 shows both the relationships. Where $P = 16$ tonnes we obtain $M_F = 293.2$ cm tonnes and $\varphi = 0.0255$ (the more precise hypothesis of interpretation).

If this determination is effected for the other loads also, we obtain the picture shown in Fig. 4. M_{st} and M_F at first increase according to linear law, then from $P'_s = 11.12$ tonnes onwards by curves. With $P = 16$ tonnes the values of M_x and $M_{st\,el}$ established above (Fig. 1h) are entered in the diagram. It may also be seen from this diagram how a reimposition of load takes place after removal of load, and that the values in the more exact hypothesis of interpretation agree well with the experimental values indicated by the lighter lines.

In Fig. 3 for $P = 16$ tonnes is given the experimental value $M_F = 307.4$ cm tonnes as the ordinate EF. The ordinates of the curve φ (M_F) are, therefore, greater than those of the curve φ (M) which appertain to the beam of comparison with $l = 120$ cm. This is due to the fact that reference should properly be made for shorter experimental beams of comparison with a span equal to the distance between the points where the moment in the central opening is zero; a fact which is confirmed by experiments carried out subsequently to the Congress with $l = 950$ cm corresponding to P_s and $l = 730$ cm corresponding to P_T.

The foregoing provides a basis for a more exact solution than has hitherto been available of the problem of the actual carrying capacity of continuous girders of structural steel, and closes a gap to which *J. Fritsche* has drawn attention in the journal Der Stahlbau, Vol. 9 (1936), page 67. In future, therefore, as is done in the „Traglastverfahren" [carrying capacity method] it will not be necessary to rely upon the overprimitive assumption of equalisation of moments.

I 7

Interpretation of Tests of the Equilibrium Load Method.

Zur Auswertung von Versuchen über das Traglastverfahren.

L'interprétation des essais sur la méthode de l'équilibre plastique.

Privatdozent Dr. F. Stüssi,
Berat. Ing., Zürich.

Prof. *Maier-Leibnitz,* in his contribution to the Preliminary Report of the Congress,[1] has collected and interpreted the results of the experiments on the equilibrium load method already made known through the technical press. These experiments include some recent work by Prof. *Maier-Leibnitz* himself[2] with which the present writer now proposes briefly to amplify. This work, in agreement with the author's experiments at Zürich,[3] does not show any complete assimilation of the moments in the span and over the supports.

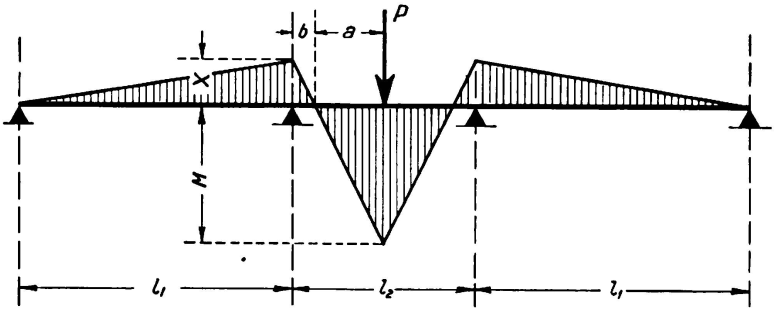


Fig. 1.
Arrangement of test.

In the case of a continuous girder as shown in Fig. 1, the conditions of equilibrium and of elasticity derived from statical practice must hold good even as regards the non-elastic region, and, in particular, the bending moment line must continue across any intermediate support. If A denotes the total angular

[1] *H. Maier-Leibnitz:* Versuche, Ausdeutung und Anwendung der Ergebnisse. I.A.B.S.E., Second Congress Berlin 1936, Preliminary Report.

[2] *H. Maier-Leibnitz:* Versuche zur weiteren Klärung der Frage der tatsächlichen Tragfähigkeit durchlaufender Träger aus Baustahl. „Stahlbau", (1936), No. 20.

[3] *F. Stüssi* and *C. F. Kollbrunner:* Beitrag zum Traglastverfahren. „Bautechnik", (1935), No. 21.

rotation in a simple beam of span $l = 1$ loaded to correspond with a triangular bending moment diagram M, and if B denotes the greater angle of rotation over a support, then the condition of elasticity may be written as

$$B_X \cdot l_1 = A_M \cdot a - A_X \cdot b. \tag{1}$$

If, as in the present case (Fig. 2), the variation in moment as the load increases is found by observation, then from Equation (1) the unknown A_M may be calculated. The values A_x and B_x are known in the elastic region, and for higher increments of loads they may be derived successively from the values of A_M

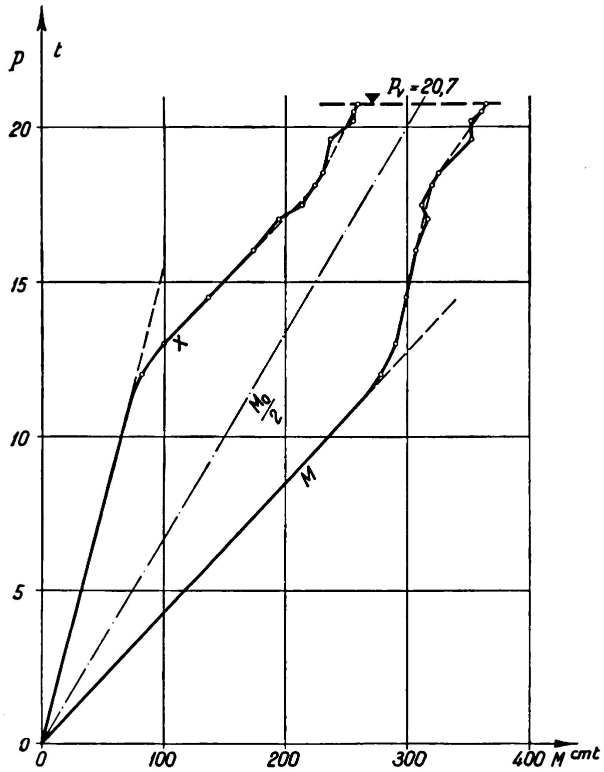


Fig. 2.
Variation of moments.
Experiments by Prof. Dr. Maier-Leibnitz.

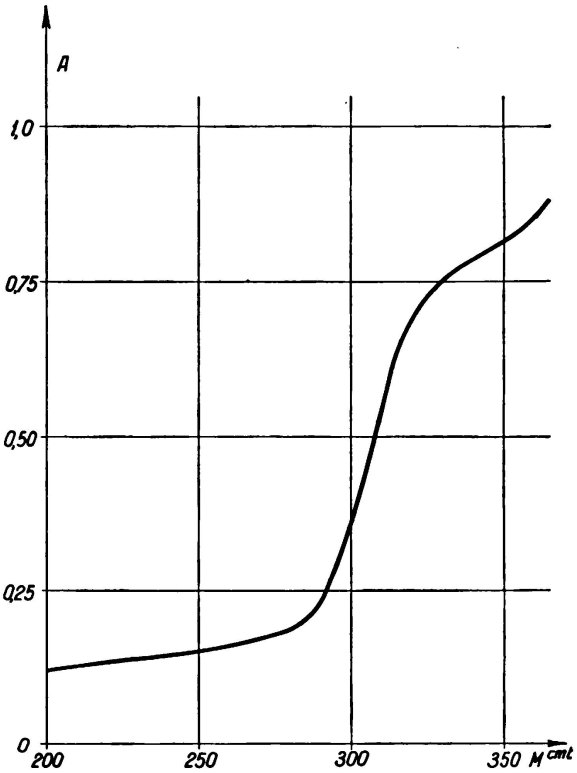


Fig. 3.
Total angular rotation A.

which correspond to smaller loads. Fig. 3 shows the variation in the total angular rotation A as calculated in this way: it will be observed that there is a clearly marked increase in the bending moments above approximately 315 cm-tonnes, that is to say in the zone which could no longer be observed in the comparative experiment with the simple beam.

In this way a single experiment allows the coefficients to be determined, enabling the variation in moment to be calculated with the help of the elastic condition of Equation (1) also for other conditions of span.

If it be desired to draw from this some conclusion as to the variation in carrying capacity, one more assumption must be made, namely, that in all cases the limit of carrying capacity will be reached when the maximum bending moment occurring under load attains to a particular limiting value. This assumption is plausible enough in itself, for unless it is justified the whole method of calculation of stresses underlying constructional practice must be invalid. The first consequence of this assumption is that we may derive from the elementary conditions of equilibrium existing in the central span, a comparison between the carrying capacity of the continuous girder (P) and that of the simply supported beam (P_o), namely

$$P : P_o = (M + X) : M \tag{2}$$

Since, however, these experiments by Prof. *Maier-Leibnitz* again indicate no complete equalisation of moment, the carrying capacity of the continuous girder is not double that of a simply supported beam. A continuous beam calculated in

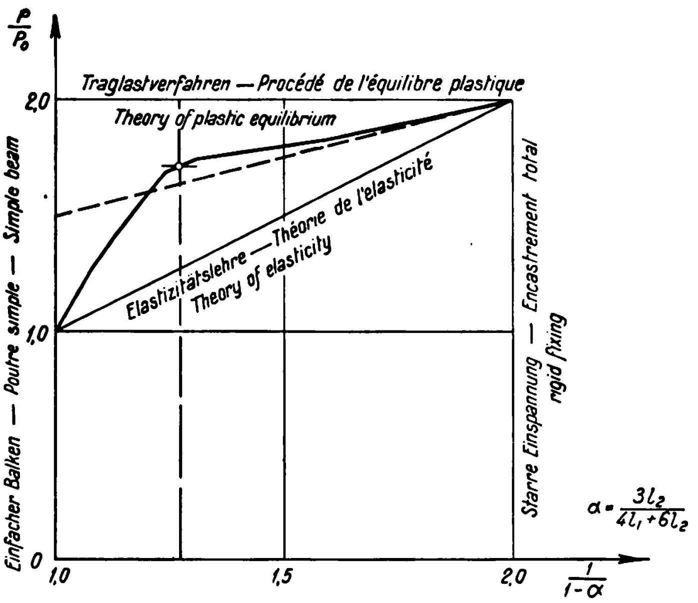


Fig. 4.

Calculated carrying capacities.

accordance with the equilibrium load method therefore possesses a smaller factor of safety against the attainment of the limiting load than is possessed by a simple beam.

Fig. 4 also shows the calculated relationship between the carrying capacities. The trend of this curve agrees broadly with that determined in the author's earlier experiments, and except in abnormal cases where the side spans are very large these values still lie somewhat above those indicated by a straight line (shown broken) which halves the difference between the equilibrium load method and the theory of elasticity. The proposal made by the present writer is, therefore, that the increase in carrying capacity of continuous beams of structural steel as calculated by the equilibrium load method, by comparison with the theory of elasticity, should be utilised, if at all, only to the extent of one half; and further, that this utilisation of the increase in factor of safety should be confined for the present to rolled joists as used in building construction.

I 8

Contribution to the Question of Utilising Plasticity in Continuous Girders Subject to Repeated Stresses.

Beitrag zur Frage der Ausnutzbarkeit der Plastizität bei dauerbeanspruchten Durchlaufträgern.

Sur la plasticité dans les poutres continues sollicitées dynamiquement.

Dr. Ing. K. Klöppel,[1]
Leiter der technisch-wissenschaftlichen Abteilung des deutschen Stahlbau-Verbandes, Berlin.

Dr. Hans Bleich,[2] making the assumption of an ideally plastic material, has written as follows: "A statically indeterminate system can be made to withstand an infinitely repeated succession of loadings provided that the indeterminate quantities in the system can be so chosen as to produce a condition of self-stressing characterised by the fact that at any point the sum of this self-stress and the maximum stress calculated from the law of elasticity remains just below the yield point." This statement requires to be checked by fatigue tests, in view of the fact that frequency of loading plays no part in the matter.

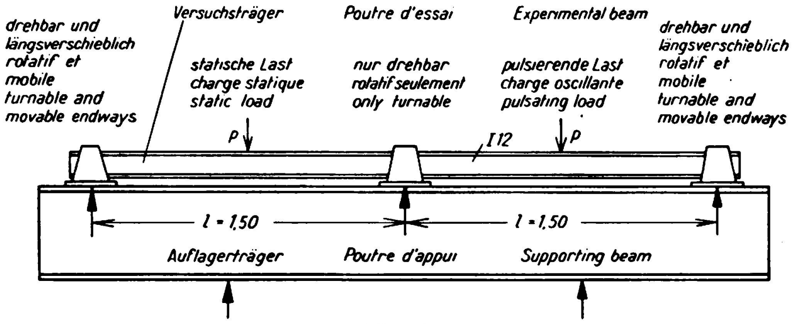


Fig. 1.

Arrangement of experiment.

For this purpose use was made of an unperforated beam (I 12) carried on three supports across spans of 1.50 m each, made of commercial structural steel (Fig. 1). All the bearings were capable of taking either a tensile or compressive load, and in addition the outer bearings had provision for longitudinal movement[3].

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[1] Abridged contribution.
[2] Der Bauingenieur, 1932, No. 19/20.
[3] The experiments were carried out in the Materialprüfungsanstalt at Stuttgart (Prof. *Graf*).

Bleich's statement rests on the assumption that a beam which is capable of carrying loads under one particular kind of loading out of many, is also necessarily capable of doing so when the types of loading are altered as many times as is desired.

The loadings applied are shown in Fig. 1. To suit the characteristics of the machine the left hand load was made steady and the right hand load made to pulsate about ten times a minute in excess of an original value of 200 kg. There was, therefore, no intermediate condition of complete freedom from load.

The magnitude of the load P was at first fixed in such a way that according to the theory of elasticity the yield point ($\sigma_F = M : W$) would be reached in the most heavily stressed section. This gave $\sigma_F = 2{,}420$ and $2{,}730$ kg per sq. cm for the respective flanges of the two beams 12 m long, from which the specimens, each about 3 m long, had been prepared.

According to Bleich, if the most favourable condition of internal stress is assumed (Fig. 2) the moment over the support and in the flange would tend to

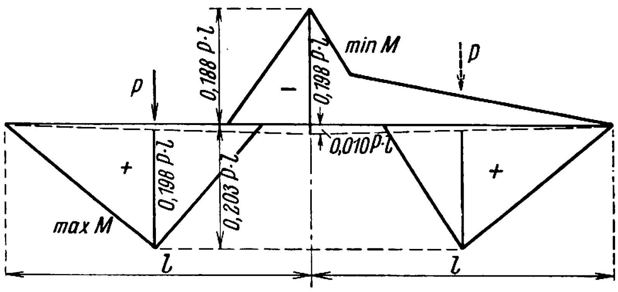

Fig. 2.

equalise, and in the determination of P the yield point would be exceeded only by about 2.5 %, because in both cases of loading an almost "natural compensation of moment" occurs.

The beam withstood 700,000 changes of load with P = 4,210 kg without showing signs of premature breakage through fatigue. The elastic deflection (which could be accurately read on a frame to within $^1/_{100}{}^{th}$ of a millimetre) corresponded with the calculated values, and the permanent deflections were practically zero. An internal stress effect was therefore ruled out.

The loads P were now increased in the same beam to such an extent that the yield point was exceeded by about 20 %. Under this loading again the beam withstood 630,000 further changes of load. The deflections increased only a little faster, in relation to the load, than had been the case with the first type of loading. The experiment was discontinued since again no fatigue breakage was to be expected. The residual deflections reached only about 15 % of their calculated values which had been arrived at on the following considerations:

In order to obtain an internal stress moment over the central support of 0.01 P · 1 it is necessary that a force of 0.01 P should be applied in the statically determinate system from which the cantilever beam is derived, and this results in a deflection of the supporting edge which is given by

$$f = \frac{0.01 \cdot P \cdot 2\,l^3}{3 \cdot E \cdot J}$$

The same deflection is produced in the statically determinate system if the deflection due to cold working in the central field amounts to $\frac{f}{2}$. Taking $E = 2100$ t/cm² and $I = 328$ cm⁴ we have

$$\frac{f}{2} = \frac{0.01\,P \cdot 150^3}{3 \cdot 2100 \cdot 328} = 0.0165\,P.$$

The cold-deformed girder, considered as a beam on three supports, is further bent elastically in accordance with the area of the moment diagram for internal stress, so that f/2 is diminished by an amount

$$\delta = \frac{0.01 \cdot P \cdot l^3}{16 \cdot EJ} = \frac{0.01 \cdot P \cdot 150^3}{16 \cdot 2100 \cdot 328} = 0.0031\,P.$$

The permanent deflections at the centre of the left hand field corresponding to the conditions of internal stress are given in millimetres by

$$\delta_{bl} = (0.165 - 0.031)\,P = \mathbf{0.134\,P.}$$

The condition of internal stress cannot occur until the yield point σ_F is reached at the centre of the field, and we then obtain

$$P \geqq \frac{W \cdot \sigma_F}{0.203 \cdot l}$$

owing to the compensating effect of the internal stress, P may be increased to

$$P^1 = \frac{0.203}{0.198}\,P = \sim 1.025\,P$$

The corresponding permanent deformation is then

$$\delta_{bl} = 0.134\,P^1$$

and this must not be allowed to increase even under loading many times repeated.

The elastic deflections for P = 1 tonne at the centre of the left hand field, when the latter alone is loaded (case A), are given by —

$$\delta_{el} = 0.734 \text{ mm}$$

and when both the fields are loaded (case B) —

$$\delta_{el} = 0.446 \text{ mm}.$$

Two of the experiments carried out will be briefly discussed. The loads P amounted to 5.04 and 5.83 tonnes. The yield point was $\sigma_F = 2.420$ kg/cm², and the section modulus $W = 53.1$ cm³. The load at which the yield point was reached had now been exceeded by 1.2 and 1.38 times, and in either case over 500,000 changes of load were made without any appearance of fatigue breakage. The deflections given in Table I were measured at the middle of the left hand field under both conditions of loading, A and B, and also obtained by calculation:

Table I.

Load	Loading	$\delta_{el} + \delta_{bl}$ measured	$\delta_{el} + \delta_{bl}$ calculated	δ_{bl} measured	δ_{bl} calculated
5.04 Tonnes	A	3.65 mm	**4.37** mm	0.18 mm	**0.67** mm
	B	2.49 „	**2.92** „		
5.83 „	A	**5.25** „	5.055 „	**1.68** „	0.775 „
	B	**4.75** „	3.375 „		

Under a load of P = 5.04 tonnes the measured values are below those calculated. The actual permanent deflection is very small, so that although the yield point is exceeded by 20 % in the flanges the conditions of internal stress has not yet been called into operation at all. With P = 5.83 tonnes the reverse is true, for in this case the actual deflections are the greater, and the difference is particularly large in the case of permanent deflections. It is apparent here — despite the contrary implication of Bleich's principle — that under loading B additional internal stresses are produced, for in the right hand field permanent deflections occur without any permanent increases. Moreover the moments over the support and in the flange are practically equal. The permanent deflections do not however continue to increase when the experiment is continued. The implication is that structures which are designed to resist bending when calculated in accordance with Bleich's principles possess an additional margin of safety, even under fatigue stresses. This conclusion is attributable to the nonhomogeneous nature of the bending stresses: if the yield point is reached at the edges of the cross section resistance to permanent deformations is still afforded by the remaining portions which are stressed only elastically, and on this account an increase in the strength of 16 % may be expected. This figure is given when W is replaced by 2 S_x (where S_x is the statical moment of one half of the cross section of the beam referred to the x-axis). This effect is further increased by the fact that, as a rule, the yield point in the web is greater than in the flanges, and yet again through phenomena of restraint and through the operation of an upper yield point. Finally, rolling stresses will tend to hinder the development of permanent deformations up to a certain stress which lies above the yield point.

Only the Case A of loading was examined, because this gives a large difference between the moment over the support and that in the span. With P = 6.28 tonnes — a value which would imply that the yield point was exceeded to the extent of 1.3 times ($\sigma_F = 27.3$ kg/mm^2) — the permanent deflection amounted to only 1.6 mm, whereas it would have to be 5.75 mm in order to correspond with a condition of internal stress such that the moments were equalised (moment over support = $^3/_2$ (0.203—0.094) P l = 0.072 P l). The elastic deflection was measured as 4.6 mm, corresponding closely with the calculated amount and being, therefore, smaller than the permanent deflection necessary for equalisation of moments. After more than one million changes of load the girder bent sideways, but no fatigue fracture occured.

The theoretical and practical knowledge now available can be applied with a view to greater economy in the design of continuous beams subject to fatigue stresses: but only on condition that such beams contain no notches — as, for instance, holes or fillet seams — and this is a limitation which to a great extent excludes the use of riveted designs and riveted connection. On the other hand, rolled beams unperforated by holes and having perfect butt welded joints free from surface notches may properly be adopted under conditions of fatigue stress, using this term in the sense of higher stress values in the "carrying capacity" method of design.

The application of that method may, however, be interfered with by premature instability of the beam. Further, it may happen that when the most favourable condition of internal stress is one which implies very large permanent deflections — as is the case in those instances which are the most important from an economic point of view — such condition of stress will be attained before the designed amount of loading has been imposed,[4] and it will then, of course, be impossible to count upon an equalisation of moments.

[4] *Stüssi* and *Kollbrunner:* Bautechnik, 1935, No. 21. *Maier-Leibnitz:* Stahlbau, 1936, No. 20. *Klöppel:* Stahlbau, 1937, No. 14/15.

I 9

Formulae for the Stability of Eccentrically Loaded Steel Columns.

Formelmäßige Lösung des Stabilitätsproblemes exzentrisch gedrückter Stahlstäbe.

Les formules de la stabilité des barres excentriquement comprimées.

Dr. Ing. K. Ježek,

Dozent an der Technischen Hochschule Wien.

The classical problem of stability of a straight column subjected to a concentric compressive load was satisfactorily solved by the researches of *Euler*, *Engesser* and *Kármán*.[1] This type of loading represents an ideal which hardly never occurs in practice, because even the smallest deviations from the assumptions underlying it — as, for instance, the unavoidable and vanishingly small eccentricities of imposition of the load, or curvatures of the column — cause additional bending which may under certain conditions result in quite a notable reduction of the carrying capacity. In a steel column subjected in this way to axial compression and bending stresses there is also the danger of the occurrence of unstable equilibrium, at any rate in cases where the load has already increased to a value involving permanent deformations. It is on account of this statical condition that the problem of determining the carrying capacity differs fundamentally from that of the buckling of a straight bar.

This special problem of stability was first examined by *Kármán* both theoretically and experimentally in connection with his well known experiments on buckling with very small eccentricities of loading.[1] Here *Kármán*, beginning from the energy-line for a particular kind of steel, developed a powerful graphical method of integration for solving the differential equation of the bending moment line, but this portion of his work remained for a long time unnoticed. The approximate method, developed some thirteen years later by *Krohn*,[2] shared the fate of *Kármán*'s investigations on account of the difficulty of the calculations involved, and still more because its results would not admit of quantitative evaluation. Some years later *Roš* and *Brunner* developed an approximate graphical method, and showed the results of this in a diagram which, for the first time, expressed quantitatively the relation between carrying capacity, eccentricities and

[1] *T. von Kármán:* Untersuchungen über Knickfestigkeit. V.D.I., No. 81, 1910.

[2] *R. Krohn:* Knickfestigkeit. Bautechnik, 1923.

slenderness for a definite law of deformation.[3] Finally *Chwalla,* following up *Kármán's* line of thought, gave the strict solution applicable to columns with any desired eccentricity of loading.[4]

All these investigations were exposed to the notable defect that results could be obtained from them only after very tedious calculation, and could be represented only in the form of a diagram or of a table of figures. When it is further remembered that a large number of types of steel and shapes of cross section exist which each require different diagrams, it will readily be understood that this circumstance not merely makes them difficult to apply in practice but may actually prevent such application; the clearest indication of this fact being that in the official regulations of nearly all countries (so far as the author is aware the only exception is the Swiss regulations) no account is taken of the new and fully established knowledge now available in reference to the design of eccentrically compressed columns of steel — obviously because no simple formula with a theoretical basis is available.

The author wishes briefly to indicate how a solution to this important problem in steel construction may be expressed as a formula. First of all, the law of deformation for the types of steel now in use may for the present purpose be replaced by an ideally plastic line, having regard to the circumstance that fixation can occur only in extremely short columns, such as hardly ever occur in practice, having a ratio of slenderness of $\lambda < 20$. The assumption that *Hooke's* law is valid as far as the yield point is amply supported by careful compressive experiments carried out by the German Steelworks Committee.[5] If, further the line of bending moment is replaced by a sine curve, the author's solution by formula[6] is at once obtained for the simplest case of a rectangular cross section, and the results correspond closely with the values obtained from the exact bending moment diagram (the maximum error being 3%).[7] From this the yield point may be derived in the usual way by means of a compression experiment.

Finally, the author has examined, under the same assumptions, the behaviour under load of eccentrically compressed steel columns in relation to the shape of cross section,[8] in reference to which some clarification is required, having regard to the thin walled sections used in steel work, concerning the conditions of stability both in and at right angles to the moment diagrams. Fig. 1 shows the critical axial stress σ_{kr} for the most frequently used type of St. 37 with an

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[3] Cf. reports of First Meeting and First Congress for Bridge and Structural Engineering, Vienna, 1928, and Paris, 1932.

[4] *E. Chwalla:* Theorie des außermittig gedrückten Stabes aus Baustahl. Stahlbau, 1934. (Summary statement of the strict graphical solution.)

[5] *W. Rein:* Versuche zur Ermittlung der Knickspannungen für verschiedene Baustähle. No. 4 of Reports of Ausschuß für Versuche im Stahlbau. J. Springer, Berlin, 1930.

[6] *K. Ježek:* Näherungsberechnung der Tragkraft exzentrisch gedrückter Stahlstäbe. Stahlbau, 1935. — Die Tragfähigkeit axial gedrückter und auf Biegung beanspruchter Stahlstäbe. Stahlbau, 1936.

[7] *K. Ježek:* Die Tragfähigkeit des exzentrisch beanspruchten und des querbelasteten Druckstabes aus einem ideal plastischen Stahl. Report of Akademie der Wissenschaften, Vienna, Math.-naturw. Kl., Abt. IIa, Vol. 143, No. 7, 1934.

[8] *K. Ježek:* Die Festigkeit von Druckstäben aus Stahl. Julius Springer, Wien 1937.

eccentricity of m = 1 (that is to say, the axial load coincides with the centre of gravity). Beyond this no equilibrium between external and internal forces is possible, depending on the slenderness λ and on the shape of cross section. It is recognised that the effect of the shape of section in small columns is important, but its importance is rapidly lost when the slenderness increases or the eccentricity decreases. The most favourable performance is given by columns of cruciform section, and the least favourable by columns of I and T cross section. In the last two cases, in particular, the critical stress is only slightly above the limiting stress σ_n for the elastic condition.

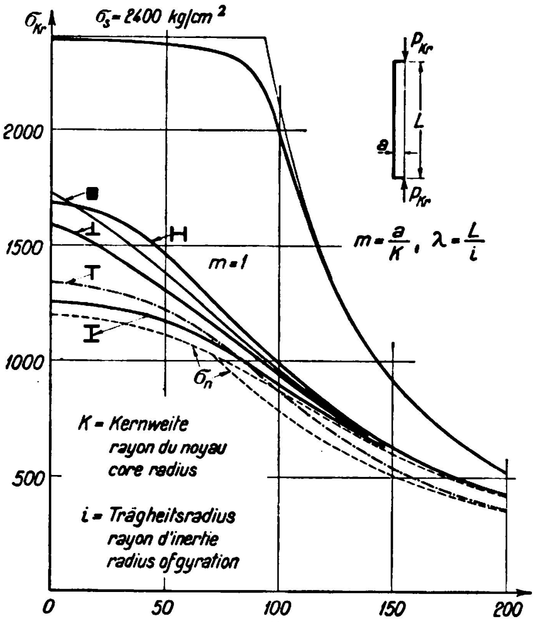


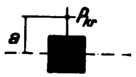
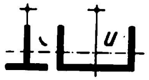
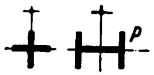
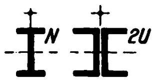
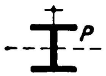
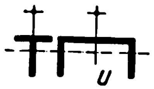
Fig. 1.

The line of critical stress drawn for m = 0,01 (the σ_k line) is almost independent of the shape of cross section, and in the case of columns of medium slenderness this clearly indicates the effect of exerted on the carrying capacity by a vanishingly small eccentricity of $^1/_{100}$ of the core radius.

Finally, it is possible, by considering columns of any desired cross section, to obtain approximate formulae as in Table 1 applicable to any type of steel, the coefficients μ_1 und μ_2 being dependent on the type of section. In the case of columns of T cross section below a particular value of axial stress a second formula, similarly derived, is to be used. This basis of calculation applies quite generally to steel columns subject to axial pressure and to bending, when the measure of eccentricity (in the sense explained) is taken as the relation between the bending moments (referred to the undeformed axis of the column) and the axial load. For further explanations the author would refer to his published papers. For $\mu_1 = 1$ and $\mu_2 = 0$ there is available the formula corresponding to the σ_n line (shown as a broken line in Fig. 1).

For the purpose of the practical design of columns which are loaded *concentrically in the "design" sense of the word,* the author would propose to adopt as a value of the unavoidable eccentricity $^1/_{100}$ of the core radius. There will then be obtained, with $m_o = 0{,}01$ and $\mu_1 = 1$, $\mu_2 = 0$, a simple *"buckling formula"* which will correspond to the fact (now accepted without question) that the limit of compression σ_s represents the limit of buckling stress, so that practically, in the case of very slender columns, we arrive back at the *Euler* formula. It should be stated that these formulae are well confirmed by the published experimental results, and that they ensure a safe estimate of carrying capacity while reducing the amount of calculation required. The problem may thus be regarded as having been satisfactorily cleared up, both from a theoretical and from a practical standpoint.

Table I. Bases of Calculation for Eccentrically Loaded Steel Columns.

Cross Section	Formula for Critical Slenderness	Region of Equilibrium	Co-efficients		Remarks
			μ_1	μ_2	
	$\lambda^2 = \frac{\pi^2 E}{\sigma_{Kr}}\left[1 - \mu_1 \frac{m\,\sigma_{Kr}}{(\sigma_s - \sigma_{Kr})}\right]\left[1 - \mu_2 \frac{m\,\sigma_{Kr}}{(\sigma_s - \sigma_{Kr})}\right]$	Unlimited $0 \leq \sigma_{Kr} \leq \sigma_s$	0.5	0.5	L = Length of column F = Area of cross section $W_{1,2}$ = Resisting moment of bending compression or bending-tension edge i = Radius of gyration $\lambda = \frac{L}{i}$ = Slenderness a = Eccentricity $m = \frac{a\,F}{W_1}$ = Measure of eccentricity σ_s = Yield point E = Modulus of elasticity σ_{Kr} = Critical stress P_{Kr} = $F \cdot \sigma_{Kr}$ = Carrying capacity
			0.5	0.5	
			0.4	0.4	
			0.9	0.1	
			0.9	0.1	
		$\frac{\sigma_{Kr}}{\sigma_s} \geq \frac{W_1 - W_2}{W_1 + W_2}$	0.8	0.2	
	$\lambda^2 = \frac{\pi^2 E}{\sigma_{Kr}}\left[1 - \mu_1 \frac{W_1 m\,\sigma_{Kr}}{W_2(\sigma_s + \sigma_{Kr})}\right]\left[1 - \mu_2 \frac{W_1 m\,\sigma_{Kr}}{W_2(\sigma_s + \sigma_{Kr})}\right]$	$\frac{\sigma_{Kr}}{\sigma_s} \leq \frac{W_1 - W_2}{W_1 + W_2}$			

I 10

The Effect of Unequal Eccentricities on the Carrying Capacity of a Steel Column.

Der Einfluß einer Ungleichartigkeit der Fehlerhebel auf die Tragfähigkeit einer Stahlstütze.

L'influence des erreurs de centrage sur la résistance des colonnes métalliques.

Dr. techn. J. Fritsche,
Professor an der Deutschen Techn. Hochschule, Prag.

In actual steel work the incidence of a load in a column is in most cases quite uncertain and difficult to determine; the assumption of concentric loading, or of a loading which is eccentric by an equal amount at each end of the column, can be regarded only as a means of creating a standard of comparison of load by reference to which the effects exerted on the compressive force by the shape of cross section, length of column and magnitude of the eccentricities may be studied. The last mentioned of these quantities is determined, in the case of a column in a framework or in a steel skeleton, by its rigid connection with the adjacent members or by constructional details of such connections, and it can be precisely stated only if what are known as the secondary stresses in the frame are determined. The calculation of the latter is in any case very tiresome and requires lengthy calculations such as cannot be expected from a designing engineer; while moreover there is room for serious question whether its influence on the safety of the structure, having regard to the plastic behaviour of the material, can properly be compared with that of the primary stresses.

The approximate magnitude of the secondary stresses and the nature of their distribution within the column must now be regarded as established,[1] and the results of theoretical investigation have been confirmed by measurements of elongation on finished work from which it is known that the eccentricities of loading at the two ends are usually different — so much so, indeed, that the line of pressure in the compression flange usually intersects with the axis of the column. The solution to this more general problem, assuming no limit to the elastic behaviour of the material, offers no particular difficulty; the question is no more than an ordinary problem of stresses, and the safety of a column stressed in this way can be perfectly well guaranteed by ascertaining that the stresses

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[1] *M. Roš:* Nebenspannungen infolge vernieteter Knotenpunktverbindungen eiserner Fachwerkbrücken. Report in Group V, Technical Commission of the Verband Schweiz. Brücken- und Eisenhochbaufabriken; June 1922.

lie within the permissible limits. It is only when the phenomena of plastic deformation are taken into account that the problem becomes complicated, giving rise to questions of critical loads; hence design by reference to a permissible stress is not a method which can be applied to all columns with equal assurance.

A predominant part in the treatment of any problem of plasticity by calculation is played by the "condition of yield", this being an analytical expression for the circumstances under which the steel changes from the elastically fixed to the plastically deformable condition. Under uniform conditions of stress, there is now agreement as to the nature of this phenomenon, but there is still room for question whether it can be assumed to operate in the same way where the condition of stress is not uniform. A newer hypothesis assumes that knowledge of the local conditions of stress is not sufficient for the prediction of yield phenomena, and that the question of risk of yielding can be decided only by considering the condition of stress over a larger region. On the basis of this new concept of "yield condition" it is now possible, by calculation, to obtain considerable knowledge as to the actual carrying capacity of a column which is subject to different accidental eccentricities of loading, taking account of the actual shape of its cross-section.

Assuming equal eccentricities at the two ends of the column the maximum moment occurs at the centre of the height and coincides, therefore, with the position of y_{max}. The carrying capacity of the column disappears if the resistance is so far weakened at this place by the sudden operation of the yield phenomenon that when an increase in the load occurs it can no longer contribute appreciably to the equilibrium between external loads and internal resistances. The lateral deflections thereupon increase very rapidly, and a new equilibrium can be reached only when the material has become set. When the eccentricities of load at the two ends are unequal y_{max} moves away from the middle of the column towards the end where the accidental eccentricity is the greater (Fig. 1). As long as it remains within the length of column l there is no important difference from the former condition, but when the ideal maximum of the elastic line falls outside the length of the column, so that the column receives its maximum bending moment at the end amounting to $P \cdot p_1$, then quite different phenomena appear (Fig. 2). The fact that the yield condition at the end of the column is satisfied then does not imply the disappearance of the carrying capacity, for the column cannot yet fail under load either laterally or in such a direction that unacceptable amounts of compression will occur along its axis. Yielding at the supported ends of the column cannot render its equilibrium unstable, for in such a case the column must for the most part retain its shape and length, no change in these magnitudes being possible without the expenditure of energy. The fact of the column being deformed over the whole of its length in an exclusively elastic way affects the deformation at the place where it has yielded, so that in this case the fulfilment of the "yield condition" means only that plastic deformation is about to take place.

As the load increases the support undergoes further deformation if the marginal conditions are different; and in view of what has been said above it is noteworthy that in such a case the carrying capacity does not reach its limit until the maximum moment at the end of the column coincides with the maximum of the elastic line, and the tangent to the former must then coincide there with the direction of the force.

It is of course difficult to give a correct and satisfactory explanation of the phenomena which now occur, and this can be done only approximately. If P_1 denotes the load which must exist at the end of the column in order to satisfy the "yield condition", then the stress at the point of yield is supplemented by

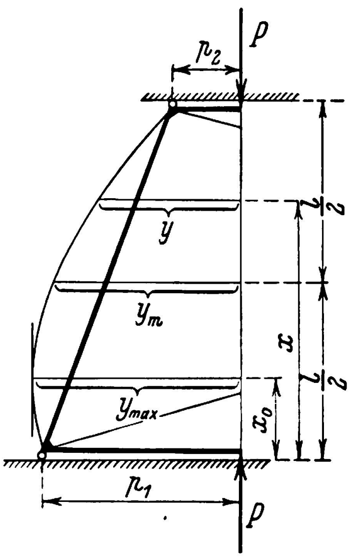

Fig. 1.

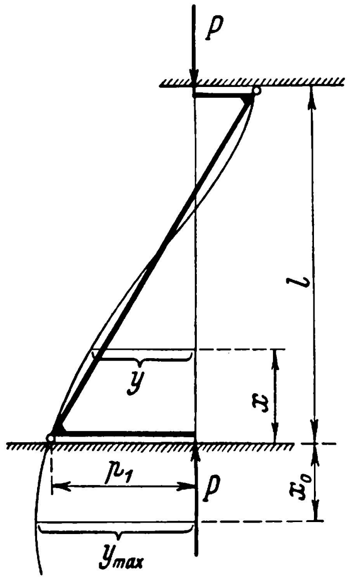

Fig. 2.

the compressive force $P - P_1$, and on the basis of the experiments carried out by *Hohenemser*[2] and *Prager*[3] (who applied torsion as far as the yield point and caused this to be followed by tension) it is to be inferred that when bending occurs as far as the yield point and is followed by compression the bending moment supportable by the cross section is gradually diminished. In that case the load in the column could be increased only to the extent that its end has formed a "yield hinge". Thus the yielding has the effect of concentrating the increase of load along a particular direction, for owing to the failure that occurs at the point of yield the moment at the end of the column due to the increase in load $(P - P_1)\, p_1$, together with a gradually increasing share of the yield moment $P_1 p_1$ already sustained, must be resisted by the structure in some other way. The loss of bending resistance at the end of the column may be expressed by imagining that the pre-existing external load at the place where yielding occurs is supplemented by two moments which adapt themselves to the curvature of the elastic line corresponding to the new marginal conditions. The increase in the end moment of the column as P increases to P_1 opposes a moment of rotation in the opposite direction equal to $(P - P_1)\, p_1$, and the reduction in the yield moment through the imposition of the longitudinal force may be brought about

[2] *K. Hohenemser:* Neuere Versuchsergebnisse über das plastische Verhalten der Metalle. Zeitschrift für angew. Math. und Mech., 1931, p. 423.

[3] *K. Hohenemser* and *W. Prager:* Beitrag zur Mechanik des bildsamen Verhaltens von Flußstahl. Zeitschrift für angew. Math. und Mech., 1932, p. 1.

by a moment ΔM which may be calculated from the assumed condition of yield. Owing to lack of space it is not possible to quote here the very extensive calculations that arise, but these are given in the present author's contribution to

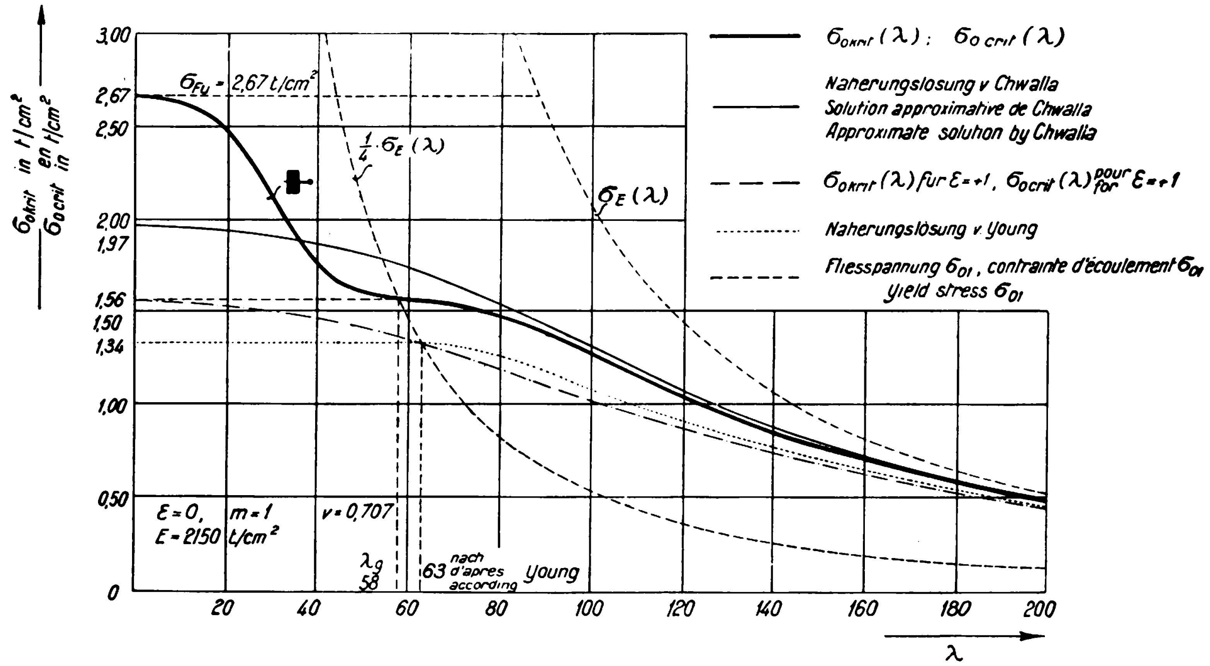

Fig. 3.

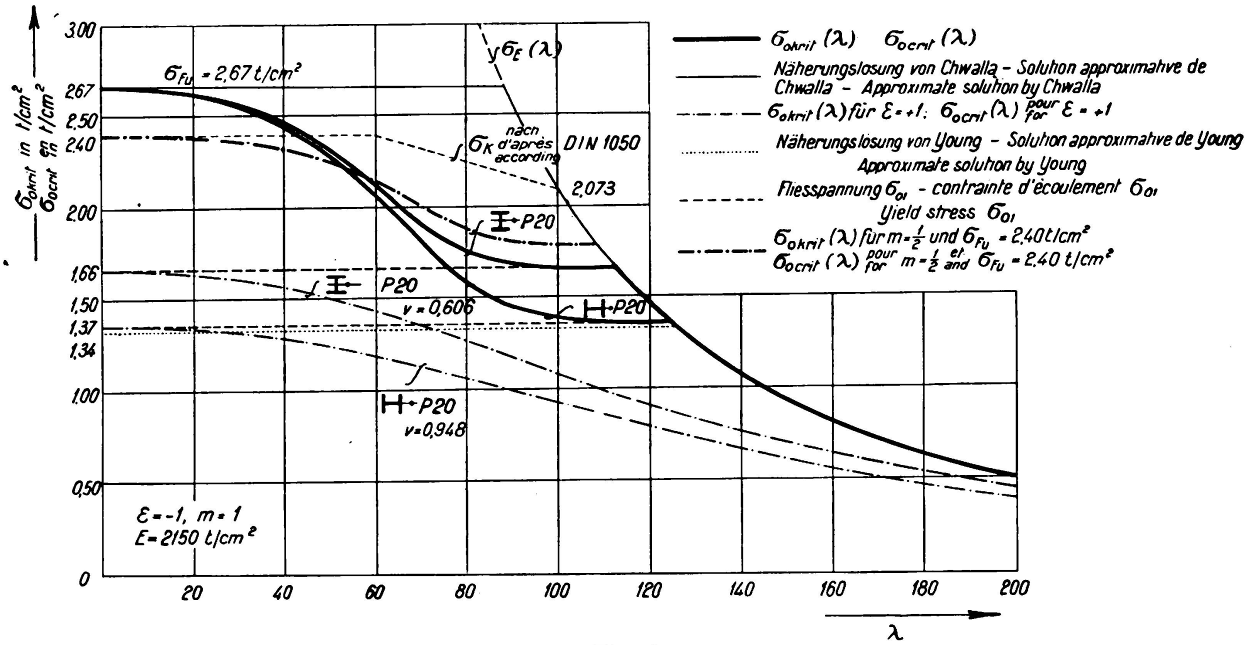

Fig. 4.

the journal, Der Stahlbau[4]. Figs. 3 and 4, taken from that publication, relate to the case where the eccentricities of loading at the two ends are equal but on opposite sides, and serve to show that the results so obtained differ quite

[4] *J. Fritsche:* Der Einfluß einer Ungleichartigkeit der Fehlerhebel auf die Tragfähigkeit außermittig gedrückter Stahlstützen. Der Stahlbau, 1936, Nos. 23 and 24.

appreciably from the cases hitherto examined in which the load is either concentric with the column or is equally eccentric at each end.

As already remarked, the conditions for a column which is built-in rigidly at the ends are such that $\varepsilon = \frac{p_1}{p_2}$ is approximately -1. The conditions where $\varepsilon > o$ or $\varepsilon = 1$ constitute rare exceptions, and moreover in such cases the eccentricities are usually small so that it is not correct to base the method of designing columns on exceptions of this kind. Where $\varepsilon = -1$ the lines $\sigma_{o\,krit}(\lambda)$ approximate very closely to the lines $\sigma_K(\lambda)$, the latter expressing the carrying capacity under concentric loading; hence it is correct, in the author's opinion, to make use of a "buckling stress line" — as is done, for instance, in the

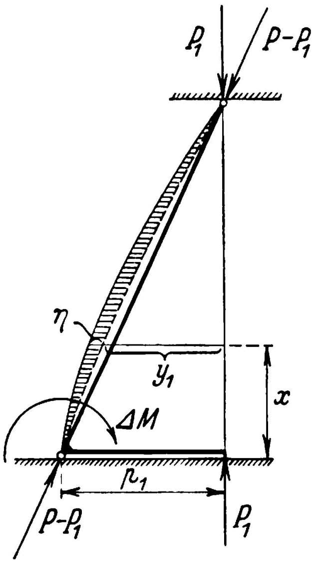

Fig. 3a.

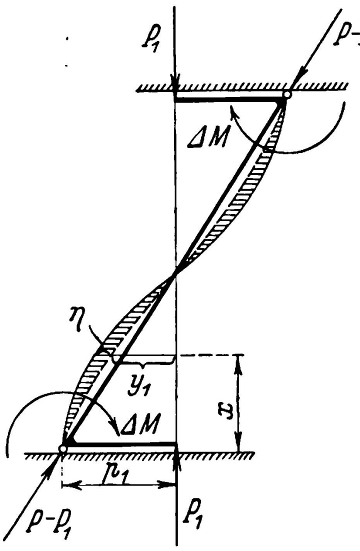

Fig. 4a.

German regulations DIN 1050. It would then be only a question of calculating maximum values for the usual eccentricities, or of estimating them from measurements, and of taking account of these when plotting the line $\sigma_{krit}(\lambda)$ for all columns. In such a case p would not normally be introduced, but the ratio, denoted by m, between p and the core radius of the cross section k. The value $m = 1$ would certainly be too high, and it would be preferable to use $m = 0.5$ in the present circumstances. In this way there would be obtained a "buckling stress line" which differs only slightly from that given in DIN 1050, but it would be advisable to deviate from the *Euler* line earlier than at 2073 kg/cm² — as, for instance, at about 1800 kg/cm².

So far as the author is aware no experiments for checking these calculated results are on record, and it is indeed not easy to carry out such experiments because in order to do so some method of supporting the experimental column must be devised which will enable the load to be centred within certain limits, as occurs in struts which are rigidly fixed at the ends. It is here a question of reproducing the varying marginal conditions of such a strut in the testing laboratory. How far this may be possible at all the author does not know; but it would be very desirable to attempt such experiments, as they would afford further insight into the actual behaviour of the compression members in a structure.

I 11

The Physics of the Tensile Breaking Test.

Zur Physik des Zerreißversuchs.

La physique de l'essai de rupture par traction.

Dr. phil. W. Späth,
Wuppertal-Barmen.

Even to-day the basis of the testing of materials continues to be the tensile breaking test. Fundamentally it appears so simple a matter to indicate the connection between imposed load and resulting deformation of the test specimen that the physical condition of the loading procedure has tended to be relegated into the background compared with questions of a practical technical nature in the arrangement of the testing devices. The interpretation of the diagrams obtained with the testing apparatus now available nevertheless throws open a whole series of questions, with which a large proportion of the literature is concerned. For instance, even to-day the significance or insignificance of the elastic limit and of the upper and lower yield points are subjects for dispute. Again, the results of fatigue tests indicate that the conventional characteristics of materials as given by the tensile breaking test stand in no definite relationship to the fatigue strength which is of so decisive importance.

For the purpose of testing a material, or whole constructional members, the parts to be tested are clamped into a testing machine and are subjected by some means or other to gradually increasing load. Thus the test specimen is caused to participate with portions of the testing machine (which may be either rigid or sprung) in a common flow of forces. Closer examination indicates that the self-vibrating effect of the machine, the compressibility of the pressure fluid, and also the error of the apparatus used for measuring the force in the testing machines which are usual at the present time, are all factors which it is improper to neglect — the elastic yield of the testing apparatus usually being much greater than the deformation of the test specimen itself.[1 2]

The effect of all this on the loading process is seen in Fig. 1. Here the line OA represents the increase in load in relation to increasing deformation of a specimen; in the loading apparatus itself there occurs a process of loading which may be represented by the straight line CA. At the point A statical equilibrium exists between the elastic load of the specimen and the elastic reaction of the loading apparatus. The specimen has deformed by the amount O B through

[1] *W. Späth:* Arch. Eisenhüttenwesen, Vol. 9, 1935/36, p. 277.

[2] *W. Späth:* Meßtechnik, Vol. XII, 1936, p. 21.

the imposition of the load AB, and the corresponding deformation of the loading device is represented by CB. Two angles α and β indicate the magnitude of the spring constants of the test specimen or of the loading apparatus respectively. If, now, a plastic deformation of the test specimen from A to D suddenly takes place, the specimen will endeavour to release its load along the line DO', and at the point E where this line cuts the load line of the testing machine the system again comes into equilibrium, for at this point the force acting upon the specimen again becomes equal to the elastic reaction of the loading device. Through the plastic yield from A to D two effects come into play; the original stress is diminished by an amount corresponding to AA', while at the same time the externally measured deformation of the specimen is increased by the amount A'E. It will at once be apparent that this process depends not only on the specimen but also, to a very large extent, on the elastic properties of the testing machine — that is to say mainly according to the inclination of the lines CA —

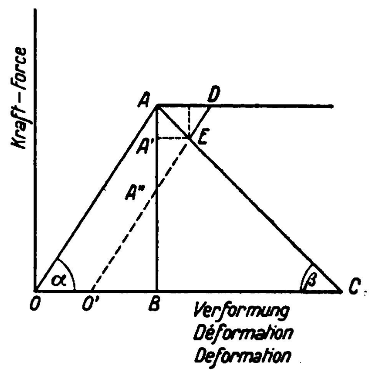

Fig. 1.

the results differ widely. In a very "soft" machine which suffers a large amount of self-deformation in order to develop the force AB, the line CA is practically horizontal in respect of very small deformations of the test specimen here in question; hence an increase in the length of the specimen by AD occurs under approximately constant stress and the externally measured increase in deformation corresponds to the amount AD. A "soft" machine of this kind may also be designated as a "delayed-action" machine since the yield, when it once begins, continues to develop under a load which remains constant. The conditions are altogether different in a "hard" machine characterised in the limit by indefinitely large spring constants as measured by the vertical AB; here the load drops from A to A" owing to yielding of the specimen, while the externally measured deformation of the specimen remains unaltered. A "hard" machine of this kind may also be referred to as a "relaxation" machine, for in such a case the initial deformation is maintained and the occurrence of yield is attended by a reduction in the load. The machines in use to-day lie between these two limits, and their indications cannot be compared as between one machine and another unless account is taken of the elasticity proper to each.

These theoretical considerations have been confirmed by a series of experiments effected by the author, and the problems in question are now being pursued in several research institutions in view of their fundamental importance for the testing of materials.

As early as in the publication by the author denoted under[1] below it was suggested that an existing testing machine might be rendered "soft" by introducing a spring into the flow of force. Experiments of this kind were carried out by *G. Welter*[3] and the result corresponded with expectation. The introduction of a spring to make the machine artificially "soft" must, according to the argument above, have the effect of allowing a yield once begun to continue while the stress remains unaltered. For instance, a material which under the usual forms of tests exhibits an upper and a lower yield point will, when placed in a machine of this kind, show no reduction of stress to the lower yield point, and this was in fact found to be the case.

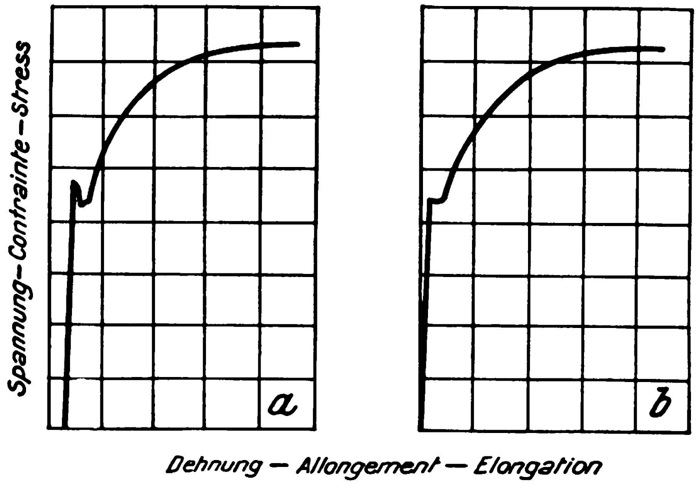

Fig. 2.

Yield as produced with an ordinary test machine (a)
„ „ „ with a test machine with increased volume of water (b).

With very heavy loads it is not feasible to introduce a spring as the dimensions would have to be excessive. In the Losenhausen works at Düsseldorf, at the suggestion of Baurat *von Bohuszewicz*, a 60-ton hydraulic machine was accordingly rendered "soft" by connecting a large hydraulic accumulator to the pressure cylinder, and the result is shown in Fig. 2. On the left is the curve obtained by working the machine in the ordinary way, in which a clear development of upper and lower yield point is discernible. When the machine was artificially rendered "soft" by being connected to the pressure accumulator it was found that a second specimen of the same material gave the curve shown on the right, wherein it will be noticed that the yielding continues to progress under constant load as a result of the elasticity of the pressure water. A large number of further implication arise as to the dependence of upper and lower yield points on the conditions of testing, but these cannot be further considered here.

Some interest, however, attaches to a series of experiments lately concluded by the author from a precisely contrary point of view. When a machine is made very "hard" it may be hoped that considerably sharper phenomena may be caused

[3] *G. Welter*: Metallwirtschaft, Vol. XIV, 1935, p. 1043.

to occur in the loaded material.[4] It will be recalled that in rotary fatigue bending machines what are known as short period experiments are frequently carried out, in which the bending of the rotating bar is measured in relation to the load. The well known machine of *Schenck* at Darmstadt (and certain other machines) provide for the application of load by means of weights, and in this way the bending line gradually deviates from the straight. If such a machine is artificially rendered "hard" by applying the load through a spring considerably "harder" than the specimen itself, there is obtained a bending curve which approximates very closely to the breakage curve with an upper and a lower yield point. The stress drops off quite definitely from the "upper" to the "lower" range of loading, and in very plastic material — such as, for instance, aluminium — the load curve as a whole is made up of a large number of these jumps of load. It was further established that the sensitivity is sufficient to obscure even the important questions of notch effect. For details reference must be made to a publication which is about to appear.[5]

From these considerations there follow a number of important consequences for the further development of testing machines, and especially it will in future be necessary to produce considerably "harder" machines by taking systematic precautions in the design. Such machines offer the invaluable advantage that they give very sharp indications of the critical limits of load governed by the materials tested in the form of clearly discernible reductions in the stress. On the other hand the testing machines which are in common use at present possess so much resilience themselves that they tend to obscure these important transitions, sometimes to such an extent that the latter are quite undetectable.

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4 *W. Späth:* Metallwirtschaft, Vol. XVI, 1937, p. 193.

5 *W. Späth:* Z.V.D.I. Vol. 81 (1937), p. 710. — See also: *W. Späth,* Physik der mechanischen Werkstoffprüfung, Springer, Berlin 1938.

I 12

The Influence of Ductility of the Steel on the Stability of Structures.

Der Einfluß der Zähigkeit des Stahles auf die Stabilität der Stahlkonstruktionen.

Rôle de la ductilité de l'acier dans la stabilité des constructions.

F. Aimond,

Docteur ès sciences, Ingénieur des Ponts et Chaussées, détaché au Ministère de l'Air, Paris.

Ductility is the property which enables a material to undergo large deformations after the limit of its elastic region has been reached. In structural mild steel the large amount of strain occurring under this condition has no appreciable effect, from a mechanical point of view, on the texture of the metal. The zone of strain which exists, in this way, beyond the elastic region is known as the plastic region.

For a long time it has been recognised that the stability of steel structures is the result of small zones of plastic strain which are produced wherever the elastic strains are great enough to exceed the elastic limit, and the outcome of this is that the maximum stress actually existing in the material is lower than that calculated by the ordinary method based on elasticity. Hence the effect of ductility in the steel is, apparently, to increase the strength by eliminating those zones in which elastic deformation is excessive: a property which, suitably generalized, is now known as adaptation.

Adaptation, however, cannot be counted upon except in systems of stationary loads, wherein such permanent strains as tend to produce fracture are very infrequently repeated. Adaptation cannot be relied on where resistance to alternating loads is expected: it is known that in resisting loads of this kind the apparent extent of the elastic region is considerable, and that for each material there is, so to speak, a true elastic region within the elastic regions commonly so called. The former is known as the endurance region.

Thus the ductility of steel plays no part in stability under alternating loads, but through this phenomenon of adaptation it plays an essential part where stability against stationary or practically stationary loads is concerned.

Owing to the law of adaptation, the ductility of the steel becomes effective in any parts of a construction where, for one reason or another, the elastic limit has been reached. Consequently the zones of plastic strain are found in the neighbourhood of all points of geometrical or mechanical discontinuity, and these are very numerous. Zones of plastic strain also occur where the elastic strains are large. In a well designed structure, however, the zones of plastic strain are usually of very limited extent. In effect the surplus strength conferred by adaptative strains is due to inequality in the distribution of stress, and to the existence of some zones which are stressed less than others; in a well designed structure these zones will necessarily be limited.

From these considerations one conclusion may immediately be drawn: while the ductility of the steel may, indeed, be a phenomenon which plays an essential part in ensuring the stability of structures, it does nothing to improve the resistance of a well designed structure, but serves only to correct errors in design or deficiencies in the homogeneity of material, or compensate for settlement of the supports. In the author's opinion it would not, therefore, be wise to base a new method of structural calculation on the exploitation of the property of plastic deformation.

To say this is not to assert that methods so based must be rejected, and indeed the author is himself accustomed to make daily use of them. For various reasons, the forms actually given to structural members are not always those which would correspond to the most efficient possible use of the material, and it is a natural procedure to make use of the property of ductility of steel as a means of partly correcting the mechanical errors which result from faulty shaping of the material. As an explanatory example, consider the case of an arch, frame, or continuous gider: the best plan, if it were possible, would be so to design its constituent members that the elastic limit under a dangerous load would be attained at all points simultaneously, and the ductility of the material would, therefore, be of no use for purposes of design; but where the designer is compelled to adopt forms which, from the point of view of mechanical efficiency, are imperfect, he should avoid falling into the error of calculating them on the elastic hypothesis under stationary loads, which would merely be adding a second mistake to the first. The parts should, on the contrary, be calculated from the hypotheses of plasticity, so as to minimise in this way the loss of efficiency that attends a faulty choice of shape.

In the author's opinion, then, methods of calculation based on the theory of plasticity are to be regarded as a last resort, and one which should not be used except in calculating elements which are mechanically inefficient; nor, of course, should it be applied in respect of any but fixed loads. From this point of view it is to be wished that the methods at present in use might be codified with a view to arriving at simple formulae capable of being applied to the commonest problems of hyperstatical systems, especially arches and frames. It ought no longer to be the practice to calculate these common structural forms, when under fixed loads or only slightly varying loads, by other methods than those depending on the law of adaptation.

The author, for his part, relies on the following rule in designing any framework subject to loads which are either stationary or may be considered as such: any system of forces and stresses which will keep in equilibrium a given mechanical medium admits of realisation if account is taken of the occurrence of adaptation.

If, now, the system depends on a certain number of arbitrary parameters, every attempt should be made so to determine these parameters as to minimise the maximum value of the stresses occuring at the various points in the system. In other words, if an equilibrium is possible from the purely statical point of view, the construction will be stable under fixed loads without the necessity for enquiring whether the system of stresses as calculated is in fact the true system.

The principle which has just been explained has served the author as a guide in all the designs of frames on which he has been engaged, and it has proved itself a particularly useful aid in connection with those structures wherein, unlike the most usual case, the stresses are determined not by the magnitude of the strains but solely by the positions of the loads and the nature of the supports.

This is particularly true of mechanical systems in two dimensions, that is to say, where the stresses are propagated practically over a surface. The properties of such systems are closely related to the mechanical properties of surfaces. Now, when the mechanical phenomena attending the equilibria of surfaces are analysed, one is rapidly led to consider balanced systems wherein a given curvature of surface includes discontinuities of the stresses in parallel elements, such discontinuities entailing sudden variations in the length of the elements. A close investigation will indicate that, owing to the elastic properties of the material, equilibria of this kind are not possible without breakage of the material. Experience shows, however, that system of this kind are in fact perfectly stable, and the explanation of this apparent paradox is to be found once again in the theory of ductility.

Owing to sudden variations in stress the linear element in a surface may suffer large deformations, or the surface may deform itself geometrically in such a way that its linear element undergoes the variation in question. Alternatively, permanent elongations may arise which are the effect of counterbalancing the deformations caused by the mechanical action of the stresses. The author is of opinion that the ductility of the steel, though its precise action may be difficult to ascertain, plays a very important part in these phenomena.

The lines of discontinuity of stress which thus appear in the equilibrium of a surface usually start from points of discontinuity in the perimeter or are merged in the latter. It is easy to eliminate lines of discontinuity due to discontinuities of perimeter, simply by rounding the angles (at any rate for purposes of calculation). Lines of discontinuity which follow the perimeter itself are more difficult to eliminate, and here ductility of material plays an essential part.

Among these lines of discontinuity of stress in a surface the most important are asymptotic lines, where present. If some of these lines are followed, the conditions of equilibrium will lead to the discovery of discontinuities of stress and hence discontinuities of elongation. If the surface is flexible enough to allow it to deform, such deformation will cause asymptotic lines of discontinuity to deviate from the perimeter of the surface by modifying the shape of the latter, and this leads us to the case where asymptotic lines of discontinuity originate at a corner of the perimeter. The ductility of the steel in the neighbourhood of this angle has the effect of overcoming the discontinuity in question and of substituting a fictitious perimeter for the true perimeter, all discontinuity being eliminated.

The study of plastic strains in the steel is perhaps even more important in regard to systems such as those which we have just examined than in regard to ordinary framed structures, because, contrary to what is true of the latter, it would be impossible in the former to construct stable systems without calling into play the power of adaptation of the material, which depends on its ductility. We are thus left with what appears, *a priori*, to be a paradox, namely an isostatic system which is justified by the theory of plasticity.