

**Zeitschrift:** IABSE congress report = Rapport du congrès AIPC = IVBH  
Kongressbericht

**Band:** 2 (1936)

**Artikel:** Fundamental principles of the theory of plasticity

**Autor:** Fritsche, J.

**DOI:** <https://doi.org/10.5169/seals-3150>

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 24.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## I 2

Fundamental Principles of the Theory of Plasticity.

Grundlagen der Plastizitätstheorie.

Principes de la théorie de la plasticité.

Dr. techn. J. Fritsche,

Professor an der Deutschen Technischen Hochschule Prag.

Whilst it was formerly held that, assuming elastic behaviour, all questions of strength of the material which concerned the structural engineer are capable of solution; it is now recognised, from comprehensive measurements on structures and tests carried to near the breaking load in the testing laboratory, that by such an ideal conception of the material, it is impossible to obtain a uniform degree of security against the setting up of dangerous conditions. Although it has long been known that purely elastic behaviours are confined within comparatively narrow limits of stress, this opinion was based on the grounds that, for reasons of safety, the stresses produced by the working loads must lie within these elastic limits, and it was thought that the greatest stress determined by the elastic theory afforded a basis for deciding the factor of safety for a structure, and, above all, the equal safety of all its parts. Accordingly a permissible stress was decided upon, with the proviso, that is should not be exceeded by the action of the applied loads.

This stress scale is very convenient for the practical dimensioning of structures, because for the designer it eliminates complicated and often disquieting questions as to the actual safety of his design; the crucial question of safety appearing then to be incorporated with the determination of permissible stress. This at once involves difficulties, since one can only introduce into the strength calculations simple data for the material — for example, the lower yield point stress  $\sigma_{Fu}$  of the steel — that has been determined from simple tension tests. The limits of proportionality might well be left out of consideration, as it has been recognised that calculations are exact enough if the purely elastic behaviour up the lower yield point is accepted.

So long as the conditions of stress set up in the structure are the same as those imposed upon the test piece, the question of safety is clear; the factor of safety is then  $n = \frac{\sigma_{Fu}}{\sigma_{perm}}$ . For judging the uniform multi-axial stress conditions, the known conditions of yield have been laid down and tested by experiment<sup>1</sup> — the hypothesis of *Mises-Huber-Hencky*, based on the comparison of the specific

<sup>1</sup> *M. Roš u. Eichinger: Versuche zur Klärung der Frage der Bruchgefahr. I. Flussstahl. (Investigations to solve the problem of rupture). — Diskussionsbericht Nr. 19 der E.M.P.A.*

deformation, energy corresponds most nearly with the facts in the case of structural steel: for the duo-axial stress conditions frequently occurring, it takes the form

$$\frac{\sigma_{Fu}}{n} = \sigma_{perm} \geq \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2} \quad (1)$$

Accordingly, all dimensioning methods based on the stress scale include the assumption that in those places where the most unfavourable stress reaches the lower yield point stress  $\sigma_{Fu}$  the yield of material sets in and thereby the structure has reached the limit of carrying capacity. Uniform stress conditions are realised in ideal lattice structures but the above assumption leads to the conclusion that a lattice structure must be deemed no longer serviceable at the moment when a member has reached the lower yield point. However, a little deliberation<sup>2</sup> would seem to show that these conclusions only apply to statically determined lattice structures whilst for statically indeterminate, considerable increases of load are still possible before the structure collapses under the load. To a still greater degree is this assumption erroneous in the case of the plasticity of a cross section stiff against bending. However, non uniform conditions of stress exist whose influence upon the yield is yet to be examined. *Maier-Leibnitz*<sup>3</sup> has indisputably shown by experiment that when the yield point is reached at the most unfavourably stressed position, the stability of the framework is in no way endangered; it was found, on the contrary, that indications of yield must have penetrated very deeply into the cross section, to have set up an increased rate of deformation for any increase of loading.

Here the theory of plasticity comes in, the purpose of which is to estimate the actual carrying capacity of a structure under consideration of the yield phenomena. From the fact that the reaching the yield point at the elastic peak stress is not accompanied by any conditions endangering the stability of the structure, the theory of plasticity discards the stress scale and with it the expression of a permissible stress, and introduces as safety the ratio of carrying capacity to working load. The method of dimensioning based on the theory of plasticity will therefore be designated in many instances as the 'Carrying capacity Method' (Method of plastic equilibrium).

## 2) Mechanism of Plastic Deformation.

Properly to grasp the idea of the influence of the yielding process on the carrying capacity, it is necessary to take an effort to conceive an idea of its nature and its physical properties. Steel is a crystalline mass and a strict consideration would imply that the deformation of material should be deduced from that of the individual crystals. However, the irregular arrangement of the individual crystals would make it impossible to carry through this conception other than with statical methods. In respect of its mechanical properties the individual

<sup>2</sup> *M. Grüning*: Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl bei beliebig häufig wiederholter Belastung. (The carrying capacity of statically indeterminate structures in steel under frequently repeated loading Berlin 1926, J. Springer.

<sup>3</sup> *Maier-Leibnitz*: Versuche mit eingespannten und einfachen Balken von I-Form aus Stahl 37. (Experiments with encastered and simply supported I-joists of Steel 37). Bautechn. 1929, Heft 20, S. 313.

crystal is distinctly anisotropic, whilst, considered mechanically, where in only portions of the material that already contain a very large number of individual crystals are tested, the crystalline mass must, in view of their irregular disposition, be regarded as quasi-isotropic.

A crystal with its specific direction in an assigned position changes elastically in the first place, as the crystal lattice is distorted by the working of external forces; when the distortion reaches a definite amount, the stability of the lattice is exhausted and slipping of layers of atoms along distinct crystallographically defined planes and directions ensues, which is to be regarded as a purely plastic process. Therefore with very close approximation the stress-strain diagram of a crystal can be likened to that of the ideally plastic body. A deviation takes place only on the occurrence of greater deformations at which consolidation to a new lattice stability begins. Since in the crystalline mass the specific direction of the individual crystals lie quite irregularly, they will slip under a fixed direction of the external load at different stress limits. The extent of the slips are of course very small, so that a very delicate measuring apparatus is required to reveal them; under these conditions, there is no steady deformation of crystalline mass, the deformation is in reality jerky.

The observations of *Kollbrunner*<sup>4</sup> provide a fine confirmation of this. The yield hypotheses of *Böker*<sup>5</sup> and *Brandtzaeg*<sup>6</sup> which unfortunately are far too little known in specialist circles, rest on the conception of plasticised islands, in a still elastic environment, which become greater and more frequent with increased loading.

In the case of carbon steels a secondary phenomenon occurs, caused by the structural arrangement of the metal itself. This consists chiefly of soft ferrite grains, which are bedded in a hard network of cementite or perlite, the latter being able to exert a powerful check on the slip of the ferrite grains. Under a certain external load, the perlitic network collapses and allows slip to occur in many of the ferrite grains, and the phenomenon appears which in the ordinary mechanical sense is understood as yield of the steel. That the yield point is not bound up with the strength properties of actually chemically pure iron is clearly shown by the observations of *Köster*<sup>7</sup>, according to which the yield point and the extent of yield can be considerably modified by the alteration in grain size, whilst breaking strength and constriction remain unaffected. To this ordinary mechanical yield process corresponds a considerable change of texture; this is proved by the recrystallisation phenomena to be observed after the yield and by these summarised as 'ageing' of the steel. After conclusion of the yield, the soft ferrite grains form the sole carrier, at the same time this internal diversion of

<sup>4</sup> C. F. *Kollbrunner*: Schichtenweises Fließen in Balken aus Baustahl. (Laminated yield of beams of building steel). III. Bd. d. Abhandlungen der I.V.B.II. Zürich 1935, S. 222.

<sup>5</sup> R. *Böker*: Die Mechanik der bleibenden Formänderungen in kristallinisch aufgebauten Körpern. (Mechanics of permanent deformations in bodies of crystalline texture). — Forschungsarbeiten auf dem Gebiete des Ingenieurwesens. Heft 175—176. Berlin 1915, V.D.I.-Verlag.

<sup>6</sup> A. *Brandtzaeg*: Failure of a material composed of non-isotropic elements. Trondhjem 1927.

<sup>7</sup> W. *Köster*, H. v. *Köckritz* und E. H. *Schulz*: Zur Kenntnis der Form der Spannungs-Dehnungs-Kurven auf Grund der Messung des zeitlichen Verlaufes der Alterung weichen Stahles. (Further contribution to the knowledge of the stress-strain diagram based on time measurements for the ageing of mild steel). Archiv für das Eisenhüttenwesen 6, 1932/33.

the external force gives rise to the great plastic deformation. It is to be assumed that such a rearrangement texture cannot be a process which in the strict sense constantly progresses with increased loading, but it is more probable that at the same time it will extend to greater areas of the material or of sections.

It is known that on the application of a mono-axial uniform stress in one direction to a test bar, an upper and lower yield point is observable in the case of mild steels (Fig. 1). As the upper limit is only a temporary condition, it is clear that in dealing with uniform stress conditions, the lower yield point must be considered. The occurrence of an upper yield point is regarded by *Moser*<sup>8</sup> as a delayed yield and by *Prager*<sup>9</sup> compared with delayed boiling, without, however, being able to give other than purely formal connections for the comparison. Its real origin is not yet satisfactorily explained; it does not seem impossible that the applied stress is favourable to a development of the lattice

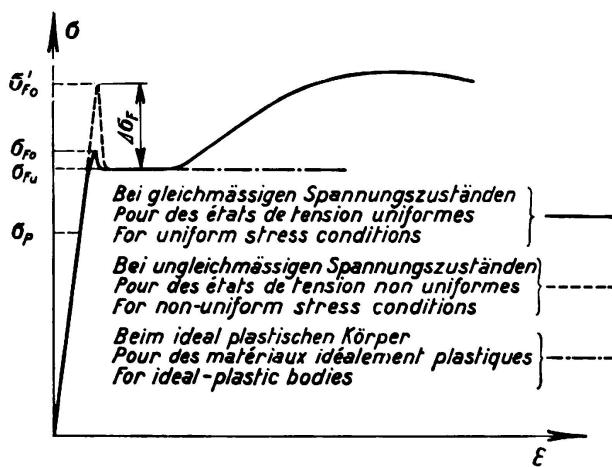


Fig. 1.

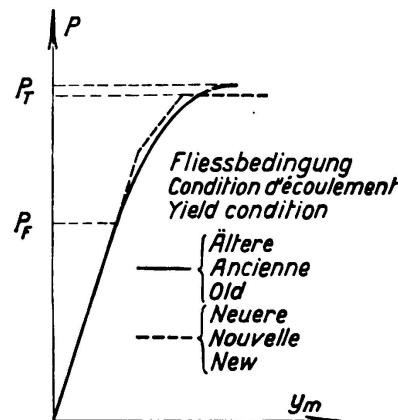


Fig. 2.

structure at the grain boundaries and an increase of the purely elastic resistance to deformation in the perlitic network. The fall of stress from the upper to the lower yield point is accompanied by the formation of the known yield lines which are a clear characteristic for every yield process; exact observation shows that they do not increase steadily but spread intermittently. All these phenomena occur in a fairly regular manner under uniform stress conditions, though it should not be overlooked that the quantitative side of the phenomena is subjected to a series of contingencies, which reveal themselves in an unavoidable diversion of the test results.

Doubt has for some time been expressed as to whether all these phenomena do not substantially alter in accordance with a definite law in the case of the application of a non-uniform stress condition; and for this reason the desirability has been expressed that the determination of yield should be confined (1) to uniform stress conditions. Experiments in this connection admit of no final

<sup>8</sup> *M. Moser*: Verein deutscher Eisenhüttenleute, Werkstoff-Aussch. Ber. 96.

<sup>9</sup> *W. Prager*: Die Fließgrenze bei behinderter Formänderung. (The yield limit for restricted deformation). Forschung auf dem Gebiete des Ingenieurwesens 1933.

opinion; one group<sup>10, 11, 12</sup> justifies the assumption that under variable stress conditions the magnitude of the upper yield point is influenced, since  $\sigma_{F_0}$  is so much the higher, the steeper the ascent of the elastic stress peak and the smaller the zone over which it extends, whilst another<sup>13</sup> cannot admit these phenomena, or at least, only to a small degree. Of the lower yield point limit it can be said that such influences do not affect it<sup>14</sup>. It seems to be certain that the raising of the yield point in the various kinds of steel varies considerably; in the case of soft steel it is greater, in hard, smaller, and it therefore seems to be present only in such steels as have already shown a well defined upper yield point in a tension test. The cause of such a raising of the upper yield point has not yet been explained in a satisfactory manner; from the standpoint of atomic forces it is difficult to understand that understressed parts of a cross section support the overstressed and can stop the yield process there. *Thum*<sup>11</sup> and *Wunderlich* have assumed that the occurrence of yield lines in small zones of any kind is not possible, inasmuch as the still elastic environment blocks the yield. The actual lower yield point must be exceeded up to a certain depth of the cross section before the stored-up energy of yield is great enough to break through the elastic grip in some manner.

With regard to these observations, two fundamentally different conceptions have been formed in the course of the development of the theory of plasticity, with regard to the conditions which determine the yield phenomena in a stressed area, according to which distinction is made between an 'old' and a 'new' condition of yield. It must at the start be acknowledged that both doubtless embody the idealising of the actual process; they correspond in certain aspects to extreme cases, and it is very probable that the actual phenomena lie between them. It is certain that without some kind of idealisation of these observations, one could not dispose of the great difficulties in comprehending their influence on the carrying capacity of structures.

The old yield condition assumes that the local condition of stress is the sole decisive factor in setting up the yield phenomena, it can therefore be based on the yield formula (1) for uniform stress conditions. For the case of bending — disregarding shear stress — the elastically stressed field is in one direction mono-axial and its expression takes the simple form  $\sigma \leq \sigma_F$ . It is clear from the foregoing that the lower yield point  $\sigma_{Fu}$  has been introduced for  $\sigma_F$  because the strength must be judged after conclusion of the locally restricted yield or because one wishes to know how much of this place can still contribute after yield to the maintenance of equilibrium between inner and outer forces. A con-

<sup>10</sup> *F. Nakanishi*: On the yield point of mild steel. World Eng. Congress, Tokyo 1929, Proc. Vol. III.

<sup>11</sup> *A. Thum* und *F. Wunderlich*: Die Fließgrenze bei behinderter Formänderung. (The yield limit for restricted deformation). Forschung auf dem Gebiete des Ingenieurwesens 1932.

<sup>12</sup> *H. Möller* und *J. Barbers*: Über die röntgenographische Messung elastischer Spannungen. (X-ray investigations into elastic stressing). Mitt. d. Kaiser-Wilh.-Inst. f. Eisenforschung, Düsseldorf 1934.

<sup>13</sup> *F. Rinagl*: Die Veröffentlichung ist noch nicht erschienen.

<sup>14</sup> *E. Siebel* und *H. F. Vieregg*: Über die Abhängigkeit des Fließbeginns von Spannungsverteilung und Werkstoff. (The dependence of yield on stress distribution and material). Mitt. Kaiser-Wilhelm-Inst. f. Eisenforschung, Düsseldorf. Abhandlung 270, 1934.

sequence of this conception, is the steadily and gradually increasing area of yield due to increasing loads and a plastic reduction of the peak stress<sup>15</sup>; finally, a completely plastic condition of the cross section is attained provided it extends right through. The stress distribution then consists of a tensile stress rectangle and a compression stress rectangle with a height of the lower yield point  $\sigma_{Fu}$ . The moment of the inner stress has increased to its greatest value  $M_r$  which cannot be exceeded; for further deformation this cross section works as a so-called plastic joint.

The 'new yield condition' maintains that all these conceptions do not correspond to the facts; that, on the contrary, the yield region spreads spasmodically in depth, and it can be assumed that to some extent at the first setting up of yield indications in the cross section, the resistance to deformation at this place is already so weakened that it can really no longer take up an increase in loading. It proceeds from the observation of the raising of the elastic peak, that by 'yield point', the upper yield point must be understood, since this alone is influenced by the non-uniformity of stress distribution. On this account, the stress distribution over the whole cross section must be introduced into the yield formula. Therefore, in the case of bending, free from longitudinal force, the increase of the yield points  $\Delta\sigma_F = \sigma_{F0} - \sigma_{Fu}$  will be essentially a function of the cross sectional shape. During the yield this increase collapses and the upper yield point goes back to the lower, without, however, the stress distribution of the complete plastic condition being necessarily attained in the sense of the older yield formula.

The difference between the two yield formulae becomes most pronounced, when any convenient quantity of deformation is considered in its relation to the load. In the case of the deflection at the centre of a simply supported beam, the old yield formula furnishes, at the junction with the straight (in the case of purely elastic deformation) a steady curve, whose tangent at the moment of concluding a possible condition of equilibrium between the inner and outer forces must be horizontal. According to the new flow formula, the straight line of the purely elastic deformation continues until the carrying capacity is reached, then abruptly changes into the horizontal direction which is maintained until hardening occurs. That such line  $y(P)$  has been actually observed, is evident from the experiments of *E. Siebel*<sup>14</sup> and *H. F. Vieregge*, thought it must be acknowledged that lines of the first type<sup>3</sup> are frequently found in literature.

According to the old yield formula, in the case of a statically indeterminate continuous girder, a steady curve is again obtained in the elastic-plastic region, which curve at its end points must have a horizontal tangent; whereas according to the new formula, the line  $y(P)$  is represented by a polygon which, never deviates far from a steady curve (Fig. 3). The break points of the polygon correspond to the instants at which a cross section is eliminated from resist once to bending, through its suddenly becoming plastic, and there must always be as many of such break points present as there are stiff corners which could

<sup>15</sup> *J. Fritsche*: Die Tragfähigkeit von Balken aus Stahl mit Berücksichtigung des plastischen Verformungsvermögens. (The carrying capacity of steel beams under consideration of plastic deformability). Der Bauingenieur 1930. Heft 49, 50 u. 51.

be replaced by plastic joints, in order to establish a labile arrangement. A restriction of this conception may be mentioned here: each statically less indeterminate intermediate system and, of course, the statically determined fundamental system must be stable in all their parts. The continuous girder with very long end spans is therefore excluded, because the degree of stability of the statically determined fundamental system which originates through the plastic state of the centre section of the middle span becomes progressively smaller and in the case of the infinitely long side spans, vanishes altogether.

In view of former experiments it is not yet possible to answer indisputably the question concerning the correctness of one or the other yield formulae,

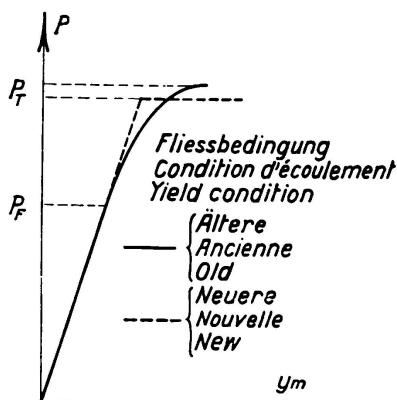


Fig. 3.

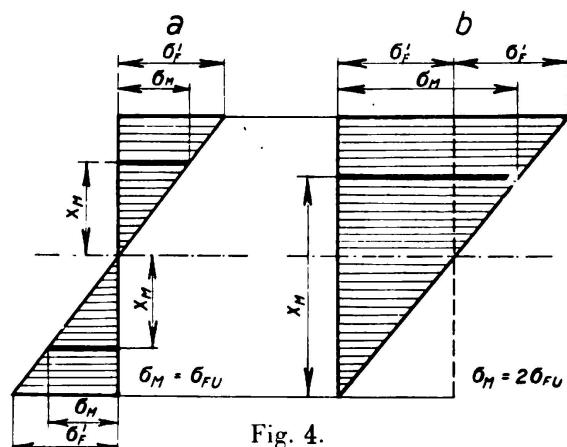


Fig. 4.

a original and b revised interpretation of the yield condition according to Kuntze.

and in discussing the most important experiments, reference will be made to the difficulties of their significance in one or the other direction. The new yield formula has of itself the great advantage of offering a simple basis for the theory of plasticity. If one adopts the standpoint of considering the raising of the yield point as not sufficiently assured by experience, the possibility remains open of conceiving the new formula of yield as a much needed approximation of the old, as it admits of a simple treatment of many problems which no longer appear soluble by the old.

### 3) The Mathematical Conception of the Different Formulae of Yield.

In the case of the usually examined mono-axial fields of stresses, the old formula of yield is based on the aforesaid condition  $\sigma \leq \sigma_F$  wherein  $\sigma_F$  represents the lower yield point. It has been applied almost exclusively as the basis of former investigations, but it has the great disadvantage that under the assumption, always permissible, that the material, as an ideally plastic body, entails an extraordinarily complicated and circuitous calculation<sup>16, 17</sup> and for

<sup>16</sup> J. Fritsche: *Arbeitsgesetze bei elastisch-plastischer Balkenbiegung*. (Laws of work for elastic-plastic bending). *Zeitschrift f. ang. Math. u. Mech.* 1931.

<sup>17</sup> K. Ježek: *Die Tragfähigkeit des exzentrisch beanspruchten und des querbelasteten Druckstabes aus einem ideal plastischen Stahle*. (The carrying capacity of transversely and eccentrically loaded columns of ideal-plastic steel). *Sitzungsberichte d. Wiener Akad. d. Wissensch., Math.-Naturw. Klasse, Abt. II a*, 143 Bd. 1934.

this reason cannot be applied to important practical problems. Apart from the calculation of the carrying load itself, which in consequence of the equalization of moments can be very quickly and easily determined, the determination of a deformation value or of the internal play of forces in the case of partial plasticity of the cross section concerned — bearing in mind their actual form — cannot be carried out.

Such problems are not only of theoretical interest: in the case of the determination of the carrying capacity of an eccentrically loaded steel column, the calculation of the deflection at the centre of the column is unavoidable, as the final carrying capacity is not necessarily attained for the most unfavourably stressed cross section becoming completely plastic, but an instability between the inner and outer forces must already set in in the partially plastic state.

Recourse has been had to a graphical integration of the differential equations concerned; nevertheless, the calculations involved still remain considerable. In considering the solution of this problem established by *Chwalla*<sup>18</sup>, *Ježek*<sup>17</sup>, *Eggenschwyler*<sup>19</sup>, and others, on the basis of the old yield formula, the question arises whether the degree of accuracy reached by the calculations justifies such a laborious treatment, especially having regard to its uncertain basis; 'accuracy' implying the concordance between calculation and experience. Added to the uncertainties of the yield formula are: — the unavoidable variations in the level of the lower yield point, which directly can deviate 10%; the considerable deviation from the accepted law of permanent flatness of cross section, which increases pari passu with plasticity; the disregarded influence of the shear stress and so forth. There exists, therefore, a pressing need for simplification of calculations in investigations of the theory of plasticity.

The first mathematical conception of the new flow formula originates with *Kuntze*<sup>20</sup>. According to this, the ultimate carrying capacity of a cross section is attained when the average resistance value  $\sigma_M$  introduced by *Kuntze* is equivalent to the lower yield point stress  $\sigma_{Fu}$  whilst the yield point  $\sigma_{Fu}$  is exceeded at the elastic stress peaks of the marginal portion. The mean resistance value  $\sigma_M$  is ascertained by dividing the stressed body into two halves by a section parallel to the edge.

'Stressed body' being understood to mean a prism with the cross section as face area, which is so bounded by an inclined section that the overall height of the body corresponds with the stress. The equal volume, or the internal equilibrium, between the over and the under stressed parts of the stressed body cannot, of course be looked upon as a physical basis for the occurrence of yield, even if one remembers that the accumulated yield energy could be proportional to the volume of the over-stressed parts of the stressed body, and that the yielding can be blocked by the residually elastic parts of the cross section

<sup>18</sup> *E. Chwalla*: Theorie des außermittig gedrückten Stabes aus Baustahl. (The theory of eccentrically loaded steel-columns). *Stahlbau* 1934, Heft 21, 22 u. 23, S. 161.

<sup>19</sup> *A. Eggenschwyler*: Die Knickfestigkeit von Stäben aus Baustahl. (The buckling strength of steel columns). *Schaffhausen* 1935, Selbstverlag.

<sup>20</sup> *W. Kuntze*: Ermittlung des Einflusses ungleichförmiger Spannungen und Querschnitte auf die Streckgrenze. (The influence of unequal stresses and the shape of sections on the yield limit). *Stahlbau* 1933, Heft 7, S. 49.

so long as its yield stemming influence prevails. They must therefore be valued as merely working hypotheses so long as a physical basis has not been established. Since the experiment will have satisfactorily demonstrated it, one may assume its validity as at least a good approximation, to this yield formula as yet unknown but theoretically indisputably founded.

The validity of *Kuntze's* yield formula is entirely dependent on experimental proof and must be amended if it fails in this aspect. Such cases occur, as will be later shown in detail, when the formula is desired to be applied for describing the carrying capacity of an eccentrically loaded steel column. In order to remain in harmony with these experiments it must be amended in the following way: — One must start with a body of stresses which comprises the non-uniformity of the field of stresses of the whole cross section, using the original reasoning of *Kuntze*, this is obtained by a previous reduction of the stress area to a zero edge stress (Fig. 4). The mean resistance value  $\sigma_M$  now divides the reduced body of stresses into two parts of equal volume. All that is now necessary in order to maintain harmony with the experiments is to put  $\sigma_M = 2\sigma_F$ , in the case of cross sections with two symmetrical axes. For joists, the increased yield point  $\sigma'_F$ <sup>21</sup> in the elastic peak stresses, accordingly works out as

$$\sigma'_F = \sqrt{\frac{2}{1 + \alpha(1 + \beta)}} \sigma_F \quad (2)$$

If  $\alpha = \frac{h}{h}$  and  $\beta = \frac{a}{b}$  (Fig. 5) this relation holds so long as the mean value of resistance falls in the flange of the cross section, provided  $x_M \geq h + t$ . It follows, therefore, that equation (2) is valid only if  $1 - \alpha(\alpha + 2\beta) \geq 0$ . If this condition is no longer fulfilled then

$$\sigma'_F = \sqrt{\frac{4\beta}{(1 + \beta) - \alpha^2(1 - \beta)}} \sigma_F \quad (3)$$

The value  $\sigma'_F$ , the raised upper yield point, has therefore the significance of a 'bending yield point', since in the case of bending, yield sets in when the greatest edge stress has reached this amount. The bending yield point is therefore not constant but is dependent upon the shape of the cross section; it is

$$\sigma'_F = \psi \sigma_F \quad \text{or} \quad \psi = \frac{\sigma'_F}{\sigma_F} = \frac{\sigma_{Fo}}{\sigma_{Fu}} \quad (4)$$

A conclusion drawn from this yield formula, but not yet proved by experiment, is that in the case of purely elastic deformation, parts of the cross section remaining without stress in proximity to the zero line, must to a certain degree react favourably to the carrying capacity of the beam, so that, e. g. the cruciform profile must have greater carrying capacity than the rectangular cross section remaining after the cutting away of the horizontal flanges, since the

<sup>21</sup> *J. Fritsche*: Der Einfluß der Querschnittsform auf die Tragfähigkeit außermittig gedrückter Stahlstützen. (The influence of the form of cross section on the carrying capacity of eccentrically loaded steel columns). Stahlbau 1936.

unstressed parts stem the flow process in the extreme fibres. For the cruciform profile (Fig. 6)

$$\sigma'_F = \sqrt{\frac{2}{1 - \frac{\alpha\beta}{1-\alpha}}} \cdot \sigma_F \quad (5)$$

for the rectangle alone with  $\alpha = 0, \beta = 1$

$$\sigma'_F = 1,414 \sigma_F \quad (6)$$

For a cross section comprising two broad flange T-sections  $10 \times 5$ , in which  $\alpha = 0.830, \beta = 0.085$ , the ratio of bending yield point to the lower yield point under tension tests is  $\psi = 1.85$ , whilst the value yielded in the case of rectangular cross section is  $\psi = 1.41$ . This cruciform profile should therefore carry about 30 % more than the corresponding rectangular cross section. Something similar applies for the joist, when it is bent in a plane at right angles to the web, only here it should be noticed that a web of any depth does

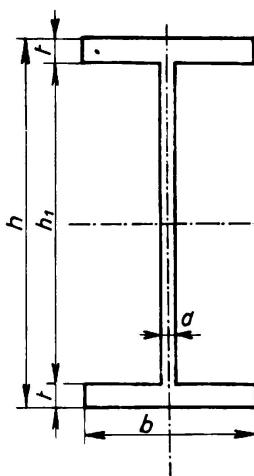


Fig. 5.

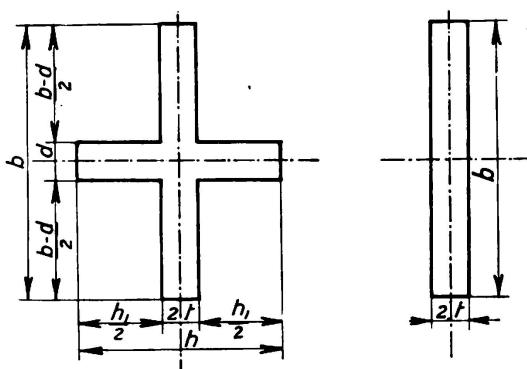


Fig. 6.

not allow an increase in the bending yield point to an indefinite extent. On the contrary, it must be assumed that only the parts of the web in the region of the flanges can stem the yield process in the extreme fibres and it is therefore advisable to include in the yield formula (Fig. 7) a portion — with, for example, the breadth of  $\frac{b}{2}$  — of the web on each side.

This theoretical result seems surprising, and it is to be hoped that it will soon be possible to test it by experiment. Perhaps such an experiment will decide the question as to the correctness of one or the other yield formula. Meanwhile, the experiments carried out with such cross sections with regard to the carrying capacity of eccentrically loaded steel columns, alone confirm the necessity for this assumption.

The second conception of the yield formula, taking into consideration the raising of the yield point at the peak stresses, emanates from *Prager*<sup>9</sup>. He takes the view concerning the yield phenomenon that the increased elastic field with

the limiting stress  $\sigma'_F$  becomes transformed in to the diagram of the complete plastic condition, with the limiting stress  $\sigma_F$ , and that this process goes on without diminishing the bending resistance of the plasticised portion (Fig. 8). If  $W$  represents the section modulus of the cross section, and  $T$  the statical moment of both halves of the cross section in respect to the neutral axis line, then of necessity  $\sigma'_F W = \sigma_F T$  and consequently

$$\sigma'_F = \frac{T}{W} \cdot \sigma_F \text{ or } \psi = \frac{\sigma_{F0}}{\sigma_{Fu}} = \frac{T}{W} \quad (7)$$

Although the *Prager* conception at first seems highly probable, critical consideration shows it also to be an idealising of the processes of yield; it is as

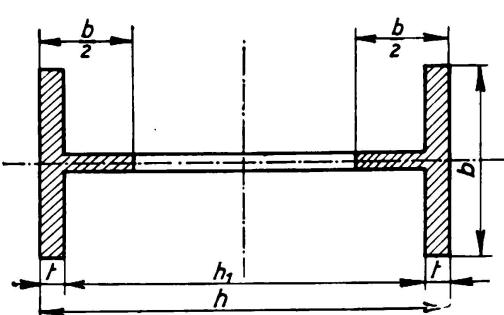


Fig. 7.

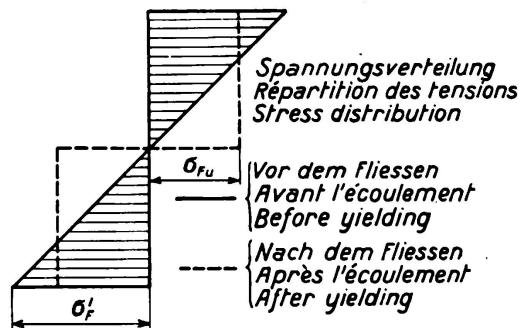


Fig. 8.

little as that of *Kuntze* to be regarded as founded on Physics and in fundamental agreement with existing phenomena. In its appropriate extension to the calculation of the carrying capacity of eccentrically loaded steel columns it unmistakably furnishes values that are too high as compared with practice, so that although a final opinion cannot yet be given, preference must nevertheless be given to *Kuntze*'s conception of the yield formula.

#### 4) Experimental Tests of the Yield Formula.

The bending tests *Thum*<sup>11</sup> and *Wunderlich* form a basis for the new yield formula. The eight tests with polished test bars of different forms of rolled I-sections (Fig. 9) furnishtaking into consideration a lower yield point of  $\sigma_F = 2.47 \text{ t/cm}^2$  in the tension test — the values of  $\sigma'_F$  calculated according to the yield formula of *Kuntze* or *Prager*, which are set out in the adjoining Table 1, together with the measured values. Disregarding tests 3 and 4, in which the greater deviations certainly admit of other explanations, the agreement with the *Kuntze* values is decidedly very satisfactory, whilst according to *Prager*, the calculated values almost throughout lie above those measured; from this the conclusion may be drawn that the stress distribution in the thoroughly plastic condition does not necessarily correspond unconditionally with the assumption of the older theory.

The experiments of *Thum* and *Wunderlich* form the most important supports of the new yield formula, and although they also seem to supply indisputable proof of an increased upper yield point, yet the supposition is not easily discarded that, in view of the observations made in the determination

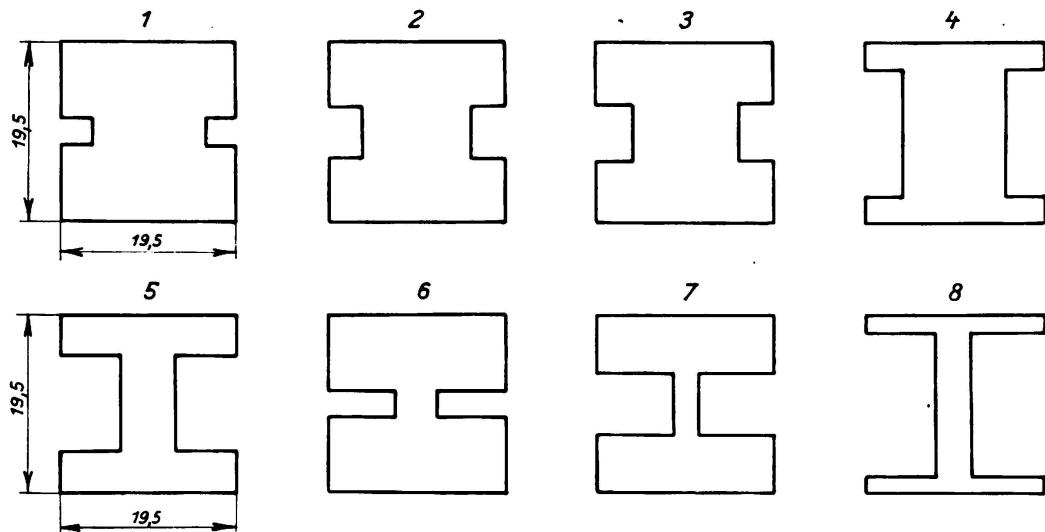


Fig. 9.

Cross sections of the 8 test beams as used by Thum and Wunderlich.

of fatigue stress, the conditioning of the surface by polishing may have had an influence on the yield point — which alone can be measured — in the surface area.

What the raising of the upper yield point causes in the different types of steel is a question only to be decided by experiments which are not yet available

Table 1. Tests of Thum and Wunderlich.  $\sigma_{Fu} = 2.47 \text{ t/cm}^2$ .

No. of test	$\sigma_{Fo}$ measured	$\sigma_{Fo}$ calculated after Kuntze	Deviation in % of the measured values	$\sigma_{Fo}$ calculated after Prager	Deviation in % of the measured values
1	3.50	3.41	+ 2.56	3.68	- 5.14
2	3.64	3.31	+ 9.07	3.61	+ 0.82
3	3.78	3.28	+ 13.20	3.60	+ 5.14
4	3.42	3.38	+ 1.17	3.45	- 0.80
5	2.91	2.96	- 1.72	3.32	- 14.10
6	3.44	3.31	+ 4.03	3.66	- 6.40
7	3.15	3.06	+ 2.96	3.43	- 8.88
8	2.61	2.72	- 4.20	3.05	- 16.84

in sufficient number to enable a final opinion to be formed. As is shown in the adjoining Table 2, the experiments of *Siebel* and *Vieregge*<sup>14</sup> with square beams show that this effect is decidedly evident only in softer types of steel, whilst it is not revealed in the case of high-grade alloyed steels. For solving the problem which of the yield formulae is the correct one, these experiments must be excluded, because the yield point has been determined by calculation, with the assumption of a certain distribution of stress, and not by direct observation at the point of flow; and the question of whether the carrying capacity is reached by gradual plasticising or by increase of the yield point in the stress peaks, remains open.

In the case of statically indeterminate supported beams, the plastic phenomena have a substantially greater influence on the play of internal forces and the laws of deformation than in the case of simply supported beams, even in view of the equalization of moments which the plastic theory requires independently of the nature of the yield formula. The assumption therefore suggests itself, that an accurate gauging of the phenomena would establish a conclusion

Table 2. Tests of Siebel and Vieregge with square beams.

No. of test	Lower yield point $\sigma_{Fu}$	ultimate strength	$\psi = \frac{\sigma_{Fo}}{\sigma_{Fu}}$	$\psi$ (theoretical)	
			measured	Kuntze	Prager
1	1.89	3.09	1.66	1.41	1.50
2	2.52	4.88	1.34	1.41	1.50
3	3.77	7.50	1.07		
4	5.46	7.10	1.05		

on the correctness of one or the other yield formula. Such tests (Fig. 10 a and b) have recently been recorded with extreme thoroughness by *Stüssi* and *Kollbrunner*<sup>22</sup> in Zurich. I have already evaluated these tests from this point of view in a work appearing in *Stahlbau*<sup>23</sup>; though in that case, the yield formula of *Kuntze* was used in its original form. In accordance with the alteration rendered necessary by the compression tests on steel stanchions, equation 2 furnishes  $\sigma'_F = 1.09 \sigma_F$  or  $\Delta \sigma_F = \sigma_{Fo} - \sigma_{Fu} = 303 \text{ kg/cm}^2$  consequently  $M_T = W \sigma'_F = 26.70 \text{ tcm}$  and  $P_T = \frac{8 \cdot 26,70}{60} = 3,56 \text{ tons}$ .

At the same time the stress at the centre of the beam of the middle bay corresponds to the increased yield point,  $\sigma'_F$  the load  $P'_F$  is with

$$\alpha = \frac{3l_2}{4l_1 + 6l_2} - \frac{3}{14}$$

$$P'_F = \frac{4M_T}{(1-\alpha)l_2} = 2,27 \text{ t}$$

The corresponding value of the moment over the supports is  $X'_F = -7.28 \text{ t/cm}$ . Exclusive of the central section of the beam, a statically determinate fundamental system remains, consisting of two simple beams with cantilever arms of the length  $\frac{b_2}{2}$  held together in the plastic joint.  $P$  can now be increased until the yielding moment  $M_T$  is also reached over the supports in the fundamental system, which is set up if  $\frac{1}{2}(P_T - P'_F) \frac{b_2}{2} = M_T - X'_F$ : from which also naturally follows the value  $P_T = 3.56 \text{ t}$ , already obtained directly.

<sup>22</sup> *F. Stüssi und Kollbrunner: Beitrag zum Traglastverfahren.* (Contribution to the theory of plastic equilibrium). *Bautechnik* 1935, Heft 21, S. 264.

<sup>23</sup> *J. Fritsche: Grundsätzliches zur Plastizitätstheorie.* (Fundamental remarks to the theory of plasticity). *Stahlbau* 1936, Heft 9, S. 65.

In the doubly statically indeterminate system, the deflection at the middle of the beam in the case of purely elastic deformation is  $f_2 = 0.872 P$ , whereas

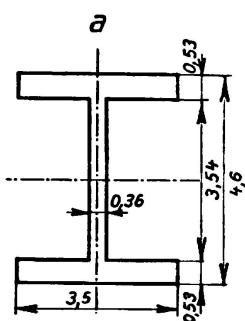


Fig. 10a.

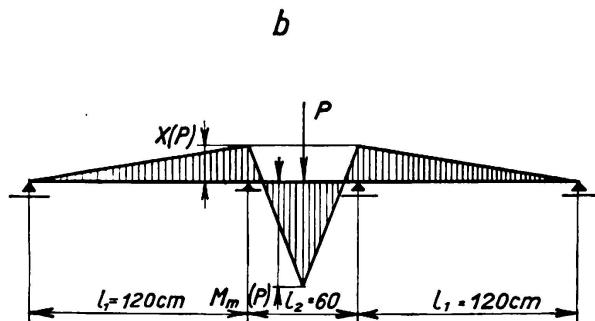
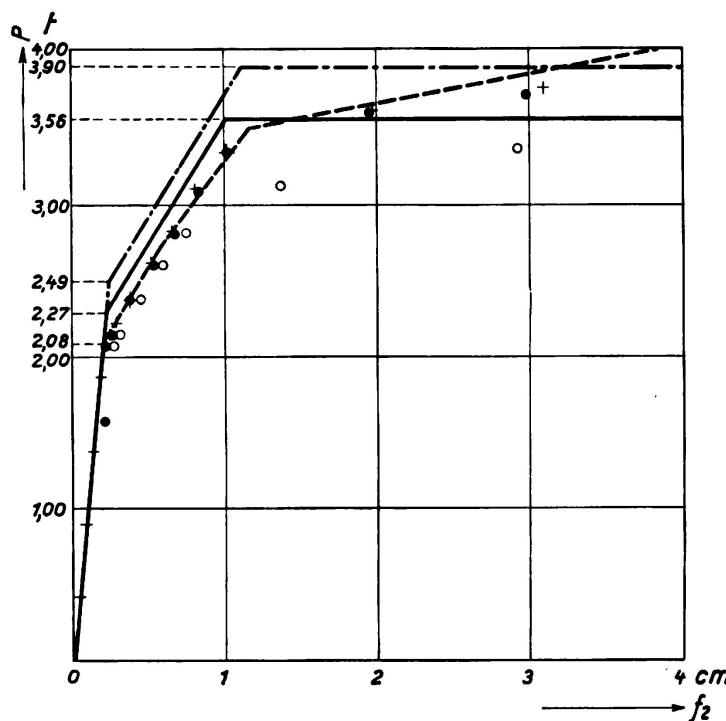


Fig. 10b.

Cross section und loading arrangement of the test beam used by Stüssi and Kollbrunner.

after the aforesaid exclusion of the elastic central section of the beam, it works out as  $f^2 = 0.198 + 0.642 (P - 2.27)$ . Stüssi and Kollbrunner have measured



- ○ + *Gemessene Werte; valeurs mesurées; measured values*
- *Auf Grund der Messungen von Stüssi u. Kollbrunner gerechnet*  
— *Calculées en se basant sur les mesures de Stüssi et Kollbrunner*  
— *Calculated, based on measurements by Stüssi and Kollbrunner*
- *Auf Grund der neueren Fließbedingung von Kuntze*  
— *D'après la nouvelle condition d'écoulement de Kuntze*  
— *Based on new yielding condition by Kuntze*
- *Auf Grund der neueren Fließbedingung von Prager*  
— *D'après la nouvelle condition d'écoulement de Prager*  
— *Based on new yielding condition by Prager*

Fig. 11.

the deflection  $f_2(P)$  and determined the course of the internal resistances  $M_m(P)$  and  $X(P)$  from the deformation of the axis of the beam by a well-considered

method of reasoning. Figs. 11 and 12 show the results of their measurement in comparison with those calculated by means of various yield formulae, and from this it becomes evident that the new *Kuntze* yield formula also best corresponds to the conditions. The sudden kink in the measured lines  $f_2(P)$ ,  $M_m(P)$ , and  $X(P)$ , on reaching  $P_F$  or  $M_F = W\sigma_F$ , indicates that, although in this case no increase of the yield point appears to have set in, yet the investigations in this direction admit of no certain significance, as the effect only occurs

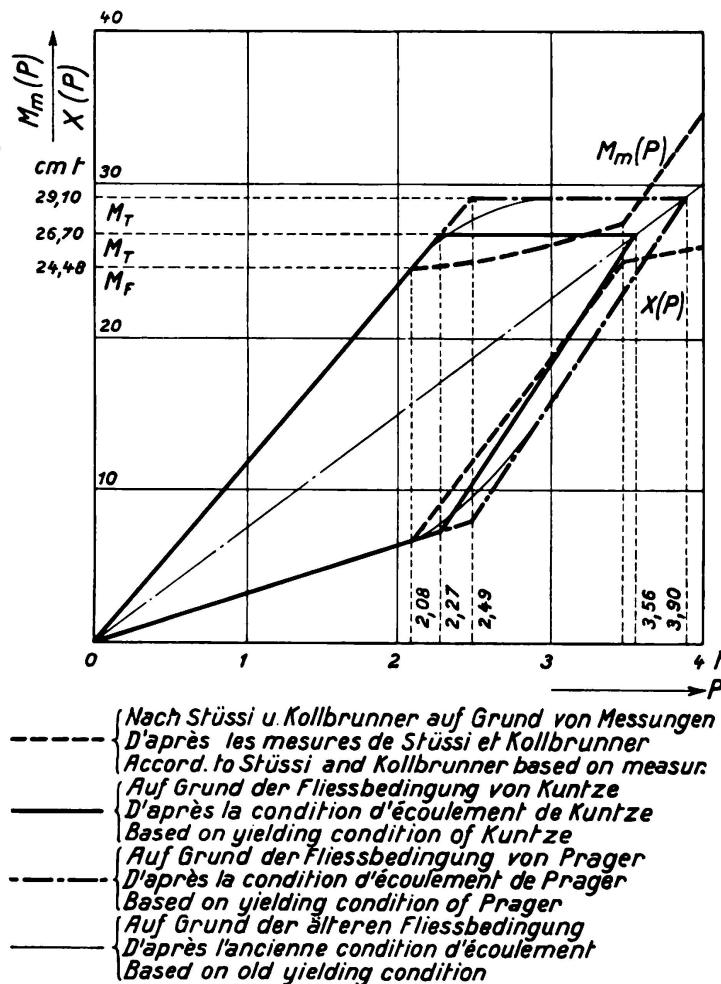


Fig. 12.

within about 10% from the  $\sigma_F$  limit and might be marked by variations in the yield point.

The investigations of *Maier-Leibnitz* previously mentioned show in general a gradual transition from the purely elastic to the elastic-plastic condition. I have already shown<sup>23</sup> that this, too, so far as it can be tested, does not contradict the *Kuntze* yield formula, the deflection polygon  $y(P)$  coincides satisfactorily with the measured lines.

As a result of this consideration of the most important experiments, all that can be said at present is: a raising of the yield point is not impossible, it requires still further experimental confirmation; so long as this is not available, the new yield formula can only be regarded as a close approximation of the

old, which should be reckoned with, because it provides an extraordinary and necessary simplification of the amount of calculation.

### 5) The Eccentrically Loaded Steel Column.

If the plastic theory founded on *Kuntze's* yield formula is practicable, it must also be capable of representing the carrying capacity of eccentrically loaded steel column in conformity with experience. This task has been dealt with lately<sup>17, 18, 19</sup> with extraordinary thoroughness on the basis of the 'older' yield formula, without, however — in view of the certainly very considerable influence of the shape of the cross section — leading to any satisfactory results<sup>24</sup>. Under the assumption of constantly spreading yield zone, we have here a pro-

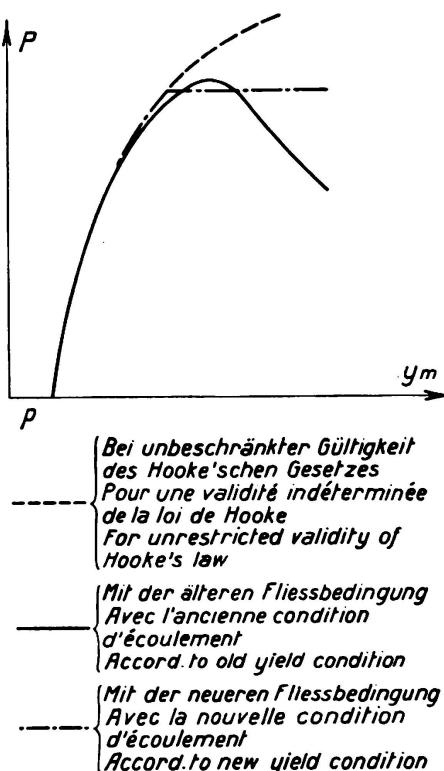


Fig. 13.

Transverse deflection of the middle of an eccentrically compressed bar.

blem of critical loads, inasmuch as the progressive plasticising of the centre of the bar more and more disturbs the equilibrium between the inner and outer forces, until at a definite depth of the yield zone, long before the complete plastic state is attained in the most dangerously stressed cross section, no stable equilibrium can any longer exist. Strictly speaking, the calculation of this critical load has been strictly possible only for the rectangular cross section, as the differential equations to be solved in the case of the ideally plastic body are very complicated and in the general case admit of only a toilsome graphical integration. The expression, 'strictly', relates solely to a purely mathematical

<sup>24</sup> E. Chwalle: Der Einfluß der Querschnittsform auf das Tragvermögen außermittig gedrückter Baustahlstäbe. (The influence of cross sectional shape on the carrying capacity of eccentrically loaded building steels). Stahlbau 1935, Heft 25 u. 26.

treatment; from the theory of strength standpoint, which strives to provide a correct description of experience, they remain, after all, approximate solutions since they rest upon a series of more or less completely realised assumptions.

With the help of the new yield formula, the solution of this problem becomes extremely simple<sup>21</sup>, whilst the actual shape of the cross section can be taken into account without special difficulty. According to this conception the operation proceeds in such a manner that its purely elastic character is retained up to the increased yield point  $\sigma_F$ . It is not until then that the yield zone suddenly extends to a very considerable depth in the cross section, and if the bar is supported in a statically determinate manner, the limit of the carrying capacity is reached at once. This way of putting it is undoubtedly an idealising of the actual phenomena, and the experiments show that small increases of load are still possible, though already significant yield traces can be observed; the sudden kink in the line  $y_m$  (P) (Fig. 13) vanishes when the yield proceeds in stages, but no substantial influence can be attributed to this phenomenon.

In the case of bending under longitudinal compression, the raised yield point  $\sigma'_F$  must depend substantially on the longitudinal stress  $\sigma_o$ , as well as on the shape of the cross section. In order to comprehend this dependence it is necessary, firstly, to consider the two limiting cases of yield point  $\sigma_o = 0$  and  $\sigma_o = \sigma_F$ .  $\sigma_o = 0$  corresponds to bending in the absence of longitudinal force for which the value  $\Delta\sigma_F(0)$  already appears to have been fixed.  $\sigma_o = \sigma_F$  is purely longitudinal stress and if the case of buckling be left out of consideration, the limit of the carrying capacity is reached. As was shown above all by the investigations of *W. Rein*<sup>25</sup>, the intervention of a moment is no longer necessary for the production of a constant increase in deformation under unchanging load. This agrees with the foregoing conception, since now the non-uniformity of the stress conditions vanishes and therefore  $\Delta\sigma_F(\sigma_F)$  must be equal to 0. A linear value for  $\Delta\sigma_F$  for intermediate values of  $\sigma_o$  is now indicated, as from experience the simplest terms often give the most suitable results. As I have already explained<sup>21</sup>; this expression corresponds to the equation

$$\sigma'_F - \sigma_o = \psi (\sigma_F - \sigma_o) = \frac{1}{\sqrt{}} (\sigma_F - \sigma_o) \quad (8)$$

wherein  $\psi$  is a factor dependent solely upon the shape of the cross section and represents the ratio of the bending yield point  $\sigma'_F$  to the lower tension yield point (Fig. 14). I also showed on that occasion that the suitable application of the *Prager*<sup>9</sup> yield formula must lead to a quadratic function  $\Delta\sigma_F(\sigma_o)$  which in the case of  $\sigma_o = \frac{\sigma_F}{3}$  reaches its maximum value  $\Delta\sigma_F = \frac{2}{3} \sigma_F$ . The *Prager* yield formula therefore leads to the conclusion, difficult to conceive, that the capacity for absorbing bending moments should increase within definite limits with increasing  $\sigma_o$ ; this affords a further explanation of the fact that this yield

<sup>25</sup> *W. Rein*: Berichte des Ausschusses für Versuche im Stahlbau, Ausgabe B, Heft 4; Versuche zur Ermittlung der Knickspannung für verschiedene Baustähle. (Test reports of the Commission for Steel Structures Edition B, No 4. Tests for determination of the buckling stress of various kinds of steel).

formula furnishes critical loads which are too high, as compared with those of the experiments<sup>26</sup>.

By means of equation (8),  $\sigma'_F$  is now determined; the failure of the stanchion is now coupled with the condition  $\sigma_1 = \sigma'_F$ . According to Fig. 15, this equation

$$\sigma_i = \sigma'_F = \sigma_o + \frac{P y_m}{W} = \sigma_o \left(1 + \frac{P}{k_i} \sec \frac{\chi l}{2}\right) \text{ is obtained,}$$

wherein  $\chi^2 \frac{P}{E J}$  and  $k_i = \frac{W_i}{F}$  which represents the cross sectional core-width corresponding to the inner fibre.

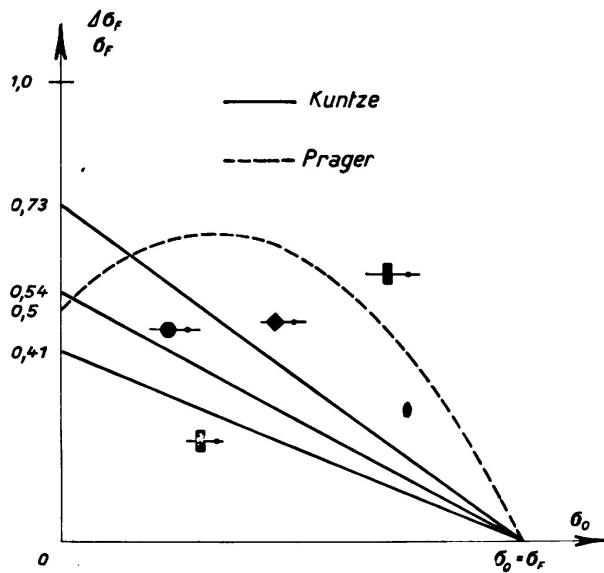


Fig. 14.

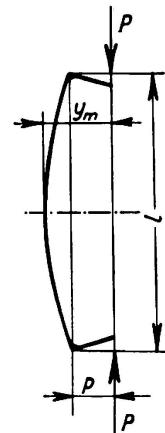


Fig. 15.

If the eccentricity  $m$  is substituted for  $\frac{P}{k_i}$  then

$$\sigma'_F = \sigma_o \left(1 + m \sec \frac{\chi l}{2}\right) \text{ or } \sigma'_F - \sigma_o = \sigma_o m \sec \frac{\chi l}{2}$$

and with the yield formula (8)

$$\sigma_o m \sec \frac{\chi l}{2} = \frac{1}{\nu} (\sigma_F - \sigma_o)$$

from which  $\sigma_{okrit}$  can now be calculated. For  $\sec \frac{\chi l}{2}$  the approximation given by Timoshenko<sup>27</sup>

$$\sec \frac{\chi l}{2} = \frac{\sigma_E + 0,234 \sigma_o}{\sigma_E - \sigma_o} \quad (9)$$

may be used with advantage, wherein  $\sigma_E$  represents the Euler stress  $\frac{\pi^2 E}{\lambda^2}$ .

<sup>26</sup> J. Fritsche: Näherungsverfahren zur Berechnung der Tragfähigkeit außermittig gedrückter Stäbe aus Baustahl. (Approximate method of calculating the carrying capacity of eccentrically loaded columns of building steel). Stahlbau 1935, Heft 18, S. 137.

<sup>27</sup> Timoshenko: Strength of Materials, Vol. II, 1931.

If the measurement of eccentricity be set down as  $m' = \nu m$  then the following quadratic equation:

$$\sigma_{o \text{ krit}}^2 (1 - 0,234 m') - \sigma_{o \text{ krit}} [\sigma_F + \sigma_E (1 + m')] + \sigma_F \sigma_E = 0 \quad (10)$$

is obtained for  $\sigma_{o \text{ krit}}^2$ .

If it is wished to represent  $\sigma_{o \text{ krit}}$  directly as a function of the ratio of slenderness  $\lambda = \frac{l}{i}$  there results

$$\sigma_{o \text{ krit}}^2 \lambda^2 (1 - 0,234 m') - \sigma_{o \text{ krit}} [\lambda^2 \sigma_F + \pi^2 E (1 + m')] + \pi^2 E \sigma_F = 0 \quad (11)$$

The solution of this quadratic equation is sometimes attended with difficulties;  $\sigma_{o \text{ krit}}$  being obtained in the form of a difference, and if both the values approach equal magnitude, greatly increased accuracy of the calculations becomes necessary and the use of the slide rule is no longer possible. In such cases the expression can be represented by the square root in the form  $\sqrt{1 - x}$ , wherein  $x$  is a very small quantity; if the square root is developed in a binomial series finishing at the second term, a sufficiently accurate approximation is often obtained with

$$\bar{\sigma}_{o \text{ krit}} = \frac{\sigma_F \sigma_E}{\sigma_F + \sigma_E (1 + m')} \quad (12)$$

whilst a better value can be obtained by including the third term of the series, in the form

$$\bar{\sigma}_{o \text{ krit}} = \bar{\sigma}_{o \text{ krit}} \left[ 1 + \sigma_{o \text{ krit}} \frac{1 - 0,234 m'}{\sigma_F + \sigma_E (1 + m')} \right] \quad (13)$$

## 6) *Reviewing the Experiments.*

The conditions established will now be compared with the abundantly available experimental results, in order to prove the correctness and the utility of the calculations built up on the new yield formula. Of primary importance in this connection are the fundamental experiments by *Roš*<sup>28</sup> with Steel Joists 22 and 32. As I have already mentioned<sup>21</sup>, the examination of these results reveals a highly satisfactory agreement between calculation and experiment. Fig. 16, which is taken from my publication in 'Stahlbau', shows distinctly how the calculated lines  $\sigma_{o \text{ krit}} (\lambda)$  represent the mean of the experiments. It is further evident that the slenderness ratio  $\lambda < 25$  must be excluded, since in such cases rigidity can already play a part and mask the actual yield phenomena. The experiments of *Roš* justify the modification of the *Kuntze* yield formula mentioned in Section 3. If the calculation does not take into account the possibility that the understressed web can at least partly restrict the flow in the edge fibres, it cannot be numerically represented.

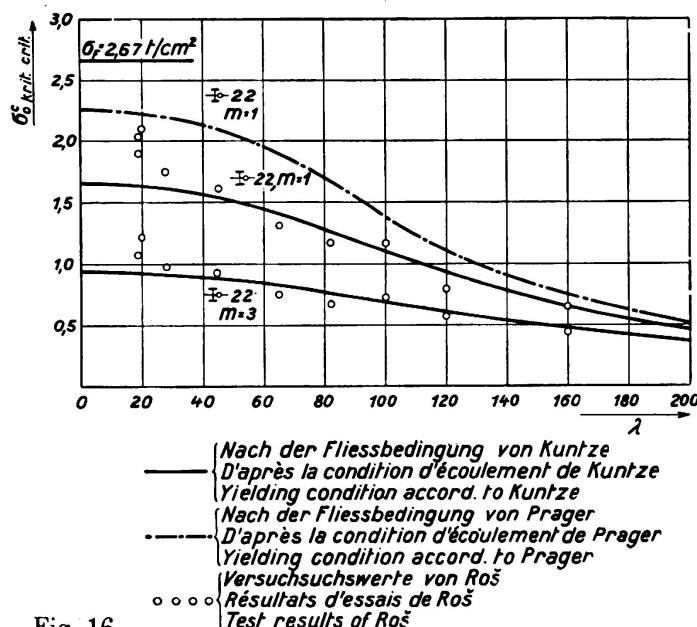
So far as I know of them from a publication by *G. Grüning*<sup>29</sup>, the experi-

<sup>28</sup> *M. Roš*: Die Bemessung zentrisch und exzentrisch gedrückter Stäbe auf Knickung. (Dimensioning of centrally and eccentrically loaded bars to buckling). Bericht der II. int. Tagung f. Brückenbau u. Hochbau, Wien 1928, S. 282.

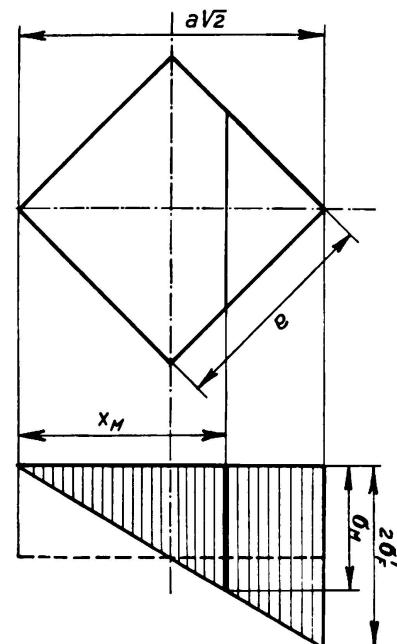
<sup>29</sup> *G. Grüning*: Knickversuche mit außermittig gedrückten Stahlstützen, Mitteilungen aus dem staatlichen Materialprüfungsamt in Berlin-Dahlem. (Buckling experiments with eccentrically loaded steel columns). *Stahlbau* 1936, Heft 3, S. 17.

ments of the 'Deutscher Stahlbauverband' furnish — as I have already been able to show<sup>21</sup> — an emphatic confirmation of this method of calculation, though it must be deemed highly desirable that further such experiments should be carried out with complicated shapes of cross sections to enable the theory to be tested in all its applications.

The experiments carried out by *A. Ostenfeld*<sup>30</sup> in the laboratories for Building Construction at the Copenhagen Institute of Technology, in the years 1928—9, and to which my attention was drawn by the courtesy of *Dr. Čížek* of Prague, are of great interest. *Ostenfeld*, like *Melan* of Prague, long before the plastic theory had made such considerations understood, took up the position that stress



From J. Fritsche: „Der Einfluß der Querschnittsform auf die Tragfähigkeit außermittig gedrückter Stahlstützen“, Stahlbau 1936.



alone cannot form a true criterion of safety and that it is necessary to refer to the ultimate stress or to the unstable equilibrium as produced by yield. He arrived at the conclusion that the true safety of eccentrically loaded steel columns can be guaranteed by comparing the permissible stress with an extreme fibre stress of the value

$$\sigma_{\text{Res}} = \sigma_0 \left( 1 + \beta m \sec \frac{\pi l}{2} \right) \quad (14)$$

set up at the least favourable point.

The value  $\beta$  cannot be understood from the mere point of view of stress based on the elastic theory; it was in the main an experimental value which was dependent upon cross section and upon the slenderness ratio and which made it possible to connect the greatest extreme fibre stress with carrying capacity. *Ostenfeld* also presented a theoretical deduction, not regarded as satis-

<sup>30</sup> A. Ostenfeld: Exzentrisch beanspruchte Säulen, Versuche mit Stahlsäulen, Querschnittsbemessung. (Eccentrically loaded columns, tests with steel columns, dimensioning of cross sections). Ingeniorvidenskabelige Skrifter A № 21. Kopenhagen 1930.

factory today, of the value  $\beta$ , in which he uses the conception corresponding to the old yield formula with its plastic reduction of the peak stresses. In order to obtain satisfactory agreement with the experimental results, he was also obliged to adopt an occasional reduction of the elastic modulus  $E$ , ranging from 10 to 20 %, which is so determined that in the condition of the attained carrying capacity the so-called secant formula

$$y_m = p \cdot \sec \frac{\chi l}{2} \quad (15)$$

is fulfilled.

For this reason no direct comparison can be made between his values  $\beta$  and the values  $\nu$  introduced here; their sense, however, is the same, since both 'correct' in an equal degree the specified eccentricity.

For the rectangular cross sections, *Ostenfeld* found  $\beta = 0.69$  whilst the theoretical value of  $\nu$  is 0.71; the agreement is unexpectedly good. In the

Table 3.  
Ostenfeld's experiments with columns of square cross section.  
 $\nu = 0.707$ .  $E = 2100 \text{ t/cm}^2$ .

No. of test	$\sigma_{Fu}$ in $\text{t/cm}^2$	$\lambda$	m	$\sigma_{o \text{ krit}}$		Deviation in % of the measured values
				calculated	measured	
1	2.44	49.2	2.15	892	912	+ 2.4
2	2.37	49.1	5.80	441	465	+ 5.1
3	2.12	72.6	2.09	732	727	- 0.7
4	2.13	72.6	5.98	370	353	- 4.8
5	2.37	99.6	2.26	672	627	- 7.2
6	2.44	98.0	6.30	370	341	- 8.5
7	2.64	123.3	2.62	581	519	- 11.9
8	2.69	123.6	6.63	349	338	- 3.3

following Table 3 his measurements on steel columns of rectangular cross section are compared with the calculated values  $\sigma_{o \text{ krit}}$ . The deviations are small, but in any case the circumstance that the theoretical values are somewhat greater than measured ones is due to the vagaries in value of the yield point  $\sigma_F$ . The value of the moment of external forces is nearly unchanged over a longer region and it is clear that yield will occur at that point at which the value of the yield point is low. This agrees with the experience that the first yield traces are not always observed to be in the centre of the column.

The second group of experiments was carried out with square sections set diagonally; for these the hypothesis of the similarity of the overloaded and underloaded stressed body, supplies for the calculation of  $\nu = \frac{\sigma_F}{\sigma'_F}$  the following relation (Fig. 17).

$$\nu^3 - 1.5 \nu^2 + 0.3124 = 0 \quad (16)$$

which is preferably solved by trial; this furnishes  $\nu = 0.58$  whilst *Ostenfeld* calculates  $\beta = 0.53$ . Having regard to the correction for  $E$ , a direct comparison of the two figures is again impossible. Consequently, in Table 4 the measured values of  $\sigma_{0\text{ krit}}$  are compared with those calculated. Here again the agreement is satisfactory as deviations of + 13.8 or - 12.6 % must be considered as reasonably within the range of accuracy obtainable. In view of the broad under-stressed parts of the cross section, the theoretically required upper yield point

Table 4.  
Ostenfeld's experiments with columns of square cross section (square set diagonally).  
 $\nu = 0.580$ .  $F = 2100 \text{ t/cm}^2$ .

No. of test	$\sigma_{Fu}$ in $\text{t/cm}^2$	$\lambda$	m	$\sigma_{0\text{ krit}}$		Deviation in % of the measured values
				calculated	measured	
1	2.63	48.3	2.44	1000	1160	+ 13.8
2	2.68	48.5	6.21	547	579	+ 5.5
3	2.15	73.8	2.51	743	713	- 4.2
4	2.20	73.9	6.11	431	456	+ 5.5
5	2.63	98.0	3.09	688	672	+ 2.4
6	2.68	98.2	7.10	421	408	- 3.2
7	2.74	122.8	2.25	694	616	- 12.6
8	2.12	124.2	6.65	330	325	- 1.5

in the elastic stress peaks when  $\sigma_0 = 0$ , works out as  $\sigma'_F = 1.73$  which from these experiments must be regarded as possible. Nevertheless, in this case, in view of the great resistance offered by the fibres, spreading into the depth, to the progressive plasticising, the older yield formula also furnishes a yield area, the growth of which increases inversely with increasing load. A very high carrying capacity would also result, since the ratio  $\frac{T}{W}$  is very large and it is seen that the old and the new yield formula proceed similarly and that in the main only differ in the intermediate stages, which lead to approximately the same final results.

The third group of tests relates to columns with circular cross section (Fig. 18). The fibre breadth is  $b_x = 2 \sqrt{x(d-x)}$ , and the yield formula reads

$$\frac{\pi d^2}{4} \cdot \sigma'_F = 2 \int_0^{x_M} \frac{4 \cdot \sigma'_F}{d} \cdot \sqrt{x(d-x)} dx$$

from which after integration we get, as equation for  $\nu$ :

$$3\pi = \frac{3}{2} \text{arc} \cdot \sin(2\nu - 1) + (8\nu^2 - 2\nu - 3) \sqrt{\nu(1-\nu)}$$

If, as is always feasible with the value of  $\nu$  now under consideration, one may take

$\arcsin(2v-1) = (2v-1) + 2\pi$  then we get

$$\frac{3}{2} (2v - 1) + (8v^2 - 2v - 3) \sqrt{v(1-v)} = 0 \quad (17)$$

The root of this equation is  $v = 0.65$ , whilst *Ostenfeld* has calculated with  $\beta = 0.58$ . Table 5 again contains the calculated and measured values for  $\sigma_{o, \text{krit}}$ . For the second time in the case of a small slenderness ratio, a value is obtained that is more than 10% too low, and it is not impossible that the explanation of

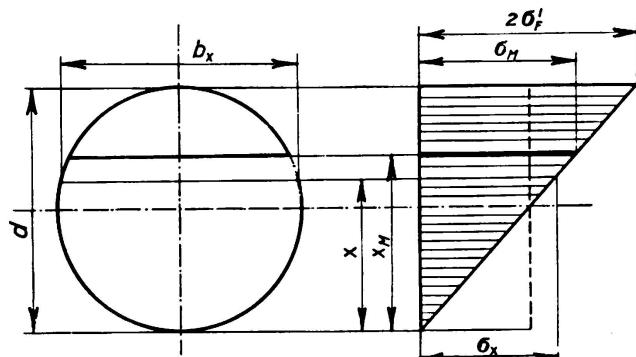


Fig. 18.

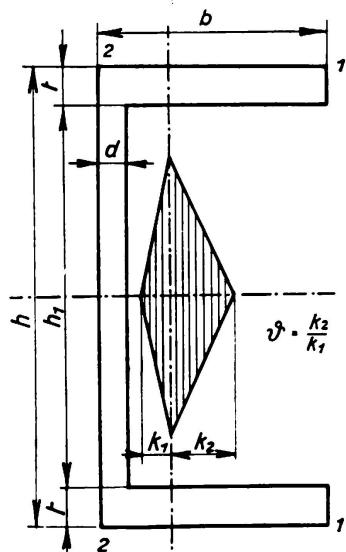


Fig. 19.

this rather great error lies in the fact that it has not been found possible completely to avoid some fixing of the column ends in the pressure heads of the knife-edged bearings. The consequent stiffening of the bar in the ends will naturally be manifest, especially in the case of shorter bars.

Table 5.

		$v = 0.650$ .		$E = 2100 \text{ t/cm}^2$ .			
No. of test	$\sigma_{\text{Fu}}$ in $\text{t/cm}^2$	$\lambda$	m	$\sigma_{\text{o krit}}$		Deviation in % of the measured values	
				calculated	measured		
1	3.25	44.2	2.70	1082	1260	+ 14.1	
2	3.25	44.4	6.20	608	668	+ 9.0	
3	3.25	95.3	2.97	775	763	- 1.6	
4	2.85	95.3	6.20	455	447	+ 1.8	

A further group of experiments concerned steel joists in which bending takes place at right angles to the plane of the web; in this case, *Ostenfeld* assumes the value  $\beta = 0.58$ , whilst the calculation according to equation 5 gives  $\nu = 0.61$ .

Of great interest are the tests with channel sections for which I have earlier

deduced the yield point under bending or the value  $v^2$ . Such an unsymmetrical shape of cross section gives quite different carrying capacities, according to which side of the centre of gravity the load is applied. If the greatest pressure is applied to the projecting flanges, a bending ensues towards the closed side of the cross section,

$$v_1 = 0.707 \sqrt{1 - \frac{\alpha \beta^2}{1 - \alpha}} \quad (18)$$

in which  $\alpha = \frac{h}{h}$  and  $\beta = \frac{d}{b}$  (Fig. 19). For the channel section 10 examined,  $\alpha = 0.83$ ,  $\beta = 0.12$ , and therefore  $v_1 = 0.682$ , whilst *Ostenfeld* in this case calculated  $\beta = 0.69$ . A comparison of the calculated and measured values is

Table 6.  
Ostenfeld's experiments with columns of steel channel section C 10.

(a) The flanges are in compression.

$$v_1 = 0.682. \quad E = 2100 \text{ t/cm}^2.$$

No. of test	$\sigma_{Fu}$ in $\text{t/cm}^2$	$\lambda$	$m = \frac{p}{k_1}$	$\sigma_0 \text{ krit}$		Deviation in % of the measured values
				calculated	measured	
1	3.04	31.9	1.95	1250	1200	— 4.2
2	3.04	31.9	5.85	592	557	— 7.4
3	2.95	56.8	1.86	1132	1070	— 5.8
4	3.23	57.4	6.06	572	510	— 12.1
5	2.95	82.0	2.01	939	875	— 7.3
6	2.95	82.0	5.76	502	479	— 4.8
7	2.94	106.0	2.12	774	707	— 9.5
8	2.94	106.0	6.00	440	406	— 8.4
9	3.04	134.0	2.85	616	567	— 8.6
10	3.17	134.0	6.32	394	360	— 9.5

afforded in Table 6, which also contains the deviation in % of the measured values.

Where the point of application of the load lies on that side of the centre of gravity on which the web is situated, the web takes the compression and bending occurs towards the open side of the cross section. So far as the carrying capacity is concerned there are here two quite different possibilities under consideration; its limit can be reached by yield in the compressed part or by yield phenomena in some part of the cross section under tension; the maximum compressive stress is  $\sigma_2 = \sigma_0 \left(1 + m \sec \frac{\alpha l}{2}\right)$  and the maximum tension stress

$$\sigma_1 = \sigma_0 \left(-1 + m \frac{k_2}{k_1} \sec \frac{\alpha l}{2}\right)$$

in which  $m = \frac{p}{k_2}$  represents the measured eccentricity.

The yield formula in the first case reads

$$\sigma'_{\text{F}} - \sigma_o = \sigma_o m \sec \frac{\chi l}{2} = \frac{1}{v_2} (\sigma_{\text{F}} - \sigma_o) \quad (19)$$

whilst that in case two can be written

$$\sigma'_{\text{F}} + \sigma_o = \sigma_o m \frac{k_2}{k_1} \sec \frac{\chi l}{2} = \frac{1}{v_1} (\sigma_{\text{F}} + \sigma_o) \quad (20)$$

$v_2$  was already calculated as  $v_2 = 0.707 \sqrt{(1+\alpha) - \alpha \beta (2-\beta)}$ ;  $v_1$  must, of course, with reference to the reduced area of stress, be of the same value as was deduced in the case of bending towards the closed side. Case 1 will occur in the case of high values of  $\sigma_o$  and small eccentricities  $p$ ; in the case 2 it is exactly the reverse. The limiting stress  $\sigma_{\text{ob}}$  at which the yield phenomena occur at both edges simultaneously is obtained by combining the two equations 19 and 20, as

$$\sigma_{\text{ob}} = \frac{v_1 \vartheta - v_2}{v_1 \vartheta + v_2} \text{ wherein } \vartheta = \frac{k_2}{k_1} \quad (21)$$

For the channel section C 10 examined,  $k_1 = 0.629$ ,  $k_2 = 1.400$  cm,  $v_1 = 0.682$ , from which results  $\sigma_{\text{ob}} = 0.253 \sigma_{\text{F}}$ . In such of the experiments of *Ostenfeld* in which the load was applied on the web side, the yield occurred in accordance with the conditions determined above, partly in the web and partly in the drawn flanges. In most of the experiments, however, the quantities  $m$  were so large as to produce the conditions of case 2. Accordingly, equation 20 expresses the critical stress as

$$\sigma_{\text{okrit}}^2 (1 + 0.234 m' \vartheta) + \sigma_{\text{okrit}} [\sigma_{\text{F}} - \sigma_{\text{E}} (1 - m' \vartheta)] - \sigma_{\text{E}} \sigma_{\text{F}} = 0 \quad (22)$$

The value for section C 10 was  $v_1 = 0.68$ , whereas *Ostenfeld* adopted  $\vartheta = 0.63$ . The experiments are very numerous, and it suffices to recalculate

Table 7.  
Ostenfeld's experiments with columns of steel channel section C 10.  
(b) The web in compression.  
 $v_2 = 0.903$ .  $v_1 = 0.682$ .  $\vartheta = 2.23$ .  $E = 2100 \text{ t/cm}^2$ .

No. of test	$\sigma_{\text{Fu}}$ in $\text{t/cm}^2$	$\lambda$	$m = \frac{p}{k_2}$	$\sigma_{\text{okrit}}$		Deviation in % of the measured values	$\sigma_{\text{ob}}$
				calculated	measured		
1	2.59	50.7	1.76	1000	1140	+ 12.1	655
2	2.57	82.2	1.95	800	776	- 3.1	650
3	2.82	108.0	1.84	698	625	- 11.7	712
4	2.71	132.4	2.03	527	498	- 5.8	685
5	2.78	82.5	1.96	830	828	- 0.3	703
6	2.97	57.4	5.79	351	382	+ 8.1	750
7	2.57	82.2	5.73	291	305	+ 4.6	650

Note: In experiment 1 yield occurred in the web under compression, in the other experiments in the flange under tension.

those with small values of  $\vartheta_m$ , as these are chiefly of interest. The results of the calculations are tabulated in Table 7. If the theoretical assumption be considered that yield can occur in the projecting flanges only if  $\sigma_{o\ krit} > \sigma_{ob}$ , it is contradicted by two of the experiments; this, of course, is not surprising, since the value  $\sigma_{ob}$  can never have the significance of a precise relative demarcation of two different phenomena, but can possess only an approximate validity. Above all, the variations in the yield point will play an important part, especially in view of the fact that in one case the yield point is attained in the web and in the other in the flange, which according to experience are quite dissimilar. It would be possible to bear these different yield points in mind in calculating  $\sigma_{ob}$ . Yield phenomenon in the web were revealed in only two parallel tests. The experimental data are shown in Table 7 under No. 1, in which case also satisfactory agreement is evident.

In conclusion, it may be said of the comprehensive tests of *Ostenfeld* that they can be regarded as an effective confirmation that the basic principles underlying the newer yield formula actually enable the carrying capacity of steel columns to be appropriately expressed with a very small amount of calculation; even though they cannot directly express its physical correctness, they nevertheless indicate its utility in connection with practical consideration of the strength of materials. A further proof of the method of calculation elaborated above is afforded by the comprehensive experiments of the American Society of Civil Engineers, which appointed a special commission of its own to investigate the carrying capacity of steel columns. The results of the experiments, which deal with very different types of cross sections, are recorded in the Proceedings of the American Society of Civil Engineers, February, 1929. Unfortunately, the space available is insufficient to discuss them here.

### Summary.

The old yield formula is based directly on the strength of material as obtained by the ordinary static tensile test in judging the local danger to yield, since the old yield formula considers only the local stress conditions responsible for yield. The new yield formula, however, gives strength values which are based solely on the nature of fields of stresses. Even if an increase of the upper yield limit within the peak stresses of a stress field is not proved certain and the old yield condition considered as a true basis, the new theory need not be disregarded in its conclusions. It can always be regarded as an approximation to the old, supplying useful results.

The new yield condition has the advantage of allowing for simple and clear modes of calculation the results of which compare well with tests. It is desired that the study of yield is carried still further.