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The Safety of Structures.

Sicherheit der Bauwerke.

Sécurité des constructions.

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I. During the last few years it has become necessary, when studying the strength of structures, to consider the plastic properties of matter.

The strength of materials had been based primarily on the hypothesis that, when the forces, to which a structure is subjected, increase, the deformations and rate of stressing increase everywhere in straight line proportion.

The pure elastic behaviour of materials does not exist except in its first approximation; this hypothesis must be abandoned in every case where the stressing conditions attain such values which may cause fracture or where at least great deformations are produced.

If the effect of the non-elastic properties of materials is favourable to the stability or not, depends on circumstances.

Contrary to the stability met with in the case of a member compressed longitudinally tending to buckle, the effect is usually regarded as increasing the stability of statically indeterminate systems.

II. The writer proposes to discuss this problem in the present paper. First of all, to arrive at the degree of safety of a structure, we have to consider how it behaves under increasing loads until fracture or a vital amount of deformation occurs.

This means replacing the modulus of elasticity E of the elastic period by instantaneous coefficients of elasticity H' corresponding to very small supplementary loads. These coefficients of elasticity H' will vary from one point to another, and will also differ according to whether the particular additional load increases or reduces the previous stress.

In this connection it may be mentioned that this consideration of instantaneous coefficients of elasticity explains the results of the researches on buckling carried out by *M. Roš* at the Zurich Polytechnic.

When a compressed member (Fig. 1) is subjected to a transverse force, for instance, the bending set up in the member increases the compression on the concave side and reduces it on the convex side. The stress-strain diagram for sections parallel to the axis is in this case a broken line (Fig. 2). The mean coefficient of elasticity which should be inserted in Euler's formula is always lower than the usual coefficient of elasticity, so that the limit can be reached when compression is still below the amount determined by the usual calculation.

Similarly, in a statically indeterminate system, the deformations may take place rapidly from the moment when the loads have caused the materials to exceed the elastic limit, and there is a risk of relying upon assumptions which are too simple for fixing the degree of safety.

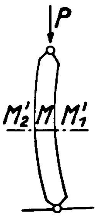

Fig. 1.

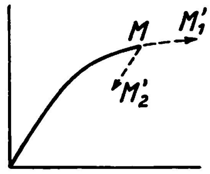

Fig. 2.

III. The following remarks may indicate the correct lines of this discussion.

The relation existing between the deformations of a member and the stresses represents a property of the material. It is an *intrinsic* law peculiar to the material combining these variables (Fig. 3).

In a structure, however, none of these variables may usually be regarded as a known quantity, and it is no more logical to say that the deformation is a function of the stress than it is to assume that the stress is a function of the deformation.

Actually, in a member AB belonging to a particular structure, equilibrium between the deformations and the stresses takes place under conditions which involve not only the elasticity of the material inside the member A B, but also the elasticity of the system outside of A B.

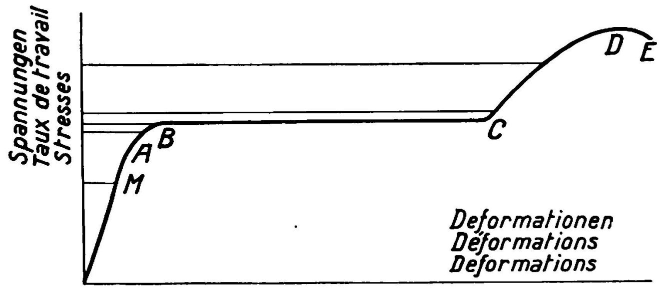

Fig. 3.

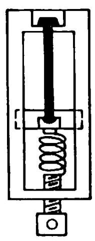

Fig. 4.

If, for instance, a test piece be stretched by means of a screw jack supplemented by a spring (Fig. 4), the pull deforms both the test piece and the spring, and the elasticity of the spring must be known before we can judge what is taking place in the test piece. For each position of the screw, we have to ascertain where the characteristic point on the stress-strain diagram (*intrinsic* curve) joins up with a curve representing the *extrinsic* law and which connects up the distance between the jaws and the stress which they transmit.

Similarly, if increasing loads be applied to a statically indeterminate system, the characteristic point referring to the member subjected to the highest stress on the *intrinsic* stress-strain diagram must be referred to a series of *extrinsic* curves (Fig. 5). These curves slope down because, in each case, the forces applied to the member in question diminish when the deformation increases.

Now consider the state of two members of the same metal and belonging to isostatic and hyperstatic systems respectively, and subjected to certain low stresses applied at the same rate. It will be assumed that the loads, and the stresses which would be applied to the members if they were not deformed per se, are proportional (Fig. 6).

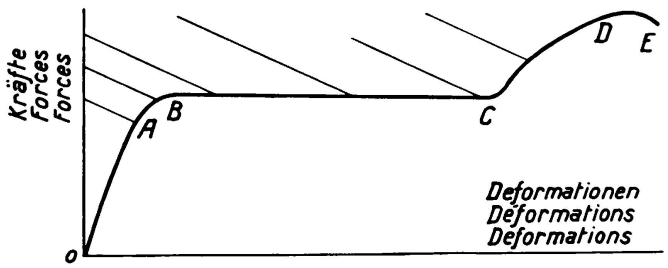

Fig. 5.

If the deformations and stresses be multiplied by a certain factor, the points M representing the state of the members will be substituted by a point P located on the elasticity line very close to the curve A B. If the load is multiplied by a lower coefficient the two members will assume equivalent positions; whereas, if the load is multiplied by a higher coefficient, the first member immediately borders on the zone C D of high permanent deformations, while the

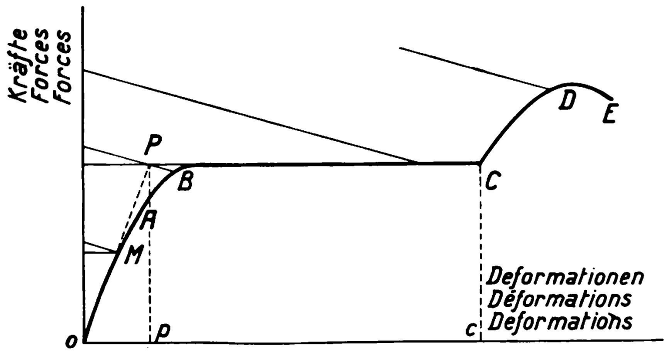

Fig. 6.

second will first of all follow the horizontal line of the elastic limit. The gain is greater, the higher the ratio

$$n = \frac{pc}{op} = \frac{\text{Deformation when Elastic Limit is exceeded}}{\text{Elastic Deformation}}.$$

It is also proportionally greater when the law of extrinsic deformation shows a rapid downward tendency.

This ratio n, which plays an essential part in the subject under discussion, may be termed the 'ductility number'.

For structural steel 42/25, the usual numerical data are as follows:

Relative deformation after line of elastic limit $= 0.027$;

Modulus of elasticity $= 22.10^3$ kg/mm^2;

Elastic limit $= 26$ kg/m^2.

Hence:

$$n = \frac{0.027 - \dfrac{26}{22.000}}{\dfrac{26}{22\,000}} = 22.$$

IV. The preceding considerations may be applied not only to the hyperstatic systems, where the distribution of the stresses is only determined with regard to the deformations, but also to bent elements of statically determinate systems. In the latter case, plasticity governs the distribution of the stresses in a section.

Actually, when the stresses increase in a hyperstatic structure or in a curved member, the zones subjected to the highest stress undergo plastic deformations from the moment they reach the elastic limit. The load conditions of the structure may then increase without the stresses increasing in the zones subjected to the highest loads.

This is the property which is termed plasticity.

The non-elastic deformation which takes place at the elastic limit is the elastic deformation at the elastic limit multiplied by a coefficient n which depends on the nature of the metal. Generally speaking, n is roughly 22.

The difference between the load at which the elastic limit is reached and the load at which it is exceeded is relatively greater, the higher the coefficient n. It is also greater when the member or element stretched is subjected to a stress which more rapidly decreases with the deformation.

Case of a Constant Load.

I. We will assume that a number of parallel bars B are secured by their ends so that their elongations are equal, and that a constant force F is applied to the combination. The characteristic point of the deformations and the rate of stressing $\frac{f_1}{s}$ describes, for a given bar B, a portion of the characteristic curve of the metal. If H is the function representing $\frac{f}{s}$ in terms of the relative elongation, the intrinsic law referred to above may be written:

$$\frac{d\,f_1}{s} = H'_1 \frac{d\,l}{l} \tag{1}$$

As regards the extrinsic law, it is obtained by adding the intrinsic laws applying for the other bars.

A simpler way is to add up, member by member, the equations (1) written for all the bars, when we get:

$$d\,F = d\,l\,\Sigma \frac{s\,H'}{l} \tag{2}$$

For the bar in question, the proportion of the additional stress absorbed by it is expressed by the relation:

$$\frac{d\,f_1}{d\,F} = \frac{s\,H'_1}{\Sigma\,s\,H'} \tag{3}$$

This quantity is obviously constant if the characteristic points of all the bars coincide, but only where all of them have been secured at their ends without any internal stress.

The case is different, however when (a) there are differences between the lengths of the various bars measured in the neutral state or between their supports, (b) certain bars have been put under tension before the other bars have been fixed, or (c) these two factors intervene.

The characteristic points will then describe the same curve without joining; but, as we saw above, the differences in length, i. e., the differences in the abscissae, will be maintained.

The additional stress given by (3) for the bar which undergoes the greatest amount of deformation, decreases when this particular bar approaches the elastic limit. The same will apply to the other bars in turn, until, the elastic limit being attained everywhere, the additional stresses are taken up by the bars most deformed.

As the factor n is very high in actual practice, equalisation will actually take place, which means that the bars subjected to the lowest loads reach the elastic limit before the stress rises again in the bars subjected to the highest loads.

Comparing this result with that derived from the theory of elasticity, it will immediately be obvious that the stretching of the metal simplifies the process of deformation. It equalizes the tensions, and makes the condition under which the elastic limit is exceeded independent of the initial conditions obtaining in the system.

As regards the exception mentioned above, it would only apply where the member subjected to the highest load exceeded the elastic limit N before the member subjected to the lowest load had. This would mean that the differences between the relative elongations would exceed the length of the horizontal B C, i. e., approximately n times the elastic extension $\frac{N}{E}$ at the elastic limit. A case of this kind may be regarded as exceptional and inadmissible.

II. The foregoing applies without restriction to the case of members braced 'in parallel' like the booms of a truss. Mistakes in erection have no effect on their capacity to resist at the elastic limit.

The same thing also applies to the members of a lattice girder arranged in the same or opposite directions, or where a lattice girder is strengthened by members fitted without tension.

III. A more general case is that in which associated members are deformed unequally, but at constant ratios. We then get:

$$\frac{d\,l_1}{l_1\,\beta_1} = \frac{d\,l_2}{l_2\,\beta_2} = d\,\alpha, \tag{4}$$

α being a variable parameter and β constants.

The geometrical construction of Fig. 7 shows how the stress rates of the different members are deduced from each other. The additional stress taken up by the member B is:

$$\frac{d\,f_1}{d\,F} = \frac{s_1 \frac{\beta_1\,H'_1}{l_1}}{\Sigma\,s \frac{\beta\,H'}{l}}. \tag{5}$$

The least deformed bar reaches the elastic limit before it is exceeded by the most deformed member, if the ratio of their elongations is lower than $(n+1)$.

Owing to the ductility factor being very high, the tensions are most usually equalised on the elastic limit line.

Fig. 8 shows how the balanced average of the deformations varies in terms of the balanced mean of the stress rates. The curve obtained is much more rounded than the actual deformation curve.

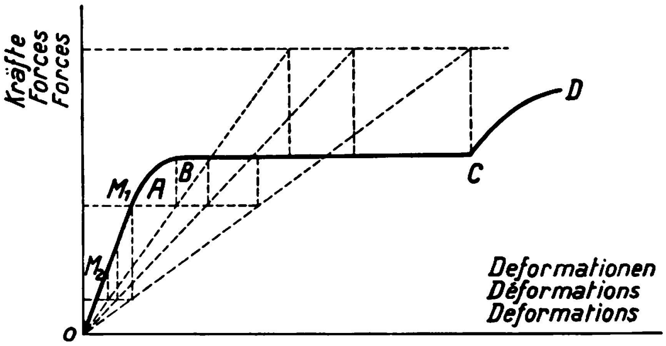

Fig. 7.

Calling k the ratio of the mean elongation to the elongation of the member supporting the highest load, the condition for which the stress rates become equal at the elastic limit is expressed by the following proximate relation:

$$\frac{1}{2k-1} < n + 1. \tag{6}$$

IV. The above considerations show that, by assuming the same stress rate for each member, we are assuming an approximation which is usually not justifiable but which becomes so at the elastic limit. Since this calculation assumes a well defined limiting stress for each member, it is taken to justify the view which holds that each member or each part possesses its own strength capacity, the capacity of the whole being the sum of the individual capacities,

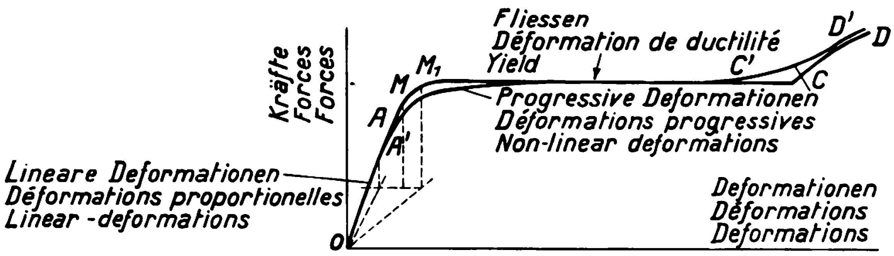

Fig. 8.

This method, which is applied particularly in the calculation of trusses, is therefore justified to a certain extent when the several elements work under conditions which are not too dissimilar.

V. It is well in this relation to verify whether allowing for plasticity in the calculations does not reduce safety.

Security signifies either a guarantee against the risk of the structure being ruined, or a guarantee against the risk of high deformations making the structure unsuitable for use. Buckling is involved in both these considerations.

In every case, safety is defined by a coefficient, and, when defining this coefficient, we cannot do better than take it as being equal to the factor by which the loads have to be multiplied in order to reach a dangerous condition.

From our previous arguments we find that, if M is the maximum fixed for the average rate of stress, the coefficient of safety after the elastic limit N is exceeded, is N/M.

VI. Ignoring the case where buckling takes place below the elastic limit, two cases may be considered:

(a) Where the buckling limit Φ is very little above the elastic limit N.

In this case, it should be noted that, as soon as the loads have exceeded rates which correspond to the level of the elastic limit, the surplus loads come exclusively on to 1, then 2, etc. .... members.

The amount by which the applied loads must be increased so that the member bearing the highest load reaches the buckling limit Φ can thus be extremely small, as will be seen from the diagram (Fig. 9) of the average stress applying in this particular case. It would thus appear that a considerable amount of caution is necessary and that, for the average stress rate, a limit should be taken, calculated as if the buckling limit were equal to the elastic limit.

(b) Where the members are all strained or their buckling limit is close to the breaking limit.

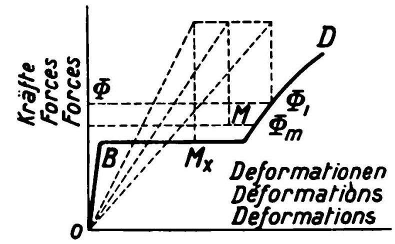

Fig. 9.

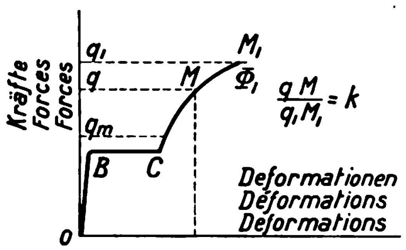

Fig. 10.

In this case it may happen that, when the characteristic point M_1 of the member bearing the highest load attains its danger limit Φ, the other characteristic points are all on the curve C Φ_1. This is generally the case when the ratio between the maximum plastic elongation and the critical elongation is lower than 2 k — 1.

The centre of gravity M of the characteristic points is then very close to the curve C Φ_1 itself (Fig. 10). The critical value of the stress rate may therefore be roughly defined as the value which corresponds, on the deformation curve, to the critical elongation multiplied by k.

This being the case, the factor of safety of an assemblage of members jointly stressed may be defined as follows:

'The conditions of resistance of the member supporting the highest load are appreciably the same as those of an imaginary member loaded at the average rate, but whose buckling or breaking limit is reduced.

'Where the buckling limit is not very much higher than the elastic limit, the strength limit of the imaginary member must be taken as being still closer to the elastic limit.

'Where the buckling limit is high, or where the members are all stretched, the dangerous elongation must be multiplied by k in order to find, on the elongation curve, the dangerous rate of stressing.

'In all cases, the factor of safety in terms of buckling or failure, must be taken as equal to the ratio between the limit thus determined and the average rate of stressing.'

From the above we may deduce that, if a hyperstatic system is similar to those which we have just examined, and if great deformations have begun to take place, the margin of safety which then obtains is lower than if the structure were statically determinate.

VII. These principles may be applied to the case of a plate drilled with rivet holes, by likening it to a group of associated fibres. The deformations are augmented locally in the ratio of 1 to 3. The ratio k therefore equals $^1/_3$.

Consequently, the breaking limit is reached at the edges of the holes, which means that cracks will develop for a critical value of the average stress rate obtained by multiplying the elongation at the breaking limit by $k = {}^1/_3$. With standard structural steels, therefore, the stress rate is roughly equal to $^4/_5$ of the breaking limit.

VIII. We shall now examine, by a few examples, the case of straight, statically indeterminate lattice girder:

(a) 45⁰ Lattice Girder with Uniform Flanges and two Supports and restrained at one End. (Fig. 11.)

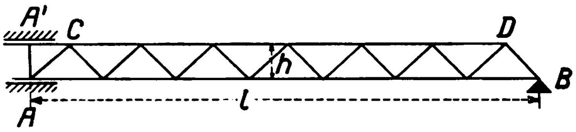

Fig. 11.

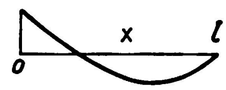

Fig. 12.

When the loads increase, the member A' C will normally be the first to reach the elastic limit. The remainder of the structure will then behave like a simple girder ACDB subjected to a force s N acting along CA'. The moment acting on the support is

$$M_0 = s\,N\,h \tag{7}$$

In the girder, the moment M is equal to

$$M = \mu - s\,N\,h\,\frac{l - x}{l} \tag{8}$$

μ being the moment of the girder on two supports. As the loads go on increasing, the deformations will also continue and attain (1) the elastic limit in the bay, and (2) the upper limit of ductility in the member A' C. Generally speaking this latter limit is reached last, and in this case the elastic limit is reached in the bay where the maximum of (6) is equal and its sign opposite to the moment on the support (7).

If the girder is uniformly loaded or supports a variable load P, the maximum moment which the most heavily loaded section of the girder has to be capable of resisting is the fraction $\frac{4}{5.83}$, or roughly $^2/_3$ of the maximum moment of the same girder, assuming it has two simple supports (Fig. 12).

It is, however, necessary to verify that the deformation of the member A' C under these conditions does not exceed the maximum deformation of ductility. This deformation is obtained by calculating the rotation ω'_0 of the neutral fibre of the girder near to A:

$$\omega'_0 = \omega_0 - \int_0^l \frac{s\,N\,h}{E\,I} \frac{(l-x)^2}{l^2}\,d\,x = \omega_0 - \frac{s\,N\,h\,l}{3\,E\,I} = \omega_0 - \frac{2\,l\,N}{3\,E\,h} \qquad (9)$$

ω_0 being the rotation of the girder resting on simple supports.

Since the lattice is assumed to be at 45^0, ω'_0 thus represents the relative elongation of A' C = h, whence we must get:

$$\omega_0 - \frac{2\,l}{3\,h}\,\frac{N}{E} < (n+1)\,\frac{N}{E} \qquad (10)$$

Calling ω_0 the value equivalent to the maximum and uniformly distributed load, we get:

$$\frac{l\,N}{3\,h\,E} - \frac{2\,l}{3\,h}\,\frac{N}{E} < (n+1)\,\frac{N}{E}$$

or:

$$\frac{l}{h} < 3\,(n+1) \qquad (11)$$

This inequality is satisfactorily met by girders of ordinary dimensions and by the usual types of steel.

(b) Girder with two rigidly connected Bays,. made up as above.

The case is the same, the section on the central support acting as a rigidly supported section.

On the other hand, if the supports are not at the same level, a correction factor is added to ω'_0 which may increase the value of the first term of (11), but is generally too small to upset inequality.

(c) Girder firmly anchored to its two Supports, or Bay rigidly connected to two other Bays; the same make-up as above.

In these two cases, the inequality of the moments on the supports or in the bay takes place by reducing to one-half, the value of the maximum moment in the bay having single supports. The condition (11) is replaced by the following, which is satisfied more easily still:

$$\frac{l}{h} < 6\,(n+1) \qquad (12)$$

Quite definitely, and apart from abnormal cases, if the girder be calculated by assuming twice the inequality of the maximum moments for the bay, and calling the maximum rate of stress Π, the coefficient of safety in terms of over-extension of the elastic limit N equals $\frac{N}{\Pi}$.

IX. We now come to the case of a solid web girder of constant and continuous section, or restrained at one at least of its supports.

The increased load brings the stress at the elastic limit into the region of the double supports, with a consequent increase in the volume which is thus rendered plastic. A plastic deformation is then set up in the middle region of the bay, but it is not clear that this phenomenon precedes the case of loading where the first plastic zone exceeds the elastic limit. To determine this particular case involves ascertaining the extent of this zone.

There is every reason to think that, in a straight section, even one partially subject to plasticity, the law of relative deformations of the longitudinal fibres remains linear with respect to the distance from the neutral fibre.

Taking O as the point of the neutral fibre, we shall suppose the stress-strain diagram plotted on any scale for the section of the girder. By projecting a point of the cross-section along x on a given straight line O D, then along the y-axis on the curve, the corresponding stress rate is obtained (Fig. 13).

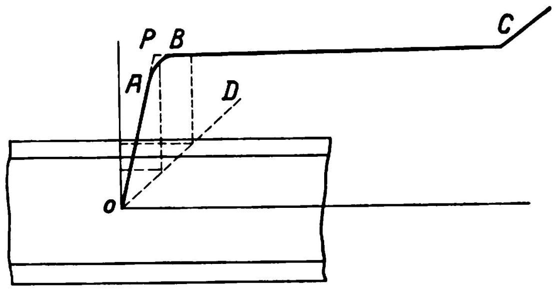

Fig. 13.

When the bending moment increases, the straight line turns in a clockwise direction. It varies from a quantity M_o to a quantity M_1 in the plastic zone, while the characteristic point shifts from B (which may be assimilated to the point P) to C. This gives us:

$$M_o = \int 2 \frac{N v^2}{h} d\sigma = N \int \frac{2 v^2}{h} d\sigma \tag{13}$$

and:

$$M_1 = \int N v \, d\sigma = N \int v \, d\sigma \tag{14}$$

v being the distance from the neutral fibre and $d\sigma$ the element of surface of the section.

The difference is equal to

$$M_1 - M_o = N \int 2 \frac{v}{h} \left(\frac{h}{2} - v\right) d\sigma \tag{15}$$

which may be reduced to the value of the integral for the web alone. If b is its thickness, then:

$$M_1 - M_o = N \frac{b h^2}{12} \tag{16}$$

The moment varies between the two values M_o and M_1 following a parabolic law and in terms of the height Z of the non-plastic portion:

$$M = M_o + \frac{N b}{12} (h^2 - Z^2) \tag{17}$$

The variation of the bending moment thus supplies the type of boundary which limits the plastic zone. It is parabolic where the shearing stress is constant, and straight where the shearing stress varies in a straight line.

The plastic volume is greatest when it is limited as shown in Fig. 14.

We get:

$$d = \frac{b h^2 N}{12 T} \tag{18}$$

Let us assume that the thickness of the web is calculated exactly to work at the elastic limit for a shearing stress equal to T, i. e., so that b h N = T.

From (18), we find that the half-length of the plastic zone attains the value

$$d + \frac{a}{2} = \frac{h}{12} + \frac{a}{2} \tag{19}$$

If the web were calculated very accurately, this would mean that the height of the girder would not have to be less than $^1/_6$ of the span if it were desired that the elastic limit be not exceeded on the support before being reached in the bay.

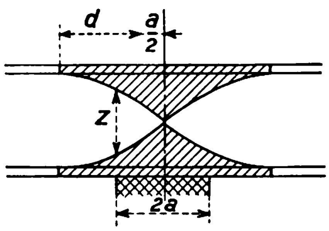
Fig. 14.

We therefore find that the length of girder bent plastically is only a small fraction of the height. The condition (11) must be replaced by a condition which is harder to satisfy:

$$\frac{b\,h^2\,N}{12T} + \frac{a}{2} > \frac{1}{3\,(n+1)} \tag{20}$$

Paradoxical as it may seem, the conclusion is that, if we are to take full advantage of the plasticity of continuous girders above their supports, the web must be strengthened to a greater extent, the thinner the section of the girder. Such reinforcement must cover approximately $^1/_{60}$ of the span of the girder.

In the contrary case the girder pivots, as it were, to the right of its support only with a thrust at the end, and a break will tend to occur in the shape of the girder if the web is too weak.

From the standpoint of plasticity, reinforcing the webs and providing stiffening uprights near the supports are particularly interesting solutions, and call for broad treatment in the region of the points of fixation and the double supports.

X. It seems desirable to consider not only the risk of the occurrence of large deformations in beams where the elastic limit is exceeded in several regions, but the risk of breakage as well, unless of course this particular point has already been taken care of in connection with the particular location of the structure.

We shall therefore now study the variation δ M of the bending moment corresponding to a slight increase of the loads. The variation in curvature is expressed by the equation:

$$\delta \left(\frac{d^2 u}{d x^2}\right) = \frac{\delta M}{H'I} \tag{21}$$

The coefficient H' is nothing more than the instantaneous coefficient of elasticity of the booms, where the girder is a webless one. In the case where the web intervenes in the bending, H' I is defined as the sum of the moments of the elements of area of the section, multiplied by the corresponding H' values.

Let us take, for example, the case of a girder restrained at both ends. The deviation of each end being nil, we get:

$$0 = \int \delta \left(\frac{d^2 u}{d x^2}\right) d\,x = \int \frac{\delta M}{H'\,I}\, d\,x \tag{22}$$

Now M is the sum of the moment ($-M_o$) on the supports and of the moment μ which would obtain if the girder were laid on simple supports. We therefore get:

$$\delta M_o \int \frac{dx}{H' I} = \int \frac{\delta \mu}{H' I} dx$$

or:

$$\delta M_o = \frac{\int \frac{\delta \mu}{H' I} dx}{\int \frac{1}{H' I} dx} \tag{23}$$

This expression should be compared with equation (3) referring to members braced in parallel. In the present case, the sections which bend are braced in series, and this explains why the coefficients H' become denominators instead of numerators.

The general equation (23) enables us to resume the discussion: *Elasticity Phase:* H' is everywhere equal to E, which gives us simply:

$$\delta M_o = \int \frac{\delta \mu}{I} dx \tag{24}$$

Plasticity Phase on the Supports: Since H' becomes nil at the supports, the adjacent regions supply, in the integrals, main terms which, in the numerator, are multiplied by very small quantities $\delta \mu$. The quotient becomes appreciably nil, thus:

$$\delta M_o \approx 0 \tag{25}$$

Bay Plasticity Phase: As H' becomes nil in a region C where the variation $\delta \mu_c$ is not nil, the quotient of the main terms of the two integrals of (23) then becomes

$$\delta M_o = \delta \mu_c$$

We therefore get $\qquad \delta M_o = -\delta M_o + \delta \mu_c = 0 \qquad (26)$

High Deformation Phase: The plastic zones move away from the parts which have exceeded the elastic limit, and we find that $\frac{\delta M_o}{\delta \mu_c}$ diminishes without dropping to the value $\frac{1}{2}$ corresponding to the case of the statically determinate girder.

Fig. 15 shows the variation in the rate of stress in terms of load.

Similarly, in the case of a doubly supported girder, the rate of variation after equalization is roughly 1.5 times what it was on the average before. This is the inverse of the relation connecting the bending moment which obtains during equalization of the moments, to the bending moment which would then obtain without the continuity.

In other words, if plasticity first attains the rate of stress in the supported sections, it will act in the opposite direction afterwards.

It will be seen, then, that the rate of stress on the support reaches the buckling limit Φ of the member under compression for an increase in load of $\frac{N+\Phi}{2\Pi}$ in the case of two double supports and $\frac{N+2\Phi}{3\Pi}$ in the case of one double support.

Consequently, by determining the maximum value Π of the stress rates in a beam comprising one or more double supports, and assuming that these rates are equalized in the highest possible number of sections, the factor of safety in terms of excess of the elastic limit N is $\frac{N}{\Pi}$.

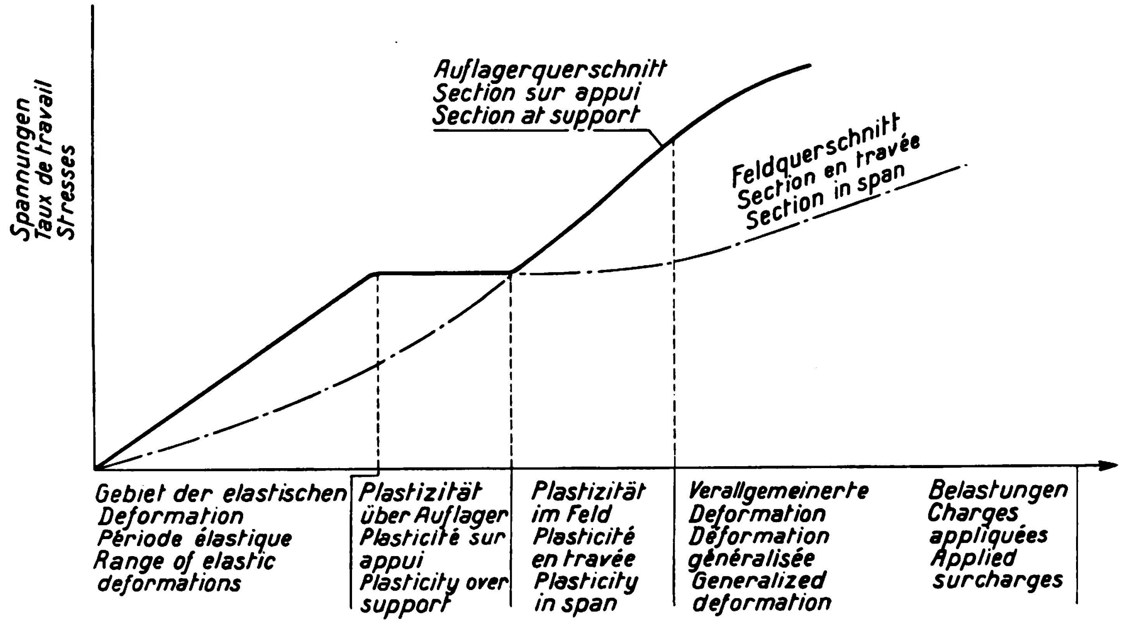

Fig. 15.

But as regards the factor of safety in terms of the buckling of the member under compression, it comes between the foregoing and the ratio $\frac{\Phi}{\Pi}$. The interval between these two factors of safety must be reduced in the same proportions as the maximum bay moment is relatively to the moment of the same bay resting on single supports.

XI. The case of statically indeterminate arches is in some respects similar to the case of continuous or fixed beams.

We shall therefore see what takes place in the section subjected to maximum stress and where the pressure curve passes through its centre of gravity. When the loads increase, the normal stress exceeds the elastic limit immediately after having attained it. The arch does not then derive any benefit from plasticity unless its lowest strength sections are under bending stress. This may be the case for certain load conditions in parabolic arches, in fixed arches with light section in $^1/_4$-span, and in two-hinged arches with light section at the crown.

Apart from this case, the elastic limit is not attained simultaneously at all points of the most heavily loaded section. Take the case of the double-boom arch: when the section of one boom is plastic, the increased load will set up an additional stress in the other boom only, which means that the pressure line for the additional loads passes through the latter boom. Generally speaking, plasticity has the effect of shifting the total pressure line to the centre of gravity of the plastic sections (Fig. 17).

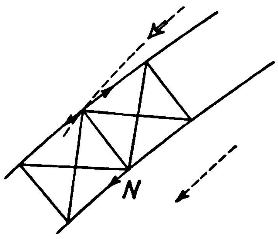

Fig. 16.

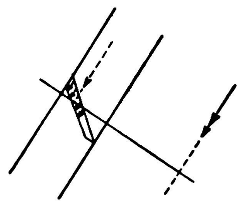

Fig. 17.

In this case, the interval between the moment when the first plastic zone appears and the moment when the entire section attains the elastic limit is equivalent to an increase in load which may be large. In the same interval, the plasticity extends along the arch.

An arch is thus capable of undergoing considerable deformation at various parts of its low-strength sections, without exceeding the elastic limit, unless the pressure line passes through the centre of these sections. During this deformation the arch behaves as if it were hinged.

Except in abnormal cases, the elastic limit is successively reached, without being exceeded, at n points, where n is the degree of static indetermination for a given load value.

In order to ascertain these n sections, as also to find the safe working stress Π for a particular load, the designer should proceed on the following hypotheses:

(1) On the assumption of elastic deformations, ascertain first the sections which are most exposed to the distribution of the particular loads (uniformly distributed load, concentrated load, etc.).

(2) Ascertain the load for which the limit of safety Π is attained in these sections.

(3) Compute the effect on the other sections, of additional loads considered separately, assuming that their line of pressure passes through the half of the sections, previously found, which comes opposite to the points of maximum load.

(4) Ascertain the load for which the total stresses, defined by (1) and (3) reach the maximum value Π.

The sections in which this maximum stress is set up, and the sections found previously, are those which are deemed to work at the limit of safety Π under the particular loads considered.

The factor of safety in terms of over-extension of the elastic limit N is then really equal to $\frac{N}{\Pi}$, since increasing the load in this ratio has the result of raising to the elastic limit the sections defined first of all, and then the others.

For any fresh increase in load, the stress rates rise in the arch as a whole, except in the zones that are subjected to plasticity in the first place.

Consequently, the breaking limit will be reached for a fresh increase in load which is less than $\frac{R}{N}$, which means that the factor of safety in terms of failure lies between the ratio $\frac{N}{\Pi}$ and the ratio $\frac{R}{\Pi}$.

XII. Great caution is necessary in generalising the above results.

The number of bars or members which may be omitted must not be taken as the degree of static indetermination of the structure as a whole. It is the latter factor alone which must be allowed for when applying the theory of plasticity.

In the cantilever system, Fig. 18, for instance, it would not be right to assume equalization of the moments on the double support B and in the bay A B, even if a diagonal member were added to render the girder A B C hyperstatic. The whole does not become hyperstatic, and the plasticity theory only applies where a bar is added above the point C.

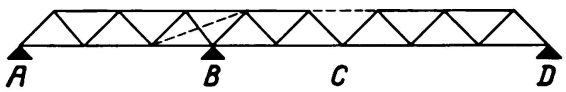

Fig. 18.

In complex systems, on the other hand, it is essential to define by degrees the sections which become successively plastic.

Care must also be taken to ensure that the zones affected by plasticity are not capable of setting up a failure in the neutral fibre before the elastic limit is reached in the other sections involved. It must also be remembered that the stress rates allowed in the calculations involving plasticity can only serve as a basis of comparison for determining the factors of safety in terms of excess of the elastic limit. In terms of failure, the factors of safety may be very little higher than the foregoing.

Variable Loads.

I. Where certain states of load recur, and where the elastic limit is not exceeded at any point, there is every reason to believe that the deformations and stresses recur in cyclic form. This view is not invalidated by experience.

We have tried to verify that this is the case when a member alternates between two limits characterised by well defined extrinsic laws, the upper limit of which is sufficient to make the member plastic.

For this purpose, a tensile test piece was secured between two jaws, one of which was fixed to a screw jack and the other to an excentric.

The excentric was first placed in the position corresponding to the maximum elongation of the test piece, and a tensile stress was applied to the other end by the jack until the region of plasticity for steel was reached. The excentric was then turned several times so as to make the length of the test piece vary between two well defined limits, so as to unstress and then stress the test piece.

In this experiment, the extrinsic law corresponding to the maximum stress in the test piece is represented by a vertical straight line on the stress-strain diagram, and not a horizontal straight line as in the classic tensile tests.

The stress in the test bar was determined by the frequency of the vibrations set up by transverse impact.

These tests gave a negative result. It would therefore appear that, when the metal has entered the plasticity phase, the internal stress is a function of the linear deformation only, provided the latter does not exceed the maximum limit previously attained (Fig. 19).

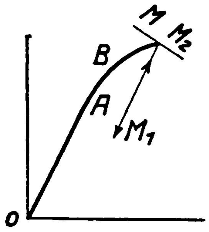

Fig 19.

Apparently, then, if given loads bring a zone into the elastic limit, the distribution of these loads does not modify the value of the tensions reached, and, consequently, the risk of the elastic limit being exceeded is no greater with reneweed loads than it is with a constant load, which means that the conclusions of the previous section stand.

II. The problem is much more delicate when it comes to computing the risk of failure. In that case, we must carefully consider what experience has recently taught us regarding failure by repeated stresses.

There is no doubt that the metals used in metallic structures are much softer than the metals which have been shown by endurance tests to be more sensitive to repeated stresses than they are to a single specified stress.

Nevertheless, failure is possible with every metal, whatever it be, when the repeated stresses attain a value lower than the figure which causes failure in an ordinary tensile test.

For reasons of prudence, the principle that we propose to follow consists, as above, in finding rules such as will coincide, for isostatic structures, with the ordinary rules of safety, and which, for hyperstatic structures, will give the same coefficients for the ratio between the permitted and the dangerous loads.

Adopting this point of view, the designer will tend to consider that an isostatic structure is satisfactory as regards stability when it gives a certain factor of safety (a) in terms of over-extension of the elastic limit, and (b) against breakage when the load is first applied, and when it is sufficiently remote from the limiting case where breakage would take place due to indefinite repetition of the overloads.

Our knowledge concerning failures by repeated stresses leads us to think that the permanent load plays a much smaller part than the live loads, which means that we ought to consider, not the maximum rate of stressing, but rather the sum of two terms:

(1) The half-amplitude of variation of the stress rate σ_2 and σ_1.

(2) The mean value $\frac{\sigma_2 + \sigma_1}{2}$ of the extreme stress rates multiplied by a coefficient α, which would be small.

According to the researches of *M. Caquot* at the Aerotechnical Laboratory, the coefficient would be of the order of $^1/_5$ th.

Safety rules for isostatic structures which would allow for the risk of failure by repeated stresses would lead the designer to compare the binomial defined above with a limit Π_f obtained by dividing the limit of endurance by a factor of safety.

Without going beyond the rules actually laid down, we should thus be inclined to put the permissible limit Π_f at not less than the limit Π_0 previously admitted for the total stress of a member subjected to alternating stresses.

We should then have to verify the following inequality:

$$\frac{\sigma_2 - \sigma_1}{2} + \alpha \frac{\sigma_2 + \sigma_1}{2} \leqslant \Pi_f \; (\geqslant \Pi_0) \tag{27}$$

When we come to consider a hyperstatic structure, we ought to lay down a similar condition; but in the absence of proof to the contrary we may consider that the fact that the repeated stress σ_2 is equal to N does not affect the risk of failure any more than it does the relation between the deformation and the forces.

Experience in this connection would be extremely useful.

In the absence of more precise information, we shall therefore assume that, in a hyperstatic structure, we must verify the condition (27) in which σ_2 coincides with the stress rate set up on the first application of the load.

Speaking more precisely, we compare with the limit of endurance itself $K\Pi_f$, the expression (27) corresponding to loads which are K times higher than the loads we are considering. Dividing the two members by K, the expression becomes:

$$\frac{\sigma_{max} - \sigma_{min}}{2} + \alpha\left(\frac{\sigma d}{K} - \frac{\sigma_{max} - \sigma_{min}}{2}\right) \leqslant \Pi_f \qquad (28)$$

where σ_d is the stress rate equivalent, in the plasticity theory, to the load considered.

σ_{max} and σ_{min} are the extreme rates of stress in accordance with Hooke's law.

The first term must be derived exclusively from the elasticity theory. The second is obtained by subtracting the results supplied by the elasticity theory and the plasticity theory.

The advantage derived by plasticity is that this latter term is lower than the value deduced from Hooke's law, where the term σ_d is replaced by σ_{max}. Since, however, the coefficient α is small, this advantage is very limited, especially if the amplitude of variation in the stress rate is high, i. e., where the overloads are high with reference to the true load.

The condition (28) may be replaced by a more difficult condition where σ_d is replaced by its limit:

$$\frac{\sigma_{max} - \sigma_{min}}{2} \leqslant \frac{\Pi_f - \alpha\,\Pi_o}{1 - \alpha} \qquad (29)$$

In short, it would seem that if, in our conception of structures, we wish to take advantage of the latitude which plasticity allows, we ought, in principle, to limit ourselves to a twofold verification.

(1) It is indispensable to verify that there is a suitable factor of safety in terms of excess of the elastic limit, and in terms of failure or buckling, when the load is first applied.

The first verification is carried out according to the rules deduced from a knowledge of the plasticity of the material, and with due regard to the above observations.

(2) We ought to verify that there is a suitable factor of safety in terms of fatigue of the members, and for this purpose the variations in the stress rates will be deduced from a knowledge of the laws of elasticity. These variations will then be compared with a given limit of endurance.

Of the two inequalities to be verified, the former will obviously be the more important if the permanent load predominates and the second inequality can be dispensed with.

On the other hand, the latter inequality will operate in cases where a structure is light in terms of the overloads — say, a longitudinal girder or a bridge member.

Briefly, then, plasticity makes the factor of safety of structures insensitive to mistakes in fitting or erecting, and to anomalies in the distribution of the stresses and the initial loads of the members, etc. It renders correct the hypothesis of the equalization of the maximum stress rates when permanent loads are involved and when the structure is well designed.

Summary.

Every structural element for which the stressing depends not only on the loads applied to the whole structure, but which is also subjected to deformations, can reach the elastic limit without the necessity of exceeding this limit immediately. From this it follows that the forces applied to statically indeterminate system can change within certain limits, without causing the stresses to exceed the elastic limit; however, it will cause stresses to increase at other places of the structure. This conclusion applies also to the distribution of stresses set up in statically determinate structures.

The method of calculation as based on the assumption of stress equalization in structures is justified; the stresses however, must not exceed the yield limit before equalization of stresses has taken place. It is for this reason that the laws of plasticity may only be applied with the utmost caution in designing arches or other structures of a high degree of statical indetermination. On the basis of these assumptions, it can be said that the hypothesis of stress equalization leads to the determination of the factor of safety. The consideration of plasticity does not increase the factor of safety in the case of frequently repeated loads.