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Theory of Thin Curved Shells not Subjected to Bending.

Einführung in die allgemeine Theorie der biegungsfreien Schalen.

Etude des voiles minces courbes ne subissant pas de flexion.

Dr. ès siences F. Aimond,

Ingénieur des Ponts et Chaussées détaché au Ministère de l'Air, Paris.

1) *Review of the general equations for statical equilibrium in rectilinear coordinates.*

Let $z = f(x,y)$ be the equation for the surface in rectilinear coordinates not necessarily rectangular. The conditions of stresses at a point m of the surface are determined by knowledge of the stresses n_1, $-n_2$, $-\Theta$ acting on the elements mm_1 and mm_2 respectively, parallel to planes zox and zoy. The stress n_1 acts on mm_2 parallel to plane zox, and n_2 stresses mm_1 parallel to plane zoy and Θ acts at the same time on mm_1 parallel to zox and on mm_2 parallel to zoy (fig. 1). Let α_1, o, γ_1 and o, β_2, γ_2 be the coefficients governing the directions of the tangents to elements mm_1 and mm_2, in other words, the projections of the unit vector upon ox, oy, oz, for each of the tangents.

Let us assume that the surface is loaded in some way, and that Xdxdy, Ydxdy and Zdxdy are the components parallel to ox, oy, oz of this load for the element $mm_1 m'm_2$, defined by parallels mm_1 and $m'm_2$ to plane zox and by parallels mm_2 and m_1m' to plane zoy. The investigation of the conditions of equilibrium of these elements leads to the following equations:

$$(1) \quad \frac{\partial \nu_1}{\partial x} + \frac{\partial \Theta}{\partial y} = X$$

$$(2) \quad \frac{\partial \Theta}{\partial x} + \frac{\partial \nu_2}{\partial y} = Y$$

$$(3) \quad r\nu_1 + 2s\Theta + t\nu_2 = \zeta$$

taking: $p = \dfrac{\partial f}{\partial x} \qquad q = \dfrac{\partial f}{\partial y} \qquad r = \dfrac{\partial^2 f}{\partial x^2} \qquad s = \dfrac{\partial^2 f}{\partial x \partial y} \qquad t = \dfrac{\partial^2 f}{\partial y^2}$

$$\zeta = Z - pX - qY$$

$$\nu_1 = n_1 \frac{\alpha_1}{\beta_2} \qquad \nu_2 = n_2 \frac{\alpha_2}{\beta_1}.$$

2) *Geometrical interpretation of the quantities contained in the general equations for equilibrium.*

The term ζ contained on the right side of equation (3) is nothing more than the projection of vector (X, Y, Z) on oz, this vector being projected parallel to the tangential plane of the surface. In order to interpret the quantities $\nu_1 - \nu_2$ and Θ, which are the unknown quantities of the equations of equilibrium, we must give a general definition of what we shall call "reduced stress". By definition, the reduced stress acting on an element of the surface is the projection on the xy plane of the elastic force which acts on this element, divided by the length of the projection of this element.

It can easily be realized that the distribution of reduced stresses around a given point follows the same laws as the real stresses and in particular the theory of *Mohr* can be applied. The quantities ν_1, ν_2, Θ are, in fact, the reduced stresses in relation to elements which are projected along parallels to axis of x and y. It will be noticed that the shear stresses Θ are maintained in projection, whereas this is not the case for the other stresses n_1, n_2.

3) *Geometrical interpretation of the general equations for equilibrium.*

Equations (1) and (2) evidently express the conditions of equilibrium, in projection on the tangential plane. Equation (3), on the contrary, expresses the equilibrium of forces normally applied on the surface. For a geometrical translation, let us take the origin 0 of the trihedron oxyz on the surface itself and direct ox and oy along the directions of two optional elements. We are then able to complete the definition of trihedron oxyz by taking arbitrarily the direction oz. Equation (3) determines a linear relation between the stresses acting on the optional elements ox and oy and the projection ζ on oz, parallel to the tangential plane, of the density of the stress applied. If we change the direction oz without any modification to ox and oy, each term of the preceding linear relation is only multiplied by the same coefficient.

We can take advantage of the indetermination of the direction of elements ox and oy to simplify the equation (3). If these elements in particular are directed according to two conjugated directions of the surface, that is, according to two directions conjugated in relation to the indicator, the coefficient of Θ becomes null and equation (3) is reduced to a linear relation between the longitudinal stresses ν_1 and ν_2. We can wonder whether it is not possible to direct the elements ox and oy in such a manner that they are eliminated from equation (3), leaving only one stress. We can at once realize that this is not possible if the surface is convex, that is, if the principal radii of curvature are in the same direction, and that on the contrary it is possible if the surface is not convex.

Let us consider the latter hypothesis and discriminate between two cases, according to whether the stress which remains in the equation (3) is an axial stress or is the shearing stress Θ. The first case is not possible unless the surface is developable, i. e. if we can consider it as being the envelope of a family of tangential planes relating to a parameter. If we consider the element ox with respect to the direction of the linear generatrix which passes through 0, the equation (3) is reduced to:

$$r\nu_1 = \zeta \tag{4}$$

The second case applies to surfaces of opposite curvatures. If we consider ox and oy with respect to the asymptotical directions, the equation (3) is reduced to:

$$2s\Theta = \zeta \tag{5}$$

Equations (4) and (5) can be immediately interpreted. Let us first examine the equation (4). It is obvious that the only stresses acting on an infinitely small element of the surface and admitting a component which is not located in the tangential plane to the surface, are the stresses projected along v_1 and equation (4) simply expresses the identity between the projections of the stresses n_1 on oz parallelly to the tangential plane and the projection of the applied load, in the same conditions.

Let us now examine equation (5): it suffices here to consider an elementary quadrangle, two consecutive sides of which are formed by asymptotical arcs transecting at 0. The longitudinal stresses n_1 and n_2 applied to this quadrangle admit a resultant in the tangential plane owing to the fact that this resultant is the geometrical summation of the resultant of the stresses n_1 and of the resultant of the stresses n_2 and that each of these two latter resultants is necessarily in the osculatory plane of an asymptotical arc, such osculatory plane concurring with the tangential plane, being given the very definition of asymptotical lines. Therefore, component ζ of the stresses applied to the surface, outside the tangential plane, depends only on shear Θ, to which it is, in fact, proportional. The coefficient of proportion, the value of which is $2s$, admits a very simple geometrical significance: it is the quotient of twice the distance from the vertex opposed to 0, in the quadrangle, to the tangential plane at 0, this distance being evaluated parallelly to the direction oz, by the product of the lengths of the asymptotical arcs which form the sides of the quadrangle.

4) *Classification of thin shells with respect to their mechanical properties.*

The above considerations lead to a classification of the thin shells into three groups. The first group covers developable surfaces, such as cylinders, cones; the second group comprises the convex surfaces, such as spheres, elliptical paraboloid, ellipsoids, polar-symmetrical hyperboloids and, generally speaking, all surfaces of double curvature, which are generated by a curve the concavity of which is directed downwards and which rest on a curved directrix, the concavity of which is also directed downwards. In the third group, we find the surfaces characterized by opposed curvatures, such as hyperbolical paraboloid, hyperboloids, conoids, all undevelopable ruled surfaces and, generally speaking, all surfaces which can be generated from a curve, the concavity of which is directed upwards and which rests on a directrix whose concavity is directed downwards.

This classification has been suggested to us by the geometrical interpretation of equation (3). Shells of the first group are those for which equation (3) can take the form (4); shells of the second group are those for which equation (3) can take the form:

$$rv_1 + tv_2 = \zeta \tag{6}$$

r and t being preceded by the same sign; shells of the third group are those for which equation (3) can take the form (5).

It should be noted that for shells of the third group, the equation (3) can also take the form (6), but r and t are then of opposite signs. It should also be noted that for shells of the second group, equation (3) can also take the form (5), where Θ represents the shear on the asymptotical lines but equation (5) is then no longer an equation with real terms, as s and t are purely imaginary expressions.

Shells of the first group are characterized by the fact that the normal component of the stress on the rectilineal generatrices is, at each point, proportional to the normal component of the density of the applied load. Shells of the second group are characterized by the fact that the purely imaginary shear stress acting on the imaginary elements of asymptotical lines is, at each point, proportional to the normal component of the density of the applied load. Shells of the third group are characterized by the fact that the shear stress acting on the elements of asymptotical lines is, at each point, proportional to the normal component of the density of the applied load.

The following difference should also be noted between shells of the second and of the third group. If we consider at a given point, the longitudinal stresses acting on two conjugated elements, then the normal component of the applied load, which can itself be considered as the bulging produced by these longitudinal stresses, is a linear form of these stresses. The related coefficients are of the same sign for surfaces of the second group, and of opposite signs for shells of the third group. It therefore follows that the carrying capacity of a shell of the second group can be considered as a result of the action of longitudinal stresses of same direction, acting on two conjugated elements, and that the carrying capacity of a shell of the third group can, in a similar way, be considered as produced by longitudinal stresses of opposite senses, acting on two conjugated elements.

As regards shells of the second group, the conjugated elements can always be chosen so that the coefficients of the corresponding stresses are equal, in the linear form which represents the normal component of the density of the applied load. Such elements will be called canonical. It can then be said that in shells of the second group, the normal component of the density of the applied load is proportional to the summation of the longitudinal stresses acting according to the directions of canonical elements.

These differences in properties just mentioned above and which distinguishing the three groups of shell from one another, are of the utmost importance as regards the kinds of supports which can be considered for the periphery of such shells, in order to achieve their equilibrium, and as regards the actual method of calculation for the stresses in the shells as functions of the conditions on the periphery.

5) *Shells of the first group.*

The study of shells of the first group is a generalization of the study of cylinders. Equation (4) gives, for each point of the shell, the normal component relating to the rectilinear generatrix passing through this point, of the stress acting on an element of this generatrix. Consequently, if we draw on the surface a family of geodesic lines intersecting the rectilinear generatrices at a constant angle, we shall know the value of the longitudinal stress which acts parallelly to these geodesic lines on the elements of the rectilinear generatrices. Equation (2) gives then, by immediate integration, the value of shear on the generatrices and geodesic lines and a second integration from formula (1) gives the longitudinal stresses acting on the elements of the geodesic lines parallelly to the generatrices.

Such a determination of the stresses is not complete unless we assume to be given, on a given curve intersecting only once each generatrix, the values of the stresses acting on the elements of the said curve. In the same way, we can also assume to be given, on two curves, each of them intersecting each generatrix only once a relation between the components of the stress acting on any element of the two curves.

6) *Shells of the second group.*

Let us consider a thin shell of the second group. We have seen that the normal component of the density of the applied load is, at each point, proportional to the summation of the longitudinal stresses acting on canonical elements. We shall now suppose that these longitudinal stresses are equal. Their value is therefore clearly determined, at each point, by the value of the normal component of the density of the load. Thus equation (3) is fulfilled. Equations (1) and (2), which express equilibrium in the tangential plane, are then fulfilled only if the tangential component of the density of the load has a determined value, which can be obtained precisely by writing the conditions of equilibrium parallelly to the tangential plane. We shall call "fundamental system of loads" every system of loads corresponding to the preceding conditions, that is, such that the longitudinal stresses acting on two conjugated elements, symmetrical to the principal directions, be equal. It then becomes obvious that any system of loads can be considered as the summation of a fundamental system and of a system which would be exclusively composed of tangential loads, which we shall call "complementary system" to the fundamental system of loads.

We are thus induced to study the complementary systems, i. e. the systems in which the applied load is tangential to the surface. In such systems, the longitudinal stresses on canonical elements are equal and therefore the stress on any element now depends only on two parameters, for instance the components of the stress which acts on one of the two preceding canonical elements. It is obvious that these two parameters can be arbitrarily chosen. It will be easily understood that we can determine two conjugated imaginary functions φ and ψ in such a manner that, when choosing as parameters two quantities which we shall call S_φ and S_ψ, the elastic forces acting on any element of the surface consist of linear forms from the differential expressions $S_\varphi\, d\psi$ and $S_\psi\, d\varphi$. The equations for equilibrium in the tangential plane then show that the partial derivative of S_φ with respect to φ and the partial derivative of S_ψ with respect to ψ are linear functions of S_φ and S_ψ. By elimination of one of the two parameters, S_ψ for example, between these two relations, we get an equation to linear partial derivatives of the second order, with imaginary characteristics, which the retained parameter S_φ must fulfil.

In order to arrive at a solution for such an equation, we can assume a given the value of S_φ and of one of its derivatives on an optional curve of the surface. provided however that, the equation having imaginary characteristics, certain conditions of analycity be fulfilled. If we observe that S_φ of the curve and one of its derivatives may be assumed, the values S_φ and S_ψ of this curve can be obtained and in consequence there of the stresses acting on any element of the curve. After making certain reservations for analycity, we now see that the stresses in the shell can be determined, provided that the stresses acting on the elements of a curve be known.

The foregoing reserves for analycity are not merely formal reserves. They correspond to a physical reality which is the following. We know that in all problems where a function used to verify an equation with imaginary characteristics is determined by means of the values it assumed — the same applies to one of its derivatives —, on a given curve, the solution is not a continuous function of the given values; in other words, by slightly varying the given values, results of any desired difference can be obtained from this function, on points arbitrarily chosen. It follows that the states of equilibrium of a convex shell. corresponding to given values of the stresses acting on a curve, are not stable with respect to the values of the stresses on this curve.

In order to arrive at stable solutions, it is necessary to consider, for the limits, conditions which are different from those we have taken. Instead of assuming the values of the stresses on a curve to be given, we shall assume on a closed curve, a given relation between the components of the stresses acting on the elements of the curve. The problem then becomes clearly determined and its solution will be a continuous function of the given values. The corresponding equilibrium will be stable. Let us suppose, for example, that we wish the normal component of the stress along the given curve to be null. The relative indetermination of the parameters S_φ and S^ψ allows us to determine them in such a manner that S_φ represents, along the given curve, the value of the normal component of the stress acting on the elements of this curve. The theory of integral equations then allows us to determine the function S_φ by a method similar to that used by Fredholm and his successors in solving problems of the same type, relating to equations with imaginary characteristics.

7) *Shells of the third order.*

Let us consider a thin shell of the third order. The value of the normal component of the density of the load determines first of all at each point of the shell the shear stresses on the asymptotical elements. Let us assume that the stresses in the shell are reduced to these shearing stresses. For this purpose it is necessary and sufficient, for the tangential component of the load which is applied to an elementary quadrangle of asymptotical arcs to balance the projection of the resultant of the tangential stresses applied to the elements of the quadrangle on the tangential plane. We shall call "fundamental system of loads", any system of loads corresponding to the preceding conditions, i. e. such that the stresses acting on the elements of asymptotics are reduced to shear stresses. It is quite obvious that any system of loads can be considered as a superposition of a fundamental system of loads and of a system which we shall again call "complementary system", exclusively composed of tangential loads.

We thus return to the study of the action of complementary systems. For this purpose, we again observe that the elastic force acting on an element of shell can be translated into a linear form of differential expressions such as $S_\varphi d\psi$ and $S_\psi d\varphi$, φ and ψ now being two real functions and S_φ and S_ψ two real parameters. The equations for equilibrium according to the tangential plane allow then of expressing the partial derivatives of S_φ with relation to φ and of S_ψ with relation to ψ, in linear functions of S_φ and S_ψ. The elimination of S_ψ between these equations leads to an equation in S_φ, linear to the partial derivatives of the second order, with real characteristics. The characteristics of such an equation to the partial derivatives are precisely the asymptotical lines.

In order to obtain a solution from the preceding equation which would be valid in an area D limited by a contour C, we shall divide this contour into to two series of arcs Γ and Γ' in such a way that from any point of D two asymptotical lines are drawn intersecting Γ only once; we shall then divide Γ into to two series of arcs Γ_1 and Γ_2 in such a manner that any broken line of asymptotical arcs joining any point from Γ_1 to a point on Γ' has its intermediate vertices on Γ_2 or on Γ' and that there is no broken line of asymptotical arcs having its ends on Γ_1 and its intermediate vertices on Γ_2. We shall obtain a single solution, valid in D, when assuming a given the value on Γ_1 of the stress acting on the elements of Γ_1 and on Γ_2, a relation between the components of the stress acting on the elements of Γ_2. The value of this solution will be given by the *Riemann* formula, successively applied to different fractional zones of zone D. No condition of analycity is here necessary and the solution arrived at is always a continuous function of the data. On the other hand, there is generally no corresponding solution to a relation between the components of the stresses acting on the different elements of the closed curve C.

When the thin shell taken into consideration is a straight-line surface, the equation to the partial derivatives of the second order can be reduced to a linear equation with partial derivatives of the first order containing only one derivative, the integration of which is immediate, as it can be considered as a linear differential equation. In the case when the thin shell consists of a straight-line surface of the second order, the determination of parameters S_φ and S_ψ is reduced to the solving of two quadratic equations.

8) *Choice of supports for thin shells of the three groups.*

The choice of the system of supports for a thin shell depends essentially on the group to which the surface belongs. We shall distinguish between two classes of supports: single supports with which the reactions depend on one parameter, and double supports, with which the reactions depend on two parameters. The components of the stresses transmitted by the shell to a simple support therefore fulfil a relation which is known a priori, whereas the components of the stresses transmitted by the shell to a double support can assume independent values. However, certain parts of the shell, on the marginal zones, shall be left without support; in such a case the contour is said to be free.

We propose to find out how the free edges, single supports and double supports must be distributed on the contour of a thin shell, so that the latter is subjected to definite and stable equilibrium conditions.

Let us consider at first the case of a thin shell of the first group. We can assume the presence of a free edge on every part of the contour which comprises no rectilinear generatrix, intersected once at the most by any generatrix. If the free edge meets all the generatrices, the distribution of stresses in the shell is determined and therefore the other edges must be equipped with double supports. The resulting system of equilibrium is stable. If, on the contrary, we consider two edges, each of them only once intersected by all the generatrices and provided with double supports, we shall again obtain a state of stable equilibrium, on condition that the rest of the contour exclusively composed of generatrices, be arranged as double supports.

Should we now consider à shell of the second group, such a shell cannot admit of free edges, as the resulting equilibrium is not stable. The whole of the periphery can, however, be arranged for single supports and the resulting equilibrium is well defined and stable.

Let us finally consider a shell of the third group and divide its contour into three series of arcs Γ_1 Γ_2 Γ', defined as indicated in 7. We can assume a free edge according to Γ_1, single supports according to Γ_2 and double supports according to Γ'. The resulting state of equilibrium is well defined and it is a stable.

9) *Geometrical properties of the shells of the third group and their geometrical calculation.*

Shells of the third group show remarkable geometrical properties which allow for an accurate graphic calculation.

Let us first interpret geometrically the parameters S_φ and S_ψ and the functions φ and ψ introduced at 7. S_φ and S_ψ are the longitudinal stresses acting on the asymptotical lines for a complementary system of loads. φ and ψ are the curvilinear coordinates of the surface for which the lines of coordinates are the asymptotical lines.

Let us replace these thin shells by a skew reticular system, the meshes of which consist of rectilinear skew quadrangles formed by chords of asymptotical lines. The system thus obtained works as does the given surface and the assimilation of the two systems to one another is legitimate if the meshes are sufficiently small. The loads applied to the reticular system must be applied to the vertices of this system according to the tangential plane to the surface.

If we apply a single force F to any vertex of the reticular system, such a force can be decomposed between two of the bars passing at this point and corresponding to two different asymptotical lines. Force F is thus transferred to two other knots of the surface, where we operate in the same manner, and so on. If we suppose that the contour of the surface is divided into three series of arcs Γ_1 Γ_2 Γ' according to the foregoing conditions and if we conveniently choose the two initial members in accordance with which the given force F has been devided, the fransfer of force F can be made as indicated without ever encountering a free edge. If we meet with a free edge on Γ_2 supposed to be arranged for single support, we can still make the division between the second member ending on the vertex considered on Γ_2 and the direction of the reaction of the single support. The operation thus carried out is called a reflection on the single support.

By continuing in the same manner, we finally transmit the force F to a whole zone of double supports. We thus obtain for the system a state of equilibrium which is consistent with the reactions at the supports, and the equilibrium will be stable. Operating in the same manner for each loaded knot of the reticular system, we determine the state of equilibrium for the complementary system of loads only by dividing forces according to the parallelogram of forces. The corresponding diagram can be easily drawn by projecting on an arbitrary plane.

The geometrical determination of the stresses mentioned above allows of considering the equilibrium of a shell of the third group as resulting from a propagation of stresses according to the asymptotical arcs and starting from the free edges so as to arrive at the double supports by reflection on the simple edges. Such behaviour is similar to the propagation by means of waves of the phenomena following the rule of linear equations to the partial derivates of the second order with real characteristics, and is also essentially due to the real nature of the characteristics of the equations governing the equilibrium of stresses in the shell under consideration.

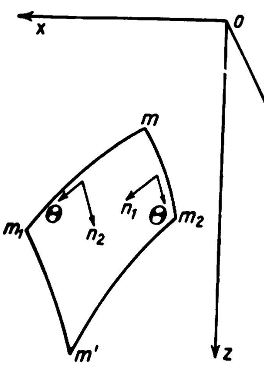

Fig. 1.

10) *Elementary examples of shells of the third group.*

The most simple example of a shell of the third group is the hyperbolic paraboloid. This shell is characterized by its property that shear along to the rectilinear generatrices, within a certain coefficient constant for the whole sur-

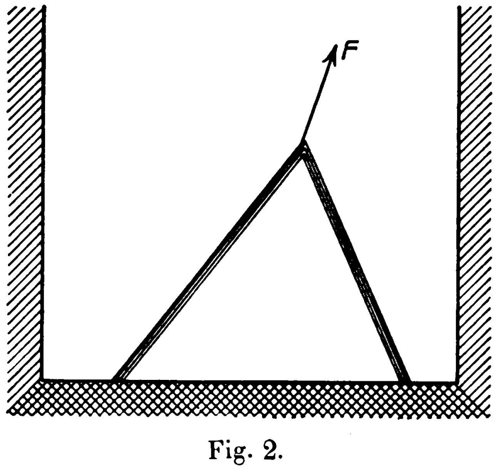

Fig. 2.
Mode of propagation of tangential stresses, in a ruled quadric surface.

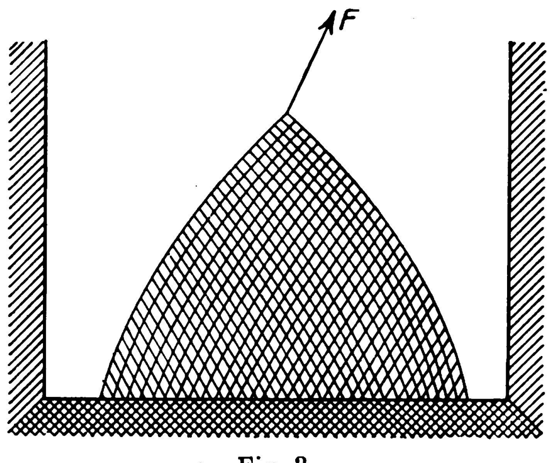

Fig. 3.
Mode of propagation of tangential stresses in any other surface of the third group.

face, is equal to the component along the axis of the paraboloid of the applied load, brought back to the unit of surface projected on an arbitrary plane not parallel to the axis. On the other hand, the stresses due to the complementary system of loads propagate each generatrix without any interference between the generatrices, so that a tengential stress applied to a small element of the shell acts only on the bands produced by the generatrices encountered. The simplest form of shell of the third group after the hyperbolie paraboloid is the hyperboloid. Just like the paraboloid, this shell has the property that stresses due to the complementary system propagate each generatrix without any interference with other generatrices. It only differs from the hyperbolic paraboloid in a more intricate expression of the coefficient of proportionality between shear and density of the applied load.

Then follow the undevelopable straight-line surfaces, and first of all the conoids. For these surfaces, the coefficient of proportionality between shear on the asymptotical lines and density of the applied load is expressed in a much more complicated form than for the preceding surfaces, but the most distinguishing character of such surfaces is that the stresses due to the complementary system propagate by opening on the surface; the unrectilinear asymptotical lines abutting on the rectilinear generatrices of the surface, so that a tangential effort applied to a small element affects a whole zone distributed over the surface, just as for the most general surfaces of the third group.

Figures (2) and (3) show the difference between straight-line quadric surfaces and other surfaces of the third group as regards the views expressed above.

11) *Conclusions.*

With the exception of the developables straight-line surfaces, such as cylinders and cones which form a very particular class of shells, all the shells with double curvature can be divided into two important classes, according to the sign of the total curvature. In these two classes, the asymptotical lines play the essential part in the transmission of tangential stresses and therefore in the determination of the nature of the reactions of supports corresponding to well defined and stable conditions of equilibrium. When the asymptotical lines are imaginary, the shell cannot admit of free edges, but can be limited by edges all arranged as simple supports. A common example of such supports is given by a tympan or flat slab of great stiffness in its own plane and without any appreciable stiffness perpendicularly to this plane. When the asymptotical lines are real, the edges of the shell are to be divided into free edges, edges with simple supports and edges with double supports, according to the well determined principles we have mentioned.

As double supports might involve difficulties as regards design, it is beneficial to reduce their importance as far as possible, and this can be done in different ways when conveniently choosing the outline of the surface.

If we strictly consider the facilities of calculation, the views expressed above show that, among the shells with double curvature, straight-line quadric surfaces are those which lead to the most elementary calculation.

Summary.

The problems offered by the design of thin curved shells of reinforced concrete consist, in the first place, of purely statical problems, independent from the theory of elasticity. We shall deal later on with these problems, as a whole, excluding such other questions as concerning the application in practise of shells under consideration of existing deformations and particularly the problem of comptability of deformations due to stresses calculated by means of ordinary statics.

We shall apply the hypothesis, generally accepted, of a uniform distribution of stresses on any transverse section in such a manner that the shell can be considered as being replaced by mid-surface of the shell.

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Shell Construction in Reinforced Concrete.

Die Flächentragwerke des Eisenbetonbaues.

Les surfaces portantes dans la construction en béton armé.

Dr. Ing. Fr. Dischinger,
Professor an der Technischen Hochschule Berlin.

Twelve years have passed since the introduction — by Messrs. Dyckerhoff & Widmann AG., in conjunction with Messrs. Zeiss, Jena — of the shell method of reinforced concrete construction, in which the distribution of loading is effected solely by elongation forces. During these twelve years this type of construction has advanced by leaps and bounds; its development was only possible after the theoretical side of these tri-dimensional structures had been greatly elaborated through extensive experiments and in a surprisingly short space of time. The result was that this theory has opened up new fields of work in monolithic reinforced concrete construction as applied to wide-spanned hall-possibilities which by far exceed those offered by the theory of crosswise reinforced and mushroom slabs. By employing shells and saw-type-roof shell structures span widths can be obtained that were formerly regarded as impossible in massive constructions. It must, however, be remembered that little more than a decade has gone by since shell construction was discovered. In this short period hundreds of thousands of square metres of large halls with spans up to 100 metres wide have been erected.

The present paper may be divided into two sections, the first of which gives a review of the development of the theory since the last Congress and the constructional progress made as demonstrated by some practical examples. The second section deals with the problem of continuous cylindrical shells or pipes.

1. Development of the Theory of Shell Construction since the Congress in 1932.

As regards the various forms of shells mentioned in this paper, we would refer to *W. Petry*'s work for the Paris Congress of 1932 (Theme II, 4). Vol. I of the 'Publications', which appeared in the same year, contains an article by *U. Finsterwalder*[1] on the problem offered by the Zeiss-Dywidag system of cylindrical barrel shell, a combination of cylindrical shell and frame supporting its edges on either side. This forms a uniform system in space which can be described as a T-beam in space in which the shell represents the flange of the beam. In contrast to the commonly known T-beam systems — in which, when the webs are widely spaced, the flange only takes up a limited amount of the compressive forces — in the case of these T-beams in space the whole shell acts as compression flange. This results from the fact that in ordinary T-beams as shown in Fig. 1a), the participation of the flange for the transmission of compressive forces N_x forcibly must be established by shear stresses N_{xy} acting between the beam and the flange. The active width is therefore a function of the length of beam. However, the compressive stresses are not evenly distributed over the whole width of the flange, since the strips of slab situated farther away from the beams fail to co-operate on account of deformations due to shear.

The action of the T-beams in space, as shown in Fig. 1b), is essentially different; for, as is clear from Eq. 2 (Section II), compressive forces N_x arise in the shells — even when no account is taken of the compressive forces $N_{x\varphi}$ acting between shell and lateral beam — which are governed by the dead weight of the shell and consequently the whole width of the shell is operative in taking up the compressive forces. This occurs to a greater extent the higher the cross-sectional curve of the shell is raised above the funicular polygon. For this reason shell structures whose cross-sectional curve follows the form of flat elliptical segments possess an essentially better girder action from cross-panel to cross-panel than circular cylindrical shells. Furthermore, the bending moments

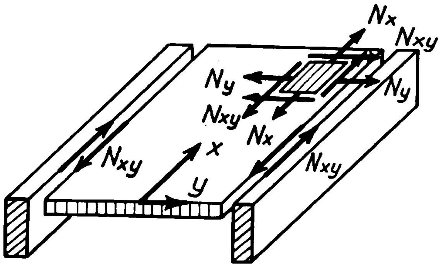

Fig. 1 a.

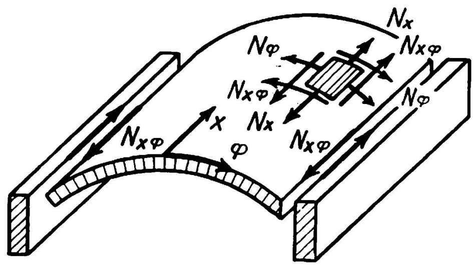

Fig. 1 b.

which arise in these steeply arched systems are much smaller because the compressive forces N_x, necessary here for the transference of the external bending moments, are for by far the greater part produced by the dead weight of the shell itself and not by shear forces $N_{x\varphi}$. The magnitude of the bending moments which occur in the direction of arch are dependent upon that portion of the compressive forces N_x which must of necessity be produced under strain by the shear forces $N_{x\varphi}$. For these reasons it becomes apparent at once that essentially smaller bending moments are caused in shell systems of steeply arched cross-sectional curve, than in circular cylindrical shells. I shall refer to this point again at a later stage.

Between shell and lateral frame there are four statically indeterminate forces acting, namely: — 1) the arch action N_φ, the transverse force Q_φ, the bending moment M_φ and the shear force $N_{x\varphi}$. Thus for both edges together we have eight statically indeterminate quantities, and consequently the basis of the shell problem must be a differential equation of the eighth order or a system of three differential equations corresponding thereto, for in accordance with the eight statically indeterminate quantities we need eight constants for the closing of the two joints between shell and lateral frame. In arriving at his solution *U. Finsterwalder* started with the assumption that when the cross panels are located at relatively large intervals, the bending moments M_x prevent the shell from transmitting loads on to the cross panels; he therefore assumed the moment M_x and the pertaining transverse force Q_x, together with the (twist moment) $M_{x\varphi}$ as being zero. In consequence of this approximation it was possible to work out the problem in the form of a differential equation of the eighth order, introducing a stress function in which the internal forces of the shell could be represented in the same manner as for Airy's stress function for panels, as derivatives of this stress function.

When the cross panels are situated at relatively small intervals in comparison with the radius of curvature of the circular cylindrical shell, *U. Finsterwalder*'s assumption $M_x = Q_x = M_{x\varphi} = 0$ are no longer permissible. It is for this reason that the author has been endeavouring to find a strictly accurate solution for circular cylindrical shells in these cases, which are of importance in the construction of halls with wide arch spans. As the shells of these wide-spanned arches must be strengthened with ribs to ensure safety against buckling, I have also extended my investigations to cover anisotropic shells.[2] Here three linear simultaneous differential equations with constant coefficients are obtained. Part-solutions of these differential equations can be arrived at by following *H. Reissner*[3] and representing the dead weight by circular functions in the form of double trigonometric series. The investigations now show that there are three possibilities for transferring loads in a closed pipe: — 1) transference of loads to the gross panels system by means of elongation forces (membrane theory); 2) transference of loads to the cross panel systems by means of bending moments M_x in the shel! (slab action): and 3) equalization of the loads of the higher harmonics by means of bending moments in the direction of the curve. This equalization in the direction of curve is only possible because there is no real vertical load resultant to correspond to the higher harmonics in relation to the whole cross section of the shell. The actual loading is transferred to the cross panel systems by actions 1) and 2). In order to fulfil the support conditions required for the two lateral beams in the case of the Zeiss-Dywidag barrelsystem, the above-mentioned part-solution must be supplemented by a solution of the homogeneous system of differential equations. The latter system is fulfilled in the same manner, introducing the exponential term $e^{m\varphi} \cos \lambda x$, as for the problem solved by *K. Miesel*[3] in 1930, which we shall discuss below. Thus the three differential equations resolve into three ordinary homogeneous equations leading to one equation of the eigth order, from whose solution we obtain the wave length and attenuation of the double oscillations issuing from the two edges of the shell. This equation of the eight order has been solved for about one hundred different cases. The values for wave length and attenuation of the oscillations obtained from it were worked up into diagrams from which the values can be read without any calculation whatever. With the assistance of the above-mentioned basic term not only the eight support conditions along the lateral edges, but also the support conditions for the cross panel systems can be satisfied.

As has already been mentioned, the problem offered by the support conditions of closed circular cylindrical pipes at the cross panels was solved as early as 1930 by *K. Miesel* for any desired variation of the support conditions. Here

Miesel also took the elasticity of the stiffening panels into account — a problem which plays an important part in submarine construction. *Finsterwalder,* too, investigated this problem in his work cited under 1) and found an approximative solution for it, again in the form of a stress function. Here, however, in contrast to the corresponding solution for Zeiss-Dywidag shells, not the quantities M_x, Q_x, $M_{\varphi x}$ but the values M_φ, Q_φ, $M_{x\varphi}$ have been neglected. With much less calculation work and for reasonably small values of the harmonics, this approximative solution coincides very well indeed with *Miesel*'s strict solution. And in our practical constructional problems there are no very high values of the harmonics involved.

The more rigid the shell is constructed as regards bending in the direction of arch, the nearer does the law of stress distribution of the N_x forces approach *Navier*'s straight line law in the case of Zeiss-Dywidag barrels, because then the work of deformation of the bending moments in the direction of arch are insignificant compared with those of the elongation forces. The thinner the shell is, however, the greater is its tendency to reduce the bending moments for correspondingly unfavourable distribution of the elongation forces. If, nevertheless, a more favourable distribution of the N_x forces is to be attained, these thin shells must be combined with correspondingly high lateral beams.

At the commencement of my expositions I pointed out that in greatly raised cross-sectional curves, as for example in flat elliptical segments, smaller bending moments are produced and a more favourable girder action obtained. The larger the cylindrical shells are made, the more necessary does it become to replace them by shells of steep curvature. Thus, for the huge halls constructed for the German Air Ministry, practically only shells with elliptical cross-sectional curves were employed, calculated according to a suggestion made by *U. Finsterwalder.* on the theory of circular cylindrical shells, in such a manner that the elliptical segment was approximated by a three-centre arch. This naturally involves very intricate calculation, as there are now four edges to cope with and the oscillations starting from these have a mutual influence on one another. It is therefore urgently necessary for these cross-sectional curves to be resolved in a strictly accurate manner. One of my assistants has succeeded in doing this, and the solution will shortly be published in the form of a dissertation.

Shell systems are frequently constructed as continuous systems over several spans. As these shell systems are extremely high in relation to their span, the moments at the support are in parts substantially influenced by deformations due to shear. This fact has already been pointed out by *W. Flügge*[4]. As is also well known, the influences of these shear deformations are deliberately neglected as being insignificant in the case of slender beams. For shell systems, however, this omission is not always permissible. In Section II of this paper I have given detailed proof of the influence of these shear deformations on the support moments, and with the assistance of *Flügge*'s three-moment equations developed a process by which the support moments for isotropic and anisotropic shell systems can be obtained for any desired span and loads, both in the direction of arch and also lengthwise.

As the width of spans in shell systems increases, so does the problem of buckling grow in importance. In this connection distinction must be made between two cases, namely: — a) Buckling of the shell in direction of arch, and b) buckling in the direction of the generatrices. The first problem was treated as early as 1914 by *R. von Mises*[5] and the second even earlier by *Lorenz*[6] and *Timoshenko*[7]. For shell systems of large arch and beam spans, however, both these problems appear in combination, so that too favourable results would be obtained if the two cases of buckling were calculated separately. This combined case of buckling, so important in the construction of cylindrical shells, was solved by *W. Flügge*[8] in 1932 and worked out in a very detailed manner and in a very practicable form. Here the influence of the combined buckling becomes apparent in an unfavourable manner. *Flügge*'s investigations also extend to the case of the anisotropic circular cylindrical shell, which is bound to be involved when it is a question of large spans. By means of a transition process *W. Flügge* shows that his equations can also be applied to the special case of buckling in slabs.

As it is assumed when deducing buckling conditions that the deformations of the shell are small in proportion to the thickness of the latter, but that, on the other hand, it is extremely difficult to adhere to this condition in practical constructional work, since very considerable deformations are set up with large span widths, it becomes necessary to reckon with very much higher factors of safety against buckling in shells than in simple arches. These safety factors can easily be attained by reinforcing the shell with ribs. These have further the advantage of being able to reduce the deformations considerably and of simultaneously taking up the bending moments of the shell as well.

During the last few years cylindrical shells have been coming steadily to the fore in practically every country. Such shells have been constructed with girder spans of up up to 60 m and arch spans of 45 m, i. e. with ground plan areas of as many as 2700 square metres. For the reasons mentioned above, elliptical cross-sectional curves have been employed for shells of large arch spans and large shell spans. On the other hand, a number of halls have been constructed with arch spans up to 100 m wide and relatively small intervals between the cross panels. Fig. 2 shows an Airplane Hangar of this type seen from the outside, with a very large arch span; Fig. 3 is an interior view of an Aircraft Hangar of large arch and girder span, for the reproduction of which I am indebted to the courtesy of the German Air Ministry. Figs. 4 and 5 illustrate the use of shells in industrial structures, Fig. 4 being an interior view of Bamberg Postal Car Hall and Fig. 5 depicting the application of cylindrical shells in the form of shed-type roofs for a plate mill in Buenos Aires.

2) *Shed-Type Roofs.*

In shed-type roofs the bent cross-sectional curve of the shells is replaced by a polygon and the shell thus replaced by a system of planes. The problem is naturally quite the same as for the cylindrical shells. Instead of differential equations we have equations of difference of the same order. Now in addition to the bending moments emanating from the shell action appear others from the slab action, since the individual slabs must first transfer their loads by bending moments to the edges of the shed-roof system, the loads being then

Fig. 2.

transferred by the action of the shell, i. e. of the shed-roof, from here to the stiffening panels by means of elongation forces. The bending moments emanating from the shell action were first considered in this problem by *E. Gruber*[9] and *G. Grüning*[10]. Both authors thereby neglected the influence of the rigidity against torsion of the lateral beams. In this respect the abovementioned works were complemented by *R. Ohlig*[11], who also took the rigidity against torsion of the lateral trusses into account in the same manner as has always been customary for shell systems. Plane systems are less economical than

Fig. 3.

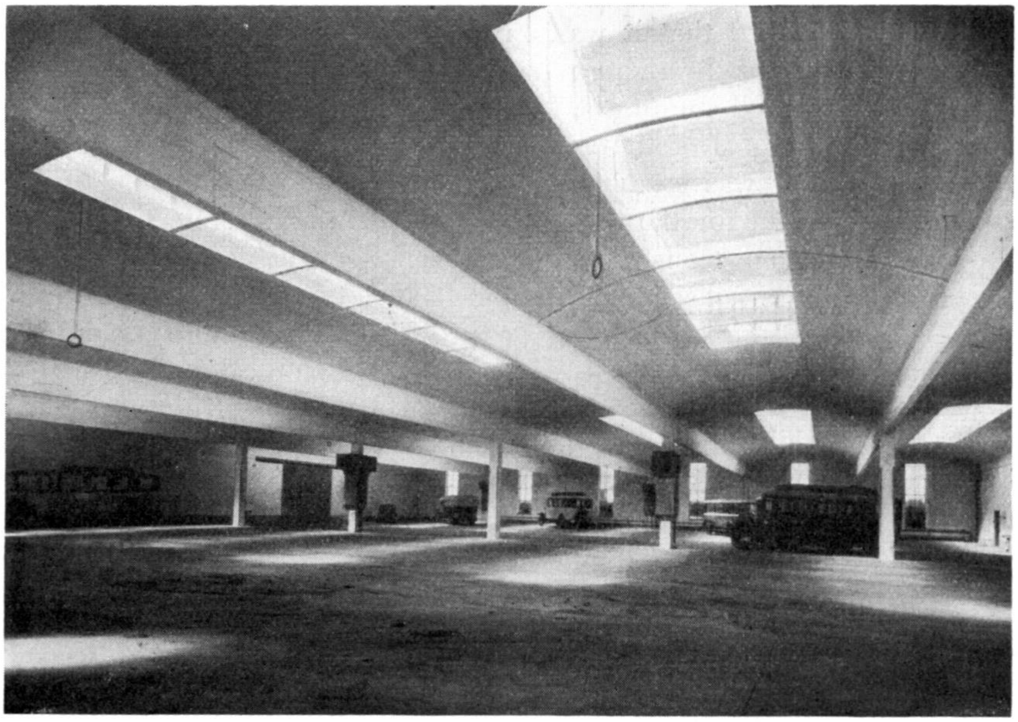

Fig. 4.
Postal motor coach garage, Bamberg

shells in consequence of their greater bending moments, so that as yet no very large structures of this type have been carried out. Of course the reason for this is also to be sought in the fact that the patents for shells and shed-roof types are the property of the one concern — Messers. Dyckerhoff & Widmann AG.

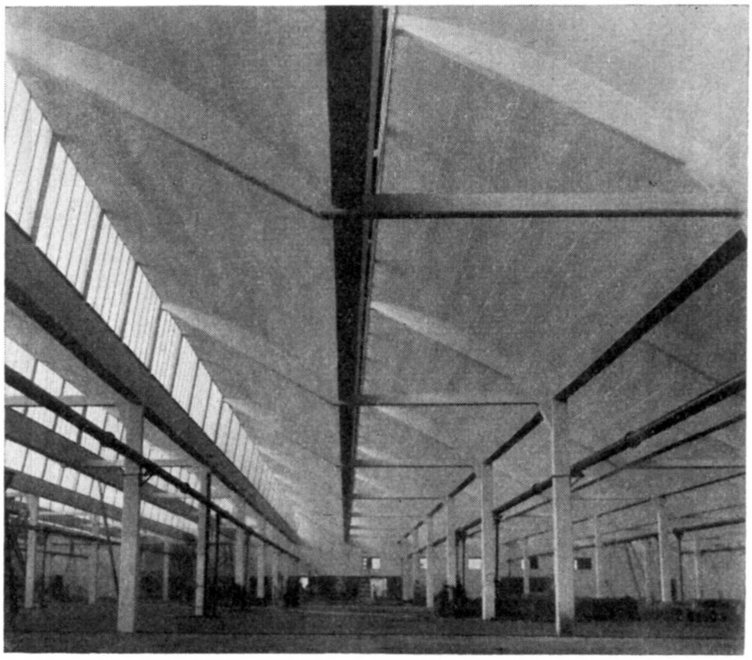

Fig. 5.

3) Polygon Domes composed of Cylindrical Shells.

It will be remembered that it was on this system that the monolithic domes of the great Market Hall in Leipzig, at present the largest in the world with a span of 76 m, and the dome of the Market Hall at Basle, 60 m span, were constructed. Their vaulting is of the so-called monastery type. Although the theory of monastery vaulting was established and published a considerable time ago,[12] the same cannot be said for the theory of cross vaulting. Architecturally beautiful domes, perfect from an acoustical point of view, can be constructed with cross vaulting. Fig. 6 shows a hectagonal dome of this type. Apart from their good acoustical qualities, these domes can be provided with beautiful and effective lighting systems by means of large windows let into the calottes through which the light is reflected right into the middle of the room from the cylindrical shells. I elaborated the theory of these domes in 1930 in a paper written for the competition organized by the Academy of Architecture. In this paper I showed that it is possible to keep the stiffening ridges free from bending moments. As the space allowed me in the present article is not sufficient for the purpose, this theory will shortly be published in a technical periodical.

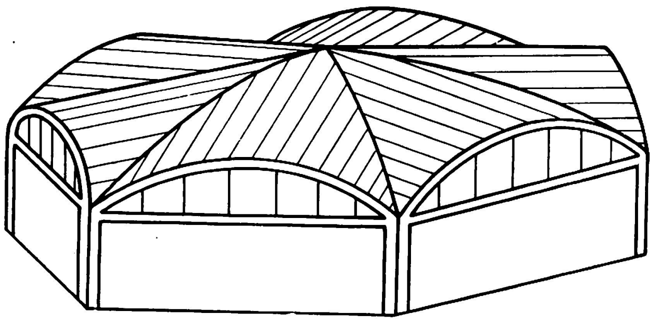

Fig. 6.

4) Shells with Double Curvature.

The membrane and the bending theory of rotary shells continuously supported on their springing has long ago been established. The following types are important forms which have since developed: — a) Rotary shells supported at a few points only, their girder action being superimposed on the dome action so that the shell can transmit its loads to pillars situated a considerable distance away. b) Rotary and translation shells with rectangular or polygon-shaped horizontal projections. c) Apse domes.

The theory of these various forms of shells with double curvature was elaborated by the author in 1930 for the competition already mentioned. The Academy had intended to issue these works in book form, but was obliged to withhold publication owing to lack of funds. I therefore abbreviated the works for publication in the 'Bauingenieur'[13]. As regards rotary shells on single columns it should be mentioned that their girder action produces the astonishing result, coinciding with the well-known slab action, that the height of girder and with it the lever arms of the internal forces are proportional to the distances between the girders in transmitting loads to the pillars. The stresses arising from the girder action are therefore independent of the girder spans.

From this it follows that with these shells, just as with polygon-shaped domes, very large girder spans can be attained. Here, however, the shells do not remain free from bending. A work by *A. Havers*[14], who employs a spheric function to treat and solve the problem of the disturbance of support conditions on a latitude circle of a spherical shell for an arbitrarily selected harmonic, has made it possible also to calculate bending moments arising in shells, knowledge which is absolutely indispensable when carrying out large constructions. It would be extremely useful to carry through the complete calculation of an example, for although this naturally takes a great deal of trouble, it would definitely show what spans can be attained with these shells and whether they posses economic advantages over the shells classified under b), in which the load is transmitted almost entirely by means of elongation forces and shell-thickness therefore depends solely on the safety against buckling. This because the permissible stresses cannot be fully utilised in these forms of shell, even when the spans are maximum. The calculation of the rotary shells with rectangular or polygon-shaped horizontal projection can be effected in a very simple manner by means of the procedure, already quoted, of the differential equations of the membranal condition of stress.

Fig. 7 shows a very flat shell of this type with a rectangular plan; it was carried out for one of the buildings of the Danzig Institute of Technology. The shell has a span of 12 m, but only a rise of 0.77 m. The ratio l/f between rise and span amounts to 15.6, being therefore smaller than for that of the flattest bridge. This illustration shows quite clearly that a shell system of this type is nothing else but a T-beam in space, but one distinguished from the ordinary T-beam in that the whole shell acts as a compressive slab. Fig. 8 depicts the employment of these shells with double curvature and rectangular plan for a clinker factory in Beocin, and also the application of the apse domes mentioned under c). As I elucidated in my article published in the 'Bauingenieur'[13] there is a state of membranal stress in these semi-spherical domes if the shell is stiffened by rings at the springing. As this type of dome can be constructed as an independent structural member and, in combination with cylindrical shells, can be used for structures which are more or less oval in plan, they prove to be a very important new structural element in the construction of large halls or airplane hangars. In the latter form of structure they are therefore frequently employed as terminal features, with spans up to 40 m. The hangar illustrated in Fig. 3, which is constructed of one longitudinal shell, is terminated at both ends by apse domes. Lastly, Fig. 9 shows another semi-apse dome of this type constructed for the Music Pavilion at Schwalbach Spa.

5) *The principle of calculating the statical balancing of masses, applied to affined shells.*

The types of shell discussed in the previous section of this paper can be calculated with the assistance of the differential equations of the state of membranal stress because the spherical shell in itself is easy to estimate mathematically. The principle of the statical balancing of masses now enables us to calculate in an extremely simple manner the forms of shell affined to them as well. I elaborated this principle in 1928 and elucidated it in the 'Handbuch für

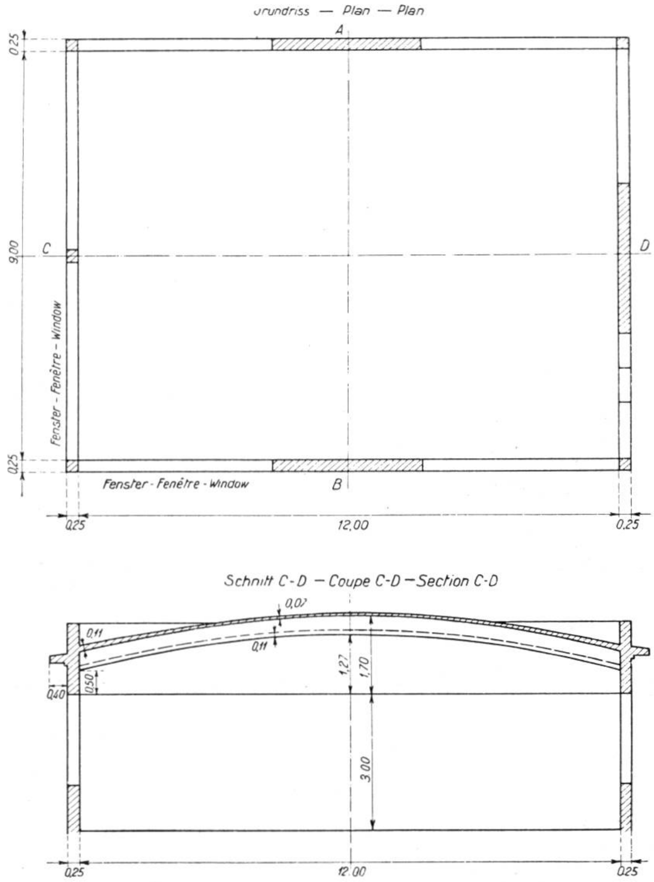

Fig. 7.

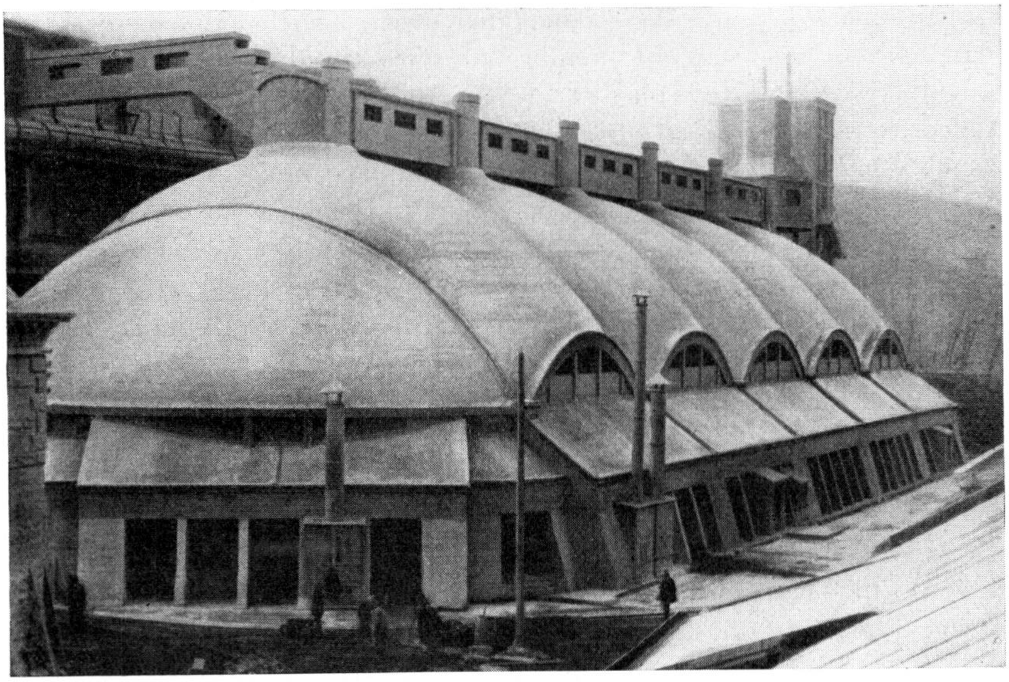

Fig. 8.

Clinker factory, Beocin

Eisenbetonbau'[15] for specific cases. Then, in 1930, I employed the differential equations for shells of any form to give a general outline of the problem: this work was written for the competition already mentioned and was subsequently published in the 'Bauingenieur'.[16] On this basis a shell with elliptical plan, for instance, can be calculated on the basis system of a rotary shell. The numerous other problems that can be solved in this manner are set forth in the above-mentioned article. It may, however, be briefly stated that affined shells can also be estimated in a simple manner.

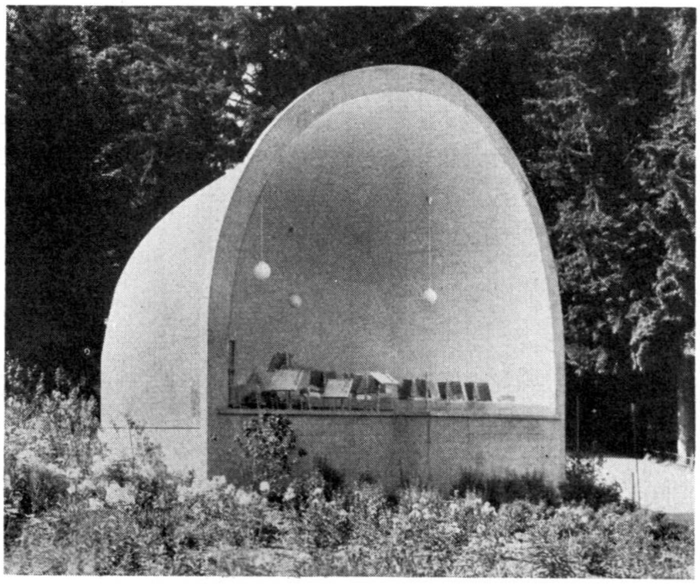

Fig. 9.

Music pavillon Spa Schwalbach

6) Shells of any curvature.

No solutions can be obtained by means of the differential equations of the membranal state of stress for shells with double curvature shaped to any type of surface, because the three partial differential equations thereby evolved cannot be integrated. We are obliged to find another means of approach and solve these equations by difference calculation.

An extremely clear and easily applied method for the solution of problems of this type was elucidated by *Pucher*[17] in 1931. This simple solution is rendered possible by showing that the three differential equations can be combined to form a single one; this is done by introducing a stress function which completely describes the stress conditions. The internal forces of the membranal state of stress can hereby be deduced in a similar manner as from Airy's stress function. As the only assumption that can be made as regards the form of surface is that of continuity, all the forms of shells used in practical construction can be calculated if the circumferential conditions are known and are compatible with the conditions of membranal stress. The method of differences should always be applied if, as mentioned above, a solution is possible with the differential equations. More recent French works follow in principle the line developed by *Pucher*. It is on this theory that the increasing employment of shell construction in France, in the form of non-developable straight-line surfaces, is based. The specific case of translation surfaces has been solved by *Flügge* in the same manner by means of equations of difference.

In conclusion I should like to mention an interesting construction illustrated in Fig. 10. It is the dome in the 'Haus des Deutschen Sports', which was built for the Olympic Games. The Project was prepared by Mr. *Marchi,* architect, and designed by Mr. *U. Finsterwalder*. The skylight is eccentrically placed to afford good ligthting for the platform. The dome in itself, however, exerts no dome action, because the separate shell sectors, which are stiffened by sturdy ribs, project beyond the springing without mutual support.

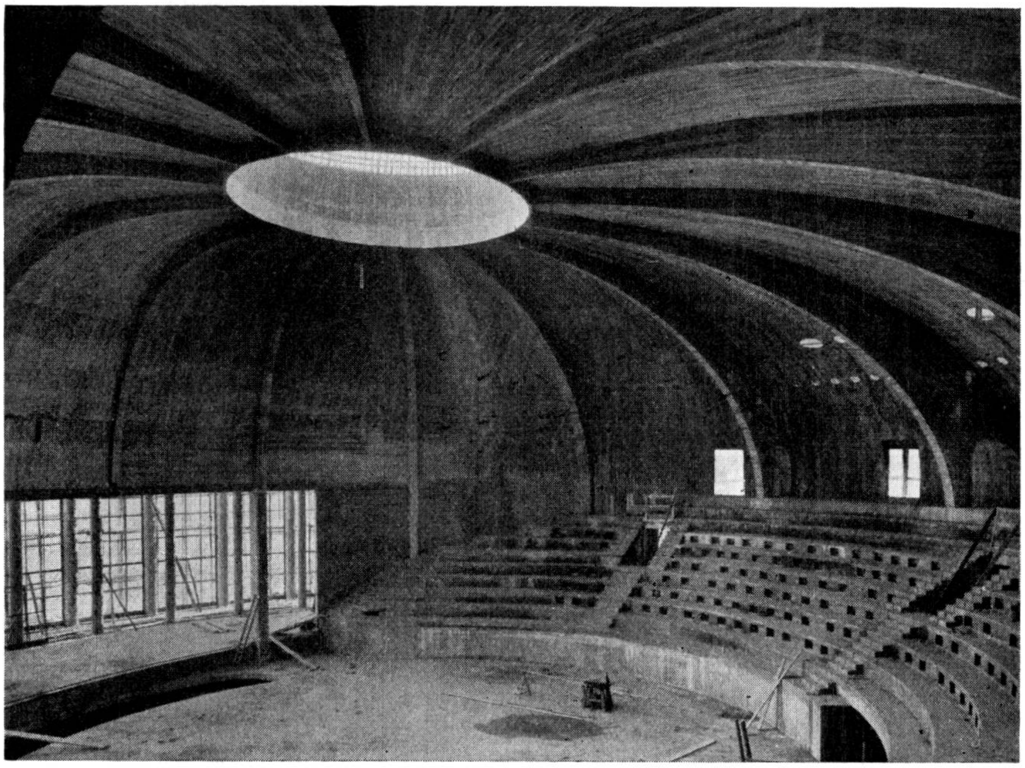

Fig. 10.

„Haus des deutschen Sports". (House of German Sport) Berlin-Reichssportfeld.

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Summary.

The theory of slender continuous beams purposely neglects the influences of deformation due to shear, since they are very small. These influences, however, cannot be neglected in the case of continuous stiffened tubes or cylindrical Zeiss-Dywidag shells. In the treatise following, a general procedure for determining these influences is given and the influences themselves are shown by examples. It is also shown that continuity conditions entirely disappear for boundary cases of small spans in relation to the tube diameter.

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IVa 3

Solid Domes, Cylindrical Reservoirs and Similar Constructions.

Massive Kuppeln, zylindrische Behälter und ähnliche Konstruktionen.

Coupoles massives, réservoirs cylindriques et constructions semblables.

Dr. techn. H. Granholm,
Dozent an der Königlichen Technischen Hochschule Stockholm.

The precise calculation of the bending stresses in a massive dome is a very difficult matter. These difficulties are brought out in a thesis [1] presented to the Royal Technical College Stockholm, and it may well be asked whether the practising engineer ever has the time and opportunity of working out the dimensions of a dome in terms of exact theories. Even to draw up the fundamental equations is a fairly complicated business, and their exact integration leads to series which are often difficult to handle, and which slowly converge. Even though their convergence is satisfactory for many wall-thicknesses (gauges), any alteration in gauge may result in this good convergence being lost. Even where the engineer has the mathetical equipment necessary for dealing with the problem, the amount of work necessary for working out a definite case of loading is much too great; and it may not be possible at all to arrive at practical methods in the way indicated by *Meissner, Bolle, Dubois, Honegger, Ekström* and others. In the case of spherical domes, for instance, integration, even in the simplest cases. gives hypergeometric series which do not constitute the proper equipment of the engineer owing to their slow convergence.

In view of these facts, it is particularly necessary that the further development of the dome theory should be based upon solutions that fully meet practical requirements, even though this involves introducing certain approximations. As *Geckeler* [2] has shown, it is possible, even with comparatively simple mathematical expedients, to arrive at a solution which differs only inappreciably from the true one, and which can be easily and conveniently employed, in cases where the wall-thickness and radius are constant. The good agreement between *Geckeler*'s theory and the exact theory may justify our discussing the former in greater detail, provided we are clear as to what approximations are introduced. A still further step in the direction of the true result is achieved by using *Blumenthal*'s and *Steuermann*'s method of asymptotic integration, which is applicable to

[1] *John Erik Ekström:* Studien über dünne Schalen von rotationssymetrischer Form und Belastung mit konstanter und veränderlicher Wandstärke. Stockholm 1932.

[2] See, inter alia, Handbuch für Eisenbetonbau, Vol, 6, Berlin. 1928.

variable wall thicknesses as well. We actually get farther with this method than we do with the methods that are based on solutions in the form of infinite series, in which connection the wall thickness was always assumed to vary in terms of a definite function if the solution had to be worked out.

Closer examination of *Geckeler*'s final equations reveals that these are of the same type as the equations for an elastically supported beam. Nor is it difficult to appreciate the physical analogy. The meridian of the dome may be regarded as a girder supported by the parallel circles or rings. As these may be compressed or expanded, they correspond, statically, to elastic supports.

When the dome is regarded in this way, its statics may be elucidated with sufficient accuracy. It is not then necessary to revert to *Meissner*'s differential equations for drawing up the equations of equilibrium, but all the necessary equations may be set out directly, simply with the aid of the theory of the elastically supported beam. For the practising engineer, this means that he need not attempt to understand the fairly complicated classic theory of the dome, but can work out the necessary equations for himself.

Geckeler's published works show that he himself has not fully appreciated the high importance of the approximations he suggested; that is, he has not understood that the dome, considered broadly, acts like a steady series of girders on elastic supports. The method of treatment which I suggest can of course be extended by regarding the meridian, not as a girder, but as an arch supported elastically by the ring elements of the dome.

By considering the dome in this way, it is possible to get a more accurate idea of the statics of the structure, and the equations obtained as the result of doing so are the same as *Meissner*'s.

It is obviously necessary to introduce this latter method of conception especially in the case of very flat domes; that is to say, where the arch effect is very manifest in the elements of the meridian, if the desired accuracy is to be achieved. The more inclined the tangent of the dome at its support, the more accurate will be the method where the meridian is regarded as a girder on an elastic support; and in the special case where the tangent of the cupola is everywhere vertical, i. e., when the dome merges into a cylinder, this particular method of considering the dome is perfectly exact.

In order to show more closely how simply the dome problem can be dealt with in this way, I have worked out a few problems and compared the results with those obtained in accordance with the strict theory. The agreement is extremely satisfactory throughout.

As our first example, we shall select a spherical concrete dome of uniform thickness, wall-thickness $\delta = 16$ cm, radius $r = 1000$ cm, angle of opening 40^0. We shall suppose the dome to be loaded with a constant fluid pressure of $p = 1.0$ kg/cm^2, and to be firmly restrained around the edge (see Fig. 1).

If the stresses in this dome be calculated in accordance with the membrane theory, we get a compressive stress at the meridian of $T_1 = \frac{p\,r}{2}$ and an annular compressive stress of $T_2 = \frac{p\,r}{2}$. These meridian and ring stresses are constant throughout the dome, and the solution in terms of the membrane theory is thus very simple. Due to these compressive stresses T_1 and T_2, the dome is compressed, so that its radius is reduced by $\frac{T \cdot r}{E \delta}$, i. e., by $\frac{pr^2}{2E}$. This reduction in radius is not very great, amounting to only 0.15 cm under the assumptions given, and for $E = 210.000$ kg/cm². As the dome is secured about its edge, it is not capable of freely altering its shape; the parts nearest to the edge will

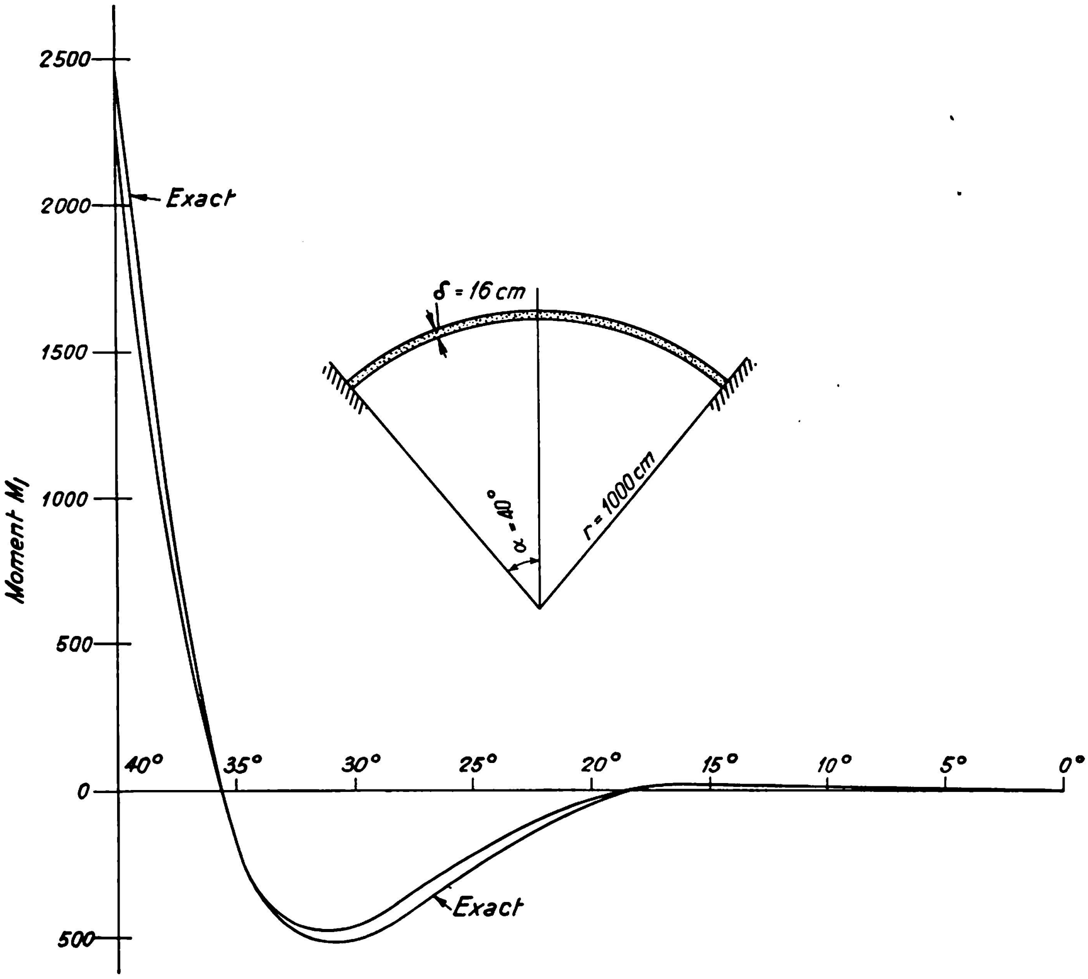

Fig. 1.

Comparison between the values of the Meridian-moments, (1) calculated according to equation and (2) according to the exact method by means of hypergeometrical series. The deviations are of no practical avail.

retain their original radius; but the farther we get from the edge, the more freely will the structure be able to move, and the more freely deformation can take place. Although the compression of the radius is fairly small in this case, certain disturbances are set up near the edges which may lead to bending moments of such magnitude that they cannot be ignored.

We shall now investigate how large moments are set up in an elastically supported girder assuming that it is deflected in accordance with the values $\frac{p\,r^2}{2E}$

calculated above. The moment and the deflection are connected by the formula

$$EJ \cdot \frac{d^2 y}{dx^2} = - M_1 \tag{1}$$

and the effect of the elastic supporting of the ring elements is expressed by the equation:

$$\frac{d^2 M_1}{dx^2} = \frac{E\delta}{r^2} \cdot y \tag{2}$$

Eliminating M_1 from these two equations, we get:

$$\frac{d^2}{dx^2}\left[EJ\frac{d^2 y}{dx^2}\right] + \frac{E\delta}{r^2} \cdot y = o \tag{3a}$$

or, assuming the bending rigidity EI to be constant and equal to $\frac{Em^2}{m^2-1} \cdot \frac{\delta^3}{12}$, we have:

$$\frac{d^4 y}{dx^4} + 4k^4 y = o$$

where

$$k^4 = \frac{3(m^2-1)}{m^2} \cdot \frac{1}{r^2 \delta^2} \tag{3b}$$

The general integral of equation (3b) can be written in the following form:

$$y = e^{-kx}(A \cos kx + B \sin kx) + e^{-kx}(C \cos kx + D \sin kx) \tag{4a}$$

which means that the deflection may be regarded as the sum of two sine vibrations, one having a damped and the other an increasing amplitude. Generally speaking, the coefficients C and D may be taken as = 0, provided the girder is not too short and that the origin is located at the point from which the disturbance proceeds. For closed domes, therefore, the integral can be written with sufficient accuracy in the following form:

$$y = e^{-kx}(A \cos kx + B \sin kx) \tag{4b}$$

Here x is the arc length of the meridian measured from the edge of the dome. In this case the arbitrary constants A and B can easily be determined from the boundary condition, so that

$$y = -\frac{pr^2}{2E\delta} \text{ and } y' = 0 \text{ for } x = 0.$$

This gives $A = B = -\frac{pr^2}{2E\delta}$, and the deflection at the meridian is therefore

$$y = -\frac{pr^2}{2E\delta} \cdot e^{-kx} (\cos kx + \sin kx).$$

By inserting in equation (1) we get the following expression for the meridian moment:

$$M_1 = \frac{\sqrt{3}}{12} pr\delta e^{-kx}(-\cos kx + \sin kx) \tag{5}$$

In this expression, the effect of the transverse compression of the material is ignored, i. e., Poisson's factor m is taken as equal to infinity.

Table I.
Values of functions $e^{-kx}\cos kx$, $e^{-kx}\sin kx$, $e^{-kx}(\cos kx - \sin kx)$ and $e^{-kx}(\cos kx + \sin kx)$.

kx	$e^{-kx}\cos kx$	$e^{-kx}\sin kx$	$e^{-kx}(\cos kx - \sin kx)$	$e^{-kx}(\cos kx + \sin kx)$
0	1.0000	0.0000	1.0000	1,0000
$\frac{\pi}{8}$	0.6239	0.2584	0.3655	0.8823
$\frac{\pi}{4}$	0.3225	0.3225	0.0000	0.6450
$\frac{3\pi}{8}$	0.1179	0.2845	— 0.1665	0.4024
$\frac{\pi}{2}$	0.0000	0.2079	— 0.2079	0.2079
$\frac{5\pi}{8}$	— 0.0536	0.1297	— 0.1833	0.0761
$\frac{3\pi}{4}$	— 0.0671	0.0671	— 0.1342	0.0000
$\frac{7\pi}{8}$	— 0.0592	0.0245	— 0.0837	— 0.0347
π	— 0.0432	0.0000	— 0.0432	— 0.0432
$\frac{9\pi}{8}$	— 0.0269	— 0.0112	— 0.0157	— 0.0381
$\frac{5\pi}{4}$	— 0.0139	— 0.0139	0.0000	— 0.0279
$\frac{11\pi}{8}$	— 0.0051	— 0.0123	0.0072	— 0.0174
$\frac{3\pi}{2}$	0.0000	— 0.0090	0.0090	— 0.0090
$\frac{13\pi}{8}$	0.0023	— 0.0056	0.0079	— 0.0033
$\frac{7\pi}{4}$	0.0029	— 0.0029	0.0058	0.0000
$\frac{15\pi}{8}$	0.0026	— 0.0011	0.0037	0.0015
2π	0.0019	0.0000	0.0019	0.0019
$\frac{17}{8}\pi$	0.0011	0.0005	0.0006	0.0016
$\frac{9}{4}\pi$	0.0006	0.0006	0.0000	0.0012
$\frac{19}{8}\pi$	0.0002	0.0005	— 0.0003	0.0007
$\frac{5}{2}\pi$	0.0000	0.0004	— 0.0004	0.0004
$\frac{21}{8}\pi$	— 0.0001	0.0003	— 0.0004	0.0002
$\frac{11}{4}\pi$	— 0.0001	0.0001	— 0.0002	0.0000
$\frac{23}{8}\pi$	— 0.0001	0.0001	— 0.0002	0.0000
3π	— 0.0001	0.0000	— 0.0001	— 0.0001

From the values of the functions $e^{-kx}\cos kx$ and $e^{-kx}\sin kx$ given in Table 1, it is an easy matter to plot equation (5) graphically. Fig. 1 shows how the meridian moment M_1 varies with the distance from the edge of the dome. The exact values obtained by *Bolle*'s method with hypergeometrical series are given by way of comparison[3]. It will be seen that the agreement between the exact results and the approximate values is surprisingly good, so that there is no occasion to make the dome problem a complicated mathematical business. For domes with a bigger angle of opening than 40°, the agreement between the exact and the approximate values is better still. Only in the case of domes whose angle of inclination to the supports is very small does the effect of the approximations achieve practical significance. Incidentally, such domes are impracticable due to the serious disturbances at the edges set up when the dome is connected to its supports.

For the calculation of the stresses in the dome, we have to consider not only the meridian moment M_1 but also the ring moments M_2 and the additions to the meridian compressive stress and ring compressive stress set up through the boundary conditions not corresponding to the assumptions of the membrane theory. These quantities, M_2, ΔT_1 and ΔT_2 can be calculated directly from the equations below. The agreement between the figures obtained by this approximation method and the exact values is also very satisfactory, as may be seen from the comparative figures given in Table 2.

It is simplest to derive the mathematical expressions for the additional stresses ΔT_1 and ΔT_2 by assuming that the meridian is a girder with an elastic support. The increase in the compressive stress at the meridian, ΔT_1, may thus be regarded as the shearing stress in the girder multiplied by $\cot \alpha$, where α is the angle of inclination of the meridian to the horizontal plane. We therefore get:

$$\Delta T_1 = \cot \alpha \cdot EJ \cdot \frac{d^3 y}{dx^3} \tag{6}$$

The increase in the ring compressive stress ΔT_2 is a measure of the elastically supporting effect of the base, and, hence, ΔT_2 is directly proportional to the deflection y of the meridian, so that

$$\Delta T_2 = \frac{E\delta}{r} \cdot y \tag{7}$$

The ring moment is most simply obtained by determining the alteration in the curvature of the rings[4], and, neglecting the effect of the transverse compression, we get:

$$M_2 = \cot \alpha \cdot \frac{EJ}{r} \cdot \frac{dy}{dx} \tag{8}$$

Inserting the equation for the deflection of the meridian, viz.,

$$y = -\frac{pr^2}{2E\delta} e^{-kx} (\cos kx + \sin kx)$$

[3] See *Ekström*, loc. cit., p. 124.

[4] See, inter alia, *Föppl:* Drang und Zwang, Vol. 2. Berlin 1928.

in equations (6), (7) and (8), we obtain the following expressions for ΔT_1, ΔT_2 and M_2:

$$\Delta T_1 = - \cot \alpha \frac{pr^2 \delta^2}{6} k^3 e^{-kx} \cos kx \tag{6a}$$

$$\Delta T_2 = - \frac{pr}{2} e^{-kx} (\cos kx + \sin kx) \tag{7a}$$

$$M_2 = \cot \alpha \cdot \frac{pr\,\delta^2}{12} k\, e^{-kx} \sin kx \tag{8a}$$

Table 2 contains the values of the meridian and ring stresses and ring moments worked out in this way, in comparison with the exact figures.

Table 2.
Comparison between the Proximate and Exact Values of the Meridian and Ring Stresses and Ring moments.

Angle of Inclination of the Meridian	$T_1 + \Delta T_1$ Proximate kg/cm	$T_1 + \Delta T_1$ Exakt kg/cm	$T_2 + \Delta T_2$ Proximate	$T_2 + \Delta T_2$ Exakt	M_2 Proximate kg cm/cm	M_2 Exakt
40°	443	439	0	0	0	0
35°	474	481	215	193	99	113
30°	503	504	437	427	62	73
25°	506	508	517	520	12	17
20°	503	504	518	523	— 8	— 10
15°	501	501	511	510	— 9	— 14
10°	499	499	501	501	— 5	— 9
5°	499	498	499	498	0	— 3

The problem worked out above relates to the simplest conceivable edge conditions. In order to show the applicability of the method for complicated edge conditions as well, I have worked out a dome connected to a circular cylinder all round, as Fig. 2. To simplify the problem to a certain extent, the water pressure on the dome was assumed to be constant. This problem has been dealt with by Ekström under the same assumptions. The calculated values for the meridian moment M_1 and the ring stress T_2 are given in Table 3, with the exact values for comparison.

The index 1 will subsequently be used for all constants of the dome, and the index 2 for all constants of the cylinder.

This design of dome is worked out as follows. When the inner dome and the cylinder are freed from each other and can deform unhindered under the effect of the load, the membrane theory shows a decrease in the radius of the dome of

$$\frac{pr_1^2}{2E\delta_1} = \frac{p \cdot 10^4}{E} \cdot 3{,}12 \text{ cm}$$

and an increase in the radius of the cylinder of

$$\frac{pr_2^2}{E\delta_2} = \frac{p \cdot 10^4}{E} \cdot 1{,}72 \text{ cm.}$$

The wall of the cylinder thereby forms a small angle to the perpendicular $= \frac{10}{E} \cdot 1.72$ (see Fig. 2).

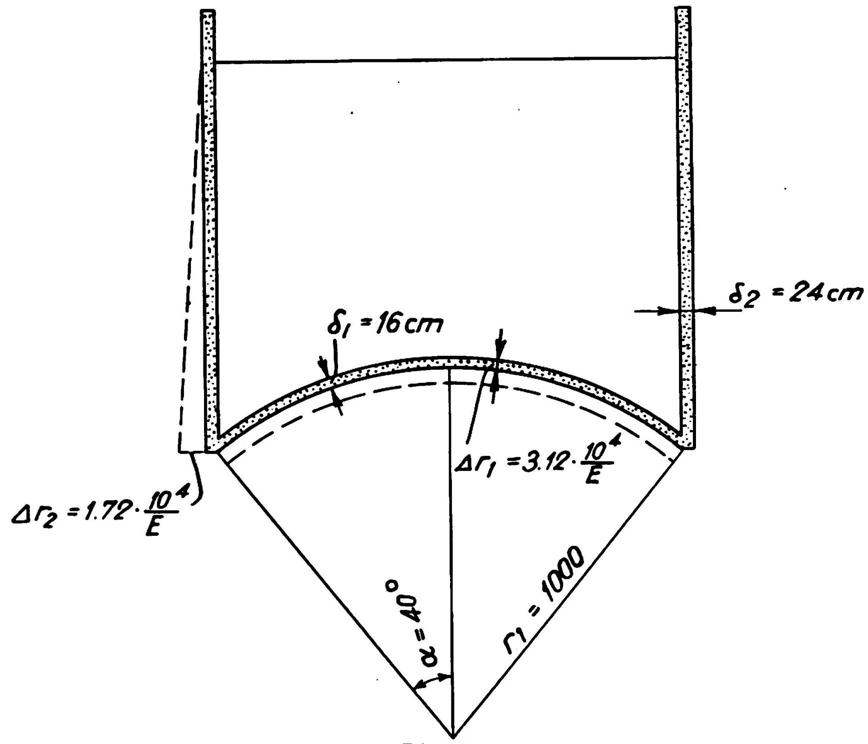

Fig. 2.

As this state of deformation is incompatible with the actual conditions of support, certain additional forces and additional moments must be introduced to satisfy the conditions of steadiness. These conditions of steadiness are as follows: The cylinder and the dome should have the same outward deflection and alteration of angle at the point of junction, and the point of junction should also be in equilibrium as regards the moments and applied forces. This involves four edge conditions, which may be expressed by means of four equations, from which all unknown deformations, moments, etc. may be determined.

To facilitate drawing up the equations, we now give the general expressions for the deflection and their derivation. We have:

$$\begin{aligned} y &= e^{-kx}\,[A \cos kx + B \sin kx] \\ y' &= k\, e^{-kx}\,[(B - A) \cos kx - (A + B) \sin kx] \\ y'' &= 2k^2\, e^{-kx}\,[- B \cos kx + A \sin kx] \\ y''' &= 2k^3\, e^{-kx}\,[(A + B) \cos kx + (B - A) \sin kx] \end{aligned} \tag{9}$$

The first condition, viz., that the deflections of the cylinder and the dome must be the same at the edge, may be expressed by the following equation:

$$- A_1 \sin 40^0 + A_2 = \frac{p \cdot 10^4}{E} (3{,}12 \sin 40^0 + 1{,}72).$$

So that the angular modifications may be the same in extent, we must get

$$k_1 (B_1 - A_1) = k_2 (B_2 - A_2) - \frac{10}{E} \cdot 1{,}72$$

and for the equilibrium of moment we get

$$k_1^2 EJ_1 B_1 = k_2^2 EJ_2 B_2.$$

The remaining condition should express the fact that the horizontal reaction due to the loading of the inner dome should be taken up by the shearing stress in the cylinder and by the shearing stress and the meridian stress in the dome; i. e.,

$$-2k_1^3 EJ_1 (A_1 + B_1) \cdot \frac{1}{\sin 40^0} - 2 k_2^3 EJ_2 (A_2 + B_2) = p \cdot 500 \cdot \cos 40^0.$$

By elimination from these four conditional equations, we get, for $p = 1$ kg/cm² the following values of the constants:

$$A_1 = -15{,}35 \cdot \frac{10^4}{E} \qquad B_1 = -7{,}16 \cdot \frac{10^4}{E}$$

$$A_2 = -6{,}13 \cdot \frac{10^4}{E} \qquad B_2 = 2{,}05 \cdot \frac{10^4}{E}.$$

This completely solves the problem. The moments, etc. can now be worked out without difficulty for any point of the cylinder and the dome. Table 3 contains a comparison of the calculated and true values for meridian moment and ring stress in the dome. The agreement is satisfactory at all points.

Table 3.

Meridian Moments and Ring Stresses of the Dome as Fig. 2.

Angle of Inclination of the Meridian	M_1 Proximate kgcm/cm	M_1 Exakt kgcm/cm	$T_2 + \Delta T_2$ Proximate kg/cm	$T_2 + \Delta T_2$ Exakt kg/cm
40°	— 5280	— 5560	— 1950	— 1930
35°	1450	2250	— 800	— 540
30°	1980	2200	401	613
25°	597	764	618	639
20°	— 6	9	572	593
15°	— 99	— 141	520	526
10°	— 54	— 80	498	498
5°	— 8	— 15	495	493

These two examples indicate that the method explained here for dealing with the problem gives results which are practically applicable and easy to find. As already mentioned, the proximate solution comes closer to the true values, the steeper the dome and the thinner the shell. This latter factor in particular is of great importance, as *Steuermann*[5] and others have pointed out. Unlike Equation (3 b), the exact equation for the outward deflection of the meridian contains not only expressions of the fourth and zero order, but also expressions with derivitives of the first, second and third degree, which,

[5] *E. Steuermann:* Some Considerations on the Calculation of Elastic Shells. International Conference for Technical Mechanics, Stockholm, 1930.

however, are all multiplied by polynomes of cot α. The significance of these expressions decreases with increasing values of α, and for $\alpha = 90^0$, i. e., for the cylinder, they drop out altogether, which means that equation (3 b) applies exactly. A reduction in the wall thickness of the dome has a similar effect on the complete differential equation. It is easy to see why this should be the case; it is simply due to the fact that, for small wall thickness, the compression at the meridian and the influence of the change in curvature are less pronounced in their effect. Put differently, this means that the work of the normal stresses due to compression of the meridian, together with the work of the meridian moment and the ring stresses may be ignored in thin-walled domes.

In the problems dealt with previously, the wall thickness was assumed to be constant throughout. Where the wall thickness δ is variable, we cannot start from equation (3b), must apply equation (3a). As the simple theory of the elastically supported girder gave sufficiently accurate results in the above casses, i. e., for constant wall thickness, there was reason for assuming that this would also be the case for variable wall thicknesses.

The theory of the elastically supported girder with variable moment of inertia and variable support has previously been studied by various researchers,[6] mainly with the aid of series. Unfortunately the results obtained are more or less useless for practical purposes. Due to the close affinity of equations (3 a) and (3 b), however, it is only natural that the solutions of both equations should have substantially the same mathematical basis. It may therefore be supposed that the solution of equation (3 a), for instance, may be written in the following form:

$$y = u e^{\pm z} (A \cos z + B \sin z) \tag{12}$$

where u and z are certain functions of x. By adopting *Blumenthal*'s "asymptotic process of integration", the functions of u and z can be ascertained, so that equation (12) represents an integral of equation (3 a) with very good approximation.

By introducing, as above, the bending rigidity of the girder $E\,J = \frac{E\,\delta^3}{12}$, we get the following expressions for the functions of u and z:

$$u = \frac{1}{\sqrt[4]{\delta^3}} \tag{13}$$

and

$$z = \sqrt[4]{3} \int \frac{dx}{\sqrt{r\delta}} \tag{14}$$

This result is obtained in the following way. Carring out the derivation of equation (3 a), and simplifying, we obtain the equation:

$$y^{IV} + p_1 y''' + p_2 y'' + p_3 y' + p_4 y = 0 \tag{15}$$

where $\quad p_1 = 6 \frac{\delta'}{\delta}$

[6] See, for instance, *Hayashi:* Theorie des Trägers auf elastischer Unterlage, Berlin, 1921.

$$p_2 = 3\left(\frac{\delta'^2}{\delta^2} + \frac{\delta''}{\delta}\right)$$
$$p_3 = 0$$
$$p_4 = \frac{12}{r^2\,\delta^2}$$

Multiplying the equations

$$\begin{aligned} v &= f(z) \\ v' &= f'\,z' \\ v'' &= f'\,z'' + f''\,z'^2 \\ v''' &= f'\,z''' + 3\,f'\,z'\,z'' + f'''\,z'^3 \\ v^{IV} &= f'\,z^{IV} + f''(4\,z'\,z''' + 3\,z''^2) + 6\,f'''\,z'^2\,z'' + f^{IV}\,z'^4, \end{aligned}$$

$\left(\text{where } f' \text{ is equivalent to } \frac{d\,f}{d\,z} \text{ and } z' \text{ to } \frac{d\,z}{d\,x}\right)$ in turn by the factors Q_4, Q_3, Q_2, Q_1, and 1, and adding them, then, when the member on the left is written as equal to zero, we obtain (1) equation:

$$v^{IV} + v'''\,Q_1 + v''\,Q_2 + v'\,Q_3 + v\,Q_4 = o \tag{16}$$

and (2), when each of the factors f', f'' and f''' are made zero:

$$\begin{aligned} &z^{IV} + z'''\,Q_1 + z''\,Q_2 + z'\,Q_3 = o \\ &(4\,z'\,z''' + 3\,z''^2) + 3\,z'\,z''\cdot Q_1 + z'^2\,Q_2 = o \\ &6\,z'^2\,z'' + z'^3\,Q_1 = o \end{aligned} \tag{17}$$

Q_1, Q_2 and Q_3 can be solved from these equations, whereas the function f (z) is determined by the remaining condition

$$f^{IV}\,z'^4 + Q_4 \cdot f = o \tag{18}$$

If the factor Q_4 is taken as being equal to $4\,z'^4$, equation (18) is then transformed into

$$\frac{d^4\,f}{dz^4} + 4\,f = o$$

that is
$$f(z) = e^{\pm z}\,(A\cos z + B\sin z) \tag{19}$$

z being determined by the condition

$$\frac{dz}{dx} = \sqrt[4]{\frac{Q_4}{4}} \tag{20}$$

Inserting $y = uv$ in equation (15) and dividing by u, we get:

$$v^{IV} + v'''\left(\frac{4\,u'}{u} + p_1\right) + v''\left(\frac{6\,u''}{u} + \frac{3\,u'}{u}\,p_1 + p_2\right)$$

$$+\,v'\left(\frac{4\,u'''}{u} + \frac{3\,u''}{u}\,p_1 + \frac{2\,u'}{u}\,p_2 + p_3\right) + v\,p_4 = o \tag{21}$$

The unknown functions Q_4 and u can be determined by equalising the coefficients for v and v‴ in equations (16) and (21). This gives us $Q_4 = p_4$ and, consequently, as equation (20):

$$z = \int \sqrt[4]{\frac{p_4}{4}}\, dx$$

$$\text{or, with} \quad p_4 = \frac{12}{r^2 \delta^2}; \quad z = \sqrt[4]{3} \int \frac{dx}{\sqrt{r\delta}} \tag{14}$$

From the condition $\frac{4u'}{u} + p_1 = Q_1$ and adopting the last of the equations (17), we get:

$$\frac{4u'}{u} = -p_1 - \frac{3}{2}(\log p_4)'$$

$$\text{or} \quad u = \frac{1}{\sqrt[4]{\delta^3}} \tag{13}$$

Summarising the result of the above calculations, the solution of equation (3 a) can be written in the following form by neglecting the expressions containing the factor e^{+z}:

$$y = \frac{1}{\sqrt[4]{\delta^3}} e^{-z} (A \cos z + B \sin z) \tag{12a}$$

in which z is determined by the condition $z = \sqrt[4]{3} \int \frac{dx}{\sqrt{r\delta}}$.

At first glance, equation (12 a) may perhaps appear involved and not very suitable for practical purposes, due to the complicated structure of the function z and of the additional factor $\frac{1}{\sqrt[4]{\delta^3}}$. The case becomes simpler in actual practice, however. The function z need never be indicated other than numerically, so that it can easily be calculated from equation (14), say, by the trapeze rule. In calculating the derivatives of equation (12 a), fairly complex expressions are obtained where no approximations are introduced. But when it is remembered that the derivations z″, z‴, u″ and u‴ are small for the dimensions involved in actual practice, and can therefore be neglected, the derivations of y are obtained in the following form:

$$\begin{aligned} y &= u\, e^{-z} (A \cos z + B \sin z) \\ y' &= u\, z'\, e^{-z} [(B - \mu A) \cos z - (A + \mu B) \sin z] \\ y'' &= 2\, u\, z'^2\, e^{-z} [-(\mu B + \gamma A) \cos z + (\mu A - \gamma B) \sin z] \\ y''' &= 2\, u\, z'^3\, e^{-z} [(A + \mu_1 B) \cos z + (B - \mu_1 A) \sin z] \end{aligned} \tag{9a}$$

$$\text{where} \quad \begin{aligned} v &= \frac{u'}{u\, z'} \\ \mu &= 1 - v \\ \mu_1 &= 1 - 3\, v. \end{aligned}$$

In cases where the wall thickness is constant, $v = 0$ and $\mu = \mu_1 = 1$, the above equations becoming exactly the same as the equations (9).

The equations (9 a) are therefore built up in the same way as the derivations for a girder of constant bending rigidity given in the equations (9). A dome of variable wall thickness can consequently be worked out in the same way and with very little more trouble than one of constant wall thickness. The examples given above (see Figs. 1 and 2) are thus typical of the case where δ is variable, and the equilibrium equations should be drawn up in the same way, but with the modifications necessitated by the difference between equations (9) and (9 a).

We have not yet considered the dome problem in cases where the girders at the meridian taper upwards and their width is nil at the apex of the dome, but have rather assumed them to be of constant width. This is only true when the dome is cylindrical; but for domes in general a certain process of approximation is inherent in this assumption. When allowing for the taper, we can, for spherical domes, express the moment of inertia of the meridian girder at a definite angular distance α from the apex by the following equation:

$$J = \frac{\delta^3}{12} \cdot \frac{\sin \alpha}{\sin \alpha_0} \tag{21}$$

With this expression for the moment of inertia, we obtain, for functions u and z:

$$u = \frac{1}{\sqrt[4]{\delta^3}} \cdot \frac{1}{\sqrt[8]{\sin \alpha}}$$

and

$$z = \sqrt[4]{3} \int \frac{1}{\sqrt{r\delta}} \cdot \sqrt[4]{\frac{\sin \alpha_0}{\sin \alpha}}\, dx.$$

The above derivations, which apply mainly to the dome problem, may of course also be applied to cylindrical tanks and similar structures, which should be regarded as special cases of the dome. The usual methods [7] for calculating such containers, based on the developments of mathematical series, may advantageously be replaced by the method given above. A special and interesting case of this particular problem is met with in the calculation of solid arched dams. The usual method of dealing with these problems was to start from equation (3 b) and introduce a mean value for the wall thickness [8].

In dealing with equation (3 a) by the above method, it is possible, without difficulty, to allow for the anisotropy of the structure which occurs in various directions and at different points. The anisotropy may be purely a phenomenon of the material, or a purely constructive anisotropy. When different quantities of reinforcing steel are inserted in different directions, for example, the apparent modulus of elasticity of the material will vary in different directions, and this we may term anisotropy of the material; while a certain constructional or design anisotropy may be introduced into a cylindrical tank or a dome by fitting reinforcing girders in the direction of the generatrix or the meridian

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[7] See, *Lorenz*: Technische Elastizitätslehre, Berlin 1913. *H. Reißner:* Beton und Eisen 7, 150, 1908. *T. Pöschl* and *K. Terzaghi:* Berechnung von Behältern, Berlin 1913.

[8] *N. Royen:* Tvärödammen vid Norrfors kraftverk (Der Damm von Tvärö am Kraftwerk Norrfors), Zeitschr. Betong, vol. 2, 1926,

(ribbed domes). In such circumstances, equation (3 a) cannot be written in the form in which it is contained in equation (15), but the coefficients p_1 to p_4 assume the following aspect:

$$\begin{aligned} p_1 &= \frac{2\,(E_1\,J)'}{E_1\,J} \\ p_2 &= \frac{(E_1\,J)''}{E_1\,J} \\ p_3 &= o \\ p_4 &= \frac{E_2\,\delta}{r^2\,E_1\,J} \end{aligned} \tag{22}$$

and the functions z and u accordingly appear in the following form:

$$z = \int \sqrt[4]{\frac{E_2\,\delta}{4\,r^2\,E_1\,J}}\,dx \tag{23}$$

and

$$u = \sqrt[8]{\frac{r^6}{E_1\,J \cdot E_2{}^3\,\delta^3}}$$

Since no mathematical expression is necessary either for u or z, the introduction of equations (22) and (23) does not make the calculations more difficult.

Summary.

By dividing the shell into two systems of intersecting beams and by applying the well-known theories of the elastically supported beam, we get a clear idea of the statical behaviour of the construction and obtain results that are sufficiently correct. As the exact theories lead to solutions in form of infinite series, which under certain conditions converge only slowly, this way of dealing with the problem offers great simplification.

IVa 4

Shell Structures with or without Stiffeners.

Versteifte oder unversteifte Flächentragwerke.

Ouvrages à parois minces reforcées ou non par des raidisseurs.

R. Vallette,

Ingénieur au Chemin de fer de l'Etat, Paris.

The question of thin shell-structures having been discussed at the Paris Congress, we only intend to study the tendencies which arose since this Congress.

We can distinguish between two types of thin shell-structures: first, structures in which the stiffness of the shell has been taken into account for the strength of the system as a whole; second, structures in which this stiffness has been completely neglected, the shell itself being then considered able to withstand only stresses acting tangentially to the surface, thus the shell behaving as a simple membrane.

Accordingly we have to consider:

1. — the stiff, thin walls and shells,
2. — the membranes.

We intend to examine in the present report the structures consisting of stiff, thin walls and shells, the structures consisting of membranes being the subject of a report by Mr. *Aimond.*

I. — Shell-Structures.

A. — Design.

Generalities.

From the beginning of reinforced concrete, the thin shells of the slab type have been taken into account for the general strength of structures. The monolithic nature of construction is, in fact, one of the important characteristics of reinforced concrete. However, a more complete use of the strength of shells has been considered later on, and these shells became the principal element of strength for structures such a load-carrying walls of reservoirs, silos, vaults and arches, etc.

Constructional applications.

1. — *Reservoirs.*

In the design of reservoirs, the load-carrying wall has been entirely employed for bottoms, cantilevers, covers, but the stiffness of such walls has been taken into account only occasionally.

2. — *Silos.*

For silos, the use of carrying walls, originally only partial, became exclusive after certain methods were known which were reported to the Paris Congress by Mr. *Freyssinet.* We have nothing to add on this subject, no further aspect having evolved since.

3. — *Vaulted structures.*

a) *Normal vaults.*

So far as normal vaults are concerned, Mr. *Freyssinet* had already shown the tendency of like development at the Paris Congress, when he stated that if he had to design now the Orly sheds, he would adopt a ribbed design, with spans of 25 m between the supporting ribs. Such a statement is all the more noteworthy as the Orly sheds[1] in their existing form (erected 1922) can be considered as the most remarkable and precursory example of self-supporting systems with numerous short spans, such as were subsequently frequently used in Central Europe.

In fact, we find sheds with bays 7,50 m wide, spanning 90 m. The resistance of the shells being hereby entirely absorbed by the general bending conditions of the whole structure (Freyssinet-Limousin, Contractors).

Since the Paris Congress, this tendency has remained and it has been

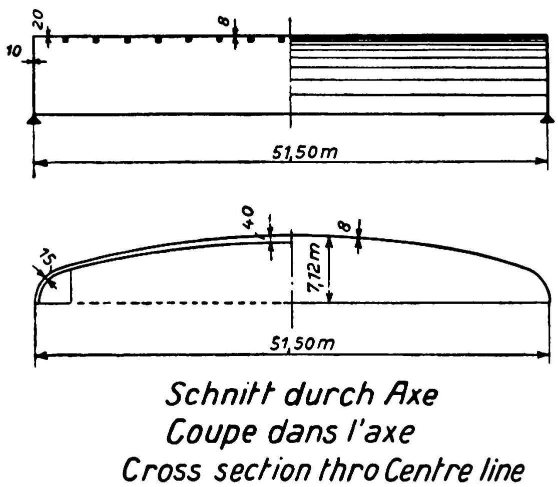

Fig. 1.

Barrel arch of 51,50 m span.

actually possible to design a reversed cradle vault of 51,50 × 51,50 m supported only at the four corners, the vault being entirely self-supporting and containing only small stiffening ribs of a purely secondary character, without any end beams contribution to the strength of the structure (Fig. 1) (Boussiron's scheme). We can consider this type of construction as the outcome of the type of roof design used in France since 1910. by several designers, using a portion of the vault itself as supporting beam (end beam), between more or less widely spaced columns. Originally, the heigth of the acting portion of the vault OA taken into account was small (Fig. 2) and a beam ON was necessary to establish

[1] Génie Civil, Sept. 22 to Oct. 6, 1923.

the required strength of the structure. Gradually, the heigth OA was increased, while the supporting action of the beam decreased. The span between columns was notably increased as well and at present, the whole of the shell is utilized for any span and without the use of supporting end beams.

b) *Other types of vaults.*

1. An alternative to the Orly sheds has been adopted for the construction of a twin-shed for airplanes at Cherbourg. The vault consists exclusively of thin shells (Fig. 3) which, as at Orly, are alone responsible for the strength of the structure: the shells became self-supporting, between widely spaced columns (Société Rabut-Subileau).

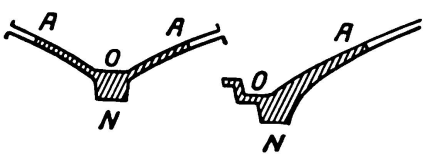

Fig. 2.
Beams supporting shells.

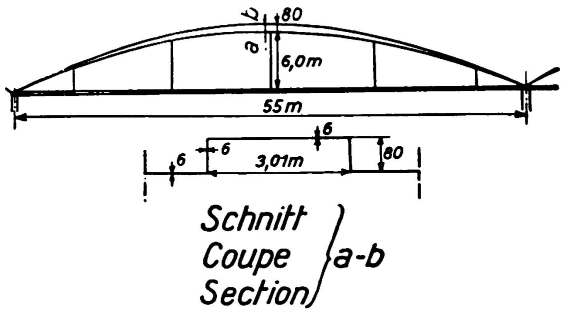

Fig. 3.
Twin hangar. Cherbourg.

2. The conoidal shaped shells (Freyssinet-Limousin) were applied for a great number of structures (Works at Montrouge, Caen, Fontenay, etc.); This type of shells has been studied by Mr. *Fauconnier* in a report published in the second volume of the "Publications" of our Association, which leaves us only to say that these shells are self-supporting, even if resting on columns far apart.

4. — *Other types of structures.*

Other types of roof shells, such as for domes with square plan, cross vaults, cloister vaults, etc., have been considered by different designers in connection with competitions for aerodrome schemes, for airplane hangars; however, with competitions for aerodrome schemes, for airplane hangars; however, such types are not sufficiently developed as yet to allow for definite conclusions.

A very remarkable construction, of quite a different nature has been erected near Paris for the testing of airplanes; it is the Aerodynamic Tunnel of Chalais-Meudon[2]. This tunnel consists of a certain number of thin walled, self-supporting elements, among which an elliptical diffuser tube of imposing dimensions (Fig. 4), which has its supports in two places only. These supports are 34 m apart whilst the self-supporting structure has walls only 7 cm thick, stiffened by ribs, 3,60 m apart (Limousin).

5. — *Conclusions.*

Concluding, we can distinguish in France between two tendencies in the development of shell construction. On the one hand, exists an undefined tendency

[2] Génie Civil, Nov. 3, 1934.

to search for new types of roof shells, without the definite trend to find a particular type. On the other hand, with such types as are now definite, a marked tendency exists for taking full advantage of the load-carrying properties of the walls. This tendency is going as far as using the whole of the section of a structure (great vaults, tunnel at Chalais-Meudon), if the span allows of it. At the same time the character of shell constructions should be maintained without the necessity of requiring end- and other beams as supporting members. This can be regarded as one of the characteristics of French constructional tendencies.

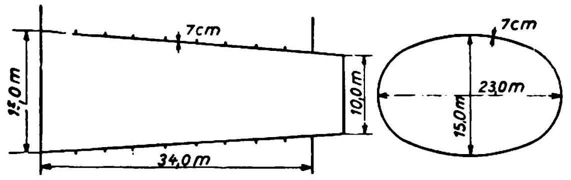

Fig. 4.

Tubular wind channel, Chalais-Meudon.

B. — *Calculation.*

1. — *Flat shells.*

As regards the calculation of flat load-carrying shells, we refer to the method suggested by Mr. L'Hermite (Génie Civil, April 29, 1933).

2. — *Self-supporting shells of different types.*

The use of the shells of a vault to act as end beams in particular, for transmitting to the supports the loads of the structure, was a feature introduced in concrete shell constructions right from the introduction of such structures.[3] These beams, making only use of a small portion of the shell, were calculated by the designers as ordinary independent straight beams leading this way to unnecessarily heavy structural members. For short spans between supports, such excess of material does not render it advisable to apply advanced methods of calculation.

For the case, otherwise rare, where the purpose of the structure demanded long spans between columns, the height of vault to be taken into account caused profiles of pronounced curvature, calling for special methods of calculations. We know that some designers (particularly Mr. Boussiron) succeeded in evolving special solutions for such problems, though they did not publish their investigations. We have ourself indicated, later on, a method[4] concerning the calculation of such a beam and which applies at the same time for calculating shells of the cradle type of any shape, supported only at the extreme ends.

This method allows to extend the theory of bending to thin shells of curved sections and permits investigations into the consequences of secondary stresses. With this method the means are given to study specially the transverse bending stresses introduced in a sectional element of the shell by tangential forces acting along the directrices of the shell. This particular mode of calculation applied for large spans was found complete and reliable. The results obtained with this

[3] Génie Civil, January 27, 1934.

[4] Génie Civil, January 27, 1934.

method are in accordance with observations made on models, of experimental shells, as well as on actual structures.

3. — *Other structures.*

The same method of calculation can be applied to structures forming complete tubes and we have shown this method which was used also for the calculation of the diffuser tunnel of Chalais-Meudon (described above) in the reports concerning this structure.[5]

With regard to domes, the normal type of which requiring but a simple method of calculation, the stiffness of the shell has to be taken into account only for concentrated loads acting in a confined zone. In most cases, the problem leads to the class of membranes on account of such systems being composed of meridians and parallel circles.

With regard to other types of stiff surfaces for roofing purposes, there can be found, in French technical literature, no statement of any method of calculation; these types are still imperfectly studied and remain a field of exploitation to the designer.

4. — *Conclusions.*

Methods for the calculation of thin walled and stiff structures have retained in France the character of simplicity which has so far been the rule for the design of reinforced concrete structures. In fact, we are dealing with materials and systems of complex and varying nature and it would be vain to search for laws and rules expressing all phenomena possible which take place in a structure under the influence of loading. It suffices to retain the principal facts which can be considered as characteristics and expressed by means of simple laws (Hooke's law, Navier's law, etc.), such laws being imperfect, but safe.

The aim is not to obtain a purely mathematical solution of a problem; it is only a question of calculating sufficiently well all important influences which appear in a given system, in order to prevent useless surplus material or noticeable underestimation. The endeavour to find such a practical solution, on the simple basis indicated, should however be guided by making use of all the possible resources of the art of calculation, with the aim to arrive at a definite solution, safe and easily applicable. It is worth pointing out that in the history of reinforced concrete structures, definitive and clarified methods of calculation have only been established after the actual execution of structures carried out by our big contractors.

Imagination, technical sense and a true conception of the internal working of a structure are inseperable foundations for the creation of new types of structures, and are sufficient to the designer for the calculation and design of any new type. It is in fact always possible to value an acting force if its action is fully perceived. It is this point which calls for the most careful investigations into the nature of the numerous parts which form a reinforced concrete structure. The solution of these problems demands that practical technical sense which makes the prerogative of a good designer.

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[5] Génie Civil, November 3, 1934.

The calculation of thin and stiff shells has followed this development and has retained the definite tendency to keep on the lines of simplicity and clearness, required for the investigation into the problems with which the analysis of such types of structures is concerned.

Summary.

A study of shell constructions under consideration of the actual stiffness.

After summarising the development of shell construction, the author shows that in France nowadays only self-supporting shells are used without the aid of any border beams, even if it is a question of systems composed of small multiple shells (Halls at Orly) or single shell constructions of wide spans, or closed shells composed of rings (wind channel at Meudon).

The author points out that shell constructions are calculated in France with the same clarity as is usual for other reinforced concrete constructions. This permits the designer to employ and develop this new type of construction in an unrestricted manner under proper consideration of all the forces.