Zeitschrift: IABSE congress report = Rapport du congrès AIPC = IVBH Kongressbericht

Band: 2 (1936)

Rubrik: I. Importance of the thoughness of steel for calculating and dimensioning steel structural work, especially when statically indeterminate

I

Importance of the Toughness ot Steel for Calculating and Dimensioning Steel Structural Work, especially when Statically Indeterminate.

Die Bedeutung der Zähigkeit des Stahles für die Berechnung und Bemessung von Stahlbauwerken, insbesondere von statisch unbestimmten Konstruktionen.

La ductilité de l'acier. Sa définition. Manière d'en tenir compte dans la conception et le calcul des ouvrages, notamment des ouvrages hyperstatiques.

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I 1

General Theory of Plasticity, Fields of Equal Yield Lines.

Allgemeine Plastizitätstheorie, Gleitlinienfelder.

Théorie générale de la plasticité. Champs des lignes de cession.

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Warschau.

Introduction.

Though the theory of plasticity has advanced with such rapid strides during recent years that a separate working meeting has been devoted at this Congress to the consideration of the results and effects of its development, yet there is still a considerable amount of unclarity attached to its basic principles. It is true that the changes that have taken place in the views of modern physics have made us revise many of our traditional ideas on the theoretical strengths of materials (and above all from a structural point of view), still the unclarity that predominated in various basic conceptions of the theory of plasticity far outweighs the effects of these modifications. This unclarity is mainly caused by unclear reasoning in connection with phenomenological facts.

The mechanical aspect of solid bodies is for the great part governed by *Hooke*'s Law. But as this law, which enabled a comparatively exhaustive theory of elastic continuum to be elaborated, is only valid up to a certain point, it has long been the endeavour of mechanical science to discover new and similar laws of general import applying to the conditions prevalent beyond this point. Unfortunately, this aim has been thwarted by numerous great difficulties, since, although the elastic behaviour of various materials — at least from a phenomenological point of view — is more or less the same, the behaviour of the material after the limit of elasticity has been passed is fundamentally influenced by its internal structure. The actual beginning of the mathematical investigation of the theory of plasticity was the employment, based on the affinity between *Mohr*'s enveloping curves for non-cohesive matter and those of various solid bodies, of methods for the calculation of solid matter which proved applicable to the theory of conditions at the limit of equilibrium of non-cohesive matter. Owing to the entirely different composition of these bodies, however, the method was bound to fail, i. e. lead to results deviating substantially from reality. For it should never be forgotten that a crystaline body must first undergo more or less elastic changes in shape before attaining the plastic condition in which elastic and plastic fields practically always exist side by side and overlap along certain areas, whereas the non-cohesive mass is usually subjected to equilibrial disturbances through attaining the "plastic condition". In treating theoretical problems of elasticity *Hencky*'s[1] distinction between "statically determinate" and "statically indeterminate" conditions of equilibrium is essential. By a statically determinate case *Hencky* means one in which the conditions necessary for equilibrium and that for plasticity are together sufficient to determine the tensile stresses at every point, while the solution of a "statically indeterminate" case necessitates the examination of deformations. When investigating the plastic conditions of materials practically all the cases encountered are bound to be "statically indeterminate" ones, since generally, in the ultimate conditions analysed, plastic fields are not to be found where large elastic areas exist, so that in the transition regions there must be compatibility between the two conditions and these can thus only be considered independently of each other. Mathematical treatment of such conditions is rendered extremely difficult by this connection. Yet no assumptions may be made which are contradictory to the actual behaviour of the materials, simply for the sake of simplifying calculation and arriving at a mathematical solution.

The most important simplification of this kind, which governs the whole mathematical theory of plasticity, is the assumption that the elastic deformations can be neglected on account of their relative smallness in comparison with the plastic ones. This assumption, which is nothing but an analogical conclusion drawn between the behaviour of amorphous and crystalline substances, is inadmissible for conditions of equilibrium in which both elastic and plastic fields exist. In the well-known work by *Haar* and *Kármán*[2] proof is already to be found that in the semi-plastic zone, i. e. in the zone in which $(\sigma_1 - \sigma_2)^2 = 4k^2$ ($2k =$ yield limit), while $(\sigma_2 - \sigma_3)^2 < 4k^2$, $(\sigma_3 - \sigma_1)^2 < 4k^2$, the plastic deformations are of the same magnitude as the elastic and it is therefore not feasible to neglect the latter for the former where both kinds of deformation occur.

All attempts hitherto made to solve problems of plastic equilibrium for crystalline bodies have, however, been more or less based on this assumption. The exceptions are few and far between. The fundamentally most important work discarding this assumption is one by *Hencky*,[3] which, on the other hand, leads to such involved mathematical investigations for the simplest case that the treatment of less straightforward cases rendered impossible with the mathematical resources at our disposal today.

In consideration of the unclarity surrounding the basic principles of the theory of plasticity, we shall now proceed to analyse briefly these basic principles and to compare the significance of the phenomena of plastic deformation of crystalline bodies, paying special attention to those phenomena termed in literature "yield patterns". — As is customary in the theory of plasticity, we shall consider processes of such a gradual nature that they may be regarded as a consequence of equilibrium conditions, so that in general there will be no need to discuss the rapidity of deformation in the various cases.

1) *Conditions necessary for plasticity.*

The first question to be answered in the theory of plasticity is: under what circumstances the yield limit of a material is passed. Before giving a brief review of the existant yield hypotheses, let us first quote a theorem by *Roš* which is very important when considering both the rupture and the yield hypotheses: — "A general theory of rupture which makes no allowance for the texture of the material is not possible owing to the fact that the behaviour of materials of different internal structure is often fundamentally different. Each material requires its own theory of rupture, a consequence of its internal structure and behaviour under deformation". The fact that this theorem had never before been formulated so precisely and that there was thus a tendency to generalise the results of experiments carried out when testing a certain material, explains the existence of so many hypotheses.

The materials used in engineering are generally crystalline substances which, though composed of individual crystals, yet behaves in a quasi-isotropic manner in consequence of the amorphous arrangement of the latter. — As regards the structure of the single crystals, in the metals used in engineering these are almost exclusively in stereometrical lattice arrangement, of which there are three kinds:

1) The simple type of lattice, which is singularly determined by stating the distance between molecular accumulations (characteristic distance);
2) the plano-centric lattice, with additional molecular accumulation in the planes of cubes;
3) the stereo-centric lattice, with central molecular concentration.

α, β and δ irons crystallise in stereo-centric, γ iron, nickel and manganese steel, as well as copper, aluminium, etc., crystallise in plano-centric lattice formation.

The type of lattice is extremely important from the technical standpoint also, as the manner of transition to the plastic state and the characteristic phenomena of the latter are decisively influenced by the crystal lattice.

The most important of the yield hypotheses are the following:

1) *Guest-Mohr*'s shear stress hypothesis[4] in the form of

$$\tau_{max} = f(\sigma_x + \sigma_y)$$

developed from *Coulomb*'s old theory of internal friction[5]. The function $f(\sigma_x + \sigma_y)$ can be adapted to the results of experiments.

2) *Beltrami*'s hypothesis of constant deformation energy, which regards a definite amount of accumulated deformation energy as a criterion for the attaining of the yield limit, but which did not tally with the results of experiments, and was newly formulated and improved by *Huber*[6] and independently by *Mises* and *Hencky*[7] to become

3) Hypothesis of constant deformation energy

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 8k^2.$$

4) *Schleicher*'s improvement on this hypothesis[8], which perhaps represents the most general form, runs

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = \sigma_e(p),$$

$$\text{whereby } p = {}^1/_3(\sigma_1 + \sigma_2 + \sigma_3).$$

The many experiments that have been carried out to test the correctness of all these hypotheses, and various others that have been entirely abandoned today,[9] have shown that for plastically deformable metals, whether they possess a pronounced yield limit or not, the *Huber-Hencky-Mises* hypothesis characterises their transition to the yield zone and subsequent behaviour in the latter, whereas for brittle materials and conditions of stressing at yield limit *Mohr*'s hypothesis gives the best mean values.

When assessing the value of these statements it must always be borne in mind that they are provisional results which may yet be altered when further research work is carried out.

2) *Yield Limit.*

The yield hypothesis is the condition that must be fulfilled by the main stresses so that yielding is attained in one point. This condition applies (an essential fact) for the state occuring immediately after the yield limit has been passed. It gives no information, however, as to the manner in which this transition takes place. Not only with regard to different metals (steel and copper) are there differences in the process of deformation, but also in the various kinds of one metal and even in absolutely similar kinds of different antecedents.

The principal difference is between metals with and without a pronounced yield limit. In the case of the latter the transition from elastic to plastic state takes place in quite constant manner, for even slight stressing is sufficient to cause plastic deformations. In the former, on the other hand, the deformations are completely reversible up to a certain point; suddenly, however, the material, which has till now offered such resistance, suddenly collapses, plastic deformations at once begin to appear and develop rapidly. Often, however, the load does not remain constant, but decreases considerably, so that there would seem to be an "upper" and a "lower" yield limit.

Bach[10] was the first to point out that this upper yield limit depends to a very great extent upon the shape of the test bar, and he recognised the nature of this limit as one of a typical symptom of instability (overturn of the loading). Modern research has gone a step further and also declared the "lower" yield limit to be a symptom of retardation such as may be seen in other branches of physics (delayed boiling, undercooling) and which are distinguished in that the change of state to be expected in accordance with the physical laws is considerably retarded and then, suddenly, sets in and develops with rapid strides. When the stress-deformation curve gives a straight line, it is considered — for instance by *Moser*[11] as a phenomenon of retardation, namely, as the expression of a retardation of the permanent deformations, brought about by internal resistance to yield. The shock-like development of plastic deformation at yield limit is then to be regarded as an outward effect of retardation. This opinion is also supported by the fact, confirmed by experience, that for steel of the same granulation and processing, the limit of proportionality approaches the yield limit, the more homogeneous the material is and the more undisturbed the state of stressing that can be produced in it.

The proper manner of regarding the nature of the yield limit is extremely important for the theory of plasticity in that it forms a basis without which it is impossible properly to judge the fundamental significance of the individual-plastic phenomena in the transition stage.

3) *Distortion Patterns.*

In smoothly polished test pieces of soft iron relatively fine relief patterns become apparent in the initial stage of deformation and become more and more corwded as stressing increases. These dull lines, which represent intersections of the more deformed layers with the polished surface, are either crested (in compression) or troughed (in tension) or else shed-roofed. These lines, called *Lüder* or *Hartmann* Lines after their first observers and to-day generally described as distortion wedges in the theory of plasticity, are the most striking distortion patterns. Their most important property is their intersection with the shear-tension trajectories. It is because of this property that these lines are regarded as extremely valuable aids in the investigation of tension conditions of solid bodies in the plastic zone.

The fields of equal yeld lines are actually distinguished by a number of important properties from a mathematical point of view, properties which make it possible completely to resolve conditions of tension in the plastic zone from the knowledge of distrotion wedges[12]. The most important of these properties is the identity of these distortion wedges with the characteristics of the condition necessary for plasticity. Proof of this identity was first brought by *Massau,* though it was put into a general form by *Reissner*[13]. In view of this property it is possible to compute unanalytically various integrals along distortion wedges, a fact which greatly facilitates the application of solutions to the actual conditions prevailing. The few existent solutions to the mathematical theory of plasticity are practically all based on this property of distortion wedges.

When judging how far the above method may be applied for the real solution of technical problems in the realm of the plasticity theory, it must, however, be considered that mathematically perfect transitions to states above their admissible limit allow *a priori* of no judgment being formed. For if we start with a mathematically defined hypothesis and within it allot definite ultimate values to definite magnitudes, this procedure is indubitably admissible from a mathematical standpoint. Physically, however, it is possible that the physical behaviour of the material is considerably altered by these ultimate values and that the factors which formed the basis of the hypothesis have lost much if not all of their validity. This is the case as regards the condition necessary for plasticity.

The condition necessary for the plasticity of a generally plastic body for two-dimensional stressing is

$$\sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2} + \sin\varphi\,\frac{\sigma_x + \sigma_y}{2} = C.$$

ρ being the angle of friction, C a value dependent on cohesion. For the non-cohesive mass on which the investigation of distortion wedges was based, $C = 0$. The appearance of a distortion wedge results primarily in disturbance of equilibrium; the reversible deformations preceding entry into the disturbance zone are quite negligible compared with the "plastic" deformations. For metals, however, $C =$ constant and $\rho = 0$. Owing to the great cohesion the appearance of distortion wedges is only a local and transitory disturbance of equilibrium, the elastic state of stressing and deformation preceding the beginning of yield phenomena is of the utmost importance as regards the character of the yield, while the magnitudinal arrangement of the plastic deformations is equal to that of the elastic deformations.

It will be clear from the above that no importance can be attached to the results of the so-called mathematical theory of plasticity when it is a question of the plasticity theory of crystalline materials, for the necessary conditions are not fulfilled. This also means that less importance must be placed on distortion wedges in the investigation of plastic conditions in metals of technical interest. They only become of any value when deformation is so far advanced that there are no longer any elastic zones whatever in the whole field. These cases are not of frequent occurrence, being principally confined to problems of processing.

Discarding the generally accepted view that distortion wedges are of great importance in the theory of plasticity, and utilising without prejudice the numerous results of experiments already carried out, it will be found that the phenomenon of distortion wedges is not connected with plastic deformation as such, but only with the character of the transition from the elastic to the plastic state. Just like the yield limit, they are typical phenomena of instability. This fact is proved by quite a number of observations, such as those of *Ludwik*[14], showing that yield lines are particularly liable to appear when the body begins to yield under decreasing stress, i. e. when the formation of distortion wedges is restricted to the downward slope of the peaks of the stress-deformation diagram. This observation has also been confirmed by *Nadai*[15] and often referred to by *von Kármán*. In this connection mention should also be made of *Nadai*'s observation that the pattern of yield lines was much more crowded when compression tests were rapidly carried out than when more leisurely tests were made. This is a further proof that instability in general, whether caused by stressing or by the texture of the actual material, favours the formation of distortion wedges. It is thus quite obvious that such formation must also be favoured by boring and notching.

In this connection reference should be made to the extremely interesting measurements of hardness taken by *Moser*[16] in distortion patterns. His results show that metals increase in hardness in the yield zone, and reveal interesting details concerning the character and process of plastic deformations. *Moser* observed that permanent deformations at first only occurred in zones (deformation wedges), only a definite degree of hardness being attained in each zone. A general increase in hardness only takes place when the whole bar (tension test) is covered with a network of deformation wedges. The reason for this phenomenon lies in a kind of "blocking" of the yield surfaces. When this phase is reached in a particular zone, a further increase of loading will result in slipping (yielding) in another zone, not yet deformed. Before the yield resistance of this next zone is overcome, the loading always increases somewhat and drops again as soon as the distortion wedges have formed. Each peak of the $\sigma - \varepsilon$ diagram therefore corresponds to a local upper yield limit at which a yield line forms under decrease of loading. — Contrasted with the phenomena which occur in steel, a copper rod revealed a steady increase in hardness from the moment when loading is applied, whereby no zonal yield lines but only a general dulling of the surface was observed.

The above tests are an obvious corroboration of the view that the yield limit of steel is a "retardation" of the yielding process; they furthermore explain the deformation wedges as a phenomenon, herewith connected, characteristic only for steels with varying yield limits.

This conception is further confirmed by the results of tests carried out by *Ititaro Takaba* and *Katuni Okuda* [17] showing that

1) the appearance of deformation wedges and the sudden break in the stress-strain line are results of one and the same occurrence, viz. the displacement in groups of large quantities of crystal granules;
2) all metals in which deformation wedges can occur belong to the stereo-centric lattice crystalline structure. It is shown that formation of deformation wedges was not observed in steels of Austenite texture, which belong to the plano-centric lattice type of structure.

It may therefore be stated that in the investigation of elastic-plastic conditions of crystalline materials — and above all of metals — the observation of fields of equal yield lines is not a suitable method, but that on the contrary the fundamental occurrences that really matter are often rendered even more unclear by the deformation wedges. This chiefly applies for the development of the true limit between the elastic and the plastic state.

4) *Ultimate Limits of the Range of Plasticity.*

Employing one of the known methods for determining the plastically deformed areas in metals — the best of which is that of recrystallisation [18] — the limits between elastic and plastic areas can be definitely ascertained (Fig. 1). The form of

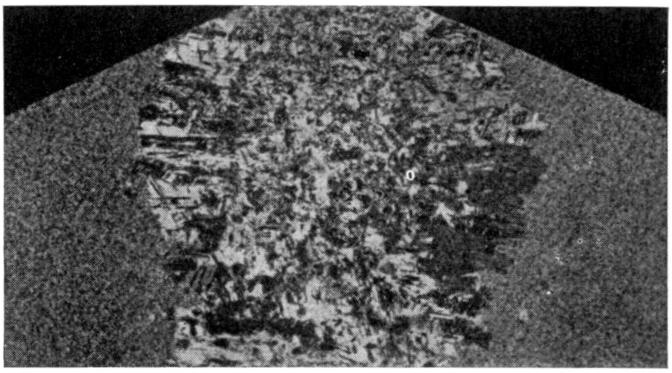


Fig. 1.

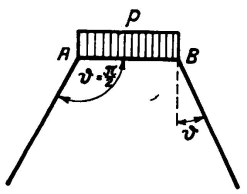
these limit lines, as all observations have indisputably proved [19], has nothing to do with corresponding deformation wedges, but is composed of those lines corresponding both to the plastic and to the elastic state of tension. The only group of lines fulfilling this stipulation are the τ_{max} = constant lines of the elastic state. This type of limit line, which is independent of that of transition from the elastic to the plastic zone, are to be observed in all elastic-plastic states and form the most essential symptom of the latter. No solutions of the plastic field of tension will correspond to reality other than those which can always be applied to the corresponding elastic field of tension along every line τ_{max} = constant. Every solution of the plastic problem must therefore be preceded by that of the elastic problem, and here it must be borne in mind that the limit between elastic and plastic range is not a fixed one, but varying as its loading varies. The latter must, however, always keep to the lines τ_{max} = constant of the elastic field.

The mathematical treatment of elastic-plastic problems under the above conditions is not easy; up to the present it has only been successfully applied in very few simple cases. A certain alleviation may, however, lie in the fact that by carrying out so-called optical investigation experiments for stresses in models in such a manner that the lines of constant difference of greatest tension appear primarily as isochromes, it becomes possible to get at the limit of plastic range *a priori*.

5) *Resistance to Penetration.*

As an example of the solution of a technical problem in the above manner we shall now treat that of resistance to penetration as a duodimensional problem. The case is of especial interest because it represents the best known example of a plastic solution with the assistance of the field of equal yield stresses, and because it was its publication that directly gave rise to the development of the modern mathematical theory of plasticity[20].

Fig. 2.



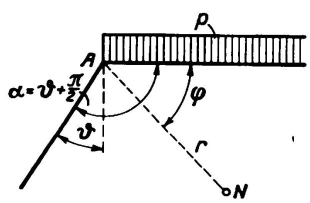


Fig. 3.

The problem is to find the uniformly distributed load stress p which (Fig. 2), acting along A B, causes yielding inside the zone under consideration. This load, which we shall define as resistance to penetration, may be represented as a function of the yield limit and of the angle of inclination of the lateral delimitation of the zone. Considering the plane state of distortion ($\varepsilon_z = 0$), the conditions necessary for yield laid down in the *Huber-Hencky-Mises* theorem are:

$$(\sigma_x - \sigma_y)^2 + 4\,\tau^2 = \frac{16}{3}\,k^2,$$

the yield limit being $\sigma = 2k$.

As the solution of the "blunt wedge" is not possible either as a plastic problem or in a closed form, assistance may be obtained by only considering the corner A and taking into account the fact that in this corner the lines τ_{max} = constant of the problem shown in Fig. 3 are tangents to the lines τ_{max} = constant of the blunt wedge. In determining the critical load it is unimportant whether we deduce the lines τ_{max} = constant themselves or their tangents.

From the elastic solution of the corner, using *Airy*'s stress function

$$F = ar^2 + br^2\varphi + cr^2 \sin 2\varphi + dr^2 \cos 2\varphi \qquad (1)$$

the four constants a, b, c, d can be obtained from the four support conditions

$$\text{for } \varphi = 0 : \sigma_t = -p, \quad \tau = 0 \text{ (without friction)}$$
$$\text{for } \varphi = \alpha : \sigma_t = 0, \quad \tau = 0$$

we get the stresses

$$\begin{aligned} \sigma_r &= p\,(Q-1) - 2P \cdot p \cdot \varphi - p \cdot P \cdot \sin 2\varphi + p \cdot Q \cos 2\varphi \\ \sigma_t &= p\,(Q-1) - 2P \cdot p \cdot \varphi + p \cdot P \cdot \sin 2\varphi - p \cdot Q \cos 2\varphi \qquad (2) \\ \tau &= p \cdot P - p \cdot P \cos 2\varphi - p \cdot Q \cdot \sin 2\varphi \end{aligned}$$

whereby

$$P = -\frac{1}{2(\alpha - \operatorname{tg}\alpha)}, \qquad Q = -\frac{1}{2(\alpha \operatorname{ctg}\alpha - 1)}.$$

With the abbreviation

$$x = \frac{\tau_{max}^2 - p^2 \cdot Q^2}{4p^2 \cdot P^2}$$

a brief calculation yields the equation of the lines τ_{max} = constant in the form of

$$y = -\frac{x}{2(x-1)}\left[\operatorname{tg}\alpha \pm \sqrt{\operatorname{tg}^2\alpha - 4x^2}\right] \qquad (3)$$

This is the equation of a pair of straight lines passing through A, which remain real as long as $\operatorname{tg}^2\alpha \geqq 4x^2$.

The principal shear stress becomes a maximum on that line for which

$$\frac{\delta \tau_{max}}{\delta \varphi} = 0.$$

This is satisfied for $\varphi = \frac{\alpha}{2}$. The calculation leading to the second deduction further shows that only for $\frac{\pi}{2} < \alpha \frac{3\pi}{2}$ along the line $\varphi = \frac{\alpha}{2}$ is a maximum value created, while for $\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ a minimum is given there. The latter values of α are, however, not of technical interest. For $\varphi = \frac{\alpha}{2}$ we get the magnitude of the principal shear stress

$$\tau_{max}^2 = p^2\,[Q^2 - 2P \cdot Q \cdot \sin\alpha + 2P^2\,(1 - \cos\alpha)]. \qquad (4)$$

Introducing the condition for yield, we obtain for the critical load stress

$$p = \sigma_F \cdot \frac{\left(\vartheta + \frac{\pi}{2}\right)\sin\vartheta + \cos\vartheta}{1 + \sin\vartheta} \qquad (5)$$

This is the relation between resistance to penetration, angle of wedge and yield limit[21].

Sachs[22] has studied the problem of resistance to penetration in metals with great thoroughness, and he, also by recrystallisation, ascertained that the plastically deformed zone is delimited by the lines τ_{max} = constant of the elastic stress area. In Fig. 4 the resistances to penetration in steel, as obtained by *Sachs*, are compared with the values deduced from Eq. 5 for various wedge angles. There is a satisfactory amount of coincidence.

The solution of the same problem, as worked out by *Prandtl* with fields of equal stress lines, gave the equation

$$p = \sigma_F \, (1 + \vartheta) \tag{6}$$

as a function of the resistance to penetration from the wedge angle and the yield limit. By way of comparison this equation has also been entered Fig. 4 with the result that coincidence is revealed solely for extremely acute wedge angles, but that the course is quite different.

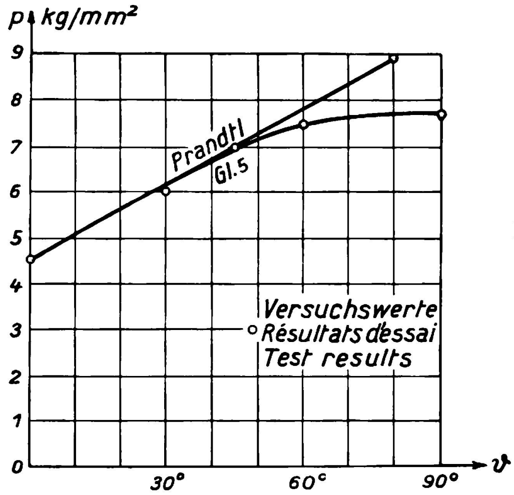


Fig. 4.

The example cited shows that the treatment of plastic problems for crystalline materials must always be based on the delimitation curves of the plastic zone. The assumption of stress lines a delimitation of this kind, and the obtaining of solutions with the assistance of the properties possessed by these stress lines will always lead to results which do not correspond to reality.

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Summary.

The fundamental principles of the general theory of plasticity are still unclear in certain essential points owing to the fact that the phenomena of yielding are not judged and evaluated in a clear and uniform manner.

The most essential conceptions of the theory of plasticity — condition for yield, yield stress limit and deformation wedges (stress lines) are therefore submitted to a brief analysis, the most important result of which is the conclusion that both the yield stress limit and the yield lines are essentially phenomena of instability dependent upon the internal structure of the material and that they specifically influence the manner of transition from elastic to plastic but are nevertheless of much less importance for general plastic deformation than is commonly believed. The more so, as both phenomena can only be observed in the case of very definite crystalline structures, namely, those of stereo-centric lattice formation, while for materials of a different type of crystalline structure the transition from elastic to plastic takes place in a continuous, uninterrupted manner.

The limit of the plastic zone is formed independently of the manner of transition by lines τ = constant in the elastic field of stress.

The example of resistance to penetration shows the distinctions to be made in the treatment of plastic problems — on the one hand from the point of view defined above, and on the other from the standpoint of the mathematical theory of plasticity, which in fact is a theory of deformation wedges — and proves that for metals the results obtained with the mathematical theory of plasticity do not correspond with reality.

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I 2

Fundamental Principles of the Theory of Plasticity.

Grundlagen der Plastizitätstheorie.

Principes de la théorie de la plasticité.

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Whilst it was formerly held that, assuming elastic behaviour, all questions of strength of the material which concerned the structural engineer are capable of solution, it is now recognised, from comprehensive measurements on structures and tests carried to near the breaking load in the testing laboratory, that by such an ideal conception of the material, it is impossible to obtain a uniform degree of security against the setting up of dangerous conditions. Although it has long been known that purely elastic behaviours are confined within comparatively narrow limits of stress, this opinion was based on the grounds that, for reasons of safety, the stresses produced by the working loads must lie within these elastic limits, and it was thought that the greatest stress determined by the elastic theory afforded a basis for deciding the factor of safety for a structure, and, above all, the equal safety of all its parts. Accordingly a permissible stress was decided upon, with the proviso, that is should not be exceeded by the action of the applied loads.

This stress scale is very convenient for the practical dimensioning of structures, because for the designer it eliminates complicated and often disquieting questions as to the actual safety of his design; the crucial question of safety appearing then to be incorporated with the determination of permissible stress. This at once involves difficulties, since one can only introduce into the strength calculations simple data for the material — for example, the lower yield point stress σ_{Fu} of the steel — that has been determined from simple tension tests. The limits of proportionality migth well be left out of consideration, as it has been recognised that calculations are exact enough if the purely elastic behaviour up the lower yield point is accepted.

So long as the conditions of stress set up in the structure are the same as those imposed upon the test piece, the question of safety is clear; the factor of safety is then $n = \frac{\sigma_{Fu}}{\sigma_{perm}}$. For judging the uniform multi-axial stress conditions, the known conditions of yield have been laid down and tested by experiment[1] — the hypothesis of *Mises-Huber-Hencky*, based on the comparison of the specific

[1] *M. Roš* u. *Eichinger:* Versuche zur Klärung der Frage der Bruchgefahr. I. Flußstahl. (Investigations to solve the problem of rupture). — Diskussionsbericht Nr. 19 der E.M.P.A.

deformation, energy corresponds most nearly with the facts in the case of structural steel: for the duo-axial stress conditions frequently occurring, it takes the form

$$\frac{\sigma_{Fu}}{n} = \sigma_{perm} \geqq \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2} \qquad (1)$$

Accordingly, all dimensioning methods based on the stress scale include the assumption that in those places where the most unfavourable stress reaches the lower yield point stress σ_{Fu} the yield of material sets in and thereby the structure has reached the limit of carrying capacity. Uniform stress conditions are realised in ideal lattice structures but the above assumption leads to the conclusion that a lattice structure must be deemed no longer serviceable at the moment when a member has reached the lower yield point. However, a little deliberation[2] would seem to show that these conclusions only apply to statically determined lattice structures whilst for statically indeterminate, considerable increases of load are still possible before the structure collapses under the load. To a still greater degree is this assumption erroneous in the case of the plasticity of a cross section stiff against bending. However, non uniform conditions of stress exist whose influence upon the yield is yet to be examined. *Maier-Leibnitz*[3] has indisputably shown by experiment that when the yield point is reached at the most unfavourably stressed position, the stability of the framework is in no way endangered; it was found, on the contrary, that indications of yield must have penetrated very deeply into the cross section, to have set up an increased rate of deformation for any increase of loading.

Here the theory of plasticity comes in, the purpose of which is to estimate the actual carrying capacity of a structure under consideration of the yield phenomena. From the fact that the reaching the yield point at the elastic peak stress is not accompanied by any conditions endangering the stability of the structure, the theory of plasticity discards the stress scale and with it the expression of a permissible stress, and introduces as safety the ratio of carrying capacity to working load. The method of dimensioning based on the theory of plasticity will therefore be designated in many instances as the 'Carrying capacity Method' (Method of plastic equilibrium).

2) *Mechanism of Plastic Deformation.*

Properly to grasp the idea of the influence of the yielding process on the carrying capacity, it is necessary to take an effort to conceive an idea of its nature and its physical properties. Steel is a crystalline mass and a strict consideration would imply that the deformation of material should be deduced from that of the individual crystals. However, the irregular arrangement of the individual crystals would make it impossible to carry through this conception other than with statical methods. In respect of its mechanical properties the individual

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[2] *M. Grüning:* Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl bei beliebig häufig wiederholter Belastung. (The carrying capacity of satically indeterminate structures in steel under frequently repeated loading Berlin 1926, J. Springer.

[3] *Maier-Leibnitz:* Versuche mit eingespannten und einfachen Balken von I-Form aus Stahl 37. (Experiments with encastred and simply supported I-joists of Steel 37). Bautechn. 1929, Heft 20, S. 313.

crystal is distinctly anisotropic, whilst, considered mechanically, where in only portions of the material that already contain a very large number of individual crystals are tested, the crystalline mass must, in view of their irregular disposition, be regarded as quasi-isotropic.

A crystal with its specific direction in an assigned position changes elastically in the first place, as the crystal lattice is distorted by the working of external forces; when the distortion reaches a definite amount, the stability of the lattice is exhausted and slipping of layers of atoms along distinct crystalographically defined planes and directions ensues, which is to be regarded as a purely plastic process. Therefore with very close approximation the stress-strain diagram of a crystal can be likened to that of the ideally plastic body. A deviation takes place only on the occurrence of greater deformations at which consolidation to a new lattice stability begins. Since in the crystalline mass the specific direction of the individual crystals lie quite irregularly, they will slip under a fixed direction of the external load at different stress limits. The extent of the slips are of course very small, so that a very delicate measuring apparatus is required to reveal them; under these conditions, there is no steady deformation of crystalline mass, the deformation is in reality jerky.

The observations of *Kollbrunner*[4] provide a fine confirmation of this. The yield hypotheses of *Böker*[5] and *Brandtzaeg*[6] which unfortunately are far too little known in specialist circles, rest on the conception of plasticised islands, in a still elastic environment, which become greater and more frequent with increased loading.

In the case of carbon steels a secondary phenomenon occurs, caused by the structural arrangement of the metal itself. This consists chiefly of soft ferrite grains, which are bedded in a hard network of cementite or perlite, the latter being able to exert a powerful check on the slip of the ferrite grains. Under a certain external load, the perlitic network collapses and allows slip to occur in many of the ferrite grains, and the phenomenon appears which in the ordinary mechanical sense is understood as yield of the steel. That the yield point is not bound up with the strength properties of actually chemically pure iron is clearly shown by the observations of *Köster*[7], according to which the yield point and the extent of yield can be considerably modified by the alteration in grain size, whilst breaking strength and constriction remain unaffected. To this ordinary mechanical yield process corresponds a considerable change of texture; this is proved by the recrystallisation phenomena to be observed after the yield and by these summarised as 'ageing' of the steel. After conclusion of the yield, the soft ferrite grains form the sole carrier, at the same time this internal diversion of

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[4] *C. F. Kollbrunner:* Schichtenweises Fließen in Balken aus Baustahl. (Laminated yield of beams of building steel). III. Bd. d. Abhandlungen der I.V.B.H. Zürich 1935, S. 222.

[5] *R. Böker:* Die Mechanik der bleibenden Formänderungen in kristallinisch aufgebauten Körpern. (Mechanics of permanent deformations in bodies of crystalline texture). — Forschungsarbeiten auf dem Gebiete des Ingenieurwesens. Heft 175—176. Berlin 1915, V.D.I.-Verlag.

[6] *A. Brandtzaeg:* Failure of a material composed of non-isotropic elements. Trondhjem 1927.

[7] *W. Köster, H. v. Köckritz* und *E. H. Schulz:* Zur Kenntnis der Form der Spannungs-Dehnungs-Kurven auf Grund der Messung des zeitlichen Verlaufes der Alterung weichen Stahles. (Further contribution to the knowledge of the stress-strain diagram based on time measurements for the ageing of mild steel). Archiv für das Eisenhüttenwesen 6, 1932/33.

the external force gives rise to the great plastic deformation. It is to be assumed that such a rearrangement texture cannot be a process which in the strict sense constantly progresses with increased loading, but it is more probable that at the same time it will extend to greater areas of the material or of sections.

It is known that on the application of a mono-axial uniform stress in one direction to a test bar, an upper and lower yield point is observable in the case of mild steels (Fig. 1). As the upper limit is only a temporary condition, it is clear that in dealing with uniform stress conditions, the lower yield point must be considered. The occurrence of an upper yield point is regarded by *Moser*[8] as a delayed yield and by *Prager*[9] compared with delayed boiling, without, however, being able to give other than purely formal connections for the comparison. Its real origin is not yet satisfactorily explained; it does not seem impossible that the applied stress is favourable to a development of the lattice

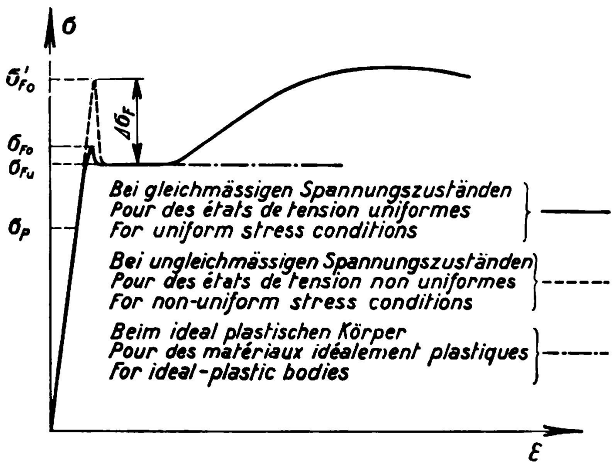

Fig. 1.

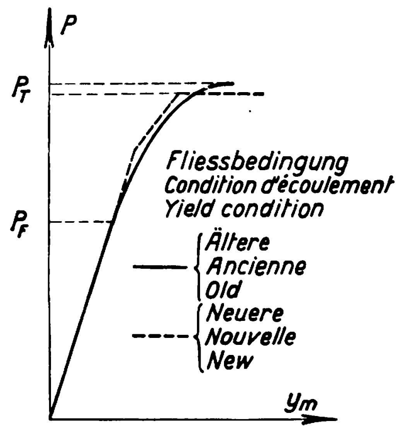

Fig. 2.

structure at the grain boundaries and an increase of the purely elastic resistance to deformation in the perlitic network. The fall of stress from the upper to the lower yield point is accompanied by the formation of the known yield lines which are a clear characteristic for every yield process; exact observation shows that they do not increase steadily but spread intermittently. All these phenomena occur in a fairly regular manner under uniform stress conditions, though it should not be overlooked that the quantitative side of the phenomena is subjected to a series of contingencies, which reveal themselves in an unavoidable diversion of the tests results.

Doubt has for some time been expressed as to whether all these phenomena do not substantially alter in accordance with a definite law in the case of the application of a non-uniform stress condition; and for this reason the desirability has been expressed that the determination of yield should be confined (1) to uniform stress conditions. Experiments in this connection admit of no final

[8] *M. Moser:* Verein deutscher Eisenhüttenleute, Werkstoff-Aussch. Ber. 96.

[9] *W. Prager:* Die Fließgrenze bei behinderter Formänderung. (The yield limit for restricted deformation). Forschung auf dem Gebiete des Ingenieurwesens 1933.

opinion; one group[10, 11, 12] justifies the assumption that under variable stress conditions the magnitude of the upper yield point is influenced, since σ_{Fo} is so much the higher, the steeper the ascent of the elastic stress peak and the smaller the zone over which it extends, whilst another[13] cannot admit these phenomena, or at least, only to a small degree. Of the lower yield point limit it can be said that such influences do not affect it[14]. It seems to be certain that the raising of the yield point in the various kinds of steel varies considerably; in the case of soft steel it is greater, in hard, smaller, and it therefore seems to be present only in such steels as have already shown a well defined upper yield point in a tension test. The cause of such a raising of the upper yield point has not yet been explained in a satisfactory manner; from the standpoint of atomic forces it is difficult to understand that understressed parts of a cross section support the overstressed and can stop the yield process there. *Thum*[11] and *Wunderlich* have assumed that the occurence of yield lines in small zones of any kind is not possible, inasmuch as the still elastic environment blocks the yield. The actual lower yield point must be exceeded up to a certain depth of the cross section before the stored-up energy of yield is great enough to break through the elastic grip in some manner.

With regard to these observations, two fundamentally different conceptions have been formed in the course of the development of the theory of plasticity, with regard to the conditions which determine the yield phenomena in a stressed area, according to which distinction is made between an 'old' and a 'new' condition of yield. It must at the start be acknowledged that both doubtless embody the idealising of the actual process; they correspond in certain aspects to extreme cases, and it is very probable that the actual phenomena lie between them. It is certain that without some kind of idealisation of these observations, one could not dispose of the great difficulties in comprehending their influence on the carrying capacity of structures.

The old yield condition assumes that the local condition of stress is the sole decisive factor in setting up the yield phenomena, it can therefore be based on the yield formula (1) for uniform stress conditions. For the case of bending — disregarding shear stress — the elastically stressed field is in one direction mono-axial and its expression takes the simple form $\sigma \leqq \sigma_F$. It is clear from the foregoing that the lower yield point σ_{Fu} has been introduced for σ_F because the strength must be judged after conclusion of the locally restricted yield or because one wishes to know how much of this place can still contribute after yield to the maintenance of equilibrium between inner and outer forces. A con-

[10] *F. Nakanishi:* On the yield point of mild steel. World Eng. Congress, Tokyo 1929, Proc. Vol. III.

[11] *A. Thum* und *F. Wunderlich:* Die Fließgrenze bei behinderter Formänderung. (The yield limit for restricted deformation). Forschung auf dem Gebiete des Ingenieurwesens 1932.

[12] *H. Möller* und *J. Barbers:* Über die röntgenographische Messung elastischer Spannungen. (X-ray investigations into elastic stressing). Mitt. d. Kaiser-Wilh.-Inst. f. Eisenforschung, Düsseldorf 1934.

[13] *F. Rinagl:* Die Veröffentlichung ist noch nicht erschienen.

[14] *E. Siebel* und *H. F. Vieregge:* Über die Abhängigkeit des Fließbeginns von Spannungsverteilung und Werkstoff. (The dependence of yield on stress distribution and material). Mitt. Kaiser-Wilhelm-Inst. f. Eisenforschung, Düsseldorf. Abhandlung 270, 1934.

sequence of this conception, is the steadily and gradually increasing area of yield due to increasing loads and a plastic reduction of the peak stress[15]; finally, a completely plastic condition of the cross section is attained provided it extends rigth through. The stress distribution then consists of a tensile stress rectangle and a compression stress rectangle with a height of the lower yield point σ_{Fu}. The moment of the inner stress has increased to its greatest value M_r which cannot be exceeded; for further deformation this cross section works as a so-called plastic joint.

The 'new yield condition' maintains that all these conceptions do not correspond to the facts; that, on the contrary, the yield region spreads spasmodically in depth, and it can be assumed that to some extent at the first setting up of yield indications in the cross section, the resistance to deformation at this place is already so weakened that it can really no longer take up an increase in loading. It proceeds from the observation of the raising of the elastic peak, that by 'yield point', the upper yield point must be understood, since this alone is influenced by the non-uniformity of stress distribution. On this account, the stress distribution over the whole cross section must be introduced in to the yield formula. Therefore, in the case of bending, free from longitudinal force, the increase of the yield points $\Delta\sigma_F = \sigma_{Fo} - \sigma_{Fu}$ will be essentially a function of the cross sectional shape. During the yield this increase collapses and the upper yield point goes back to the lower, without, however, the stress distribution of the complete plastic condition being necessarily attained in the sense of the older yield formula.

The difference between the two yield formulae becomes most pronounced when any convenient quantity of deformation is considered in its relation to the load. In the case of the deflection at the centre of a simply supported beam, the old yield formula furnishes, at the junction with the straight (in the case of purely elastic deformation) a steady curve, whose tangent at the moment of concluding a possible condition of equilibrium between the inner and outer forces must be horizontal. According to the new flow formula, the straight line of the purely elastic deformation continues until the carrying capacity is reached, then abruptly changes in to the horizontal direction which is maintained until hardening occurs. That such line y(P) has been actually observed, is evident from the experiments of *E. Siebel*[14] and *H. F. Vieregge,* thought it must be acknowledged that lines of the first type[3] are frequently found in literature.

According to the old yield formula, in the case of a statically indeterminate continuous girder, a steady curve is again obtained in the elastic-plastic region, which curve at its end points must have a horizontal tangent; whereas according to the new formula, the line y(P) is represented by a polygon which, never deviates far from a steady curve (Fig. 3). The break points of the polygon correspond to the instants at which a cross section is eliminated from resist once to bending, through its suddenly becoming plastic, and there must always be as many of such break points present as there are stiff corners which could

[15] *J. Fritsche:* Die Tragfähigkeit von Balken aus Stahl mit Berücksichtigung des plastischen Verformungsvermögens. (The carrying capacity of steel beams under consideration of plastic deformability). Der Bauingenieur 1930. Heft 49, 50 u. 51.

be replaced by plastic joints, in order to establish a labile arrangement. A restriction of this conception may be mentioned here: each statically less indeterminate intermediate system and, of course, the statically determined fundamental system must be stable in all their parts. The continuous girder with very long end spans is therefore excluded, because the degree of stability of the statically determined fundamental system which originates through the plastic state of the centre section of the middle span becomes progressively smaller and in the case of the infinitely long side spans, vanishes altogether.

In view of former experiments it is not yet possible to answer indisputably the question concerning the correctness of one or the other yield formulae,

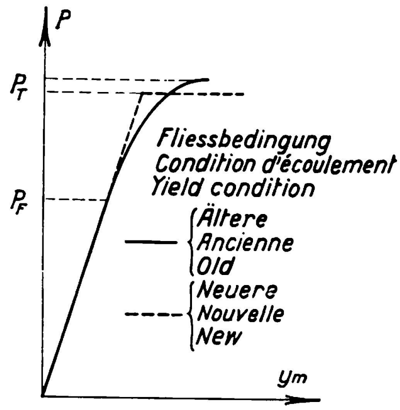

Fig. 3.

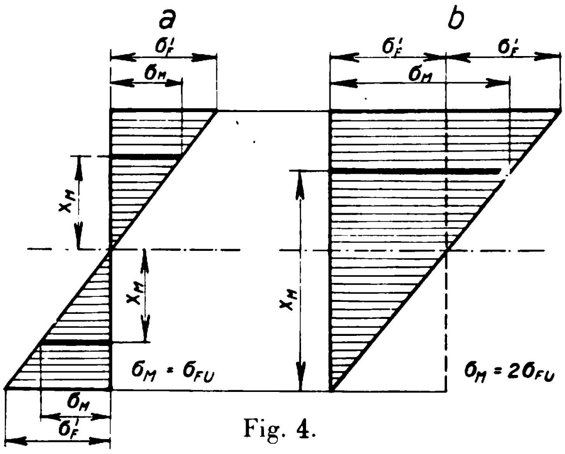

Fig. 4.

a original and b revised interpretation of the yield condition according to Kuntze.

and in discussing the most important experiments, reference will be made to the difficulties of their significance in one or the other direction. The new yield formula has of itself the great advantage of offering a simple basis for the theory of plasticity. If one adopts the standpoint of considering the raising of the yield point as not sufficiently assured by experience, the possibility remains open of conceiving the new formula of yield as a much needed approximation of the old, as it admits of a simple treatment of many problems which no longer appear soluble by the old.

3) *The Mathematical Conception of the Different Formulae of Yield.*

In the case of the usually examined mono-axial fields of stresses, the old formula of yield is based on the aforesaid condition $\sigma \leqq \sigma_F$ wherein σ_F represents the lower yield point. It has been applied almost exclusively as the basis of former investigations, but it has the great disadvantage that under the assumption, always permissible, that the material, as an ideally plastic body, entails an extraordinarily complicated and circuitous calculation[16, 17] and for

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[16] *J. Fritsche:* Arbeitsgesetze bei elastisch-plastischer Balkenbiegung. (Laws of work for elastic-plastic bending). Zeitschrift f. ang. Math. u. Mech. 1931.

[17] *K. Ježek:* Die Tragfähigkeit des exzentrisch beanspruchten und des querbelasteten Druckstabes aus einem ideal plastischen Stahle. (The carrying capacity of transversely and eccentrically loaded columns of ideal-plastic steel). Sitzungsberichte d. Wiener Akad. d. Wissensch., Math.-Naturw. Klasse, Abt. II a, 143 Bd. 1934.

this reason cannot be applied to important practical problems. Apart from the calculation of the carrying load itself, which in consequence of the equalization of moments can be very quickly and easily determined, the determination of a deformation value or of the internal play of forces in the case of partial plasticity of the cross section concerned — bearing in mind their actual form — cannot be carried out.

Such problems are not only of theoretical interest: in the case of the determination of the carrying capacity of an eccentrically loaded steel column, the calculation of the deflection at the centre of the column is unavoidable, as the final carrying capacity is not necessarily attained for the most unfavourably stressed cross section becoming completely plastic, but an instability between the inner and outer forces must already set in in the partially plastic state.

Recourse has been had to a graphical integration of the differential equations concerned; nevertheless, the calculations involved still remain considerable. In considering the solution of this problem established by *Chwalla*[18], *Ježek*[17], *Eggenschwyler*[19], and others, on the basis of the old yield formula, the question arises whether the degree of accuracy reached by the calculations justifies such a laborious treatment, especially having regard to its uncertain basis; 'accuracy' implying the concordance between calculation and experience. Added to the uncertainties of the yield formula are: — the unavoidable variations in the level of the lower yield point, which directly can deviate 10%; the considerable deviation from the accepted law of permanent flatness of cross section, which increases pari passu with plasticity; the disregarded influence of the shear stress and so forth. There exists, therefore, a pressing need for simplification of calculations in investigations of the theory of plasticity.

The first mathematical conception of the new flow formula originates with *Kuntze*[20]. According to this, the ultimate carrying capacity of a cross section is attained when the average resistance value σ_M introduced by Kuntze is equivalent to the lower yield point stress σ_{Fu} whilst the yield point σ_{Fu} is exceeded at the elastic stress peaks of the marginal portion. The mean resistance value σ_M is ascertained by dividing the stressed body into two halves by a section parallel to the edge.

'Stressed body' being understood to mean a prism with the cross section as face area, which is so bounded by an inclined section that the overall height of the body corresponds with the stress. The equal volume, or the internal equilibrium, between the over and the under stressed parts of the stressed body cannot, of course be looked upon as a physical basis for the occurrence of yield, even if one remembers that the accumulated yield energy could be proportional to the volume of the over-stressed parts of the stressed body, and that the yielding can be blocked by the residually elastic parts of the cross section

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18 *E. Chwalla:* Theorie des außermittig gedrückten Stabes aus Baustahl. (The theory of eccentrically loaded steel-columns). Stahlbau 1934, Heft 21, 22 u. 23, S. 161.

19 *A. Eggenschwyler:* Die Knickfestigkeit von Stäben aus Baustahl. (The buckling strength of steel columns). Schaffhausen 1935, Selbstverlag.

20 *W. Kuntze:* Ermittlung des Einflusses ungleichförmiger Spannungen und Querschnitte auf die Streckgrenze. (The influence of unequal stresses and the shape of sections on the yield limit). Stahlbau 1933, Heft 7, S. 49.

so long as its yield stemming influence prevails. They must therefore be valued as merely working hypotheses so long as a physical basis has not been established. Since the experiment will have satisfactorily demonstrated it, one may assume its validity as at least a good approximation, to this yield formula as yet unknown but theoretically indisputably founded.

The validity of *Kuntze*'s yield formula is entirely dependent on experimental proof and must be amended if it fails in this aspect. Such cases occur, as will be later shown in detail, when the formula is desired to be applied for describing the carrying capacity of an eccentrically loaded steel column. In order to remain in harmony with these experiments it must be amended in the following way: — One must start with a body of stresses which comprises the non-uniformity of the field of stresses of the whole cross section, using the original reasoning of *Kuntze*, this is obtained by a previous reduction of the stress area to a zero edge stress (Fig. 4). The mean resistance value σ_M now divides the reduced body of stresses into two parts of equal volume. All that is now necessary in order to maintain harmony with the experiments is to put $\sigma_M = 2\,\sigma_F$, in the case of cross sections with two symmetrical axes. For joists, the increased yield point σ'_F[21] in the elastic peak stresses, accordingly works out as

$$\sigma'_F = \sqrt{\frac{2}{1 + \alpha\,(1 + \beta)}}\;\sigma_F \qquad (2)$$

If $\alpha = \frac{h}{h}$ and $\beta = \frac{a}{b}$ (Fig. 5) tis relation holds so long as the mean value of resistance falls in the flange of the cross section, provided $x_M \geqq h + t$. It follows, therefore, that equation (2) is valid only if $1 - \alpha\,(\alpha + 2\,\beta) \geqq 0$. If this condition is no longer fulfilled then

$$\sigma'_F = \sqrt{\frac{4\,\beta}{(1 + \beta) - \alpha^2\,(1 - \beta)}}\;\sigma_F \qquad (3)$$

The value σ'_F, the raised upper yield point, has therefore the significance of a 'bending yield point', since in the case of bending, yield sets in when the greatest edge stress has reached this amount. The bending yield point is therefore not constant but is dependent upon the shape of the cross section; it is

$$\sigma'_F = \psi\,\sigma_F \quad \text{or} \quad \psi = \frac{\sigma'_F}{\sigma_F} = \frac{\sigma_{Fo}}{\sigma_{Fu}} \qquad (4)$$

A conclusion drawn from this yield formula, but not yet proved by experiment, is that in the case of purely elastic deformation, parts of the cross section remaining without stress in proximity to the zero line, must to a certain degree react favourably to the carrying capacity of the beam, so that, e. g. the cruciform profile must have greater carrying capacity than the rectangular cross section remaining after the cutting away of the horizontal flanges, since the

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21 *J. Fritsche:* Der Einfluß der Querschnittsform auf die Tragfähigkeit außermittig gedrückter Stahlstützen. (The influence of the form of cross section on the carrying capacity of eccentrically loaded steel columns). Stahlbau 1936.

unstressed parts stem the flow process in the extreme fibres. For the cruciform profile (Fig. 6)

$$\sigma'_F = \sqrt{\frac{2}{1 - \frac{\alpha\beta}{1-\alpha}}} \cdot \sigma_F \qquad (5)$$

for the rectangle alone with $\alpha = 0$, $\beta = 1$

$$\sigma'_F = 1{,}414\, \sigma_F \qquad (6)$$

For a cross section comprising two broad flange T-sections 10×5, in which $\alpha = 0.830$, $\beta = 0.085$, the ratio of bending yield point to the lower yield point under tension tests is $\psi = 1.85$, whilst the value yielded in the case of rectangular cross section is $\psi = 1.41$. This cruciform profile should therefore carry about 30 % more than the corresponding rectangular cross section. Something similar applies for the joist, when it is bent in a plane at right angles to the web, only here it should be noticed that a web of any depth does

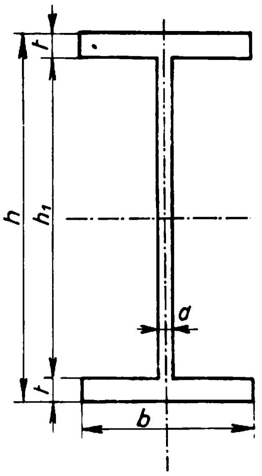


Fig. 5.

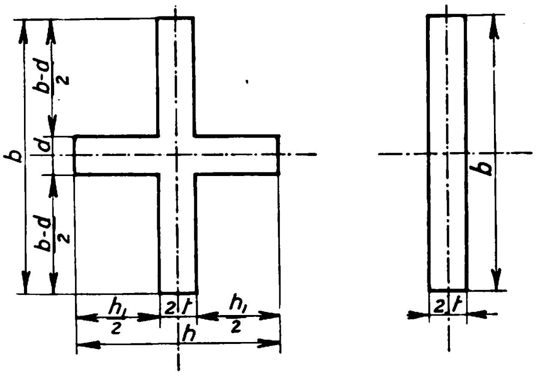


Fig. 6.

not allow an increase in the bending yield point to an indefinite extent. On the contrary, it must be assumed that only the parts of the web in the region of the flanges can stem the yield process in the extreme fibres and it is therefore advisable to include in the yield formula (Fig. 7) a portion — with, for example, the breadth of $\frac{b}{2}$ — of the web on each side.

This theoretical result seems surprising, and it is to be hoped that it will soon be possible to test it by experiment. Perhaps such an experiment will decide the question as to the correctness of one or the other yield formula. Meanwhile, the experiments carried out with such cross sections with regard to the carrying capacity of eccentrically loaded steel columns, alone confirm the necessity for this assumption.

The second conception of the yield formula, taking into consideration the raising of the yield point at the peak stresses, emanates from *Prager*[9]. He takes the view concerning the yield phenomenon that the increased elastic field with the limiting stress σ'_F becomes transformed in to the diagram of the complete plastic condition, with the limiting stress σ_F, and that this process goes on without diminishing the bending resistance of the plasticised portion (Fig. 8). If W represents the section modulus of the cross section, and T the statical moment of both halves of the cross section in respect to the neutral axis line, then of necessity $\sigma'_F\ W = \sigma_F\ T$ and consequently

$$\sigma'_F = \frac{T}{W} \cdot \sigma_F \quad \text{or} \quad \psi = \frac{\sigma_{Fo}}{\sigma_{Fu}} = \frac{T}{W} \tag{7}$$

Although the *Prager* conception at first seems highly probable, critical consideration shows it also to be an idealising of the processes of yield; it is as

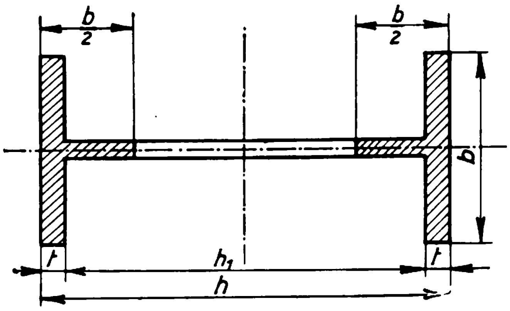


Fig. 7.

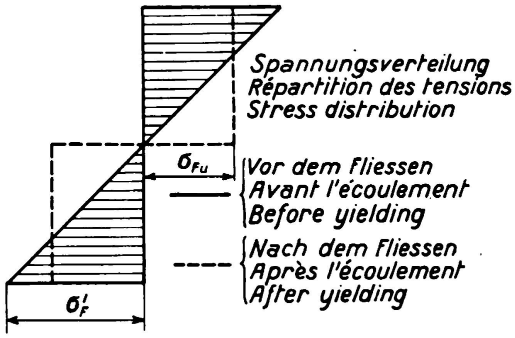


Fig. 8.

little as that of *Kuntze* to be regarded as founded on Physics and in fundamental agreement with existing phenomena. In its appropriate extension to the calculation of the carrying capacity of eccentrically loaded steel columns it unmistakably furnishes values that are too high as compared with practice, so that although a final opinion cannot yet be given, preference must nevertheless be given to *Kuntze*'s conception of the yield formula.

4) *Experimental Tests of the Yield Formula.*

The bending tests *Thum*[11] and *Wunderlich* form a basis for the new yield formula. The eight tests with polished test bars of different forms of rolled I-sections (Fig. 9) furnishtaking into consideration a lower yield point of $\sigma_F = 2.47$ t/cm² in the tension test — the values of σ'_F calculated according to the yield formula of *Kuntze* or *Prager*, which are set out in the adjoining Table 1, together with the measured values. Disregarding tests 3 and 4, in which the greater deviations certainly admit of other explanations, the agreement with the *Kuntze* values is decidedly very satisfactory, whilst according to *Prager*, the calculated values almost throughout lie above those measured; from this the conclusion may be drawn that the stress distribution in the thoroughly plastic condition does not necessarily correspond unconditionally with the assumption of the older theory.

The experiments of *Thum* and *Wunderlich* form the most important supports of the new yield formula, and although they also seem to supply indisputable proof of an increased upper yield point, yet the supposition is not easily discarded that, in view of the observations made in the determination

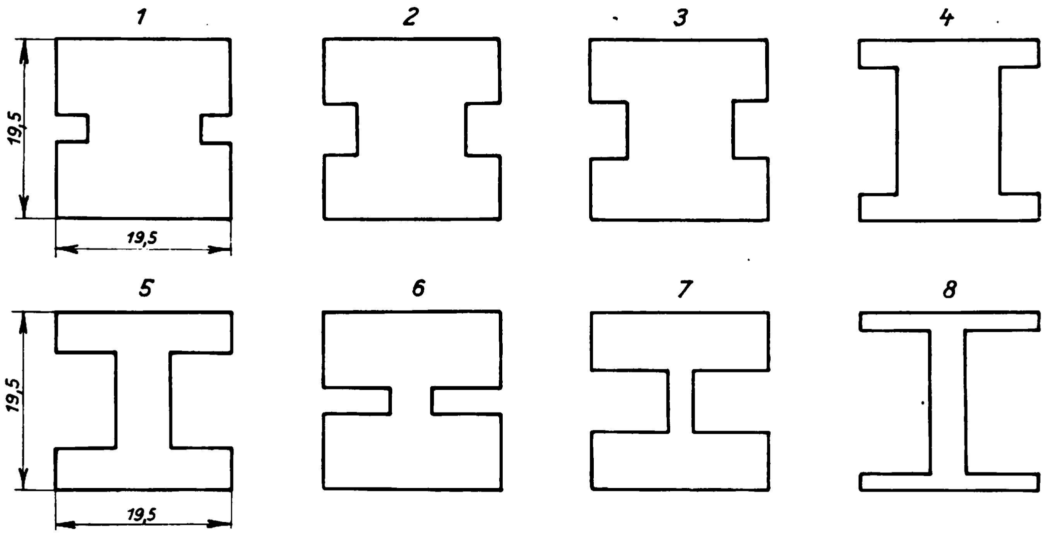


Fig. 9.
Cross sections of the 8 test beams as used by Thum and Wunderlich.

of fatigue stress, the conditioning of the surface by polishing may have had an influence on the yield point — which alone can be measured — in the surface area.

What the raising of the upper yield point causes in the different types of steel is a question only to be decided by experiments which are not yet available

Table 1. Tests of Thum and Wunderlich. $\sigma_{Fu} = 2.47$ t/cm².

No. of test	σ_{Fo} measured	σ_{Fo} calculated after Kuntze	Deviation in % of the measured values	σ_{Fo} calculated after Prager	Deviation in % of the measured values
1	3.50	3.41	+ 2.56	3.68	— 5.14
2	3,64	3.31	+ 9.07	3.61	+ 0.82
3	3.78	3.28	+ 13.20	3.60	+ 5.14
4	3.42	3.38	+ 1.17	3.45	— 0.80
5	2.91	2.96	— 1.72	3.32	— 14.10
6	3.44	3.31	+ 4.03	3.66	— 6.40
7	3.15	3.06	+ 2.96	3.43	— 8.88
8	2.61	2.72	— 4.20	3.05	— 16.84

in sufficient number to enable a final opinion to be formed. As is shown in the adjoining Table 2, the experiments of *Siebel* and *Vieregge*[14] with square beams show that this effect is decidedly evident only in softer types of steel, whilst it is not revealed in the case of high-grade alloyed steels. For solving the problem which of the yield formulae is the correct one, these experiments must be excluded, because the yield point has been determined by calculation, with the assumption of a certain distribution of stress, and not by direct observation at the point of flow; and the question of whether the carrying capacity is reached by gradual plasticising or by increase of the yield point in the stress peaks, remains open.

In the case of statically indeterminate supported beams, the plastic phenomena have a substantially greater influence on the play of internal forces and the laws of deformation than in the case of simply supported beams, even in view of the equalization of moments which the plastic theory requires independently of the nature of the yield formula. The assumption therefore suggests itself, that an accurate gauging of the phenomena would establish a conclusion

Table 2. Tests of Siebel and Vieregge with square beams.

No. of test	Lower yield point σ_{Fu}	ultimate strength	$\psi = \frac{\sigma_{Fo}}{\sigma_{Fu}}$ measured	ψ (theoretical) Kuntze	Prager
1	1.89	3.09	1.66	1.41	1.50
2	2.52	4.88	1.34	1.41	1.50
3	3.77	7.50	1.07		
4	5.46	7.10	1.05		

on the correctness of one or the other yield formula. Such tests (Fig. 10a and b) have recently been recorded with extreme thoroughness by *Stüssi* and *Kollbrunner*[22] in Zurich. I have already evaluated these tests from this point of view in a work appearing in Stahlbau[23]; though in that case, the yield formula of *Kuntze* was used in its original form. In accordance with the alteration rendered necessary by the compression tests on steel stanchions, equation 2 furnishes $\sigma'_F = 1.09\ \sigma_F$ or $\Delta\sigma_F = \sigma_{Fo} - \sigma_{Fu} = 303$ kg/cm² consequently $M_T = W\sigma'_F = 26.70$ tcm and $P_T = \frac{8 \cdot 26{,}70}{60} = 3{,}56$ tons.

At the same time the stress at the centre of the beam of the middle bay corresponds to the increased yield point, σ'_F the load P'_F is with

$$\alpha = \frac{3\,l_2}{4l_1 + 6l_2} - \frac{3}{14}$$

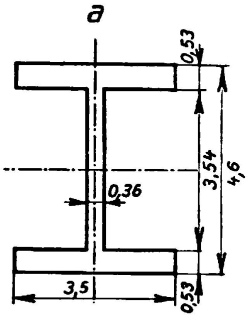
$$P'_F = \frac{4\,M_T}{(1-\alpha)l_2} = 2{,}27\ \text{t}$$

The corresponding value of the moment over the supports is $X'_F = -7.28$ t/cm. Exclusive of the central section of the beam, a statically determinate fundamental system remains, consisting of two simple beams with cantilover arms of the length $\frac{b_2}{2}$ held together in the plastic joint. P can now be increased until the yielding moment M_T is also reached over the supports in the fundamental system, which is set up if $\frac{1}{2}(P_T - P'_F)\,\frac{b_2}{2} = M_T - X'_F$: from which also naturally follows the value $P_T = 3.56$ t, already obtained directly.

[22] *F. Stüssi* und *Kollbrunner:* Beitrag zum Traglastverfahren. (Contribution to the theory of plastic equilibrium). Bautechnik 1935, Heft 21, S. 264.

[23] *J. Fritsche:* Grundsätzliches zur Plastizitätstheorie. (Fundamental remarks to the theory of plasticity). Stahlbau 1936, Heft 9, S. 65.

In the doubly statically indeterminate system, the deflection at the middle of the beam in the case of purely elastic deformation is $f_2 = 0.872$ P, whereas



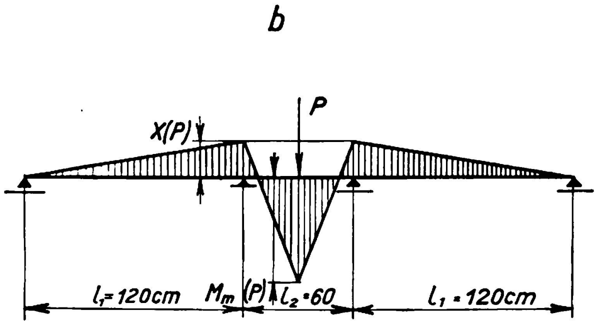


Fig. 10a. Fig. 10b.

Cross section und loading arrangement of the test beam used by Stüssi and Kollbrunner.

after the aforesaid exclusion of the elastic central section of the beam, it works out as $f^2 = 0.198 + 0.642$ (P — 2.27). *Stüssi* and *Kollbrunner* have measured

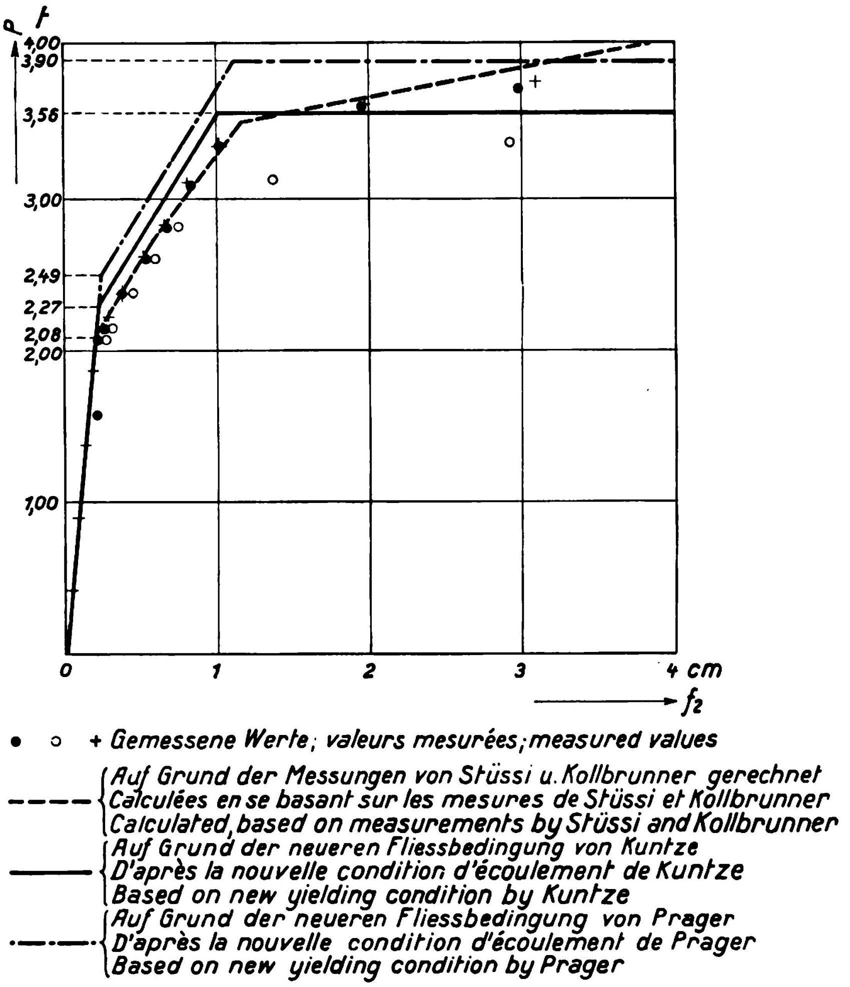


Fig. 11.

the deflection f_2(P) and determined the course of the internal resistances Mm(P) and X(P) from the deformation of the axis of the beam by a well-considered method of reasoning. Figs. 11 and 12 show the results of their measurement in comparison with those calculated by means of various yield formulae, and from this it becomes evident that the new *Kuntze* yield formula also best corresponds to the conditions. The sudden kink in the measured lines $f_2(P)$, Mm(P), and X(P), on reaching P_F or $M_F = W\sigma_F$, indicates that, although in this case no increase of the yield point appears to have set in, yet the investigations in this direction admit of no certain significance, as the effect only occurs

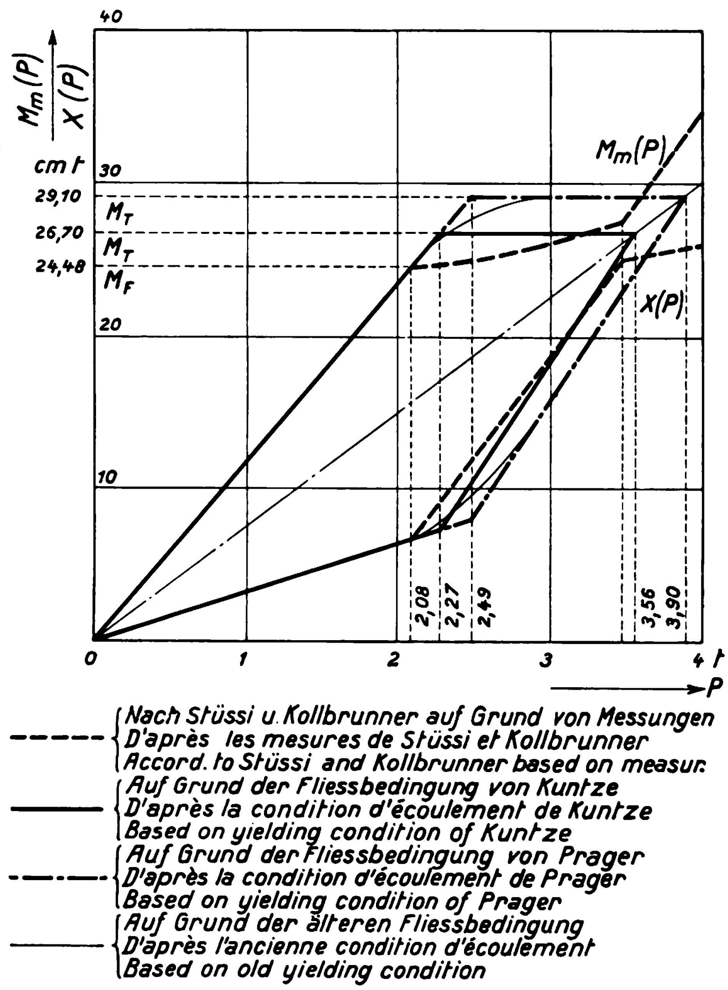


Fig. 12.

within about 10% from the σ_F limit and might be marked by variations in the yield point.

The investigations of *Maier-Leibnitz* previously mentioned show in general a gradual transition from the purely elastic to the elastic-plastic condition. I have already shown[23] that this, too, so far as it can be tested, does not contradict the *Kuntze* yield formula, the deflection polygon y(P) coincides satisfactorily with the measured lines.

As a result of this consideration of the most important experiments, all that can be said at present is: a raising of the yield point is not impossible, it requires still further experimental confirmation; so long as this is not available, the new yield formula can only be regarded as a close approximation of the old, which should be reckoned with, because it provides an extraordinary and necessary simplification of the amount of calculation.

5) *The Eccentrically Loaded Steel Column.*

If the plastic theory founded on *Kuntze*'s yield formula is practicable, it must also be capable of representing the carrying capacity of eccentrically loaded steel column in conformity with experience. This task has been dealt with lately[17, 18, 19] with extraordinary thoroughness on the basis of the 'older' yield formula, without, however — in view of the certainly very considerable influence of the shape of the cross section — leading to any satisfactory results[24]. Under the assumption of constantly spreading yield zone, we have here a pro-

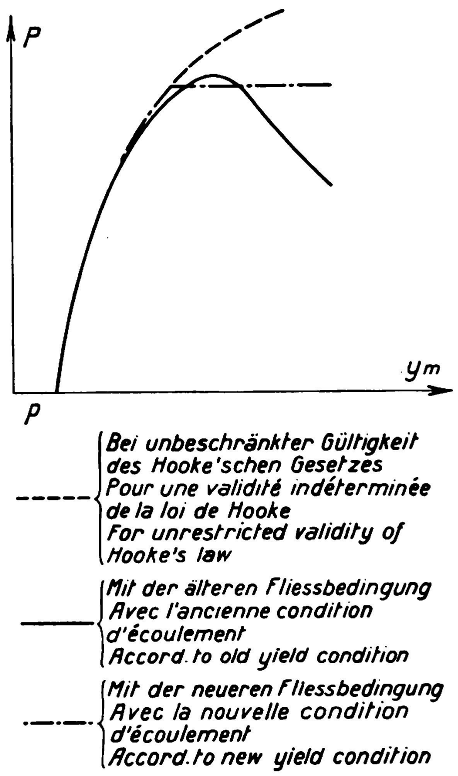


Fig. 13.
Transverse deflection ot the middle of an excentrically compressed bar.

blem of critical loads, inasmuch as the progressive plasticising of the centre of the bar more and more disturbs the equilibrium between the inner and outer forces, until at a definite depth of the yield zone, long before the complete plastic state is attained in the most dangerously stressed cross section, no stable equilibrium can any longer exist. Strictly speaking, the calculation of this critical load has been strictly possible only for the rectangular cross section, as the differential equations to be solved in the case of the ideally plastic body are very complicated and in the general case admit of only a toilsome graphical integration. The expression, 'strictly', relates solely to a purely mathematical

[24] *E. Chwalla:* Der Einfluß der Querschnittsform auf das Tragvermögen außermittig gedrückter Baustahlstäbe. (The influence of cross sectional shape on the carrying capacity of eccentrically loaded building steels). Stahlbau 1935, Heft 25 u. 26.

treatment; from the theory of strength standpoint, which strives to provide a correct description of experience, they remain, after all, approximate solutions since they rest upon a series of more or less completely realised assumptions.

With the help of the new yield formula, the solution of this problem becomes extremely simple[21], whilst the actual shape of the cross section can be taken into account without special difficulty. According to this conception the operation proceeds in such a manner that its purely elastic character is retained up to the increased yield point σ_F. It is not until then that the yield zone suddenly extends to a very considerable depth in the cross section, and if the bar is supported in a statically determinate manner, the limit of the carrying capacity is reached at once. This way of putting it is undoubtedly an idealising of the actual phenomena, and the experiments show that small increases of load are still possible, though already significant yield traces can be observed; the sudden kink in the line y_m (P) (Fig. 13) vanishes when the yield proceeds in stages, but no substantial influence can be attributed to this phenomenon.

In the case of bending under longitudinal compression, the raised yield point σ'_F must depend substantially on the longitudinal stress σ_0, as well as on the shape of the cross section. In order to comprehend this dependence it is necessary, firstly, to consider the two limiting cases of yield point $\sigma_0 = 0$ and $\sigma_0 = \sigma_F$. $\sigma_0 = 0$ corresponds to bending in the absence of longitudinal force for which the value $\Delta\sigma_F(o)$ already appears to have been fixed. $\sigma_0 = \sigma_F$ is purely longitudinal stress and if the case of buckling be left out of consideration, the limit of the carrying capacity is reached. As was shown above all by the investigations of *W. Rein*[25], the intervention of a moment is no longer necessary for the production of a constant increase in deformation under unchanging load. This agrees with the foregoing conception, since now the non-uniformity of the stress conditions vanishes and therefore $\Delta\sigma_F(\sigma_F)$ must be equal to 0. A linear value for $\Delta\sigma_F$ for intermediate values of σ_0 is now indicated, as from experience the simplest terms often give the most suitable results. As I have already explained[21]; this expression corresponds to the equation

$$\sigma'_F - \sigma_0 = \psi(\sigma_F - \sigma_0) = \frac{1}{\nu}(\sigma_F - \sigma_0) \qquad (8)$$

wherein ψ is a factor dependent solely upon the shape of the cross section and represents the ratio of the bending yield point σ'_F to the lower tension yield point (Fig. 14). I also showed on that occasion that the suitable application of the *Prager*[9] yield formula must lead to a quadratic function $\Delta\sigma_F(\sigma_0)$ which in the case of $\sigma_0 = \frac{\sigma_F}{3}$ reaches its maximum value $\Delta\sigma_F = \frac{2}{3}\sigma_F$. The *Prager* yield formula therefore leads to the conclusion, difficult fo conceive, that the capacity for absorbing bending moments should increase within definite limits with increasing σ_0; this affords a further explanation of the fact that this yield

[25] *W. Rein:* Berichte des Ausschusses für Versuche im Stahlbau, Ausgabe B, Heft 4; Versuche zur Ermittlung der Knickspannung für verschiedene Baustähle. (Test reports of the Commission for Steel Structures Edition B, N° 4. Tests for determination of the buckling stress of various kinds of steel).

formula furnishes critical loads which are too high, as compared with those of the experiments[26].

By means of equation (8), σ'_F is now determined; the failure of the stanchion is now coupled with the condition $\sigma_1 = \sigma'_F$. According to Fig. 15, this equation

$$\sigma_i = \sigma'_F = \sigma_o + \frac{P\,y_m}{W} = \sigma_o\left(1 + \frac{p}{k_i}\sec\frac{\varkappa\,l}{2}\right) \text{ is obtained,}$$

wherein $\varkappa^2 \frac{P}{E\,J}$ and $k_i = \frac{W_i}{F}$ which represents the cross sectional core-width corresponding to the inner fibre.

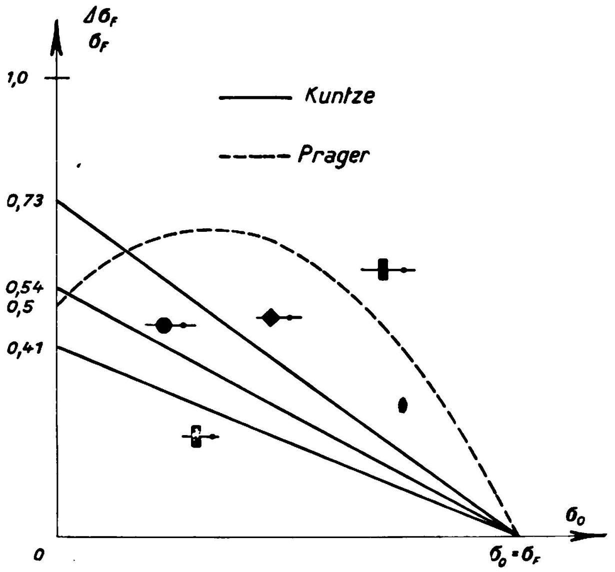


Fig. 14.

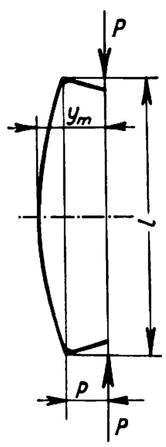


Fig. 15.

If the eccentricity m is substituted for $\frac{p}{k_i}$ then

$$\sigma'_F = \sigma_o\left(1 + m\sec\frac{\varkappa\,l}{2}\right) \quad\text{or}\quad \sigma'_F - \sigma_o = \sigma_o\, m \sec\frac{\varkappa\,l}{2}$$

and with the yield formula (8)

$$\sigma_o\, m \sec\frac{\varkappa\,l}{2} = \frac{1}{\nu}\left(\sigma_F - \sigma_o\right)$$

from which $\sigma_{o\,krit}$ can now be calculated. For $\sec\frac{kl}{2}$ the approximation given by *Timoshenko*[27]

$$\sec\frac{\varkappa\,l}{2} = \frac{\sigma_E + 0{,}234\,\sigma_o}{\sigma_E - \sigma_o} \tag{9}$$

may be used with advantage, wherein σ_E represents the Euler stress $\frac{\pi^2 E}{\lambda^2}$.

[26] *J. Fritsche:* Näherungsverfahren zur Berechnung der Tragfähigkeit außermittig gedrückter Stäbe aus Baustahl. (Approximate method of calculating the carrying capacity of eccentrically loaded columns of building steel). Stahlbau 1935, Heft 18, S. 137.

[27] *Timoshenko:* Strength of Materials, Vol. II, 1931.

If the measurement of eccentricity be set down as m'=vm then the following quadratic equation:

$$\sigma^2_{o\ krit}\,(1 - 0{,}234\ m') - \sigma_{o\ krit}\,[\sigma_F + \sigma_E\,(1 + m')] + \sigma_F\,\sigma_E = 0 \qquad (10)$$

is obtained for $\sigma^2_{o\ krit}$.

If it is wished to represent $\sigma_{o\ krit}$ directly as a function of the ratio of slenderness $\lambda = \frac{l}{i}$ there results

$$\sigma^2_{o\,krit}\,\lambda^2\,(1 - 0{,}234\ m\ ) - \sigma_{o\,krit}\,[\lambda^2\,\sigma_F + \pi^2\,E\,(1 + m')] + \pi^2\,E\,\sigma_F = 0 \quad (11)$$

The solution of this quadratic equation is sometimes attended with difficulties; $\sigma_{o\ krit}$ being obtained in the form of a difference, and if both the values approach equal magnitude, greatly increased accuracy of the calculations becomes necessary and the use of the slide rule is no longer possible. In such cases the expression can be represented by the square root in the form $\sqrt{1 - x}$, wherein x is a very small quantity; if the square root is developed in a binomial series finishing at the second term, a sufficiently accurate approximation is often obtained with

$$\bar{\sigma}_{o\,krit} = \frac{\sigma_F\,\sigma_E}{\sigma_F + \sigma_E\,(1 + m')} \qquad (12)$$

whilst a better value can be obtained by including the third term of the series, in the form

$$\bar{\bar{\sigma}}_{o\,krit} = \bar{\sigma}_{o\,krit}\left[1 + \sigma_{o\,krit}\,\frac{1 - 0{,}234\ m'}{\sigma_F + \sigma_E\,(1 + m')}\right] \qquad (13)$$

6) *Reviewing the Experiments.*

The conditions established will now be compared with the abundantly available experimental results, in order to prove the correctness and the utility of the calculations built up on the new yield formula. Of primary importance in this connection are the fundamental experiments by *Roš*[28] with Steel Joists 22 and 32. As I have already mentioned[21], the examination of these results reveals a highly satisfactory agreement between calculation and experiment. Fig. 16, which is taken from my publication in 'Stahlbau', shows distinctly how the calculated lines $\sigma_{o\ krit}\,(\lambda)$ represent the mean of the experiments. It is further evident that the slenderness ratio $\lambda < 25$ must be excluded, since in such cases rigidity can already play a part and mask the actual yield phenomena. The experiments of *Roš* justify the modification of the *Kuntze* yield formula mentioned in Section 3. If the calculation does not take into account the possibility that the understressed web can at least partly restrict the flow in the edge fibres, it cannot be numerically represented.

So far as I know of them from a publication by *G. Grüning*[29], the experi-

[28] *M. Roš*: Die Bemessung zentrisch und exzentrisch gedrückter Stäbe auf Knickung. (Dimensioning of centrically and eccentrically loaded bars to buckling). Bericht der II. int. Tagung f. Brückenbau u. Hochbau, Wien 1928, S. 282.

[29] *G. Grüning:* Knickversuche mit außermittig gedrückten Stahlstützen, Mitteilungen aus dem staatlichen Materialprüfungsamte in Berlin-Dahlem. (Buckling experiments with eccentrically loaded steel columns). Stahlbau 1936, Heft 3, S. 17.

ments of the 'Deutscher Stahlbauverband' furnish — as I have already been able to show[21] — an emphatic confirmation of this method of calculation, though it must be deemed highly desirable that further such experiments should be carried out with complicated shapes of cross sections to enable the theory to be tested in all its applications.

The experiments carried out by *A. Ostenfeld*[30] in the laboratories for Building Construction at the Copenhagen Institute of Technology, in the years 1928—9, and tho which my attention was drawn by the courtesy of *Dr. Čižek* of Prague, are of great interest *Ostenfeld,* like *Melan* of Prague, long before the plastic theory had made such considerations understood, took up the position that stress

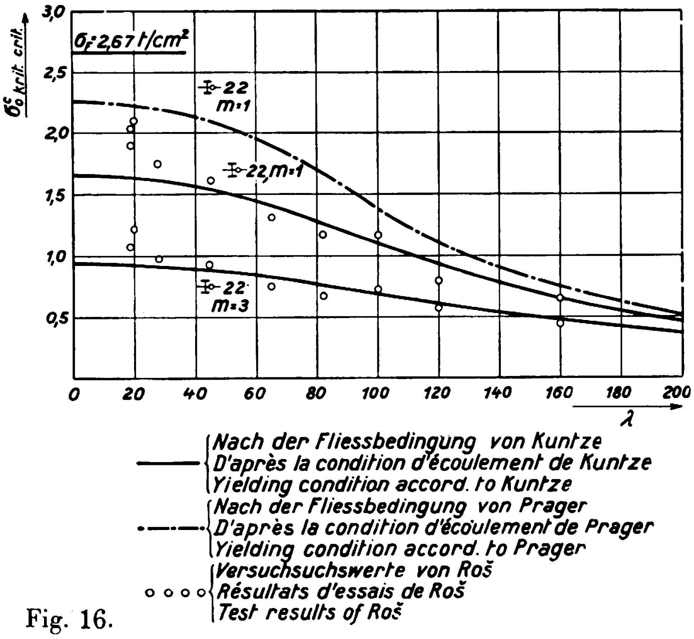


Fig. 16.

FromJ.Fritsche:„DerEinflußderQuerschnittsform auf dieTragfähigkeit außermittig gedrückter Stahlstützen“, Stahlbau 1936.

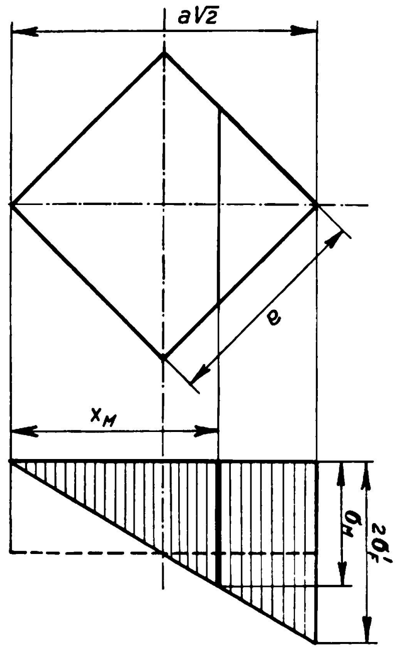


Fig. 17.

alone cannot form a true criterion of safety and that it is necessary to refer to the ultimate stress or to the unstable equilibrium as produced by yield. He arrived at the conclusion that the true safety of eccentrically loaded steel columns can be guaranteed by comparing the permissible stress with an extreme fibre stress of the value

$$\sigma_{Res} = \sigma_0 \left(1 + \beta\, m \sec \frac{\varkappa\, l}{2}\right) \tag{14}$$

set up at the least favourable point.

The value β cannot be understood from the mere point of run of stress based on the elastic theory; it was in the main an experimental value which was dependent upon cross section and upon the slenderness ratio and which made it possible to connect the greatest extreme fibre stress with carrying capacity. *Ostenfeld* also presented a theoretical deduction, not regarded as satis-

[30] *A. Ostenfeld:* Exzentrisch beanspruchte Säulen, Versuche mit Stahlsäulen, Querschnittsbemessung, (Eccentrically loaded columns, tests with steel columns, dimensioning of cross sections). Ingeniorvidenskabelige Skrifter A N° 21. Kopenhagen 1930.

factory today, of the value β, in which he uses the conception corresponding to the old yield formula with its plastic reduction of the peak stresses. In order to obtain satisfactory agreement with the experimental results, he was also obliged to adopt an occasional reduction of the elastic modulus E, ranging from 10 to 20 %, which is so determined that in the condition of the attained carrying capacity the so-called secant formula

$$y_m = p \cdot \sec \frac{\varkappa l}{2} \tag{15}$$

is fulfilled.

For this reason no direct comparison can be made between his values β and the values ν introduced here; their sense, however, is the same, since both 'correct' in an equal degree the specified eccentricity.

For the rectangular cross sections, *Ostenfeld* found $\beta = 0.69$ whilst the theoretical value of ν is 0.71; the agreement is unexpectedly good. In the

Table 3.

Ostenfeld's experiments with columns of square cross section.

$\nu = 0.707$. $E = 2100$ t/cm².

No. of test	σ_{Fu} in t/cm²	λ	m	$\sigma_{o\,krit}$ calculated	$\sigma_{o\,krit}$ measured	Deviation in % of the measured values
1	2.44	49.2	2.15	892	912	+ 2.4
2	2.37	49.1	5.80	441	465	+ 5.1
3	2.12	72.6	2.09	732	727	— 0.7
4	2.13	72.6	5.98	370	353	— 4.8
5	2.37	99.6	2.26	672	627	— 7.2
6	2.44	98.0	6.30	370	341	— 8.5
7	2.64	123.3	2.62	581	519	— 11.9
8	2.69	123.6	6.63	349	338	— 3.3

following Table 3 his measurements on steel columns of rectangular cross section are compared with the calculated values $\sigma_{o\,krit}$. The deviations are small, but in any case the circumstance that the theoretical values are somewhat greater than measured ones is due to the vagaries in value of the yield point σ_F. The value of the moment of external forces is nearly unchanged over a longer region and it is clear that yield will occur at that point at which the value of the yield point is low. This agrees with the experience that the first yield traces are not always observed to be in the centre of the column.

The second group of experiments was carried out with square sections set diagonally; for these the hypothesis of the similarity of the overloaded and underloaded stressed body, supplies for the calculation of $\nu = \frac{\varrho_F}{\sigma'_F}$ the following relation (Fig. 17).

$$\nu^3 - 1{,}5\,\nu^2 + 0{,}3124 = 0 \tag{16}$$

which is preferably solved by trial; this furnishes $\nu = 0.58$ whilst *Ostenfeld* calculates $\beta = 0.53$. Having regard to the correction for E, a direct comparison of the two figures is again impossible. Consequently, in Table 4 the measured values of $\sigma_{0\,krit}$ are compared with those calculated. Here again the agreement is satisfactory as deviations of $+13.8$ or -12.6 % must be considered as reasonably within the range of accuracy obtainable. In view of the broad under-stressed parts of the cross section, the theoretically required upper yield point

Table 4.

Ostenfeld's experiments with columns of square cross section (square set diagonally).

$\nu = 0.580. \qquad F = 2100$ t/cm².

No. of test	σ_{Fu} in t/cm²	λ	m	$\sigma_{0\,krit}$ calculated	$\sigma_{0\,krit}$ measured	Deviation in % of the measured values
1	2.63	48.3	2.41	1000	1160	+ 13.8
2	2.68	48.5	6.21	547	579	+ 5.5
3	2.15	73.8	2.51	743	713	— 4.2
4	2.20	73.9	6.11	431	456	+ 5.5
5	2.63	98.0	3.09	688	672	+ 2.4
6	2.68	98.2	7.10	421	408	— 3.2
7	2.74	122.8	2.25	691	616	— 12.6
8	2.12	124.2	6.65	330	325	— 1.5

in the elastic stress peaks when $\sigma_0 = 0$, works out as $\sigma'_F = 1.73$ which from these experiments must be regarded as possible. Nevertheless, in this case, in view of the great resistance offered by the fibres, spreading into the depth, to the progressive plasticising, the older yield formula also furnishes a yield area, the growth of which increases inversely with increasing load. A very high carrying capacity would also result, since the ratio $\frac{T}{W}$ is very large and it is seen that the old and the new yield formula proceed similarly and that in the main only differ in the intermediate stages, which lead to approximately the same final results.

The third group of tests relates to columns with circular cross section (Fig. 18). The fibre breadth is $b_x = 2\sqrt{x(d-x)}$, and the yield formula reads

$$\frac{\pi d^2}{4} \cdot \sigma'_F = 2 \int_0^{x_M} \frac{4 \cdot \sigma'_F}{d} \cdot \sqrt{x(d-x)}\, dx$$

from which after integration we get, as equation for ν:

$$3\pi = \frac{3}{2} \operatorname{arc} \cdot \sin(2\nu - 1) + (8\nu^2 - 2\nu - 3)\sqrt{\nu(1-\nu)}$$

If, as is always feasible with the value of ν now under consideration, one may take

$$\text{arc} \cdot \sin (2\nu - 1) = (2\nu - 1) + 2\pi \quad \text{then we get}$$

$$\frac{3}{2}(2\nu - 1) + (8\nu^2 - 2\nu - 3)\sqrt{\nu(1-\nu)} = 0 \tag{17}$$

The root of this equation is $\nu = 0.65$, whilst *Ostenfeld* has calculated with $\beta = 0.58$. Table 5 again contains the calculated and measured values for $\sigma_{0\,\text{krit}}$. For the second time in the case of a small slenderness ratio, a value is obtained that is more than 10 % too low, and it is not impossible that the explanation of

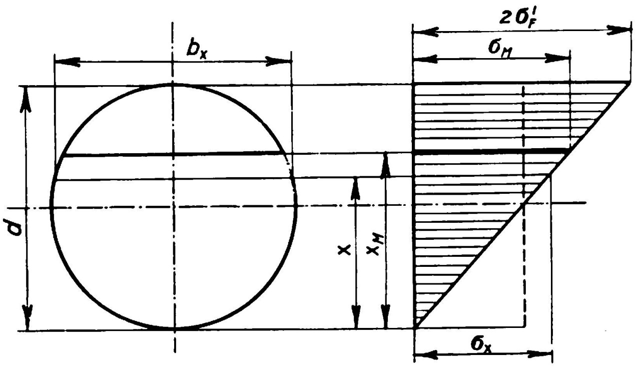


Fig. 18.

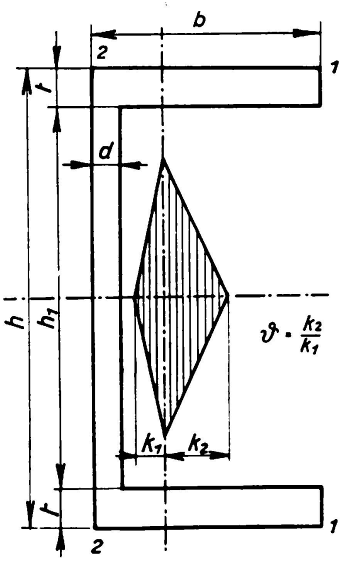


Fig. 19.

this rather great error lies in the fact that it has not been found possible completely to avoid some fixing of the column ends in the pressure heads of the knife-edged bearings. The consequent stiffening of the bar in the ends will naturally be manifest, especially in the case of shorter bars.

Table 5.

Ostenfeld's experiments with columns of circular cross section.

$\nu = 0.650$. $E = 2100$ t/cm².

No. of test	σ_{Fu} in t/cm²	λ	m	$\sigma_{0\,\text{krit}}$ calculated	$\sigma_{0\,\text{krit}}$ measured	Deviation in % of the measured values
1	3.25	44.2	2.70	1082	1260	+ 14.1
2	3.25	44.4	6.20	608	668	+ 9.0
3	3.25	95.3	2.97	775	763	— 1.6
4	2.85	95.3	6.20	455	447	+ 1.8

A further group of experiments concerned steel joists in which bending takes place at right angles to the plane of the web; in this case, *Ostenfeld* assumes the value $\beta = 0.58$, whilst the calculation according to equation 5 gives $\nu = 0.61$.

Of great interest are the tests with channel sections for which I have earlier deduced the yield point under bending or the value v^{21}. Such an unsymmetrical shape of cross section gives quite different carrying capacities, according to which side of the centre of gravity the load is applied. If the greatest pressure is applied to the projecting flanges, a bending ensues towards the closed side of the cross section,

$$v_1 = 0{,}707 \sqrt{1 - \frac{\alpha\beta^2}{1-\alpha}} \tag{18}$$

in which $\alpha = \frac{h}{h}$ and $\beta = \frac{d}{b}$ Fig. 19). For the channel section 10 examined, $\alpha = 0.83$, $\beta = 0.12$, and therefore $v = 0.682$, whilst *Ostenfeld* in this case calculated $\beta = 0.69$. A comparison of the calculated and measured values is

Table 6.

Ostenfeld's experiments with columns of steel channel section [10.

(a) The flanges are in compression.

$v_1 = 0.682$. $E = 2100$ t cm².

No. of test	σ_{Fu} in t/cm²	λ	$m = \frac{p}{k_1}$	σ_0 krit calculated	σ_0 krit measured	Deviation in % of the measured values
1	3.04	31.9	1.95	1250	1200	— 4.2
2	3.04	31.9	5.85	592	557	— 7.4
3	2.95	56.8	1.86	1132	1070	— 5.8
4	3.23	57.4	6,06	572	510	— 12.1
5	2.95	82.0	2.01	939	875	— 7.3
6	2.95	82.0	5.76	502	479	— 4.8
7	2.94	106.0	2.12	774	707	— 9.5
8	2.94	106.0	6.00	440	406	— 8.4
9	3.04	134.0	2.35	616	567	— 8.6
10	3.17	134.0	6.32	394	360	— 9.5

afforded in Table 6, which also contains the deviation in % of the measured values.

Where the point of application of the load lies on that side of the centre of gravity on which the web is situated, the web takes the compression and bending occurs towards the open side of the cross section. So far as the carrying capacity is concerned there are here two quite different possibilities under consideration; its limit can be reached by yield in the compressed part or by yield phenomena in some part of the cross section under tension; the maximum compressive stress is $\sigma_2 = \sigma_0 \left(1 + m \sec \frac{\varkappa l}{2}\right)$ and the maximum tension stress

$$\sigma_1 = \sigma_0 \left(-1 + m \frac{k_2}{k_1} \sec \frac{\varkappa l}{2}\right)$$

in which $m = \frac{p}{k_2}$ represents the measured eccentricity.

The yield formula in the first case reads

$$\sigma'_F - \sigma_0 = \sigma_0 \, m \sec \frac{\varkappa l}{2} = \frac{1}{\nu_2} (\sigma_F - \sigma_0) \tag{19}$$

whilst that in case two can be written

$$\sigma'_F + \sigma_0 = \sigma_0 \, m \frac{k_2}{k_1} \sec \frac{\varkappa l}{2} = \frac{1}{\nu_1} (\sigma_F + \sigma_0) \tag{20}$$

ν_2 was already calculated as $\nu_2 = 0.707 \sqrt{(1+\alpha) - \alpha\beta(2-\beta)}$; ν_1 must, of course, with reference to the reduced area of stress, be of the same value as was deduced in the case of bending towards the closed side. Case 1 will occur in the case of high values of σ_0 and small eccentricities p; in the case 2 it is exactly the reverse. The limiting stress σ_{ob} at which the yield phenomena occur at both edges simultaneously is obtained by combining the two equations 19 and 20, as

$$\sigma_{oG} = \frac{\nu_1 \vartheta - \nu_2}{\nu_1 \vartheta + \nu_2} \quad \text{wherein } \vartheta = \frac{k_2}{k_1} \tag{21}$$

For the channel section [10 examined, $k_1 = 0.629$, $k_2 = 1.400$ cm, $\nu_1 = 0.682$, from which results $\sigma_{ob} = 0.253 \, \sigma_F$. In such of the experiments of *Ostenfeld* in which the load was applied on the web side, the yield occured in accordance with the conditions determined above, partly in the web and partly in the drawn flanges. In most of the experiments, however, the quantities m were so large as to produce the conditions of case 2. Accordingly, equation 20 expresses the critical stress as

$$\sigma^2_{o\,krit} (1 + 0{,}234 \, m'\vartheta) + \sigma_{o\,krit} [\sigma_F - \sigma_E (1 - m'\vartheta)] - \sigma_E \sigma_F = 0 \tag{22}$$

The value for section [10 was $\upsilon_1 = 0.68$, whereas *Ostenfeld* adopted $\rho = 0.63$. The experiments are very numerous, and it suffices to recalculate

Table 7.

Ostenfeld's experiments with columns of steel channel section [10.

(b) The web in compression.

$\nu_2 = 0.903$. $\nu_1 = 0.682$. $\vartheta = 2.23$. $E = 2100$ t/cm².

No. of test	σ_{Fu} in t/cm²	λ	$m = \frac{p}{k_2}$	$\sigma_{o\,krit}$ calculated	$\sigma_{o\,krit}$ measured	Deviation in % of the measured values	σ_{ob}
1	2.59	50.7	1.76	1000	1140	+ 12.1	655
2	2.57	82.2	1.95	800	776	— 3.1	650
3	2.82	108.0	1.84	698	625	— 11.7	712
4	2.71	132.4	2.03	527	498	— 5.8	685
5	2.78	82.5	1.96	830	828	— 0.3	703
6	2.97	57.4	5.79	351	382	+ 8.1	750
7	2.57	82.2	5.73	291	305	+ 4.6	650

Note: In experiment 1 yield occured in the web under compression, in the other experiments in the flange under tension.

those with small values of ϑm, as these are chiefly of interest. The results of the calculations are tabulated in Table 7. If the theoretical assumption be considered that yield can occur in the projecting flanges only if $\sigma_{o\,krit} > \sigma_{o\,b}$, it is contradicted by two of the experiments; this, of course, is not surprising, since the value σ_{ob} can never have the significance of a precise relative demarcation of two different phenomena, but can possess only an approximate validity. Above all, the variations in the yield point will play an important part, especially in view of the fact that in one case the yield point is attained in the web and in the other in the flange, which according to experience are quite dissimilar. It would be possible to bear these different yield points in mind in calculating σ_{ob} Yield phenomenon in the web were revealed in only two parallel tests. The experimental data are shown in Table 7 under No. 1, in which case also satisfactory agreement is evident.

In conclusion, it may be said of the comprehensive tests of *Ostenfeld* that they can be regarded as an effective confirmation that the basic principles underlying the newer yield formula actually enable the carrying capacity of steel columns to be appropriately expressed with a very small amount of calculation; even though they cannot directly express its physical correctness, they nevertheless indicate its utility in connection with practical consideration of the strength of materials. A further proof of the method of calculation elaborated above is afforded by the comprehensive experiments of the American Society of Civil Engineers, which appointed a special commission of its own to investigate the carrying capacity of steel columns. The results of the experiments, which deal with very different types of cross sections, are recorded in the Proceedings of the American Society of Civil Engineers, February, 1929. Unfortunately, the space available is insufficient to discuss them here.

Summary.

The old yield formula is based directly on the strength of material as obtained by the ordinary static tensile test in judging the local danger to yield, since the old yield formula considers only the local stress conditions responsible for yield. The new yield, formula, however, gives strength values which are based solely on the nature of fields of stresses. Even if an increase of the upper yield limit within the peak stresses of a stress field is not proved certain and the old yield condition considered as a true basis, the new theory need not be disregarded in its conclusions. It can always be regarded as an approximation to the old, supplying useful results.

The new yield condition has the advantage of allowing for simple and clear modes of calculation the results of which compare well with tests. It is desired that the study of yield is carried still further.

I 3

Yield Limits and Characteristic Deflection Lines.

Über Fließgrenzen und Biegekennlinien.

Sur les limites d'écoulement et les diagrammes de flexion.

Dr. Ing. F. Rinagl,
Professor an der Technischen Hochschule Wien.

For editorial reasons this report appears in the appendix.

Aus redaktionellen Gründen erscheint das Referat im Anhang.

Pour cause rédactionelle ce rapport est publié en annexe.

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I 4

Theorie of Statically Indeterminate Systems.

Theorie statisch unbestimmter Systeme.

Théorie des systèmes hyperstatiques.

Dr. Ing. E. Melan,
Professor an der Technischen Hochschule Wien.

A) Ideal-Plastic Material.

A) Idealplastischer Baustoff.

A) Matériau idéalement plastique.

The object of statics as applied to steel structures is to determine the internal stresses and deformations of systems composed of slender members.

Three groups of equations are available for solving these problems, viz., (1) those describing the conditions of equilibrium, (2) those dealing with the geometrical conditions, and (3) those expressing the relation between internal stresses and deformations.

For statically determined systems, the first group of equations alone is sufficient for determining all internal stresses, whereas the two other groups are only necessary for calculating the deformations. In the case of statically undetermined structures, however, it is not possible to arrive at the internal stresses with the aid of only the first group of equations. The two other groups of equations must be used as well, and, in addition, the relations between deformations and stresses must be known.

The usual text-book theories assume full validity of Hooke's law, which says that the deformation of a bar is proportional to the axial stress and, further, that a change in the angle of contingency for a curved bar is proportional to the bending moment with regard to a particular point.

It has been stated elsewhere that the validity of Hooke's law is limited — a fact proved by experience — and that the linear relation between stress and deformation is only true within certain limits. These limits are by no means fixed. Actually, they are dependent on the momentary deformations already existing, as well as being a function of the rate at which the load changes. It is also essential to know whether the stress increases or decreases outside the range for which Hooke's law is applicable, and this of course does not make it easy to analyse the actual conditions obtaining.

To obtain rehults which really correspond with the actual conditions, Hooke's law must be replaced, outside the range for which it is valid, by fresh assumptions which are more in accordance with test results.

For statically determined structures this, as already indicated, only affects the calculation of the deformations, whereas, for statically undetermined structures, the substitution of Hooke's law by some other relation leads to new internal stresses. In order to obtain really practical and useful results, it will be necessary more or less to idealise the assumptions in terms of the actual conditions. The usual thing is to assume that the speed at which the change in load takes place has no influence, or, in other words, that the change of stresses is gradual (as usually assumed in statics) provided no "elasto-kinematic" considerations are included. Diagram (1a) may be considered as the basis of this theory, whereas, if we are satisfied with a less accurate reproduction of the real facts, the stress-strain diagram (1b) indicates the flow of the stresses and deformations. Over and above this, however, the theory admits of a further simplification as shown in diagram (1c), which illustrates the case for the "ideal-plastic" material. It is on such assumption that most of the following examinations are based, and this assumption will be dealt with in detail in the second part of this paper. It should be noted that it coincides with the theory postulated by Messrs. *Haar-Karman, v. Mises,* and *Hencky*[1] for stresses in the

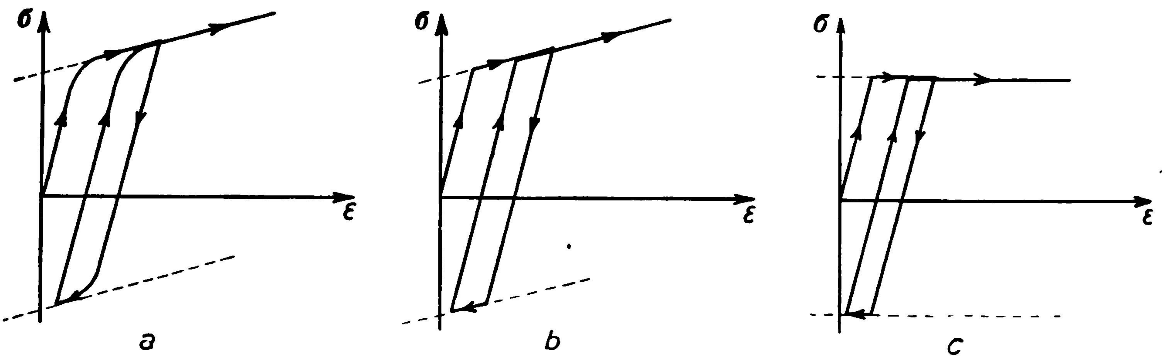

Fig. 1.

direction of the three axes, provided the necessary application and specialisation is made to suit the stress conditions in one direction only.

The results of axial stress and deformation tests can be applied as they stand for tensile members of a lattice girder. For compressive members, the load limit will as a rule be defined by the buckling load. If diagram (1c) should be applied for negative stresses as well, then we must assume that, under the influence of the buckling load, any shortening of the axis of a member is possible (e. g., shortening due to deflection of the axis), and that the buckling load always retains its critical value, which can never be exceeded. When the member is unstressed, we again have linear proportion between axial stress and deformation.

The theory outlined above is less important when applied to the ideal-plastic lattice girder than it is for system comprising members stiff against bending. The general investigations actually show that systems composed of an ideal-plastic material possess this remarkable property — that stress-peaks which would occur in a perfectly elastic material are reduced and distributed to places of lower stresses. Expressed differently, this means that there must be a certain

[1] *A. Haar* and *Th. v. Karman*: "Theory of Stresses in Plastic and Sandy Materials". Nachr. d. kgl. Ges. der Wiss., Göttingen 1909.

v. Mises: "Mechanics of Solids in the Plasto-deformable State", ibid. 1913.

H. Hencky: "Theorie of Plastic Deformations and the Stresses which they subsequently set up in the Material". Ztschr. für angew. Math. und Mech., 1924.

reserve of material which, however, in a carefully calculated and suitably dimensioned lattice girder, is less than it would be in a solid continuous beam over several spans with constant cross-section. The main reason why the fundamental investigations have nevertheless been carried out on lattice girders is that, in contrast to systems composed of members stiff against bending, the analytical investigation leads to equations having a limited number of variables, and this means that the complexity of the problem is greatly simplified for the less mathematical reader. In principle, the generalisation of the problem for an infinite number of variables is possible, so that the results obtained from lattice girders can be employed to solve problems for systems having members stiff against bending.

When investigating systems with members resistant to bending and made of ideal-plastic material, it is usual to assume, with sufficient accuracy, the same coordination between moment and radius of curvature as was done in Fig. 1c for stress and deformation. It should be noted, however, that these two assumptions do not fully tally with each other. Information on this point and on other questions connected with it is provided by a interesting paper by *Eisenmann* in "Stahlbau", 1933. Another interesting paper to which we may refer at this juncture is one entitled "The Behaviour of statically undetermined Steel Structures beyond the Elastic Limit", published by *W. Prager* in the "Bauingenieur", 1933. In this paper, the author only gives one definite example to show that, in statically undetermined systems, the deformations are affected much more than the moments are by the assumption of a special law of elasticity, and that the simplified assumption, previously mentioned, of the relationship between internal moment and alteration in curvature can be adopted without objection.

Attention is further drawn to two important facts which, as the result of substituting Hooke's law by a non-linear equation, is not stressed sufficiently in many of the published papers.

Due to the non-linearity of the equations, the validity of the law of superposition is lost, or, expressed more accurately, it remains valid only for the internal stresses of statically determined systems. With the exception of this latter case, it is therefore impossible to plot influence lines. It is also impossible to define the stress conditions in statically undetermined systems unless the previous stressing or the "previous history" is known. It is precisely to this particular point that special attention should be paid in all cases where a single loading is involved, as in the papers of Messrs. *J. Fritsche, Schaim, Kazinczy* and *Girkmann*[2], all of which deal with special systems of ideal-plastic material.

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[2] *J. Fritsche*: „Die Tragfähigkeit von Balken aus Stahl mit Berücksichtigung des plastischen Verformungsvermögens". Bautechnik, 1930.

G. H. Schaim: „Der Durchlaufträger unter Berücksichtigung der Plastizität". Stahlbau, 1930.

G. v. Kazinczy: „Statisch unbestimmte Tragwerke unter Berücksichtigung der Plastizität". Stahlbau, 1931, and „Bemessung unvollkommen eingespannter Stahl-I-Deckenträger unter Berücksichtigung der plastischen Formänderungen". Proc. of International Association for Bridge and Structural Engineering, Vol. 2, 1933—34.

K. Girkmann: „Bemessung von Rahmentragwerken unter Zugrundelegung eines idealplastischen Stahles". Sitzungsber. der Akademie der Wiss., Vienna 1931, and „Über die Auswirkung der Selbsthilfe des Baustahles in rahmenartigen Stabwerken", Stahlbau, 1932.

In all these investigations, it is tacitly assumed that all external stresses increase proportionally from nil to their final value. The same remark applies to investigations dealing with repeated loading (stressing), where it is assumed that the load varies in the same way between two limiting values. Mention should be made here of the fundamental work of *M. Grüning* „Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl mit beliebig häufig wiederholter Belastung" (The Loading Capacity of Statically Undetermined Steel Structures subject to any number of repeated Loadings), Berlin 1926. These investigations show that, with repeated loading oscillating between two fixed values, the system finally undergoes deformations which are also limited. *Grüning's* investigations are based on the less specialised case of Fig. 1a. Alongside these special researches should be ranked the special work of Mr. *J. Fritsche* on continuous beams under repeated loading, based on ideal-plastic material („Bauingenieur", 1932).

Mr. *Grüning's* results have been extended — for ideal-plastic material only — by Mr. *H. Bleich* (cf. „Bauingenieur", 1932). In this paper, Mr. *Bleich* shows that, even under variable loads, i. e., not a mere oscillation of the external stresses between definite limits, it is possible, under certain conditions, and by suitable dimensioning, to make the structure work like one of perfectly elastic material, provided that a sufficient number of changes in load have taken place. This hypothesis, dealt with in detail in the second part of this paper, forms the basis for calculating statically undetermined systems of ideal-plastic material. The method built up on this basis, and for which the term *„Traglastverfahren"* ("Theory of plastic equilibrium") has come into use, will be explained in further detail in the paper to be read by Mr. *F. Bleich,* unless the reader chooses to consult other sources, say, Mr. *F. Bleich's* book „Stahlhochbauten", Berlin 1932.

The above rather incomplete summary of the published literature dealing with the theory of statically undetermined systems, must be supplemented by two papers which deal with the theory of these systems from a more general viewpoint. These are: Mr. *J. Fritsche's* paper in the *Zeitschrift für angewandte Mathematik und Physik,* 1931, and a paper by Mr. *Hohenemser* in *Ingenieur-archiv,* 1931, and both writers take the potential energy of such systems as their starting point. On the other hand, the writer employs in his investigations now to be dealt with, and which aim at a general proof of Mr. *Bleich's* theory, the conditions of equilibrium in conjunction with the law of elasticity for ideal-plastic material. It is well to state here that the undetermined condition of the deformations, which, in my opinion, has not previously been observed, and which necessitate certain reservations as regards the validity of Mr. *Bleich's* theorem, is due to the special assumption of the stress-strain law beyond the range for which *Hooke's* law is valid. If the diagram is replaced by Fig. 1a or 1b, the undefined solutions will disappear and also the considerations which they involve. This simplification is gained, of course, at the expense of a new material constant, viz., the inclination of the straight lines at the boundary in Fig. 1a and 1b.

1) The proof adduced by Mr. *H. Bleich* in support of his theory of statically undetermined systems of ideal-plastic material is not very simple, and is worked

out for no more than twice statically undetermined systems. An extensions of the idea he develops does not seem possible for higher statically undetermined structures on account of the time required for calculation.

In what follows, a proof of general validity for any systems of multiple static undetermined structures will be established. This proof is comparatively easy to demonstrate, provided a clear conception has been reached as to the fundamental properties of ideal-plastic material. In addition, the more general discussions preceding the demonstration of the theory will be calculated to give a better idea of the nature of such structures. Only a few simple principles of the theory of linear equation systems are required, and these can be found in any algebra text-book. To avoid having to deal with an infinite number of variables, the following considerations will be confined to lattice girders. The transition to an infinite number of variables, such as would be involved when dealing with systems composed of members stiff against bending, is similar to that used in the theory of integral equations, and presents no real difficulties.

It will be remembered that the linear deformation of a bar of ideal-plastic material is expressed by the relation:

$$\Delta_S = v + \rho \cdot S \tag{1}$$

Here ρ is a constant depending on the length of bar, the modulus of elasticity, and the cross-section of the bar, v is the so-called "permanent" deformation, and S the axial stress. If S comes within the interval $T' < S < T$, where T' and T are the yield points under compression and tension, then v is constant. Only when $S = T'$ or $S = T$ is it possible for v to vary; the value v can only either increase or decrease if $S = T$ or $S = T'$ respectively. The bar is said to "yield" or "flow". Values like $S < T'$ and $S > T$ are excluded. The relation just outlined is illustrated in Fig. 2, the time being plotted on the abscissae. In the upper diagram, the ordinates represent the stress in the bar, and in the lower diagram, the deformation (elongation). In the time intervals $t_1 < t < t_2$, $t_3 < t < t_4$ and $t_5 < t$, the bar is perfectly elastic, whereas it flows in the other intervals. This assumes, of course, that dv/dt is always finite, and that the permanent deformations corresponding to a finite rate of flow do not change suddenly.

2) In a statically undetermined lattice girder two types of members may be differentiated: (1) those which are absolutely necessary and which, if cut through, cause the whole system to become movable, and (2) those which are not necessary, i. e., one or more of which may be cut through without rendering the system movable. According to definition there must, in a v-times statically undetermined system, be at least one group of maximum v members which, if cut through, would leave still one immovable system, which is defined as a statically determined basic system in the statically undetermined system we are considering.

We shall now consider this particular lattice girder at a given moment, where the external load has a definite value. Due to previous loadings, the several members have already undergone permanent deformations v, so that the stresses S occur in these members. If the system were perfectly elastic, then, according to the usual theory of statically undetermined systems, the stresses B would have been set up in the members. S, B and v are then linked up by the following system of equations:

$$S_i + \sum q_{ik} v_k = B_i \qquad (i = 1, 2, \ldots r) \tag{2}$$

The significance of q_{ik} will be readily appreciated; it represents the stress in the bar i when, for the unloaded system (all B' s = 0) all permanent deformations v are nil with the exception of member k. In this member, $v_k = -1$.

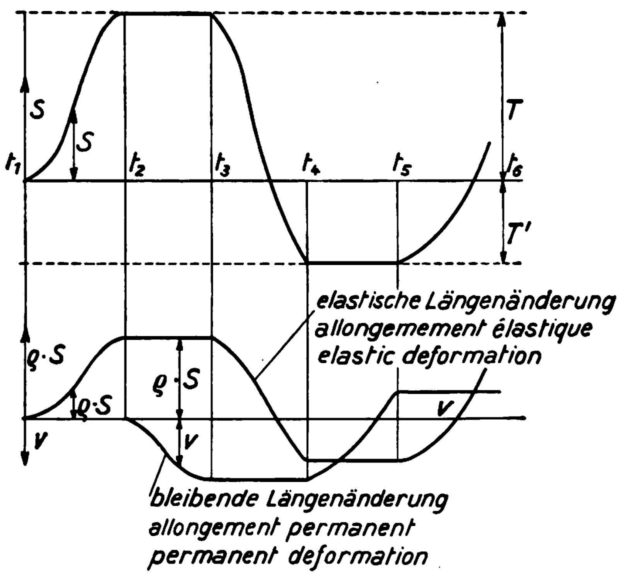

Fig. 2.

Inserting these values in equation (2) we immediately get:

$$S_i = q_{ik}$$

This tells us that, for necessary bars, $q_{ik} = 0$, and since, as will be shown, $q_{ik} = q_{ki}$, the summation in equation (2) applies only for all unnecessary members 1, 2, . . . r. The matrix $(q_{11} \ldots q_{rr})$ is symmetrical, and hence $q_{ik} = q_{ki}$; this follows from the theorem of axial stresses in members of a lattice girder, which is analogous to *Maxwell*'s theorem of the reciprocity of deformation. This theorem, which can be easily proved by solving the equations for elasticity, says that the axial force in a member i produced by a permanent deformation $v_k = -1$ of member k, hence q_{ik}, is equal to the axial force in member k produced by a deformation $v = -1$ of member i, so that $q_{ik} = q_{ki}$. This theorem naturally only applies to statically undetermined systems and only to the members in it which are not necessary, as otherwise it becomes trivial, as otherwise $q_{ik} = q_{ki} = 0$.

Writing $\sigma_1 = B_i - S_i$, we obtain from (2) the following type of equation:

$$\sigma_i = \sum q_{ik} v_k \qquad (i = 1, 2, \ldots r) \tag{3}$$

The system of the values σ_1 is termed a system of coercive forces; the $-\sigma_1$ are the stresses in members of an unloaded system which, owing to the previous loadings, have already undergone the permanent deformations v_k.

Multiplying each of the equations of (3) in turn by v_i, and adding, we obtain the "quadratic form"

$$J = \sum_i \sigma_i v_i = \sum_i \sum_k q_{ik} v_i v_k \tag{4}$$

In adducing the proof, it is necessary that J be always essentially positive in character, that is to say, which ever way the values of v_k are chosen, J should never be negative. The simplest way of demonstrating this is to start from the principle of virtual deformations which, for a lattice girder, may be written

$$\sum S \Delta s = \sum P \delta$$

The summation on the right applies for all external forces, and that on the left for all members of the system. The forces S and P represent a system of equilibrium. The deformation Δs can be arbitrary, compatible only with the geometrical relation, which means that, for au ν-times statically undetermined system, only up to ν-values can be selected arbitrarily. A compatible system of Δs values is formed by the values $v + \rho S$, and also by the values ρB, which are deformations of ideal-plastic and perfectly elastic material respectively. The difference $v + \rho (S - B) = v - \rho\sigma$ will certainly also give a compatible system of deformations. The quantities δ finally indicate the displacement of the points of attack of the external forces P in the direction of these forces when all members of the system have undergone the deformation Δs.

If, therefore, we write $\Delta s = v - \rho\sigma$ and apply the principle of virtual deformations (1) for the ideal-plastic and (2) for the perfectly elastic system, noting, however, hat the same external loads P apply to the forces S and B, whereas the δ are identical in both cases, corresponding to the values $v - \rho\sigma$, we then obtain, by subtracting the two equations:

$$\sum (v - \rho\sigma)\, \sigma = 0$$

and hence

$$J = \sum v\sigma = \sum \rho\sigma^2 \geqq 0 \tag{4a}$$

thus proving our assertion that J cannot assume negative values, since $\rho = s/EF$ is never negative and $\sum \rho\sigma^2$ is for positive terms only. The value J can only disappear when all $\sigma = 0$; but it must not be inferred from this that v must be nil too under all circumstances.

Assuming the forces in the members for the ideal-plastic system have, at a given moment, attained the values S and that, at this particular moment, the forces in the members 1, 2 .... μ have reached the yield point where $S_1 = T_1$, $S_2 = T_2 \ldots . S_\mu = T_\mu$, then if the external force alters, S will become $S + \Delta S$, v will become $v + \Delta V$, and B will become $B + \Delta B$. In accordance with the system of equations (2)

$$(S_i + \Delta S_i) + \sum q_{ik} (v_k + \Delta v_k) = (B_i + \Delta B_i)$$

and, subtracting equation (2) from this gives us

$$\Delta S_i + \sum q_{ik} \Delta v_k = \Delta B_i \quad (i = 1, 2, \ldots \mu) \tag{5}$$

or

$$\Delta \sigma_i = \sum q_{ik} \Delta v_k \tag{6}$$

where $\Delta\sigma_i$ indicates the change in the coercive forces. The sum $\Sigma q_{ik} \Delta v_k$ is applicable to no more than the first μ-members, since, according to the definition of an ideal-plastic material, the only members which can yield are those for which $S = T'$ or $S = T$. This assumes, of course, that all ΔB are chosen so small that no other members than the first μ-members have reached the yield point. As long as one and the same members yield, we speak of a certain "phase", and a new phase starts if other members start or cease to yield. For the phase immediately following the particular moment we are considering, equation (5) applies, where, however, in accordance with the properties of the ideal-plastic material, the following alternative should be noted for the first μ-members.

If $\Delta S_i = 0$, then Δv_i has not the opposite sign to $S_i = T_i$ or $S_i = T'_i$;
or if $\Delta v_i = 0$, then ΔS_i has not the same sign as $S_i = T_i$ or $S_i = T'_i$.

$$(i = 1, 2 \ldots . \mu) \tag{5a}$$

In this connection only the first μ-equations of (5) are of interest, since S_j $(\mu < j \leqq r)$ can be determined without difficulty from the remaining equations, provided the values Δv_k have been worked out from the first μ-equations. If, as we assumed, one and the same phase occurs — and this can always be attained where the value ΔB is small enough, the expression $S_j + \Delta S_j \leqq T_j$ or T'_j certainly applies, and we need not trouble about these equations at present.

3) Applying the principle of virtual deformation, it can easily be shown that the alternative equation (5a) is just sufficient to determine the ΔS of equation (5), viz., from

$$\Delta S_i + \Sigma q_{ik} \Delta v_k = \Delta B_i \qquad (i = 1, 2, \ldots r)$$

provided that they exist at all. Unless the ΔS were unequivocal, then values $S' = S + \Delta S'$ and also $S'' = S + \Delta S''$ would be possible for the external loads $P = P + \Delta P$, and alterations in the length of a member would be $v' = \rho S'$ and $v'' + \rho S''$. If the principle of virtual deformations is applied in its four possible combinations, we obtain:

$$\Sigma S'' (v'' + \rho S'') = \Sigma \overline{P} \delta'' \qquad \Sigma S' (v'' + \rho S'') = \Sigma \overline{P} \delta''$$

$$\Sigma S'' (v' + \rho S') = \Sigma \overline{P} \delta' \qquad \Sigma S' (v' + \rho S') = \Sigma \overline{P} \delta'$$

Herein, δ'' and δ' are displacements of the points of intersection which are coordinated to the values $(v'' + \rho S'')$ and $(v' + \rho S')$ respectively.

Subtracting from the difference of the two equations on the right the difference of the two equations on the left, we get:

$$\Sigma (S'' - S') (v'' - v') + \rho (S'' - S')^2 = 0$$

or

$$\Sigma (\Delta S'' - \Delta S') (\Delta v'' - \Delta v') + \Sigma (\Delta S'' - \Delta S')^2 \rho = 0. \tag{7}$$

The summation $\Sigma (\Delta S'' - \Delta S')^2 \rho$ cannot be negative on account of the quadratic terms, and the same applies to $\Sigma (\Delta S'' - \Delta S') (\Delta v'' - \Delta v')$. If in both cases a member is yielding, it can only be for $\Delta S'' = \Delta S' = 0$, while if a member is not yielding in either case, then $\Delta v'' = \Delta v' = 0$, which means that the corresponding summations disappear in the above equation. If a member yields in one case and not in the other, then $\Delta S'' = 0$, and $\Delta v''$ has not the opposite sign to $S = T$, while $\Delta v' = 0$ und $\Delta S'$ has not the same sign as $S = T$. In this case, the particular term is reduced to $- \Delta S' \Delta v''$, wherein $\Delta S'$ and $\Delta v''$ will never have the same sign, so that $= \Delta S' \Delta v''$ will not be negative. If S'' and S' are different, then (7) consists only of non-negative terms, and can only disappear if $S'' = S'$. A very definite phase can only follow if the values ΔB (or, more precisely, if dB/dt in the interval of time dt immediately following the moment under consideration) are given. It should be noted, however, that these arguments only prove the singularity of the axial forces in members, and not the permanent deformations. As a matter of fact, there may, under certain conditions, be multiple solutions for Δv.

In the phase considered above, at the beginning of which the first λ members have reached the yield point, only the first members ought actually to yield. What we have to do, then, is to solve the system of equations

$$\sum_{1}^{\lambda} q_{ik} \Delta v_k = \Delta B_i = \Delta \sigma_i \quad (i = 1, 2, \ldots \lambda) \tag{8a}$$

for the yielding members only, and

$$\Delta S_j + \sum_{1}^{\lambda} q_{jk} \Delta v_k = \Delta B_j \quad (\lambda < j \leqq r) \tag{8b}$$

for the non-yielding members.

According to a well-known theorem in algebra, the equations (8a) supply singular solutions only when the matrix $(q_{11} \ldots q_{\lambda\lambda})$ is not singular, in other words, if the determinant

$$\begin{vmatrix} q_{11} \cdots\cdots q_{1\lambda} \\ \cdots\cdots\cdots\cdots \\ q_{\lambda 1} \cdots\cdots q_{\lambda\lambda} \end{vmatrix}$$

does not disappear. Sufficient, but not necessary to establish this, is the fact that $(q_{11} \ldots q_{\mu\mu})$ is not singular. In this case, if all $\Delta B = 0$, the only possible solutions are that all $\Delta v = 0$, and this at the same time is the necessary and sufficient condition for the disappearance of the quadratic form

$$J = \sum \sum q_{ik} \Delta v_i \Delta v_k.$$

But if $(q_{11} \ldots q_{\lambda\lambda})$ is singular, and if this matrix has, say, the range $\vartheta < \lambda$, which means that not only the determinant disappears, but all sub-determinants with more than ϑ rows, while at least one determinant of ϑ lines can be different from nil, the Δv are no longer of singular value. These solutions can be written introducing an arbitrary constant c_p:

$$\Delta v_i = \Delta v_i^* + \sum_{p=\vartheta+1}^{\lambda} c_p \mathfrak{v}_{ip} \tag{9}$$

where the Δv_i^* are the solutions of the system

$$\sum q_{ik}\,\Delta v_k^* = \Delta B_i = \Delta\sigma_i \qquad (i = 1, 2, \dots \vartheta)$$

while

$$\Delta v_k^* = 0 \text{ for } k = \vartheta + 1,\ \vartheta + 2, \dots \lambda$$

As there is at least one non-singular matrix $(q_{11} \dots q\ \ )$, the solutions Δv_*^i are definitely determined. The $\mathfrak{v}_{ip}$ are the solutions of the following homogeneous system of equations:

$$\sum q_{ik}\,\mathfrak{v}_{kp} = 0 \qquad (i = 1, 2, \dots \lambda)$$

which has as solutions $\lambda - \vartheta$ different "fundamental systems" $\mathfrak{v}_{kp}$ $(p = \vartheta + 1, \vartheta + 2, \dots \lambda)$. It can be shown that these also satisfy the remaining equations

$$\sum q_{ik}\,\mathfrak{v}_{kp} = 0 \qquad (i = \lambda + 1, \lambda + 2 \dots r).$$

This follows from the fact that the rectangular matrix

$$\begin{matrix} q_{11} \cdots\cdots q_{1\lambda} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ q_{r1} \cdots\cdots q_{r\lambda} \end{matrix} \qquad (r > \lambda)$$

can only have the range ϑ. This means that all sub-determinants formed from these elements with more than ϑ lines and rows can only have the range ϑ where the corresponding quadratic term $J = \Sigma\Sigma q_{ik}\, v_i v_k$ is essentially positive and the main sub-determinant of the above matrix is also of range ϑ only.

There are again two possibilities to consider: either it is impossible to chose for the general solution the arbitrary values for c_p in such a way that $\Sigma c_p\,\mathfrak{v}_{ip}$ has the signs prescribed by the alternative (5a). If, then, $\Delta v_i = \Delta v_i^* + \Sigma c_p\,\mathfrak{v}_{ip}$ has the required signs, $\Delta v_i = \Delta v_i^* + \Sigma(c_p + \delta c_p)\,\mathfrak{v}_{ip}$ will also possess the same signs, provided δc has been selected sufficiently small. In this case, the Δv_i are not of a singular value, but lie between certain finite limits. If, however $\Sigma c_p\,\mathfrak{v}_{ip}$ alone fulfils the required condition with regard to sign, then $\Sigma k c_p\,\mathfrak{v}_{ip}$ with any positive value of k has the same sign, and if k becomes infinite, the solution $\Delta v_i = \Delta v_i^* + k\,\Sigma c_p\,\mathfrak{v}_{ip}$ also assumes infinite values.

The following relations also provide additional support in proof of the theorem already postulated. If the quantities z_i and w_i are connected by the equation:

$$z_i = \sum_{k=1}^{r} q_{ik}\, w_k \qquad (i = 1, 2, \dots r),$$

then

$$\sum_{k=1}^{r} z_k\,\mathfrak{v}_{kp} = 0. \tag{10}$$

The correctness of the above equation will be evident by inserting in it the values $z_k = \sum\limits_i q_{ki}\, w_i$. This gives us:

$$\sum_k z_k\,\mathfrak{v}_{kp} = \sum_k \sum_i q_{ki}\, w_i\,\mathfrak{v}_{kp},$$

and, by interchanging the order of summation, $\sum_i w_i \sum_k q_{ik} \mathfrak{v}_{kp}$. Since, according to definition $\sum_k q_{ik} \mathfrak{v}_{kp} = 0$, then also $\sum_k z_k \mathfrak{v}_{kp} = 0$.

The following observation is added for the sake of completeness. If we employ the principle of virtual displacements with the deformations $\Delta B \rho$ which are regarded as compatible with the geometrical relations of the lattice girder, and use (1) the axial force ΔS due to the loading ΔP, and (2) the axial forces ΔB of the members, and if we then subtract the two equations, we then get for $\sigma = \Delta S - \Delta B$ the equation:

$$\sum \Delta\sigma \Delta B \rho = 0.$$

Hence it follows that $\Delta\sigma$ and ΔB can never have the same sign for all members, because, if they had, as ρ would always be positive, the summation would also be positive and could not disappear as required. This is the very typical expression for what in practice is more sub-consciously termed "self-adaptation" of the material of ideal-plastic nature, where a reduction in stresses at certain places sets up an increase in stresses at other places.

4) The results obtained up to now may become clearer to the engineer if we study the lattice truss in a particular phase, introducing, however, certain alterations.

With the particular lattice truss we have been studying, we imagine a new system being formed by having cut through such members as were not necessarily required. At these cuts we will assume the incorporation of such mechanisms as only allow, for the two faces of the gap, movements in a particular direction. With some of the cut members it should only be possible to have movements increasing the gap. This could be achieved, say, by cutting the members at right angles to the axis and making the two ends simply abut without being joined. For other members, the cuts will be assumed to be such that only a movement closing the gap can occur, which could be done by replacing the particular cut member by a rope.

If such a systems is charged with the loads ΔP, the new loads in the members are ΔS, which, together with the loads ΔB previously obtained, are connected by the equation:

$$\Delta S_i + \sum q_{ik} \Delta v_k = \Delta B_i \quad (i = 1, 2, \ldots r).$$

The Δv represent the displacements between the faces of the cuts in the first μ-members. For these μ-members the equations are as under:

$$\Delta S_i + \sum_1^{\mu} q_{ik} \Delta v_k = \Delta B_i \quad (i = 1, 2, \ldots \mu),$$

with the alternatives for

$\Delta B = 0$, in which case $\Delta v \gtrless 0$

or $\Delta S \gtrless 0$, in which case $\Delta v = 0$,

where the sign of inequality is governed by the particular kind of mechanism above explained. These are the same equations with the same alternatives as for the system of ideal-plastic material previously considered and for which the equations (5) and (5a) were developed.

We may therefore say that the original lattice girder of ideal-plastic material, at the beginning of the particular phase, may be regarded as being replaced by another girder in which the members which would have reached the yield point in the original girder are regarded as replaced by such mechanisms as above described. If this new girder is loaded with the increment of the external forces corresponding with the particular phase, then the stresses ΔS thus set up in the members are identical to the increments of the stresses in the original girder for the same phase. If, under these conditions, the cut members should not have the singular matrix $(q_{11} \ldots q_{\mu\mu})$, this means that the new girder (irrespective of the auxiliary mechanisms) is immovable, since the homogeneous system of equation

$$\sum q_{ik} \mathfrak{v}_{kp} = 0$$

does not in this case admit of any other solution than that all $\mathfrak{v} = 0$. We also find that the system

$$\Delta S_i + \sum q_{ik} \Delta v_k = \Delta B_i$$

must always give, for any values of ΔB, singularly determined solutions for ΔS and Δv. But if the matrix $(q_{11} \ldots q_{\mu\mu})$ has a singular value and at the same time range ϑ, then two cases are possible. Either the new girder becomes "self-locked" through these mechanisms, and the solutions of the homogeneous equations do not possess the signs proper to the machanisms. The system is immovable and suits any values of ΔB, although the values Δv are not singular, but lie between two finite limits. Or the new girder system may be movable despite the mechanisms, and may give solutions of the homogeneous system of equations with the proper signs. In this case, should solutions be possible, the ΔB can no longer be arbitrarily chosen, while the Δv may even assume infinite values. Simple kinematic consideration will make matters clear in specific cases. All these facts are explained by the simple example shown in Fig. 3. The members d_1 and d_2 are assumed to have reached the yield point, in which case the truss is as shown in Fig. 3a. This new system is then immovable, whatever the nature of the mechanisms incorporated at the cuts. The matrix $\begin{matrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{matrix}$ is by no means singular, and which ever way the load varies in the succeeding phase, equilibrium will be maintained. But if the members d_1 und d_2 are stressed up to the yield limit (Fig. 3b), then the matrix $\begin{matrix} q_{11} & q_{13} \\ q_{31} & q_{33} \end{matrix}$ is singular in value, because the girder system is movable. When both members are stressed to the limits of compression T' or tension T, the corresponding mechanisms (in this case ropes) prevent movability and, with any change in loading, equilibrium remains established because the system is self-locked. If, however, one member reaches the limit of compression and the other the limit of tension (Fig. 3c) and if a change in load occurs which causes both members actually to yield, then the values ΔB must be conditional and such as to make equilibrium and the solution of the equations possible. At the same time, however, the Δv may assume infinite values.

5) After these preliminary remarks, we now proceed to prove the theorem just postulated. It says that, for a statically undetermined system of ideal-plastic nature subjected to variable loading the permanent deformations produced by sufficiently repeated changes in loading tend towards limited values. This assumes that the stresses in members of a perfectly elastic material lie within a previously determined interval $B_i^{max} \geqq B_i \geqq B_i^{min}$ and that such a system of coercive forces $\bar{\sigma}$ can be indicated where, for all members,

$$-\bar{\sigma}_i + B_i^{max} \leqq T_i \text{ and } -\bar{\sigma}_i + B_i^{min} \geqq T'_i$$

If we put $B_i^{max} - T_i = t_i$ and $B_i^{min} - T'_i = t'_i$ then a system of coercive forces $\bar{\sigma}$ must exist for which

$$t'_i \leqq \bar{\sigma}_i \leqq t_i.$$

Supposing we had indicated a system of $\bar{\sigma}$ values, and now form the values $\overline{B}_i^{max} = \bar{\sigma}_i + T_i$ or $\overline{B}_i^{min} = \bar{\sigma}_i + T_i$, then obviously the theorem, if correct at all, must also be applicable to variable loads B_i, for which

$$\overline{B}_i^{max} \geqq B_i \geqq \overline{B}_i^{min},$$

where $\overline{B}_i^{max} \geqq B_i^{max}$ and $B_i^{min} \geqq \overline{B}_i^{min}$ (cf. Fig. 4).

Whe shall now deal with a lattice girder after it has undergone a series of changes in load, and shall assume that the coercive forces $\sigma^{(\varphi)}$ obtain after

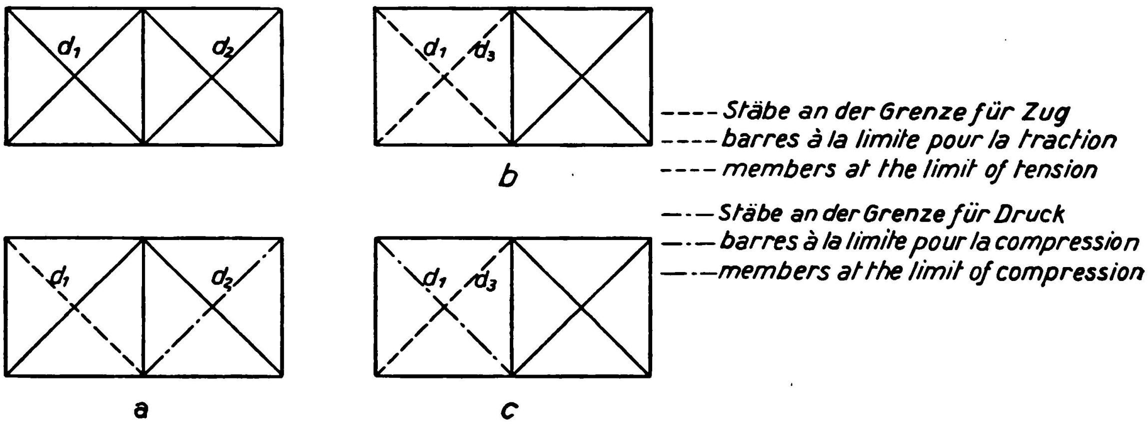

Fig. 3.

the expiration of φ phases. At the beginning of the (φ + 1st) phase, members for which $\sigma^{(\varphi)} > \bar{\sigma}$ can only yield, if at all, when the stress in the merber is $S = T'$. Values $S = T$ are, however, excluded; for even if $B = \overline{B}^{max}$, the stress in the member $S = \overline{B}^{max} - \sigma$ is smaller than $T = \overline{B}^{max} - \bar{\sigma}$.. Consequently, a member for which $\sigma^{(\varphi)} > \bar{\sigma}$, can only undergo negative permanent deformations $\Delta v^{(\varphi+1)}$ during the (φ + 1st) phase. For a member which yields during this phase, the condition $\sigma > \bar{\sigma}$ must obtain during the whole of the phase. Similarly, $\sigma^{(\varphi+1)} > \bar{\sigma}$ must apply for the end of the (φ + 1st) phase or the beginning of the (φ + 2nd) phase. Conversely, $\Delta v^{(\varphi+1)} > 0$ for members for which $\sigma^{(\varphi)} < \bar{\sigma}$ and also $\sigma^{(\varphi+1)} \leqq \bar{\sigma}$. Summarizing, we thus find for a member yielding during the (φ + 1st) phase

$$\left.\begin{array}{l} z^{(\varphi)} = \bar{\sigma} - \sigma^{(\varphi)} > 0, \text{ and also } \Delta v^{(\varphi+1)} > 0 \text{ and further} \\ z^{(\varphi+1)} = z^{(\varphi)} - \Delta\sigma^{(\varphi+1)} \geqq 0 \\ \text{and where} \\ z^{(\varphi)} = \bar{\sigma} - \sigma^{(\varphi)} < 0, \text{ then } \Delta v^{(\varphi+1)} < 0; \text{ and further} \\ z^{(\varphi+1)} = z^{(\varphi)} - \Delta\sigma^{(\varphi+1)} \leqq 0. \end{array}\right\} \quad (11)$$

Obviously no fresh changes $\Delta\sigma$ of the coercive forces can occur in subsequent phases, provided $z^{(\varphi)} = \bar{\sigma} - \sigma^{(\varphi)} = 0$ for all the members. Our proof therefore falls into two parts: we show (1) that the condition $\bar{\sigma} - \sigma^{(\varphi)} = z^{(\varphi)} = 0$ will actually obtain for all members, and (2) that the permanent deformations $\bar{v} = \Sigma \Delta v$ which have developed up to that point can only assume finite values.

Between the quantities $\bar{\sigma}$ and the permanent deformations $\bar{v}$ due to $\bar{\sigma}$ there obviously exists the system of equations

$$\bar{\sigma}_i = \sum q_{ik} \bar{v}_k \qquad (i = 1, 2, \ldots r),$$

whereas, for the coervice forces $\sigma^{(\varphi)}$ with the deformation $v^{(\varphi)}$ at the beginning of the ($\varphi + 1^{st}$) phase, we have the equations

$$\sigma_i^{(\varphi)} = \sum q_{ik} \bar{v}_k^{(\varphi)} \qquad (i = 1, 2, \ldots r).$$

The above summations apply for all members, although it is possible, (as for the following sums) that some of the quantities $\bar{v}_k$ or v become nil in particular cases. By also writing $w^{(\varphi)} = \bar{v} - v^{(\varphi)}$ and $z^{(\varphi)} = \bar{\sigma} - \sigma^{(\varphi)}$, we get

$$z_i^{(\varphi)} = \sum q_{ik} w_k^{(\varphi)},$$

and, similarly, for the end of the ($\varphi + 1^{st}$) phase,

$$z_i^{(\varphi+1)} = \sum q_{ik} w_k^{(\varphi+1)}$$

whence we immediately confirm that

$$z_i^{(\varphi+1)} = z_i^{(\varphi)} - \Delta\sigma_i^{(\varphi+1)} \text{ and } w_i^{(\varphi+1)} = w_i^{(\varphi)} - \Delta v_i^{(\varphi+1)}. \qquad (12)$$

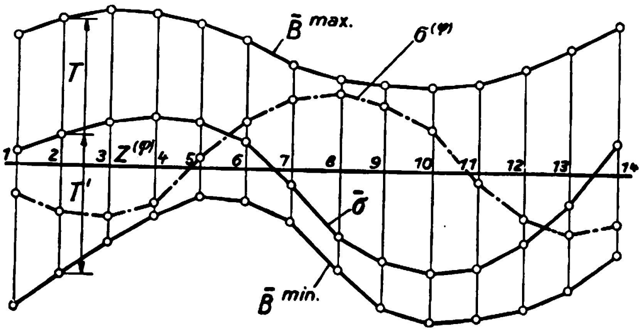

Fig. 4.

Taking the difference between the two quadratic forms

$$\Delta K^{(\varphi+1)} = K^{(\varphi+1)} - K^{(\varphi)} = \sum (z^{(\varphi+1)} w^{(\varphi+1)} - z^{(\varphi)} w^{(\varphi)})$$

and inserting $z^{(\varphi+1)}$ and $w^{(\varphi+1)}$ as per equation (12), we get:

$$\Delta K^{(\varphi+1)} = - \sum z^{(\varphi)} \Delta v^{(\varphi+1)} - \sum \Delta\sigma^{(\varphi+1)} w^{(\varphi)} + \sum \Delta v^{(\varphi+1)} \Delta\sigma^{(\varphi+1)}.$$

The first two sums are equal, as will be seen when introducing into the first the term $z^{(\varphi)} = \Sigma q_{ik}\, w^{(\varphi)}$ and into the second the term $\Delta\sigma^{(\varphi+1)} = \Sigma q_{ik}\, \Delta v_k{}^{(\varphi+1)}$, since in both cases we obtain the bilinear form

$$\sum\sum q_{ik}\, \Delta v_i^{(\varphi+1)}\, w_i^{(\varphi)},$$

and hence

$$\Delta K^{(\varphi+1)} = \sum(-2\, z^{(\varphi)} + \Delta\sigma^{(\varphi+1)})\, \Delta v^{(\varphi+1)} = -\sum(z^{(\varphi+1)} + z^{(\varphi)})\, \Delta v^{(\varphi+1)}.$$

According to previous deductions, $z^{(\varphi+1)}$ never has the opposite sign to, and $z^{(\varphi)}$ has always the same sign as, $\Delta v^{(\varphi+1)}$, so that $\Delta K^{(\varphi+1)}$ can never become positive either. Consequently, the expression for K can never increase from phase to phase. K decreases if the permanent deformations change, but K remains constant if the system of all v-values remains unaltered during a phase. Since K, if it changes, can only diminish, but never be negative as per equation (4) or (4a), it must eventually become nil. If, however, K has reached nil, then z will certainly also be nil, and no fresh changes in $\Delta\sigma$ can take place from this moment.

We still have to show that the permanent deformations $\bar{v} = \Sigma\, \Delta v$ are finite. So long as the terms $\Delta z^{(\varphi)} = \sigma^{(\varphi)} - \sigma^{(\varphi+1)}$ are finite, the Δv will also be finite, with the exception of the case where a matrix of singular value occurs and for which the prime solutions $\sum_{\varphi} c_p\, \mathfrak{v}_{ip}$ satisfy the conditions laid down as to sign. Only in this case could infinite values of Δv occur for finite values of $\Delta z^{(\varphi)}$. In view, however, of our assumption as to the existence of a system of coercive forces $\bar{\sigma}$, this case is only possible where all $z = 0$. The values z_i must always have the same sign as Δv_i, in the present case the same sign as $\sum_p c_p\, \mathfrak{v}_{ip}$. Now $\sum_p \sum_i c_p\, \mathfrak{v}_{ip}\, z_i = \sum_p c_p \sum_i z_i\, \mathfrak{v}_{ip} = 0$, because $\sum_i z_i\, \mathfrak{v}_{ip}$ is already nil as per equation (10). Accordingly, z_i and $\Sigma c_p\, \mathfrak{v}_{ip}$ cannot have the same sign for all values if i, as otherwise $\sum_p \sum_i c_p\, \mathfrak{v}_{ip}\, z_i$ would certainly be positive and not nil. Generally speaking, then, it is not possible for the case to arise where the quantities Δv would increase beyond all limits. Only where all z are nil, i. e., the state of the coercive forces $\bar{\sigma}$ is already attained, can Δv assume infinite values, when the load B has attained the basic values $\bar{B}^{max}$ or $\bar{B}^{min}$ at μ places in a v-times statically undetermined system whose matrix $(q_{11} \ldots q_{\mu\mu})$ is singular and for which the corresponding new system is not self-locked. To exclude this possibility, it would be necessary to stipulate that the values B could go as near as possible to the limits $\bar{B}^{max}$ or $\bar{B}^{min}$ without actually attaining these limits, or to attain these limits only during an infinitely short interval of time dt, during which, for a finite rate of yield, no finite permanent deformation could develop.

In reality, such permanent deformation going beyond all limits will not obtain, as the stiffening of the material will prevent them.

In the introduction, it was mentioned that the foregoing proof can also be demonstrated in a similar way for systems with members stiff against bending. It is also highly probable that the above deductions made for lattice girders are applicable to solid structures of plastic material, provided a suitable definition is found for the properties of the ideal-plastic material.

B) Material with Linear Range of Hardening.

B) Baustoff mit linearem Verfestigungsbereich.

B) Matériau à zone de solidification linéaire.

In the paper on the theory of statically indeterminate systems consisting of an ideal-plastic structural material, the author called attention to the fact that the selection of the appropriate law of elasticity is of the utmost importance. In the following he has therefore set himself the task of considering what formulation must be given to *H. Bleich's* theorem when a structural material with a linear range of hardening is employed. It thus becomes a question of generalising *M. Grüning's* researches for any desired change of loading. Here, as in the previous paper mentioned, a new foundation will be applied to the theory which, in spite of its general character as against the line of reasoning hitherto brought forward, yet affords a considerable simplification of the latter.

The following investigations thus deal with statically indeterminate structures subjected to any variable loading and consisting of such material which follows

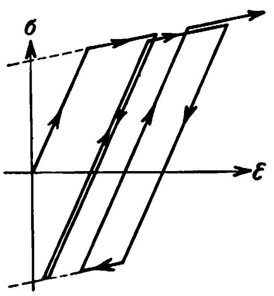

Fig. 1.

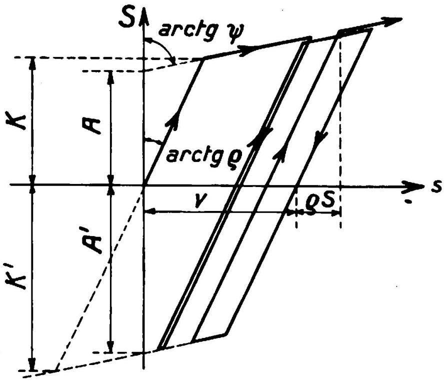

Fig. 2.

the stress diagram is shown in Fig. 1. With a view to simplifying calculation a somewhat more extended idealisation has been introduced in contrast to the assumptions made by *M. Grüning*.

If we, as the two authors mentioned have done, confine our investigations to latticed work, so that we can proceed with a finite number of variables, the forces S in the members and the deformations s of the latter are inter-related through the same connection as tension and elongation, i. e. as in Fig. 2. Accordingly, the deformations s occurring for a definitive force S in a member is given by the relation

$$s = v + \rho S \tag{1}$$

whereby we call v the "permanent" and ρS the "elastic" deformations. $\rho = l/EF$ is a constant, naturally always positive, depending on the length of bar l, its cross section F and modulus of elasticity E. As is clear from Fig. 2, v remains unchanged as long as S lies within the interval

$$cv + K' < S < cv + K, \tag{2}$$

where by, as can easily be confirmed, the meaning of the constants is

$$c = \frac{1}{\psi - \rho}, \quad K' = A' \frac{\psi}{\psi - \rho} \quad \text{and} \quad K = A \frac{\psi}{\psi - \rho}$$

c being certainly positive, K′ negative and K positive.

Should S attain one of these ultimate values, v will only remain constant during the following time interval if the absolute value of S does not increase, i. e. if the increment ΔS is not preceded by the same sign as $S = cv + K'$ or $S = cv + K$ respectively. If, however, the absolute value of S increases, ΔS is then preceded by the same sign as S and Δv will also increase in its absolute value, so that the increment Δv can be expressed by the equation

$$\Delta S = c \Delta v \tag{2a}$$

where Δv, since c is positive, must always be preceded by the same sign as ΔS and incidentally S.

If *Hooke*'s Law can be applied unrestrictedly, we can now, in what follows, regard the material as being perfectly elastic. In this case the forces B in the members would ensue for a certain loading, just as they do when following the usual theory of statically indeterminate structures. These B forces are related to the forces S in the members, which are found in our structure for the same loading, in terms of the equation.

$$S_i + \Sigma q_{ik} v_k = B_i \tag{3}$$

The significance of q_{ik} is easy to recognise. In the unstressed, perfectly elastic system (where all $B = 0$), q_{ik} represents the force in member i, in case of all $v =$ zero and for only a single member k has been permanently deformed to the extent $v_k = -1$. It can be shown in a very simple manner that $q_{ik} = q_{ki}$, for by a theorem similar to *Maxwell's* the force S_i in a member is identical with q_{ki}. Hereby, the force S_i is produced by $v_k = -1$ or in other words q_{ik} is equal to the force S_k in member K which is produced by a deformation $v_i = -1$ of the member i.

Thus the matrix

$$\begin{matrix} q_{11} \cdots q_{1r} \\ \cdots\cdots \\ q_{r1} \cdots q_{rr} \end{matrix}$$

is symmetrical; it has, however, another pronounced property, namely that the "quadratic form"

$$J = \Sigma\Sigma q_{ik} v_i v_k \geqq 0$$

is "semi-definitely positive", i. e., however v_i is chosen, the term for J cannot be negative under any circumstances whatever. The proof of this, which is not difficult to find on the principle of virtual deformations, may be drawn from the paper mentioned under (3).

Should the external forces change, the forces B in the member of a perfectly elastic system will alter by ΔB, the forces S in the members of an elastic-plastic system by ΔS and the permanent deformations v by Δv.

As long as ΔB keep within sufficiently small limits, the Δv deformations will occur only in identical members. We then speak of a definitive "Phase". A new phase thus begins when the permanent deformations begin to change in other members not hitherto involved, or when Δv disappears for certain members.

We now assume that at the end of a certain phase (φ) the values $S^{(\varphi)}$, $B^{(\varphi)}$ and $v^{(\varphi)}$ are given and that the first μ members of the lattice-work structure have just reached the ultimate values $S^{(\varphi)} = cv^{(\varphi)} + K$, and $S^{(\varphi)} = cv^{(\varphi)} + K'$ respectively. If $B^{(\varphi)}$ now alters by $\Delta B^{(\varphi+1)}$ in the following $(\varphi + 1)^{th}$ phase, $S^{(\varphi)}$ will generally change for all members by $\Delta S^{(\varphi+1)}$; new permanent deformations, however, can only have occurred, if at all, in the first μ members. Thus the equation system

$$S^{(\varphi)} + \Delta S^{(\varphi+1)} + \sum q_{ik} [v_k^{(\varphi)} + \Delta v_k^{(\varphi+1)}] = B_i^{(\varphi)} + \Delta B_i^{(\varphi+1)} \tag{4}$$

applies, and when the equations (3) are deducted herefrom, the result obtained is

$$\Delta S_i^{(\varphi+1)} + \sum q_{ik} \Delta v_k^{(\varphi+1)} = \Delta B_i^{(\varphi+1)} \quad (i = 1.2 - r) \tag{5}$$

Here it is sufficient to consider the first μ equations, which, however, contain 2μ unknowns, namely the values ΔS and Δv. It can now be demonstrated that the alternative existing for the first members, namely either:

ΔS preceded by a sign opposite to that of S, then
$\Delta v = 0$, (6)

or ΔS preceded by the same sign as S, then Δv, in accordance with equation (2a) $\Delta S = c\Delta v$, thus preceded by the same sign as ΔS and also S is just necessary and sufficient for the obtaining of definite results from the equation system (5). The proof can also be brought in the same manner as for ideal-plastic constructional material (see paper quoted under (3).

Now if new deformations really occur in the first members from the μ members mentioned during the $(\varphi + 1)$ phase, the equation it involves the solving of the system equation

$$c_i \Delta v_i^{(\varphi+1)} + \sum q_{ik} \Delta v_k^{(\varphi+1)} = \Delta B_i^{(\varphi+1)} \quad (i = 1, 2 \ldots \lambda) \tag{7}$$

deduced from (5) by introducing $\Delta S_i = c_i \Delta v_i$. Here of course Δv_i $(i = 1 \cdot 2 \ldots \lambda)$ be preceded by the same sign as $S_i^{(\varphi)} = c_i v_i^{(\varphi)} + K_i$ and $S_i^{(\varphi)} = c_i v_i^{(\varphi)} + K'_i$ respectively, at the end phase (φ). In this connection the quadratic form

$$J = \Sigma\Sigma q_{ik} v_i v_k + \Sigma c_i v_i^2$$

belonging to the matrix

$$\begin{pmatrix} q_{11} + c_1 \cdots\cdots & q_{1\lambda} \\ \cdots\cdots\cdots\cdots & \cdots \\ q_{\lambda 1} \cdots\cdots & q_{\lambda\lambda} + c_\lambda \end{pmatrix}$$

ist most certainly "positively definite", i.e., however the v are chosen, J has a positive value and can only vanish when all the v are zero. For even if the first sum of the v having another value than zero should vanish, J will in any case become positive in view of the second sum. Thus, on a well-known algebraic theorem concerning linear systems of equation, we have for all systems of ΔB_i only a single system of solution for Δv_k. In this case, too, the only solution to "all$\Delta B_i = 0$" is in particular all $\Delta v_i = 0$. When the Δv have been deduced from the equations (7), $\Delta S_i^{(\varphi+1)} = c_i \Delta v_i^{(\varphi+1)}$ yields the forces in the members $i = 1 \cdot 2 \ldots \lambda$. For the remaining members

$$\Delta S_j^{(\varphi+1)} = \Delta B_j^{(\varphi+1)} - \sum q_{ik} \Delta v_k^{(\varphi+1)} \quad (j = \lambda + 1, \lambda + 2, \ldots \mu)$$

holds good, where by for the members $\lambda + 1$, $\lambda + 2, \ldots \mu$, $\Delta S_j^{(\varphi+1)}$ must not be preceded by the same sign as $S_j^{(\varphi)}$. For the members $\mu + 1$, $\mu + 2 \ldots r$ the equation

$$K'_i + c_i\,(v_i^{(\varphi)} + \Delta v_i^{(\varphi+1)}) \leqq S_i^{(\varphi)} + \Delta S_i^{(\varphi+1)} \leqq c_i\,(v_i^{(\varphi)} + v_i^{(\varphi+1)} + K_i$$

must finally apply. We would add that the values $s - \sigma_i^{(\varphi)}$, which we shall call "coercive forces", and which are given by

$$\sigma_i^{(\varphi)} = -\Sigma q_{ik}\, v_k^{(\varphi)}$$

represent the forces in the members of a perfectly elastic structure when the individual members have undergone their permanent longitudinal deformations $v_k^{(\varphi)}$. Naturally the expression for $\Delta\sigma_i^{(\varphi+1)} = \Sigma q_{ik} \Delta v_k^{(\varphi+1)}$ also holds good.

We shall now demonstrate that for our system the theorem analogous to *H. Bleich's* theorem for systems of an idealplastic material runs as follows:

In a latticed framework in which the connection described in Fig. 2 exists between force in member and longitudinal deformation, certain finite ultimate values $\bar{v}$ of the permanent longitudinal deformation will occur for any variable loading after an adequate number of load in repetition; no further change takes place in these values on further change of loading if the condition

$$B_i^{max} - B_i^{min} \leqq Ki - K'_i.$$

is satisfied for every member. Here B^{max} and B^{min} signify the maximum and minimum values of the forces in members of a perfectly elastic material. The system will thus behave after the permanent longitudinal deformations $\bar{v}$ have been attained, like a structure composed of a perfectly elastic material.

This theorem can be proved in a manner similar to that shown by the author in the treatise mentioned under (3) for an ideal-plastic structural material. If we substitute the sign of une quality for that of equality, the theorem must also remain valid for the ultimate case

$$\bar{B}_i^{max} - \bar{B}_i^{min} = K_i - K'_i;$$

it is then bound to hold good for the smaller values $B_i^{max} - B_i^{min}$ as well.

In Fig. 3 the forces $\bar{B}'^{max}$ and $\bar{B}^{min}$ in the members have been applied at the points 1, 2, ... r. The condition at the end of the φth. phrase is also shown — at that moment the values $B_i^{(\varphi)}$ may be present. In accordance with the permanent longitudinal deformations $v_i^{(\varphi)}$ just existant (now apparent) the system of coercive forces $\sigma_i^{(\varphi)}$ may be present. The actual forces in the structure under consideration are now given by $S_i^{(\varphi)} B_i^{(\varphi)} - \sigma_i^{(\varphi)}$. In the following phase $(\varphi + 1)$ only those bars can undergo further permanent longitudinal deformations $v_i^{(\varphi)} + \Delta V_i^{(\varphi+1)} = v_i^{(\varphi+1)}$ in which $B_i^{(\varphi)}$ coincides with a limit of the interval $\sigma_i^{(\varphi)} + c_i v_i^{(\varphi)} + K_i$ or $\sigma_i^{(\varphi)} + c_i v_i^{(\varphi)} + K'_i$. In this case the extent of this interval is according to assumptions.

$$[\sigma_i^{(\varphi)} + c_i v_i^{(\varphi)} + K_i] - [\sigma_i^{(\varphi)} + c_i v_i^{(\varphi)} + K'_i] = K_i - K'_i = B'^{max}_i - B^{min}_i.$$

This is for instance only possible in the specially illustrated case of the member j for its upper limit, when $B_j^{max} - [\sigma_j^{(\varphi)} + c_j v_j^{(\varphi)} + K_j] > 0$, while the lower limit cannot be attained by $B_j^{(\varphi)}$. Thus in the $(\varphi + 1)^{th}$ phase for member j only positive $\Delta v_j^{(\varphi+1)}$ are possible.

Let $\bar{\sigma}$ be that system of coercive forces belonging to the aforementioned $\bar{v}$ and eventually appearing after a sufficient number of loading repetitions have taken place. Now Fig. 3 at once gives

$$\bar{\sigma}_i + c_i \bar{v}_i = \bar{B}^{max} - K_i = \bar{B}^{min} - K'_i = \bar{D}_i$$

for each member i. Since $\bar{\sigma}_i = \sum q_{ik} \bar{v}_k$, the equation system

$$c_i \bar{v}_i + \sum q_{ik} \bar{v}_k = \bar{D}_i$$

is obtained for $\bar{v}$ and from foregoing elucidations it is certain that this equation system must contain definite solutions $\bar{v}$ for any values of $\bar{D}$. Such a system of

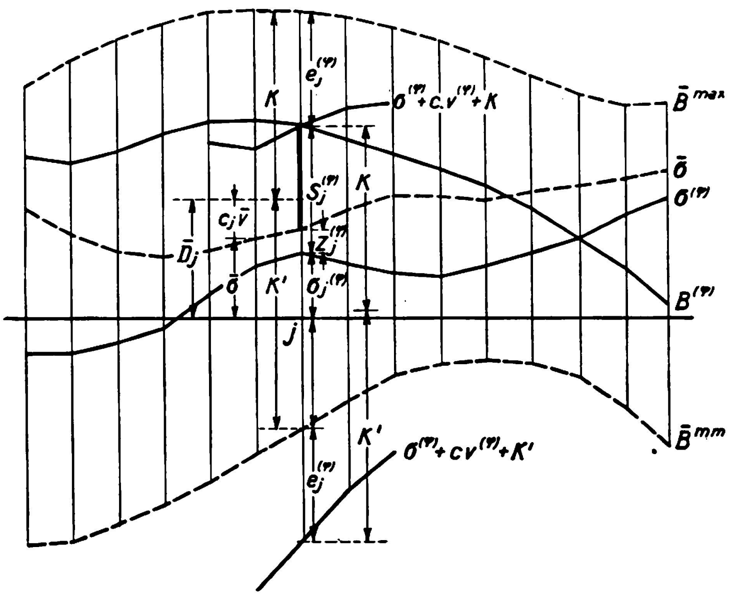

Fig. 3.

$\bar{v}_i$ and $\bar{\sigma}_i$ is therefore bound to exist. Finally, if we consider the value $e_j^{(\varphi)}$, which according to the diagram is calculated

$$e_j^{(\varphi)} = \bar{D}_j - [\sigma_j^{(\varphi)} + c_j v_j^{(\varphi)}]$$

and resolves positively for the member j, we can determine the following alternative:

either $\Delta v_i^{(\varphi+1)} > 0$, in which case $e_i^{(\varphi)} > 0$ and at the end of the $(\varphi + 1)^{th}$ phase $e_i^{(\varphi+1)} \geqq 0$,

or $\Delta v_i^{(\varphi+1)} < 0$, in which case $e_i^{(\varphi)} < 0$ and at the end of the $(\varphi + 1)^{th}$ phase $e_i^{(\varphi+1)} \leqq 0$.

Our theorem is obviously proved when we show that after a sufficient number of loading repetitions, possibly at the expiration of ε phases, $e^{(\varphi+\varepsilon)} = \bar{e} = \Theta$ must vanish for all the members. For if this is correct for all the members, the force S_i can no longer exceed the limits of the interval

$$\bar{\sigma}_i + c_i \bar{v}_i + K'_i \leqq S_i \leqq \bar{\sigma}_i + c_i \bar{v}_i + K_i$$

which then coincide with B_i^{max} and B_i^{min}. To show that as φ increases $e_i^{(\varphi)}$ approaches zero for all the members, we write

$$\bar{v}_i - v_i^{(\varphi)} = w_i^{(\varphi)}$$

and also $$\bar{\sigma}_i - \sigma_i^{(\varphi)} = z_i^{(\varphi)}$$

so that $$\Delta w_i^{(\varphi+1)} = w_i^{(\varphi+1)} - w_i^{(\varphi)} = - \Delta v_i^{(\varphi+1)}$$

and $$\Delta z_i^{(\varphi+1)} = z_i^{(\varphi+1)} - z_i^{(\varphi)} = - \Delta \sigma_i^{(\varphi+1)}$$

now operate and, in consequence of the equation systems

$$\sum q_{ik} \bar{v}_k = \bar{\sigma}_i \text{ and } \sum q_{ik} v_k^{(\varphi)} = \sigma_i^{(\varphi)}$$

the systems

$$\sum q_{ik} w_k^{(\varphi)} = z_i^{(\varphi)} \text{ and } \sum q_{ik} \Delta w^{(\varphi+1)} = \Delta z^{(\varphi+1)}$$

also exist. If we consider the quadratic form

$$J^{(\varphi)} = \Sigma(z_i^{(\varphi)} + c_i w_i^{(\varphi)}) \, w_i^{(\varphi)} = \Sigma\Sigma q_{ik} w_i^{(\varphi)} \, w_k^{(\varphi)} + \Sigma c_i w_i^{(\varphi)2}$$

which, as has already been demonstrated, is always positive for any values of w and can only vanish for all $w = 0$, and further $J^{(\varphi+1)}$ at the end of the $(\varphi + 1)^{th}$ phase

$$J^{(\varphi+1)} = \sum (z_i^{(\varphi+1)} + c_i w_i^{(\varphi+1)}) \, w_i^{(\varphi+1)}$$

then the increment of J during the $(\varphi + 1)^{th}$ phase becomes

$$\begin{aligned} \Delta J^{(\varphi+1)} &= \sum [z_i^{(\varphi+1)} + c_i w_i^{(\varphi+1)} w_i^{(\varphi+1)}] - [z_i^{(\varphi)} + c_i w_i^{(\varphi)}] \, w_i^{(\varphi)} \\ &= \sum z_i^{(\varphi)} \Delta w_i^{(\varphi+1)} + \Delta z_i^{(\varphi+1)} w_i^{(\varphi)} + \Delta z_i^{(\varphi+1)} \Delta w_i^{(\varphi+1)} \\ &\quad + c_i (2 w_i^{(\varphi)} + \Delta w_i^{(\varphi+1)} + \Delta w_i^{(\varphi+1)\,2}) \end{aligned}$$

Now, however, $\Sigma z_i^{(\varphi)} \Delta w_i^{(\varphi+1)} = \Sigma \Delta z_i^{(\varphi+1)} w_i^{(\varphi)}$, as can easily be confirmed by introducing $z_i^{(\varphi)} = \Sigma q_{ik} w_k^{(\varphi)}$ and $\Delta z^{(\varphi+1)} = \Sigma q_{ik} \Delta w_k^{(\varphi+1)}$ upon which the bilinear form $\Sigma\Sigma_{ik} w_i^{(\varphi)} \Delta w_k^{(\varphi+1)}$ arises for both cases, so that we now get

$$\Delta J^{(\varphi+1)} = \sum [2 z_i^{(\varphi)} + \Delta z_i^{(\varphi+1)} + c_i (2 w_i^{(\varphi)} + w_i^{(\varphi+1)})] \, \Delta w_i^{(\varphi+1)}$$

Finally we have

$$z_i^{(\varphi)} = \bar{\sigma}_i - \sigma_i^{(\varphi)} = (\bar{D}_i - c_i \bar{v}_i) - (\bar{D}_i - c_i v_i^{(\varphi)} - e_i^{(\varphi)}) = e_i^{(\varphi)} - c_i w_i^{(\varphi)}$$

and also

$$z_i^{(\varphi+1)} = e_i^{(\varphi+1)} - c_i w_i^{(\varphi+1)}$$

and from this

$$\Delta J^{(\varphi+1)} = - \sum (e_i^{(\varphi+1)} + e_i^{(\varphi)}) \, \Delta v_i^{(\varphi+1)}$$

But as $e_i^{(\varphi+1)}$ and $e_i^{(\varphi)}$ must be preceded by the same sign as $\Delta v_i^{(\varphi+1)}$, it follows that $\Delta J^{(\varphi+1)}$ must always be negative. As a matter of fact the term for J, which is always positive, will decrease from phase to phase and only remain constant if all the members retain the same amount of longitudinal deformation during a phase. Thus, J must become zero after expiration of a sufficient number of phases. This makes $w^{(\varphi+\varepsilon)} = 0$ absolutely necessary, however, and in consequence $v^{(\varphi+\varepsilon)} - \bar{v} = 0$, which also gives $\bar{\sigma} - \sigma^{(\varphi+\varepsilon)} = 0$. There thus actually arises a system of coercive forces in which the permanent longitudinal deformation cannot change any more and from this point on the structure behaves as if it were composed of perfectly elastic material.

In comparison with structures of ideal-plastic material, it becomes clear — a significant fact — that the investigation of systems of coercive force is of no importance whatever; on the contrary, in this case it is quite sufficient if the condition $B^{max} - B^{min} \leqq K - K'$ is satisfied for each member. Moreover, owing to the term $c_i\, w_i^2$ occurring in the quadratic form, it is impossible for semi-definite terms to appear which, with respect to the singularity of their solutions, necessitate separate investigation as in the case of ideal-plastic material. On the contrary, it is certain that to every and finite value of $\bar{\sigma} - \sigma^{(\varphi)}$, only finite and singular values of $\bar{v} - v^{(\varphi)}$ can belong. It is naturally another question whether the total deformations finally occurring have not already attained inadmissible values and do not perhaps lie beyond the ultimate rupture point elongation. This question can just as pertinently be asked in the case of ideal-plastic material. A general answer is difficult to give.

Summary.

The first part of this paper gives a brief survey of the published literature and results in connection with the subject discussed. The second part discusses the essential properties of ideal-elastic lattice girders and gives a general proof of Mr. *H. Bleich's* theorem. This theorem, which forms the basis for calculating such systems and, in government regulations, is referred to as the „Traglastverfahren" or "the theory of plastic equilibrium", is broadly based on the fact that a certain positive quadratic form can always only diminish, i. e., become nil once only. Beyond the known results, certain limitations have been set forth for the validity of Mr. *H. Bleich*'s theorem, which consist in excluding certain cases of loading as soon as a system has become fully elastic, unless such cases of loading are limited to an infinitely short interval of time where the rate of yield is finite in value.

I 5

Carrying Capacity of Trussed Steel Work.

Tragfähigkeit von Fachwerkträgern.

Résistance des poutres réticulées.

Dr. Ing. E. Kohl,

Professor an der Technischen Hochschule Hannover.

Some fundamental questions relating to the strength of trussed steel constructions shall be explained by a simple example forming the basis of the following investigations.

1) *Fixed positions of loading.*

The truss according to fig. 1 represents a simply supported structure. The member U_3 in the bottom chord and the diagonals D_2 and D_5 are composed of ⅃L 70.70.7. All other members consist of ⅃L 90.90.9. In table 1 are shown the forces S_o in the members of the structure due to a concentrated load

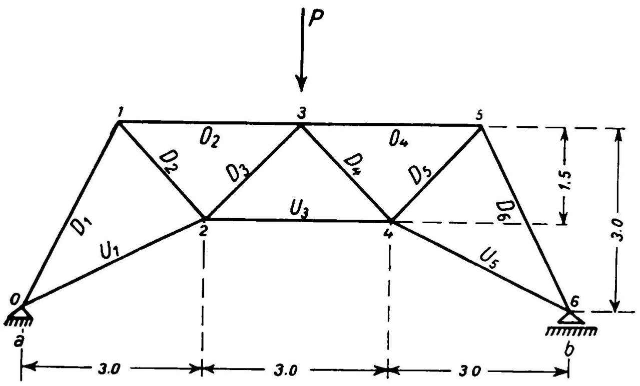

Fig. 1.

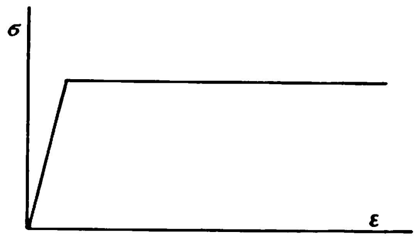

Fig. 2.

P acting in point 3, excluding the small influences due to dead weight of the structure. Considering a deduction for rivet holes of 20 mm diameter in tension members and based on a yield limit $\sigma_s = 2{,}4\ t/cm^2$ and buckling stresses σ_k according to the regulations of the German State Railways, the ultimate strengths of the members exhausting the carrying capacity of the members, are expressed by the following terms

$$S_{Gr} = F_n \cdot \sigma_s \text{ for members in tension,}$$

$$S_{Gr} = F \cdot \sigma_k \text{ for members in compression.}$$

The respective values are shown in the second colum in table I. The third row of this table shows the ultimate values of P_{Gr} of the load P which would, for an unrestricted validity of Hook's law, produce the ultimate values of strength in the members of the structure.

Table I.

Member	Force in members S_o	Ultimate Force S_{Gr} For tension	compression	Ultimate value of load P_{Gr}
U_1	+ 0,3727 P	+ 70 t	— 43,4 t	70/0,3727 = 188 t
U_3	+ 1,5 P	+ 41,8 t	— 19,5 t	41,8/1,5 = 27,9 t
O_2	— 1,0 P	+ 70 t	— 53 t	53 t
D_1	— 0,7453 P	+ 70 t	— 43,4 t	43,4/0,7543 = 58,2 t
D_2	+,0,9427 P	+ 41,8 t	— 39 t	41,8/0,9427 = 44,4 t
D_3	— 0,707 P	+ 70 t	— 69,8 t	69,8/0,707 = 98,6 t

The carrying capacity of the structure is ruled by the smallest value of $P_{Gr} = 27{,}9$ t for which load the yield point is reached first by member U_3. Assuming pin-jointed connections for all intersection points and accepting the stress-strain diagram of fig. 2, the strength of the structure is completely exhausted if only one member is stressed up to yield point, as otherwise unlimited elongations of member U_3 would be possible without any increase in load.

Through the introduction of a tie Z connecting the two bearing points a and b the play of forces and with it the strength of the structure are completely altered. According to the rules for hyperstatic systems, we receive for the tensile force in the tie, if S_a indicate the forces in the members due to $X_a = -1$, in the isostatic system: $Z = X_a = \frac{\delta_{a\,o}}{\delta_{a\,a}}$.

The numerator in this equation assumes the value

$$EF_c \sum S_o S_a s \frac{F_a}{F} = 34.5 \text{ P}$$

and the denominator

$$EF_c \sum S_a^2 s \cdot \frac{F_c}{F} = 50.64 + l_z \cdot \frac{F_c}{F_z}$$

where l_z represents the length and F_z the section of the tie respectively.

The following cases shall receive consideration:

a) The cross section of the tie shall remain constant over the length of 9 m (fig. 3 a),

b) The tie shall be composed of two angles 70.70.7 for a—c and b—d and F_z shall be the cross section of the tie for c—d. Fig. 3 b.

Assuming also for hyperstatic systems the carrying capacity of the structure to be dependent on one member only if stressed to yield point, it is obvious that based on the cross sections adopted for the members of the structure, the tie is entirely responsible for the strength of the structure. This, provided that the cross section of the tie does not go beyond a certain fixed value. Within this range the strength of the structure depends entirely on the cross

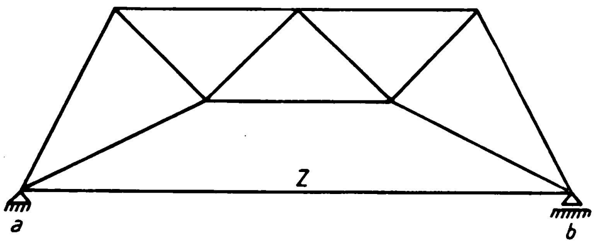

Fig. 3a.

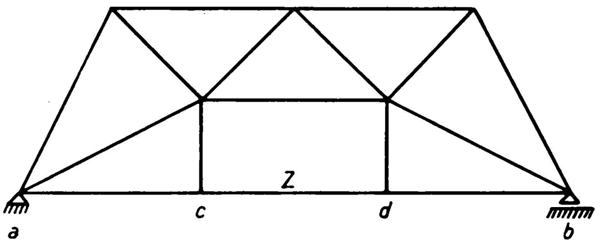

Fig. 3b.

section of the tie. This dependence of the ultimate load P'_{Gr} on the cross section F_z can be expressed by the following conditions:

$$Z = F_z \cdot \sigma_s{}^1 = \frac{34{,}5\ P'_{Gr}}{\dfrac{50{,}64 + l_z\ F_c}{F_z}} \quad \text{for case a}$$

$$Z = F_z \cdot \sigma_s = \frac{34{,}5\ P'_{Gr}}{\dfrac{60{,}54 + l_z\ F_c}{F_z}} \quad \text{for case b}$$

hence for $l_z = 9$ m and $l_z = 3$ m respectively and $F_c = 31\ cm^2$ we receive

$$P'_{Gr} = 19{,}40 + 3{,}52\ F_z \text{ for case a} \tag{1a}$$

$$P'_{Gr} = 6{,}47 + 4{,}21\ F_z \text{ for case b} \tag{1b}$$

If a weak tie is chosen it will be seen that the carrying capacity of the isostatic truss can decrease because of the tie, as for $F_z \longrightarrow 0$; $P'_{Gr} \longrightarrow 19{,}4$ t respectively $P'_{Gr} \longrightarrow 6{,}47$ t compared with 27,9 t for the isostatic structure. Only if $F_z = 2{,}42\ cm^2$ and $F_z = 5{,}1\ cm^2$ respectively (fig. 5) does the statically indeterminate system with tie, assume the same strength as the corresponding structure without tie. The intended reinforcement of the structure by means of a tie proves in fact, apart from other deficiencies, to be a considerable weakening of the structure. It is, however, not reasonable, to think that the actual strength of the structure will fall below the strength of the isostatic system, provided the cross sections are kept unchanged. We have here an obvious contradiction which has its explanation only in the definition of the loading capacity of a structure.

The foregoing deductions lead to the conclusion that the carrying capacity

[1] No deduction is made for weakening of the cross section, the tie is assumed to be an aye bar.

of a once statically indeterminate system is by no means exhausted if a single member is stressed up to yield point; the same scale of judgment as regards the strength or ultimate loading capacity of isostatic systems cannot be applied for hyperstatic systems. After the yield point is reached by one superfluous member, the deformation depends entirely on the remaining members of the isostatic system. These deformations cannot go on growing indefinitely. Therefore, rupture or inadmissibly large deformations are not possible as long as at least one member of the remaining statically determinate system does assume yield stresses. Up to this point an increase in loading of the structure can be effected without endangering the safety of the structure. In general, as shown by *Grüning* in his wellknown treatise "The strength of hyperstatic systems in steel under consideration of frequently repeated loadings" the limit of the carrying capacity for an n-times statically indeterminate system is reached if at least $n + 1$ members are stressed up to yield point.

To define the actual limit of strength of the truss with tie it is necessary to determine the forces in the members of the isostatic system due to an exterior

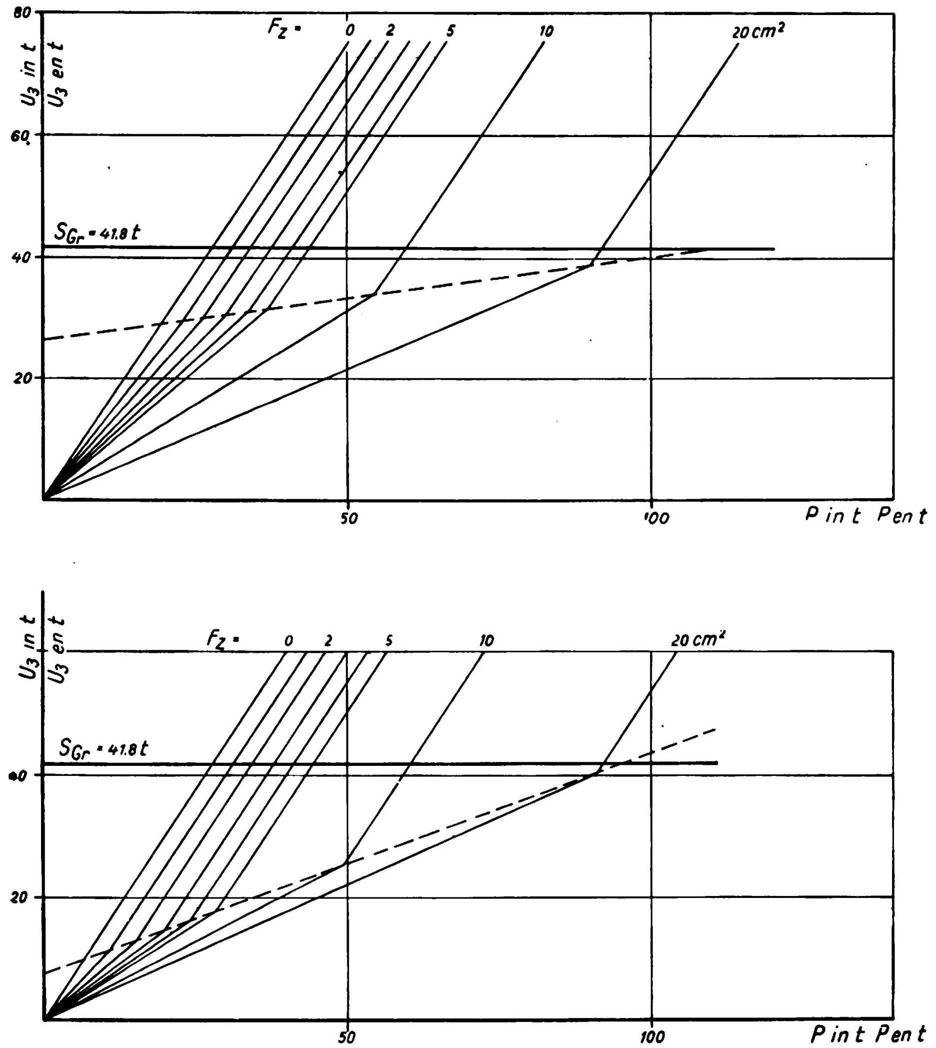

Fig. 4.

Force in member U_3 in relation to load P and cross sectional area of tie F_z.

load P. Under consideration of a constant force $F_z \cdot \sigma_s$ in the tie, but independent of P, the following general equation can be applied:

$$S = S_o - S_a \cdot F_z \cdot \sigma_s \qquad (2)$$

Fig. 4 shows diagramatically the dependence of the force in member U_3, on P, for various values of F_z. Under the assumption of an unrestricted validity of Hook's law we should have

$$U_3 = 1.5\,P - 2 \cdot \frac{34.5\,P}{50.64 + l_z \cdot \frac{F_c}{F_z}}$$

At the moment when the tie is stretched, the forces in all members increase and for U_3 the following value would be obtained:

$$U_3 = 1{,}5\ P - 2 \cdot F_z \cdot \sigma_s = 1{,}5\ P - 4{,}8 \cdot F_z$$

The conditions for rupture are based on the following equations:

$$U_3 = 26{,}4 + 0{,}137\ P \quad \text{for case a}$$
$$U_3 = 7{,}36 + 0{,}362\ P \quad \text{for case b}$$

The ultimate strength of U_3 is 41,8 t; based on this value it is possible in fig. 4 to measure, for any value F_z, on the abscissae the ultimate strength which brings the stress in U_3 up to yield point.

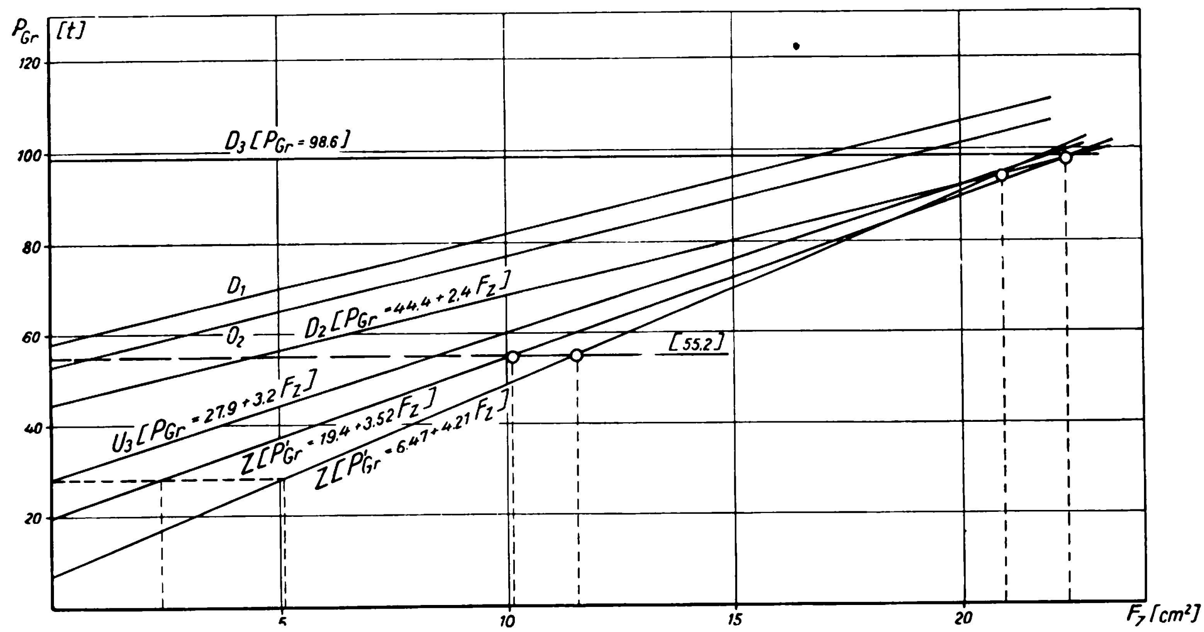

Fig. 5.

Ultimate loads P_{Gr} in relation to cross section, F_z of tie.

These conditions are shown better still and simultaneously for all members of the structure in fig. 5. The ultimate strength P_{Gr} for each particular member can be calculated by using the following formula

$$S = S_o - S_a \cdot F_z \cdot \sigma_s = S_{Gr}, \qquad (3)$$

giving in relation to F_z the following values:

$$
\begin{array}{llll}
\text{for } U_1 & (S_a = +1{,}49067) & P_{Gr} = 188 & +\ 9{,}6\ F_z \\
\text{for } U_3 & (S_a = +2 \qquad\ ) & P_{Gr} = 27{,}9 & +\ 3{,}2\ F_z \\
\text{for } O_2 & (S_a = -1 \qquad\ ) & P_{Gr} = 53 & +\ 2{,}4\ F_z \\
\text{for } D_1 & (S_a = -0{,}7453\ ) & P_{Gr} = 58{,}2 & +\ 2{,}4\ F_z \\
\text{for } D_2 & (S_a = +0{,}9427\ ) & P_{Gr} = 44{,}4 & +\ 2{,}4\ F_z \\
\text{for } D_3 & (S_a = 0 \qquad\quad) & P_{Gr} = 98{,}6 &
\end{array}
\qquad (4)
$$

The equations 4 apply for case a as well as for case b. These values of ultimate strength are shown in fig. 5 for all members of the system as a function of the cross sectional area of the tie. The ultimate strength increases for all members (not however, at the same rate) if the cross sectional area F_z of the tie increases, with the exception of D_3 for which the force is independent of the arrangement for the tie.

Up to a certain value of F_z we receive, apart from the tie, the smallest value of ultimate strength in member U_3 of the bottom chord, hence the carrying capacity of the structure is defined by the following equation:

$$P_{Gr} = 27{,}9 + 3{,}2\ F_z \qquad (5)$$

which coincides for $F_z = 0$ with the isostatic system.

The same value of ultimate load $P_{Gr} = 93{,}9$ t is received for D_2 and U_3 if the cross section F_z measures 20,6 cm². For higher values of F_z and increased loading the yield point for member D_2 is reached earlier. The validity of equations 4 ceases for $F_2 = 22{,}3$ cm² in case a and for $F_z = 20{,}9$ cm² in case b respectively, as for $P = 97{,}9$ t and 94,5 t respectively the yield limit is reached simultaneously in the tie and in the diagonals D_2 and D_5. An increased carrying capacity or safety does not exist although assumed for hyperstatic systems. This fact cannot be altered by increasing the section of the tie as for $F_z > 22{,}3$ cm² and $> 20{,}9$ cm² respectively the two diagonals D_2 and D_5 will still be stretched. The remaining members therefore form a labile system. The relation between the carrying capacity P_{Gr} of the hyperstatic system and the ultimate load P'_{Gr} characterised by yielding of the tie, is given by the following terms:

$$\frac{27{,}9 + 3{,}2\ F_z}{19{,}4 + 3{,}52\ F_z} \text{ for case a, } \quad \frac{27{,}9 + 3{,}2\ F_z}{6{,}47 + 4{,}21\ F_z} \text{ for case b.}$$

The results of these calculations are shown in fig. 6.

The property of hyperstatic systems called "self-help" or "stress distribution" does not develop in every case and under all circumstances; this could only occur under the condition of an excess of sectional area for particular members only. The present case does not give a noticeably increased value of carrying capacity due to section above 20 cm² for the tie, as a load of 90 to 95 t is simultaneously stressing several members of the structure up to yield point.

If the member D_3, whose force is independent of Z, were composed of 2 ∟ 70 · 70 · 7, the ultimate strength would be about 39 t, and the ultimate load 39/0,707 = about 55,2 t. But already with $F_z = 10{,}15$ cm² or 11,55 cm² respectively (fig. 5) the carrying capacity is exhausted, due to simultaneous buckling of the diagonals D_3 and D_4 and the tie being stressed to yield point.

an increase in section of the tie cannot increase the carrying capacity in this case, as the ultimate load $P_{Gr} = 55{,}2$ t remains decisive for D_3 and D_4.

It remains to be considered whether an increase in carrying capacity for the hyperstatic system is possible if a member in compression reaches the critical stress before any other member. Contrary to the member in tension, which in hyperstatic systems still remains a useful member even after the yield limit is reached,

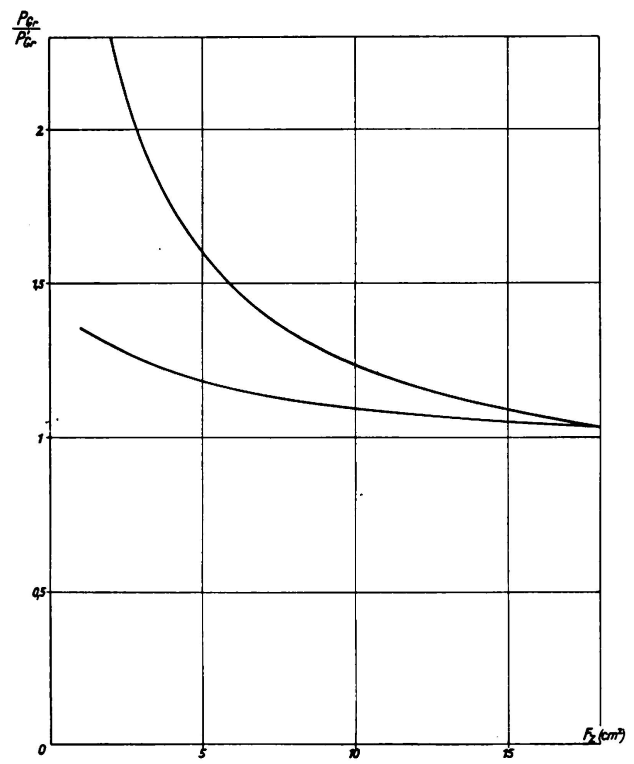

Fig. 6.

a member in compression is rendered useless at the moment of buckling. The buckling of a compression member takes place under the definite condition of bending and displacement of panel points, which can develop in isostatic systems without being influenced by other members. In hyperstatic systems the displacement of the ends of a member in compression is dependent on deformations of the remaining stable system having a tendency to counteract the bending of the buckling member. The condition of buckling for a compression member in a hyperstatic system is distinctly different from the condition of buckling for a single bar, these points have been explained by *Grüning* in his treatise, above mentioned.

2) *Variable positions of loading.*

In the following the case a load P acting at point 2 shall be examined with the object finding a reply to the question what will be the least section of the tie required to increase the carrying capacity of the isostatic system up to a definite value of P_{Gr} (= permissible load $\times$ safety factor). The calculation is based on the system shown in fig. 3b. The forces S_o of the structure without tie and the ultimate loads for each member are shown in table 2.

Table II.

member	Force in member	Ultimate load P_{Gr}
U_1	+ 0,4969 P	141 t
U_3	+ 1,0 P	41,8 t
U_5	+ 0,2485 P	282 t
O_2	— 1,333 P	39,7 t
O_4	— 0,666 P	79,5 t
D_1	— 0,9938 P	43,6 t
D_2	+ 1,257 P	33,25 t
D_3	+ 0,4713 P	148 t
D_4	— 0,4713 P	148 t
D_5	+ 0,6285 P	66,5 t
D_6	— 0,4969 P	87,2 t

The force in the tie has the following value

$$Z = X_a = \frac{29{,}57\ P}{60{,}54 + \frac{93}{F_z}}$$

The yield limit in the tie is obtained if

$$P'_{Gr} = 7{,}55 + 4{,}91 \cdot F_z \qquad (6)$$

and as long as the tie represents the first and highest stressed member the ultimate stressing values for the members U_3, O_2, D_1 and D_2 are defined by the following equations (see fig. 7),

$$\begin{aligned} &\text{for } U_3: && P_{Gr} = 41{,}8 + 4{,}8\ F_z \\ &\text{for } O_2: && P_{Gr} = 39{,}7 + 1{,}8\ F_z \\ &\text{for } D_1: && P_{Gr} = 43{,}6 + 1{,}8\ F_z \\ &\text{for } D_2: && P_{Gr} = 33{,}25 + 1{,}8\ F_z \end{aligned} \qquad (7)$$

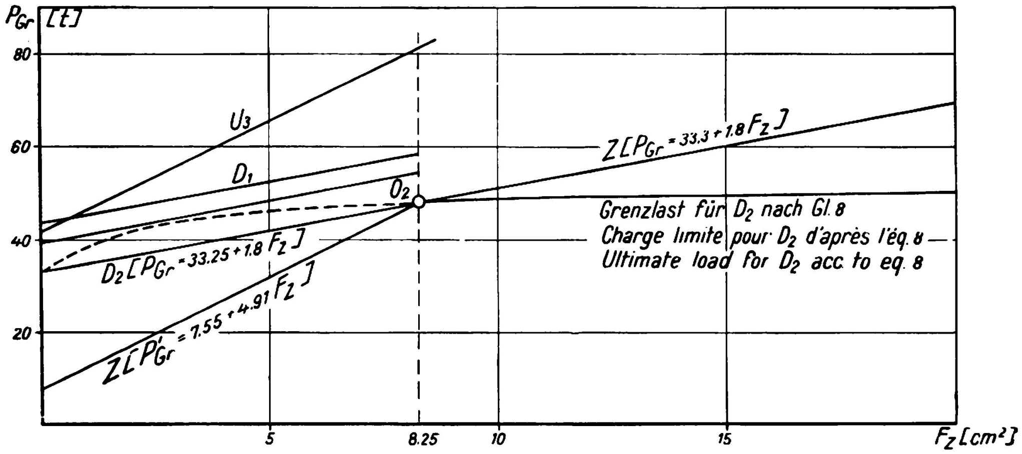

Fig. 7.

all other members do not require to be considered as their stresses are considerably below the ultimate values.

The yield limit is reached simultaneously in the tie for $F_z = 8{,}25$ cm² and in member D_2, due to a load $P = 48{,}05$ t. The carrying capacity is defined by the last of the equations 4, if $F_z < 8{,}25$ cm² and for $F_z > 8{,}25$ cm² the equations (7) become superfluous, on account of D_2 being the highest stressed member and as such yielding first. The ultimate load for D_2, as a function of F_z, in the hyperstatic system assumes the value

$$D_2 = 1{,}257\,P - 0{,}9427 \cdot \frac{29{,}57\,P}{60{,}54 + \frac{93}{F_z}} = S_{Gr} = 41{,}8$$

hence

$$P'_{Gr} = \frac{60{,}54\,F_z + 93}{1{,}15\,F_z + 2{,}8} \tag{8}$$

The results are diagramatically shown in fig. 7. The ultimate load for D_2 changes but little with an increasing value of F_z; it reaches, for instance 48,05 t for $F_z = 8{,}25$ cm² and 50,4 t if $F_z = 20$ cm². The same conditions of equilibrium apply for the other members of the structure, if $F_z > 8{,}25$ cm² and from the moment when D_2 reaches the yield limit, as for the statically determinate system with load P in point 2 and force $F_{D_2} \cdot \sigma_s$ in points 1 and 2 under elimination of member D_2. These forces can be calculated from the "self-stressing conditions" superposing the forces S_o (see table II) using the following equation:

$$S = S_o + (F_{D_2} \cdot \sigma_s - D_{2o}) \cdot \frac{S_a}{D_{2a}} \tag{9}$$

With this equation the following quantities are received for the various members:

$$\begin{aligned}
U_1 &= +\,66 - 1{,}49\,P \\
U_3 &= +\,88{,}8 - 1{,}666\,P \\
U_5 &= +\,66 - 1{,}74\,P \\
O_2 &= -\,44{,}4 \\
O_4 &= -\,44{,}4 + 0{,}666\,P \\
D_1 &= -\,33{,}2 \\
D_2 &= +\,41{,}8 \\
D_3 &= +\,0{,}4713\,P \\
D_4 &= -\,0{,}4713\,P \\
D_5 &= +\,41{,}8 - 0{,}6285\,P \\
D_6 &= -\,33{,}2 + 0{,}4969\,P \\
Z &= -\,44{,}4 + 1{,}333\,P
\end{aligned} \tag{10}$$

These values are independent of F_z. The validity of equation (10) starts for $F_z > 8{,}25$ cm² and $P \geqq P'_{Gr}$ according to equation (8), since the ultimate load for D_2 depends on the cross sectional area of the tie. The ultimate load at which the tie yields, and which therefore is a criterion for the carrying capacity, if $F_z > 8{,}25$ cm², results from the last equation of N° 10:

$$-44{,}4 + 1{,}33\, P_{Gr} = F_z \cdot \sigma_s$$

hence

$$P_{Gr} = 33{,}3 + 1{,}8\, F_z \tag{11}$$

The equations N° 10 allow calculation of the ultimate loads of the other members. The forces in O_2, D_1 and D_2 are constant and independent of P. The smallest values for P_{Gr} are received for members in the bottom chord and are:

for U_1 out of $+66 \quad -1{,}49\ P_{Gr} = -43{,}4$ hence $P_{Gr} = 73{,}5$ t
for U_3 out of $+88{,}8 - 1{,}666\ P_{Gr} = +41{,}8$ hence $P_{Gr} = 65$ t
for U_5 out of $+66 \quad -1{,}74\ P_{Gr} = -43{,}4$ hence $P_{Gr} = 63$ t

The external load $P = 63$ t therefore represents the highest load which the system can carry, whatever the respective cross section F_z of the tie, provided only that

$$33{,}3 + 1{,}8\, F_z \geqq 63$$
$$\text{or } F_z \geqq 16{,}5 \text{ cm}^2.$$

An increase in sectional area of the tie does not prevent the members D_2 and U_5 being stressed simultaneously to yield point, due to $P = 63$ tons.

The tie, for the purpose of solving the problem mentioned at the beginning, shall be dimensioned in such a way that the carrying capacity for the isostatic truss could be increased from 27,9 to 45 t or 60 t respectively. The dimensioning shall follow the principle of equal safety for all members, according to the ideas developed previously, if the external load shall act in point 3 as well as in point 2.

The required section of the tie F_z for an ultimate load of 45 t can be taken from fig. 5 or 7 or equation 5 or 7 respectively:

$F_z = 5{,}35$ cm² for the load in point 3,
$F_z = 6{,}5$ cm² for the load in point 2,

the higher value of the two to be used for dimensioning of the tie.

If the carrying capacity were be exhausted on the tie reaching the yield limit, the required section would be (according to fig. 5 and 7, equations 1b and 6) $F_z = 9{,}15$ cm² or $F_z = 7{,}65$ cm² respectively.

If the load is to be increased up to 60 t, the load in point 3 would demand a sectional aera of the tie of 10 cm² (compared with 12,7 cm² according to equation 1b) and the load in point 2 a section of $F_z = 40{,}8$ cm². This latter section is decisive. If the carrying capacity is considered as exhausted, due to a member being stressed up to yield limit, it will be found that an increase in the carrying capacity up to 60 t due to the arrangement of a tie as in the present case, would not be possible as for the load in position 2, the ultimate value for D_2 remains permanently less than 60 t according to equation (8). (See fig. 7.)

3) *The limits of carrying capacity due to deformations.*

The equations (4) for ultimate loads producing the ultimate forces in members are independent of the form of the tie as shown for instance in fig. 3a or 3b. The carrying capacity of such systems is identical for one and the same ultimate load causing the stress in the bottom chord member U_3, for a given value F_z, to reach the yield limit. The question requires to be considered whether the carrying capacity of both systems is the same and actually independent of the form of the tie. To solve this problem it is necessary to investigate the deformations of the system. Provided that only the tie is stressed up to yield limit, the deformations of the system depend entirely on the deformations of the members forming the remaining stable system. The displacement between a and b for system 3a can be expressed by the following term:

$$E F_c \, \Delta \, ab = \sum S \cdot S_a \cdot s \frac{F_c}{F}$$

wherein the summation applies for all members with the exception of the tie. For

$$S = S_o - S_a \, F_z \, \sigma_s$$

we receive

$$E F_c \, \Delta \, ab = \delta_{ao} - F_z \, \sigma_s \cdot \sum S_a^2 \cdot s \frac{F_c}{F}$$

$$E F_c \, \Delta ab = 34{,}5 \, P - 121{,}5 \, F_z.$$

If the load is increased in such a way as to produce the ultimate strength of U_3, we receive with $P = 27{,}9 + 3{,}2 \, F_z$

$$E F_c \, \Delta ab = 963 - 11{,}1 \, F_z$$

and with

$$E = 2100 \text{ t/cm}^2, \quad F_c = 31 \text{ cm}^2$$

$$\Delta ab = 1{,}48 - 0{,}017 \, F_z \text{ (in cm)} \qquad (12)$$

Correspondingly we receive for system 3b with a similar calculation an elongation of c — d of:

$$\Delta cd = 1{,}48 - 0{,}0536 \, F_z \text{ (in cm)} \qquad (13)$$

The permissible elongation within the range of yield, in other words the elongation for which rupture will occur, cannot be defined properly. As we are concerned only with fundamental deduction we are permitted to assume these values to 5 ‰ and we accordingly receive the permissible elongations of the tie as under

for $l_z = 9$ m $\quad \Delta \, l = 4{,}5$ cm
for $l_z = 3$ m $\quad \Delta \, l = 1{,}5$ cm

It will be seen that for the ultimate load, the actual displacement Δ a b or Δ c d respectively, previous to the yielding of U_3, is less than the permissible elongation within the range of yield. Therefore the deformation in both cases is of no importance as regards the carrying capacity of the systems.

The equation (4) would remain valid for a tie having a central portion composed of two ∟ 70.70.7 over the length of 10 cm only and the displacements of the ends of the central portion of l_z would be for

$$\sum S_a^2 \, s \frac{F_c}{F} = 65{,}33:$$

$$\Delta = 1{,}48 - 0{,}0713 \, F_z \text{ (in cm).} \qquad (14)$$

The permissible elongation within the range of yield is only 0,05 cm but is considerably exceeded due to deformations and other reasons before the member U_3 is stressed up to yield limit. Thus we have the possibility that a member being stressed first up to yield limit may fracture before another member becomes capable of taking over a portion of the load. A compensation of forces among the members cannot develop, unless incomplete.

The correctness of the statement has been proved by tests carried out with a continuous trussed girder in steel in the test house in Hannover[2].

The estimation of the carrying capacity of hyperstatic systems according to the ideas developed above is based on the assumption that the deformations of stretched members remain within certain limits of the range of yield, an assumption which can be considered as fulfilled in general.

4) *Temperature changes (elastic supports).*

Apart from the influence of load P acting in point 3 the variation of temperature from $\pm t^0$ shall be considered. The ultimate load P'_{Gr} of the tie will be influenced by such a change in temperature. In case of an increase or decrease in temperature of 15^0 respectively we receive for system 3 a

$$P'_{Gr} = 19{,}4 + 3{,}52\ F_z \pm 3{,}06.$$

The equations (4) still apply. The change in temperature influences only the values of deformations. A simple consideration shows that the deformations of stretched members can go on increasing indefinitely provided the tie is subject to repeated changes of temperature. The value of the load P remaining little under the ultimate load U_3, defined by equation 4, and assuming an increase in temperature of t^0 for the members of the system above the temperature of the tie, we find that the conditions of equilibrium remain unchanged. The tie is subject only to an additional elongation of $\varepsilon \cdot t \cdot l_z$. If the tie is subject to an increase in temperature, the stress in member U_3 will reach the yield limit and with the subsequent variation of temperature between $-t^0$ and $+t^0$ the tie and the member U_3 of the bottom chord will be subjected alternately to new additional elongations, which are not subject to restrictions. In this connection we wish to draw attention to the results obtained by tests carried out in Hannover[2] where the influences of uneven temperature in the chords of a girder were studied through the lifting and lowering of the extreme supports.

In case the temperature variations have to be considered, the values of deformations must be taken into account for judging the carrying capacity of a system. Elastic displacements of supports are of the same importance, but unelastic displacements (subsidence of the soil) are without influence on the carrying capacity.

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[2] *Grüning-Kohl:* Tragfähigkeitsversuche an einem durchlaufenden Fachwerkbalken aus Stahl. „Der Bauingenieur" 1933, p. 67/72 (Versuchsreihe II).

Summary.

The carrying capacity of a hyperstatic system is not dependent as a rule on the critical stress of a single member, as for isostatic systems. A few fundamental questions important for the judging of the carrying capacity have been explained by the example of a truss with tie. The actual ultimate load (safe actual load × safety factor) has been calculated for various positions of loads in relation to the cross sectional area of the tie. The investigations have revealed that an increased carrying capacity or safety does not always exist with hyperstatic systems as expected, compared with isostatic systems.

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I 6

The Safety of Structures.

Sicherheit der Bauwerke.

Sécurité des constructions.

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I. During the last few years it has become necessary, when studying the strength of structures, to consider the plastic properties of matter.

The strength of materials had been based primarily on the hypothesis that, when the forces, to which a structure is subjected, increase, the deformations and rate of stressing increase everywhere in straight line proportion.

The pure elastic behaviour of materials does not exist except in its first approximation; this hypothesis must be abandoned in every case where the stressing conditions attain such values which may cause fracture or where at least great deformations are produced.

If the effect of the non-elastic properties of materials is favourable to the stability or not, depends on circumstances.

Contrary to the stability met with in the case of a member compressed longitudinally tending to buckle, the effect is usually regarded as increasing the stability of statically indeterminate systems.

II. The writer proposes to discuss this problem in the present paper. First of all, to arrive at the degree of safety of a structure, we have to consider how it behaves under increasing loads until fracture or a vital amount of deformation occurs.

This means replacing the modulus of elasticity E of the elastic period by instantaneous coefficients of elasticity H' corresponding to very small supplementary loads. These coefficients of elasticity H' will vary from one point to another, and will also differ according to whether the particular additional load increases or reduces the previous stress.

In this connection it may be mentioned that this consideration of instantaneous coefficients of elasticity explains the results of the researches on buckling carried out by *M. Roš* at the Zurich Polytechnic.

When a compressed member (Fig. 1) is subjected to a transverse force, for instance, the bending set up in the member increases the compression on the concave side and reduces it on the convex side. The stress-strain diagram for sections parallel to the axis is in this case a broken line (Fig. 2). The mean coefficient of elasticity which should be inserted in Euler's formula is always lower than the usual coefficient of elasticity, so that the limit can be reached when compression is still below the amount determined by the usual calculation.

Similarly, in a statically indeterminate system, the deformations may take place rapidly from the moment when the loads have caused the materials to exceed the elastic limit, and there is a risk of relying upon assumptions which are too simple for fixing the degree of safety.

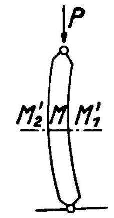

Fig. 1.

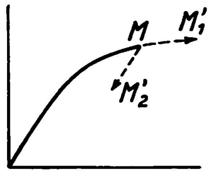

Fig. 2.

III. The following remarks may indicate the correct lines of this discussion.

The relation existing between the deformations of a member and the stresses represents a property of the material. It is an *intrinsic* law peculiar to the material combining these variables (Fig. 3).

In a structure, however, none of these variables may usually be regarded as a known quantity, and it is no more logical to say that the deformation is a function of the stress than it is to assume that the stress is a function of the deformation.

Actually, in a member AB belonging to a particular structure, equilibrium between the deformations and the stresses takes place under conditions which involve not only the elasticity of the material inside the member A B, but also the elasticity of the system outside of A B.

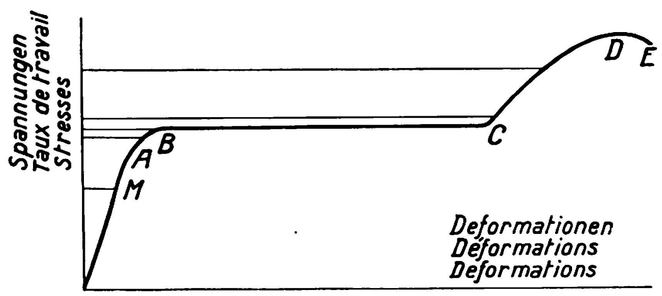

Fig. 3.

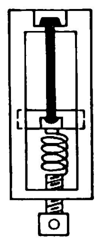
Fig. 4.

If, for instance, a test piece be stretched by means of a screw jack supplemented by a spring (Fig. 4), the pull deforms both the test piece and the spring, and the elasticity of the spring must be known before we can judge what is taking place in the test piece. For each position of the screw, we have to ascertain where the characteristic point on the stress-strain diagram (*intrinsic* curve) joins up with a curve representing the *extrinsic* law and which connects up the distance between the jaws and the stress which they transmit.

Similarly, if increasing loads be applied to a statically indeterminate system, the characteristic point referring to the member subjected to the highest stress on the *intrinsic* stress-strain diagram must be referred to a series of *extrinsic* curves (Fig. 5). These curves slope down because, in each case, the forces applied to the member in question diminish when the deformation increases.

Now consider the state of two members of the same metal and belonging to isostatic and hyperstatic systems respectively, and subjected to certain low stresses applied at the same rate. It will be assumed that the loads, and the stresses which would be applied to the members if they were not deformed per se, are proportional (Fig. 6).

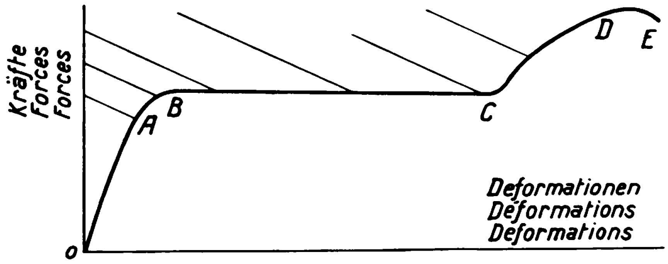

Fig. 5.

If the deformations and stresses be multiplied by a certain factor, the points M representing the state of the members will be substituted by a point P located on the elasticity line very close to the curve A B. If the load is multiplied by a lower coefficient the two members will assume equivalent positions; whereas, if the load is multiplied by a higher coefficient, the first member immediately borders on the zone C D of high permanent deformations, while the

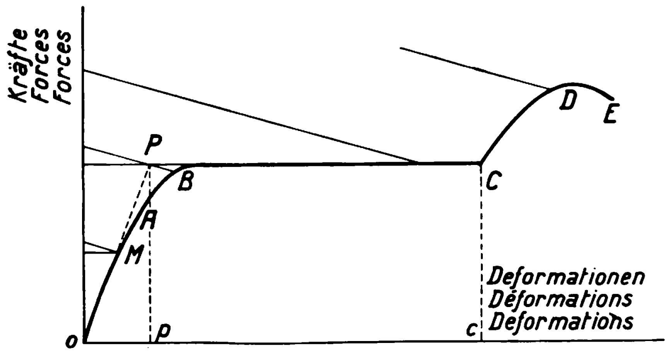

Fig. 6.

second will first of all follow the horizontal line of the elastic limit. The gain is greater, the higher the ratio

$$n = \frac{pc}{op} = \frac{\text{Deformation when Elastic Limit is exceeded}}{\text{Elastic Deformation}}.$$

It is also proportionally greater when the law of extrinsic deformation shows a rapid downward tendency.

This ratio n, which plays an essential part in the subject under discussion, may be termed the 'ductility number'.

For structural steel 42/25, the usual numerical data are as follows:

Relative deformation after line of elastic limit = 0.027;

Modulus of elasticity = 22.10^3 kg/mm²;

Elastic limit = 26 kg/m².

Hence:

$$n = \frac{0.027 - \frac{26}{22.000}}{\frac{26}{22\,000}} = 22.$$

IV. The preceding considerations may be applied not only to the hyperstatic systems, where the distribution of the stresses is only determined with regard to the deformations, but also to bent elements of statically determinate systems. In the latter case, plasticity governs the distribution of the stresses in a section.

Actually, when the stresses increase in a hyperstatic structure or in a curved member, the zones subjected to the highest stress undergo plastic deformations from the moment they reach the elastic limit. The load conditions of the structure may then increase without the stresses increasing in the zones subjected to the highest loads.

This is the property which is termed plasticity.

The non-elastic deformation which takes place at the elastic limit is the elastic deformation at the elastic limit multiplied by a coefficient n which depends on the nature of the metal. Generally speaking, n is roughly 22.

The difference between the load at which the elastic limit is reached and the load at which it is exceeded is relatively greater, the higher the coefficient n. It is also greater when the member or element stretched is subjected to a stress which more rapidly decreases with the deformation.

Case of a Constant Load.

1. We will assume that a number of parallel bars B are secured by their ends so that their elongations are equal, and that a constant force F is applied to the combination. The characteristic point of the deformations and the rate of stressing $\frac{f_1}{s}$ describes, for a given bar B, a portion of the characteristic curve of the metal. If H is the function representing $\frac{f}{s}$ in terms of the relative elongation, the intrinsic law referred to above may be written:

$$\frac{d\,f_1}{s} = H'_1 \frac{d\,l}{l} \tag{1}$$

As regards the extrinsic law, it is obtained by adding the intrinsic laws applying for the other bars.

A simpler way is to add up, member by member, the equations (1) written for all the bars, when we get:

$$d\,F = d\,l\,\Sigma \frac{s\,H'}{l} \tag{2}$$

For the bar in question, the proportion of the additional stress absorbed by it is expressed by the relation:

$$\frac{d\,f_1}{d\,F} = \frac{s\,H'_1}{\Sigma\,s\,H'} \tag{3}$$

This quantity is obviously constant if the characteristic points of all the bars coincide, but only where all of them have been secured at their ends without any internal stress.

The case is different, however when (a) there are differences between the lengths of the various bars measured in the neutral state or between their supports, (b) certain bars have been put under tension before the other bars have been fixed, or (c) these two factors intervene.

The characteristic points will then describe the same curve without joining; but, as we saw above, the differences in length, i. e., the differences in the abscissae, will be maintained.

The additional stress given by (3) for the bar which undergoes the greatest amount of deformation, decreases when this particular bar approaches the elastic limit. The same will apply to the other bars in turn, until, the elastic limit being attained everywhere, the additional stresses are taken up by the bars most deformed.

As the factor n is very high in actual practice, equalisation will actually take place, which means that the bars subjected to the lowest loads reach the elastic limit before the stress rises again in the bars subjected to the highest loads.

Comparing this result with that derived from the theory of elasticity, it will immediately be obvious that the stretching of the metal simplifies the process of deformation. It equalizes the tensions, and makes the condition under which the elastic limit is exceeded independent of the initial conditions obtaining in the system.

As regards the exception mentioned above, it would only apply where the member subjected to the highest load exceeded the elastic limit N before the member subjected to the lowest load had. This would mean that the differences between the relative elongations would exceed the length of the horizontal B C, i. e., approximately n times the elastic extension $\frac{N}{E}$ at the elastic limit. A case of this kind may be regarded as exceptional and inadmissible.

II. The foregoing applies without restriction to the case of members braced 'in parallel' like the booms of a truss. Mistakes in erection have no effect on their capacity to resist at the elastic limit.

The same thing also applies to the members of a lattice girder arranged in the same or opposite directions, or where a lattice girder is strengthened by members fitted without tension.

III. A more general case is that in which associated members are deformed unequally, but at constant ratios. We then get:

$$\frac{d\, l_1}{l_1\, \beta_1} = \frac{d\, l_2}{l_2\, \beta_2} = d\, \alpha, \tag{4}$$

α being a variable parameter and β constants.

The geometrical construction of Fig. 7 shows how the stress rates of the different members are deduced from each other. The additional stress taken up by the member B is:

$$\frac{d\, f_1}{d\, F} = \frac{s_1 \frac{\beta_1 H'_1}{l_1}}{\Sigma\, s \frac{\beta H'}{l}}. \tag{5}$$

The least deformed bar reaches the elastic limit before it is exceeded by the most deformed member, if the ratio of their elongations is lower than $(n + 1)$.

Owing to the ductility factor being very high, the tensions are most usually equalised on the elastic limit line.

Fig. 8 shows how the balanced average of the deformations varies in terms of the balanced mean of the stress rates. The curve obtained is much more rounded than the actual deformation curve.

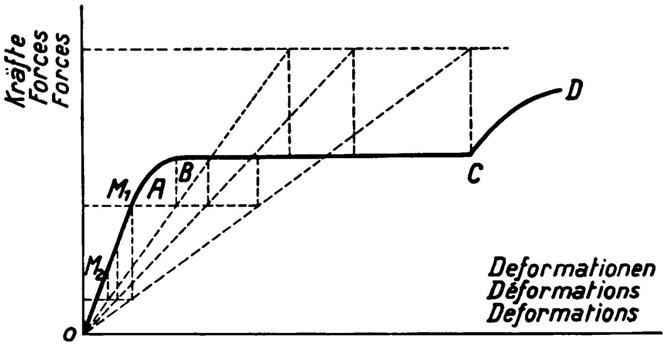

Fig. 7.

Calling k the ratio of the mean elongation to the elongation of the member supporting the highest load, the condition for which the stress rates become equal at the elastic limit is expressed by the following proximate relation:

$$\frac{1}{2k - 1} < n + 1. \tag{6}$$

IV. The above considerations show that, by assuming the same stress rate for each member, we are assuming an approximation which is usually not justifiable but which becomes so at the elastic limit. Since this calculation assumes a well defined limiting stress for each member, it is taken to justify the view which holds that each member or each part possesses its own strength capacity, the capacity of the whole being the sum of the individual capacities,

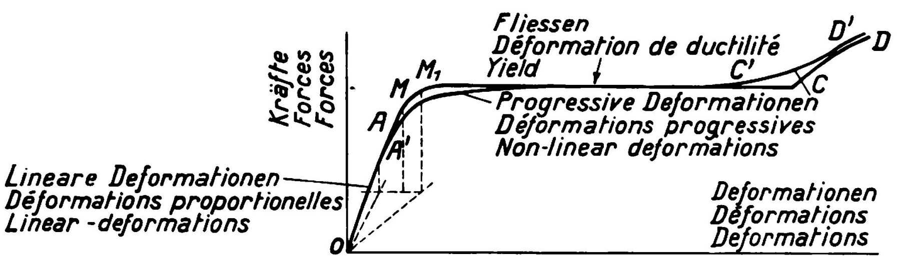

Fig. 8.

This method, which is applied particularly in the calculation of trusses, is therefore justified to a certain extent when the several elements work under conditions which are not too dissimilar.

V. It is well in this relation to verify whether allowing for plasticity in the calculations does not reduce safety.

Security signifies either a guarantee against the risk of the structure being ruined, or a guarantee against the risk of high deformations making the structure unsuitable for use. Buckling is involved in both these considerations.

In every case, safety is defined by a coefficient, and, when defining this coefficient, we cannot do better than take it as being equal to the factor by which the loads have to be multiplied in order to reach a dangerous condition.

From our previous arguments we find that, if M is the maximum fixed for the average rate of stress, the coefficient of safety after the elastic limit N is exceeded, is N/M.

VI. Ignoring the case where buckling takes place below the elastic limit, two cases may be considered:

(a) Where the buckling limit Φ *is very little above the elastic limit* N.

In this case, it should be noted that, as soon as the loads have exceeded rates which correspond to the level of the elastic limit, the surplus loads come exclusively on to 1, then 2, etc. .... members.

The amount by which the applied loads must be increased so that the member bearing the highest load reaches the buckling limit Φ can thus be extremely small, as will be seen from the diagram (Fig. 9) of the average stress applying in this particular case. It would thus appear that a considerable amount of caution is necessary and that, for the average stress rate, a limit should be taken, calculated as if the buckling limit were equal to the elastic limit.

(b) Where the members are all strained or their buckling limit is close to the breaking limit.

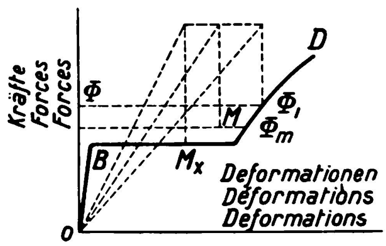

Fig. 9.

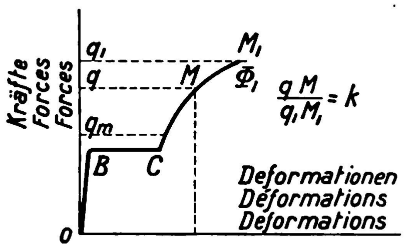

Fig. 10.

In this case it may happen that, when the characteristic point M_1 of the member bearing the highest load attains its danger limit Φ, the other characteristic points are all on the curve C Φ_1. This is generally the case when the ratio between the maximum plastic elongation and the critical elongation is lower than 2 k — 1.

The centre of gravity M of the characteristic points is then very close to the curve C Φ_1 itself (Fig. 10). The critical value of the stress rate may therefore be roughly defined as the value which corresponds, on the deformation curve, to the critical elongation multiplied by k.

This being the case, the factor of safety of an assemblage of members jointly stressed may be defined as follows:

'The conditions of resistance of the member supporting the highest load are appreciably the same as those of an imaginary member loaded at the average rate, but whose buckling or breaking limit is reduced.

'Where the buckling limit is not very much higher than the elastic limit, the strength limit of the imaginary member must be taken as being still closer to the elastic limit.

'Where the buckling limit is high, or where the members are all stretched, the dangerous elongation must be multiplied by k in order to find, on the elongation curve, the dangerous rate of stressing.

'In all cases, the factor of safety in terms of buckling or failure, must be taken as equal to the ratio between the limit thus determined and the average rate of stressing.'

From the above we may deduce that, if a hyperstatic system is similar to those which we have just examined, and if great deformations have begun to take place, the margin of safety which then obtains is lower than if the structure were statically determinate.

VII. These principles may be applied to the case of a plate drilled with rivet holes, by likening it to a group of associated fibres. The deformations are augmented locally in the ratio of 1 to 3. The ratio k therefore equals $^1/_3$.

Consequently, the breaking limit is reached at the edges of the holes, which means that cracks will develop for a critical value of the average stress rate obtained by multiplying the elongation at the breaking limit by $k = {}^1/_3$. With standard structural steels, therefore, the stress rate is roughly equal to $^4/_5$ of the breaking limit.

VIII. We shall now examine, by a few examples, the case of straight, statically indeterminate lattice girder:

(a) 45⁰ Lattice Girder with Uniform Flanges and two Supports and restrained at one End. (Fig. 11.)

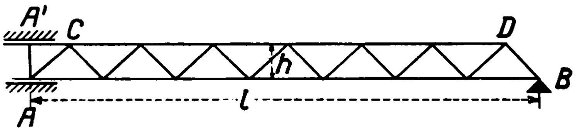

Fig. 11.

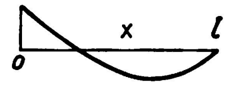

Fig. 12.

When the loads increase, the member A' C will normally be the first to reach the elastic limit. The remainder of the structure will then behave like a simple girder ACDB subjected to a force s N acting along CA'. The moment acting on the support is

$$M_0 = s\,N\,h \tag{7}$$

In the girder, the moment M is equal to

$$M = \mu - s\,N\,h\,\frac{l - x}{l} \tag{8}$$

μ being the moment of the girder on two supports. As the loads go on increasing, the deformations will also continue and attain (1) the elastic limit in the bay, and (2) the upper limit of ductility in the member A' C. Generally speaking this latter limit is reached last, and in this case the elastic limit is reached in the bay where the maximum of (6) is equal and its sign opposite to the moment on the support (7).

If the girder is uniformly loaded or supports a variable load P, the maximum moment which the most heavily loaded section of the girder has to be capable of resisting is the fraction $\frac{4}{5.83}$, or roughly $^2/_3$ of the maximum moment of the same girder, assuming it has two simple supports (Fig. 12).

It is, however, necessary to verify that the deformation of the member A' C under these conditions does not exceed the maximum deformation of ductility. This deformation is obtained by calculating the rotation ω'_0 of the neutral fibre of the girder near to A:

$$\omega'_o = \omega_o - \int_0^l \frac{s\,N\,h}{E\,I}\,\frac{(l-x)^2}{l^2}\,d\,x = \omega_o - \frac{s\,N\,h\,l}{3\,E\,I} = \omega_o - \frac{2\,l\,N}{3\,E\,h} \qquad (9)$$

ω_o being the rotation of the girder resting on simple supports.

Since the lattice is assumed to be at 45^0, ω'_o thus represents the relative elongation of A' C = h, whence we must get:

$$\omega_o - \frac{2\,l}{3\,h}\,\frac{N}{E} < (n+1)\,\frac{N}{E} \qquad (10)$$

Calling ω_o the value equivalent to the maximum and uniformly distributed load, we get:

$$\frac{l\,N}{3\,h\,E} - \frac{2\,l}{3\,h}\,\frac{N}{E} < (n+1)\,\frac{N}{E}$$

or:

$$\frac{l}{h} < 3\,(n+1) \qquad (11)$$

This inequality is satisfactorily met by girders of ordinary dimensions and by the usual types of steel.

(b) Girder with two rigidly connected Bays, made up as above.

The case is the same, the section on the central support acting as a rigidly supported section.

On the other hand, if the supports are not at the same level, a correction factor is added to ω'_o which may increase the value of the first term of (11), but is generally too small to upset inequality.

(c) Girder firmly anchored to its two Supports, or Bay rigidly connected to two other Bays; the same make-up as above.

In these two cases, the inequality of the moments on the supports or in the bay takes place by reducing to one-half, the value of the maximum moment in the bay having single supports. The condition (11) is replaced by the following, which is satisfied more easily still:

$$\frac{l}{h} < 6\,(n+1) \qquad (12)$$

Quite definitely, and apart from abnormal cases, if the girder be calculated by assuming twice the inequality of the maximum moments for the bay, and calling the maximum rate of stress Π, the coefficient of safety in terms of over-extension of the elastic limit N equals $\frac{N}{\Pi}$.

IX. We now come to the case of a solid web girder of constant and continuous section, or restrained at one at least of its supports.

The increased load brings the stress at the elastic limit into the region of the double supports, with a consequent increase in the volume which is thus rendered plastic. A plastic deformation is then set up in the middle region of the bay, but it is not clear that this phenomenon precedes the case of loading where the first plastic zone exceeds the elastic limit. To determine this particular case involves ascertaining the extent of this zone.

There is every reason to think that, in a straight section, even one partially subject to plasticity, the law of relative deformations of the longitudinal fibres remains linear with respect to the distance from the neutral fibre.

Taking O as the point of the neutral fibre, we shall suppose the stress-strain diagram plotted on any scale for the section of the girder. By projecting a point of the cross-section along x on a given straight line O D, then along the y-axis on the curve, the corresponding stress rate is obtained (Fig. 13).

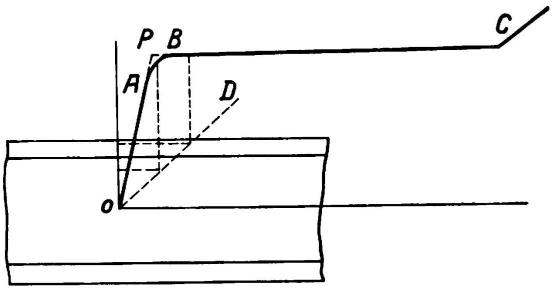

Fig. 13.

When the bending moment increases, the straight line turns in a clockwise direction. It varies from a quantity M_o to a quantity M_1 in the plastic zone, while the characteristic point shifts from B (which may be assimilated to the point P) to C. This gives us:

$$M_o = \int 2 \frac{N v^2}{h} d\sigma = N \int \frac{2 v^2}{h} d\sigma \tag{13}$$

and:

$$M_1 = \int N v d\sigma = N \int v d\sigma \tag{14}$$

v being the distance from the neutral fibre and $d\sigma$ the element of surface of the section.

The difference is equal to

$$M_1 - M_o = N \int 2 \frac{v}{h} \left(\frac{h}{2} - v\right) d\sigma \tag{15}$$

which may be reduced to the value of the integral for the web alone. If b is its thickness, then:

$$M_1 - M_o = N \frac{b h^2}{12} \tag{16}$$

The moment varies between the two values M_o and M_1 following a parabolic law and in terms of the height Z of the non-plastic portion:

$$M = M_o + \frac{N b}{12} (h^2 - Z^2) \tag{17}$$

The variation of the bending moment thus supplies the type of boundary which limits the plastic zone. It is parabolic where the shearing stress is constant, and straight where the shearing stress varies in a straight line.

The plastic volume is greatest when it is limited as shown in Fig. 14.

We get:

$$d = \frac{b h^2 N}{12 T} \tag{18}$$

Let us assume that the thickness of the web is calculated exactly to work at the elastic limit for a shearing stress equal to T, i. e., so that b h N = T.

From (18), we find that the half-length of the plastic zone attains the value

$$d+\frac{a}{2}=\frac{h}{12}+\frac{a}{2} \qquad (19)$$

If the web were calculated very accurately, this would mean that the height of the girder would not have to be less than $^1/_6$ of the span if it were desired that the elastic limit be not exceeded on the support before being reached in the bay.

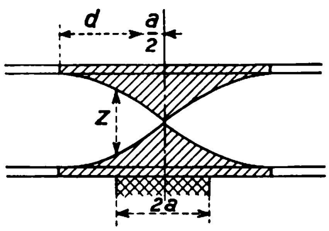

Fig. 14.

We therefore find that the length of girder bent plastically is only a small fraction of the height. The condition (11) must be replaced by a condition which is harder to satisfy:

$$\frac{b\,h^2\,N}{12T}+\frac{a}{2}>\frac{l}{3\,(n+1)} \qquad (20)$$

Paradoxical as it may seem, the conclusion is that, if we are to take full advantage of the plasticity of continuous girders above their supports, the web must be strengthened to a greater extent, the thinner the section of the girder. Such reinforcement must cover approximately $^1/_{60}$ of the span of the girder.

In the contrary case the girder pivots, as it were, to the right of its support only with a thrust at the end, and a break will tend to occur in the shape of the girder if the web is too weak.

From the standpoint of plasticity, reinforcing the webs and providing stiffening uprights near the supports are particularly interesting solutions, and call for broad treatment in the region of the points of fixation and the double supports.

X. It seems desirable to consider not only the risk of the occurrence of large deformations in beams where the elastic limit is exceeded in several regions, but the risk of breakage as well, unless of course this particular point has already been taken care of in connection with the particular location of the structure.

We shall therefore now study the variation δ M of the bending moment corresponding to a slight increase of the loads. The variation in curvature is expressed by the equation:

$$\delta\left(\frac{d^2 u}{d x^2}\right)=\frac{\delta M}{H' I} \qquad (21)$$

The coefficient H' is nothing more than the instantaneous coefficient of elasticity of the booms, where the girder is a webless one. In the case where the web intervenes in the bending, H' I is defined as the sum of the moments of the elements of area of the section, multiplied by the corresponding H' values.

Let us take, for example, the case of a girder restrained at both ends. The deviation of each end being nil, we get:

$$0=\int\delta\left(\frac{d^2 u}{d x^2}\right)d x=\int\frac{\delta M}{H' I}\,d x \qquad (22)$$

Now M is the sum of the moment ($-M_o$) on the supports and of the moment μ which would obtain if the girder were laid on simple supports. We therefore get:

$$\delta M_o \int \frac{dx}{H' I} = \int \frac{\delta \mu}{H' I} dx$$

or:

$$\delta M_o = \frac{\int \frac{\delta \mu}{H' I} dx}{\int \frac{1}{H' I} dx} \tag{23}$$

This expression should be compared with equation (3) referring to members braced in parallel. In the present case, the sections which bend are braced in series, and this explains why the coefficients H' become denominators instead of numerators.

The general equation (23) enables us to resume the discussion: *Elasticity Phase:* H' is everywhere equal to E, which gives us simply:

$$\delta M_o = \int \frac{\delta \mu}{I} dx \tag{24}$$

Plasticity Phase on the Supports: Since H' becomes nil at the supports, the adjacent regions supply, in the integrals, main terms which, in the numerator, are multiplied by very small quantities $\delta \mu$. The quotient becomes appreciably nil, thus:

$$\delta M_o \approx 0 \tag{25}$$

Bay Plasticity Phase: As H' becomes nil in a region C where the variation $\delta \mu_c$ is not nil, the quotient of the main terms of the two integrals of (23) then becomes

$$\delta M_o = \delta \mu_c$$

We therefore get $\delta M_o = -\delta M_o + \delta \mu_c = 0$ (26)

High Deformation Phase: The plastic zones move away from the parts which have exceeded the elastic limit, and we find that $\frac{\delta M_o}{\delta \mu_c}$ diminishes without dropping to the value $\frac{1}{2}$ corresponding to the case of the statically determinate girder.

Fig. 15 shows the variation in the rate of stress in terms of load.

Similarly, in the case of a doubly supported girder, the rate of variation after equalization is roughly 1.5 times what it was on the average before. This is the inverse of the relation connecting the bending moment which obtains during equalization of the moments, to the bending moment which would then obtain without the continuity.

In other words, if plasticity first attains the rate of stress in the supported sections, it will act in the opposite direction afterwards.

It will be seen, then, that the rate of stress on the support reaches the buckling limit Φ of the member under compression for an increase in load of $\frac{N + \Phi}{2 \Pi}$ in the case of two double supports and $\frac{N + 2 \Phi}{3 \Pi}$ in the case of one double support.

Consequently, by determining the maximum value Π of the stress rates in a beam comprising one or more double supports, and assuming that these rates are equalized in the highest possible number of sections, the factor of safety in terms of excess of the elastic limit N is $\frac{N}{\Pi}$.

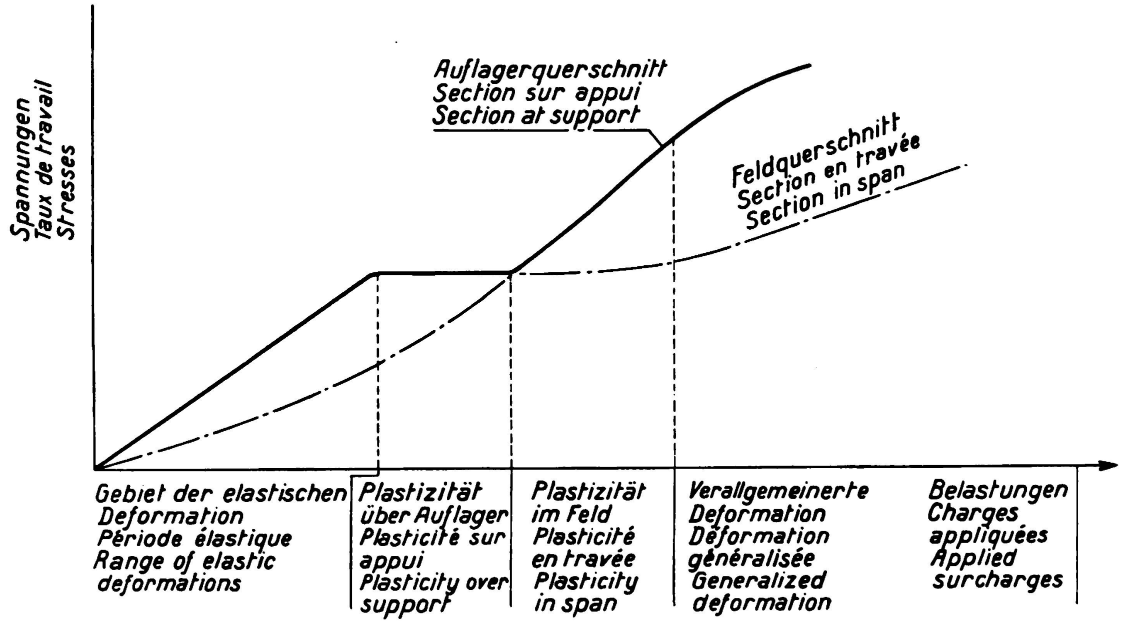

Fig. 15.

But as regards the factor of safety in terms of the buckling of the member under compression, it comes between the foregoing and the ratio $\frac{\Phi}{\Pi}$. The interval between these two factors of safety must be reduced in the same proportions as the maximum bay moment is relatively to the moment of the same bay resting on single supports.

XI. The case of statically indeterminate arches is in some respects similar to the case of continuous or fixed beams.

We shall therefore see what takes place in the section subjected to maximum stress and where the pressure curve passes through its centre of gravity. When the loads increase, the normal stress exceeds the elastic limit immediately after having attained it. The arch does not then derive any benefit from plasticity unless its lowest strength sections are under bending stress. This may be the case for certain load conditions in parabolic arches, in fixed arches with light section in $^1/_4$-span, and in two-hinged arches with light section at the crown.

Apart from this case, the elastic limit is not attained simultaneously at all points of the most heavily loaded section. Take the case of the double-boom arch: when the section of one boom is plastic, the increased load will set up an additional stress in the other boom only, which means that the pressure line for the additional loads passes through the latter boom. Generally speaking, plasticity has the effect of shifting the total pressure line to the centre of gravity of the plastic sections (Fig. 17).

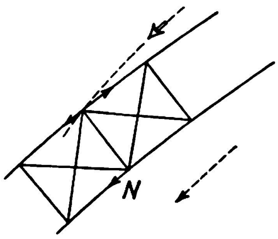

Fig. 16.

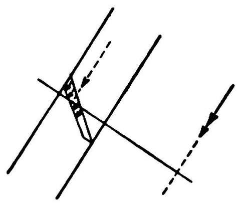

Fig. 17.

In this case, the interval between the moment when the first plastic zone appears and the moment when the entire section attains the elastic limit is equivalent to an increase in load which may be large. In the same interval, the plasticity extends along the arch.

An arch is thus capable of undergoing considerable deformation at various parts of its low-strength sections, without exceeding the elastic limit, unless the pressure line passes through the centre of these sections. During this deformation the arch behaves as if it were hinged.

Except in abnormal cases, the elastic limit is successively reached, without being exceeded, at n points, where n is the degree of static indetermination for a given load value.

In order to ascertain these n sections, as also to find the safe working stress Π for a particular load, the designer should proceed on the following hypotheses:

(1) On the assumption of elastic deformations, ascertain first the sections which are most exposed to the distribution of the particular loads (uniformly distributed load, concentrated load, etc.).

(2) Ascertain the load for which the limit of safety Π is attained in these sections.

(3) Compute the effect on the other sections, of additional loads considered separately, assuming that their line of pressure passes through the half of the sections, previously found, which comes opposite to the points of maximum load.

(4) Ascertain the load for which the total stresses, defined by (1) and (3) reach the maximum value Π.

The sections in which this maximum stress is set up, and the sections found previously, are those which are deemed to work at the limit of safety Π under the particular loads considered.

The factor of safety in terms of over-extension of the elastic limit N is then really equal to $\frac{N}{\Pi}$, since increasing the load in this ratio has the result of raising to the elastic limit the sections defined first of all, and then the others.

For any fresh increase in load, the stress rates rise in the arch as a whole, except in the zones that are subjected to plasticity in the first place.

Consequently, the breaking limit will be reached for a fresh increase in load which is less than $\frac{R}{N}$, which means that the factor of safety in terms of failure lies between the ratio $\frac{N}{\Pi}$ and the ratio $\frac{R}{\Pi}$.

XII. Great caution is necessary in generalising the above results.

The number of bars or members which may be omitted must not be taken as the degree of static indetermination of the structure as a whole. It is the latter factor alone which must be allowed for when applying the theory of plasticity.

In the cantilever system, Fig. 18, for instance, it would not be right to assume equalization of the moments on the double support B and in the bay A B, even if a diagonal member were added to render the girder A B C hyperstatic. The whole does not become hyperstatic, and the plasticity theory only applies where a bar is added above the point C.

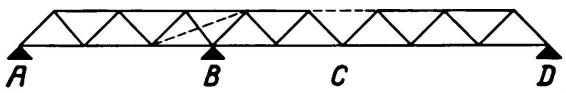

Fig. 18.

In complex systems, on the other hand, it is essential to define by degrees the sections which become successively plastic.

Care must also be taken to ensure that the zones affected by plasticity are not capable of setting up a failure in the neutral fibre before the elastic limit is reached in the other sections involved. It must also be remembered that the stress rates allowed in the calculations involving plasticity can only serve as a basis of comparison for determining the factors of safety in terms of excess of the elastic limit. In terms of failure, the factors of safety may be very little higher than the foregoing.

Variable Loads.

I. Where certain states of load recur, and where the elastic limit is not exceeded at any point, there is every reason to believe that the deformations and stresses recur in cyclic form. This view is not invalidated by experience.

We have tried to verify that this is the case when a member alternates between two limits characterised by well defined extrinsic laws, the upper limit of which is sufficient to make the member plastic.

For this purpose, a tensile test piece was secured between two jaws, one of which was fixed to a screw jack and the other to an excentric.

The excentric was first placed in the position corresponding to the maximum elongation of the test piece, and a tensile stress was applied to the other end by the jack until the region of plasticity for steel was reached. The excentric was then turned several times so as to make the length of the test piece vary between two well defined limits, so as to unstress and then stress the test piece.

In this experiment, the extrinsic law corresponding to the maximum stress in the test piece is represented by a vertical straight line on the stress-strain diagram, and not a horizontal straight line as in the classic tensile tests.

The stress in the test bar was determined by the frequency of the vibrations set up by transverse impact.

These tests gave a negative result. It would therefore appear that, when the metal has entered the plasticity phase, the internal stress is a function of the linear deformation only, provided the latter does not exceed the maximum limit previously attained (Fig. 19).

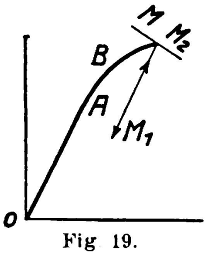

Fig 19.

Apparently, then, if given loads bring a zone into the elastic limit, the distribution of these loads does not modify the value of the tensions reached, and, consequently, the risk of the elastic limit being exceeded is no greater with reneweed loads than it is with a constant load, which means that the conclusions of the previous section stand.

II. The problem is much more delicate when it comes to computing the risk of failure. In that case, we must carefully consider what experience has recently taught us regarding failure by repeated stresses.

There is no doubt that the metals used in metallic structures are much softer than the metals which have been shown by endurance tests to be more sensitive to repeated stresses than they are to a single specified stress.

Nevertheless, failure is possible with every metal, whatever it be, when the repeated stresses attain a value lower than the figure which causes failure in an ordinary tensile test.

For reasons of prudence, the principle that we propose to follow consists, as above, in finding rules such as will coincide, for isostatic structures, with the ordinary rules of safety, and which, for hyperstatic structures, will give the same coefficients for the ratio between the permitted and the dangerous loads.

Adopting this point of view, the designer will tend to consider that an isostatic structure is satisfactory as regards stability when it gives a certain factor of safety (a) in terms of over-extension of the elastic limit, and (b) against breakage when the load is first applied, and when it is sufficiently remote from the limiting case where breakage would take place due to indefinite repetition of the overloads.

Our knowledge concerning failures by repeated stresses leads us to think that the permanent load plays a much smaller part than the live loads, which means that we ought to consider, not the maximum rate of stressing, but rather the sum of two terms:

(1) The half-amplitude of variation of the stress rate σ_2 and σ_1.

(2) The mean value $\frac{\sigma_2 + \sigma_1}{2}$ of the extreme stress rates multiplied by a coefficient α, which would be small.

According to the researches of *M. Caquot* at the Aerotechnical Laboratory, the coefficient would be of the order of $^1/_5$ th.

Safety rules for isostatic structures which would allow for the risk of failure by repeated stresses would lead the designer to compare the binomial defined above with a limit Π_f obtained by dividing the limit of endurance by a factor of safety.

Without going beyond the rules actually laid down, we should thus be inclined to put the permissible limit Π_f at not less than the limit Π_0 previously admitted for the total stress of a member subjected to alternating stresses.

We should then have to verify the following inequality:

$$\frac{\sigma_2 - \sigma_1}{2} + \alpha \frac{\sigma_2 + \sigma_1}{2} \leqslant \Pi_f \; (\geqslant \Pi_0) \qquad (27)$$

When we come to consider a hyperstatic structure, we ought to lay down a similar condition; but in the absence of proof to the contrary we may consider that the fact that the repeated stress σ_2 is equal to N does not affect the risk of failure any more than it does the relation between the deformation and the forces.

Experience in this connection would be extremely useful.

In the absence of more precise information, we shall therefore assume that, in a hyperstatic structure, we must verify the condition (27) in which σ_2 coincides with the stress rate set up on the first application of the load.

Speaking more precisely, we compare with the limit of endurance itself $K\,\Pi_f$, the expression (27) corresponding to loads which are K times higher than the loads we are considering. Dividing the two members by K, the expression becomes:

$$\frac{\sigma_{max} - \sigma_{min}}{2} + \alpha \left(\frac{\sigma\, d}{K} - \frac{\sigma_{max} - \sigma_{min}}{2}\right) \leqslant \Pi_f \qquad (28)$$

where σ_d is the stress rate equivalent, in the plasticity theory, to the load considered.

σ_{max} and σ_{min} are the extreme rates of stress in accordance with Hooke's law.

The first term must be derived exclusively from the elasticity theory. The second is obtained by subtracting the results supplied by the elasticity theory and the plasticity theory.

The advantage derived by plasticity is that this latter term is lower than the value deduced from Hooke's law, where the term σ_d is replaced by σ_{max}. Since, however, the coefficient α is small, this advantage is very limited, especially if the amplitude of variation in the stress rate is high, i. e., where the overloads are high with reference to the true load.

The condition (28) may be replaced by a more difficult condition where σ_d is replaced by its limit:

$$\frac{\sigma_{max} - \sigma_{min}}{2} \leqslant \frac{\Pi_f - \alpha\, \Pi_o}{1 - \alpha} \qquad (29)$$

In short, it would seem that if, in our conception of structures, we wish to take advantage of the latitude which plasticity allows, we ought, in principle, to limit ourselves to a twofold verification.

(1) It is indispensable to verify that there is a suitable factor of safety in terms of excess of the elastic limit, and in terms of failure or buckling, when the load is first applied.

The first verification is carried out according to the rules deduced from a knowledge of the plasticity of the material, and with due regard to the above observations.

(2) We ought to verify that there is a suitable factor of safety in terms of fatigue of the members, and for this purpose the variations in the stress rates will be deduced from a knowledge of the laws of elasticity. These variations will then be compared with a given limit of endurance.

Of the two inequalities to be verified, the former will obviously be the more important if the permanent load predominates and the second inequality can be dispensed with.

On the other hand, the latter inequality will operate in cases where a structure is light in terms of the overloads — say, a longitudinal girder or a bridge member.

Briefly, then, plasticity makes the factor of safety of structures insensitive to mistakes in fitting or erecting, and to anomalies in the distribution of the stresses and the initial loads of the members, etc. It renders correct the hypothesis of the equalization of the maximum stress rates when permanent loads are involved and when the structure is well designed.

Summary.

Every structural element for which the stressing depends not only on the loads applied to the whole structure, but which is also subjected to deformations, can reach the elastic limit without the necessity of exceeding this limit immediately. From this it follows that the forces applied to statically indeterminate system can change within certain limits, without causing the stresses to exceed the elastic limit; however, it will cause stresses to increase at other places of the structure. This conclusion applies also to the distribution of stresses set up in statically determinate structures.

The method of calculation as based on the assumption of stress equalization in structures is justified; the stresses however, must not exceed the yield limit before equalization of stresses has taken place. It is for this reason that the laws of plasticity may only be applied with the utmost caution in designing arches or other structures of a high degree of statical indetermination. On the basis of these assumptions, it can be said that the hypothesis of stress equalization leads to the determination of the factor of safety. The consideration of plasticity does not increase the factor of safety in the case of frequently repeated loads.

I 7

Test Results, their Interpretation and Application.

Versuche, Ausdeutung und Anwendung der Ergebnisse.

Essais; signification et application des résultats.

Dr. Ing. H. Maier-Leibnitz,
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A. Simply supported full-web beams (plated beams).

In order to be able to interpret the test results of continuous beams, the behaviour of a simply supported beam of span l, carrying a point load P in mid-span shall be explained first. (Fig. 1, see [1].)[1] For this purpose an I-beam 14 cm · 14 cm was used and the following properties were found by measuring the actual cross section: Sectional area: $F = 43.2\ cm^2$; moment of

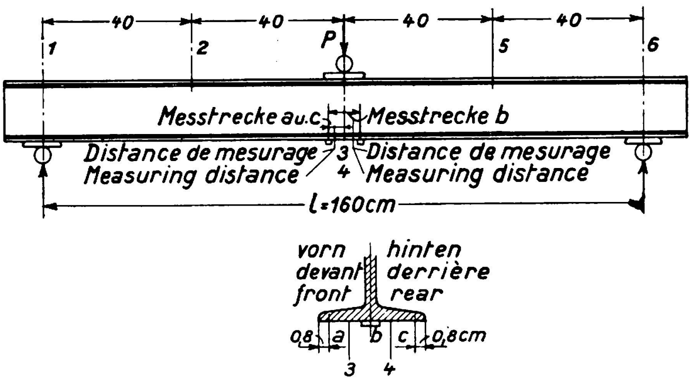


Fig. 1.

inertia $J = 1525\ cm^4$; modulus of section $W = 214\ cm^3$. Four test pieces cut from the flanges of this joist gave an average yield point stress of $\sigma_s = 2.437\ t/cm^2$. In Fig. 2 are shown the total and permanent deflections in mid-span and in Fig. 3 the average values of total and permanent elongations and straining at the points a and c of the bottom flange in mid-span. During the execution of tests it was noticed that scaling occurred in mid-span on the underside of the compression flange for a point load of $P = 12.8\ t$ For a load $P_v = 17.15\ t$ the beam buckled sideways, leading to complete failure.

The term *carrying capacity* of a beam can be interpreted variously. Based on the assumption that a beam is of no further use if it begins to have permanent deformations, this *carrying capacity* $[P_T]$ could be expressed

[1] The numbers given in [ ]-brackets refer to publications listed at the end.

as the value of P for which the extreme fibres at the point of max. bending moment are stressed up to yield point. If the load P is only slightly increased,

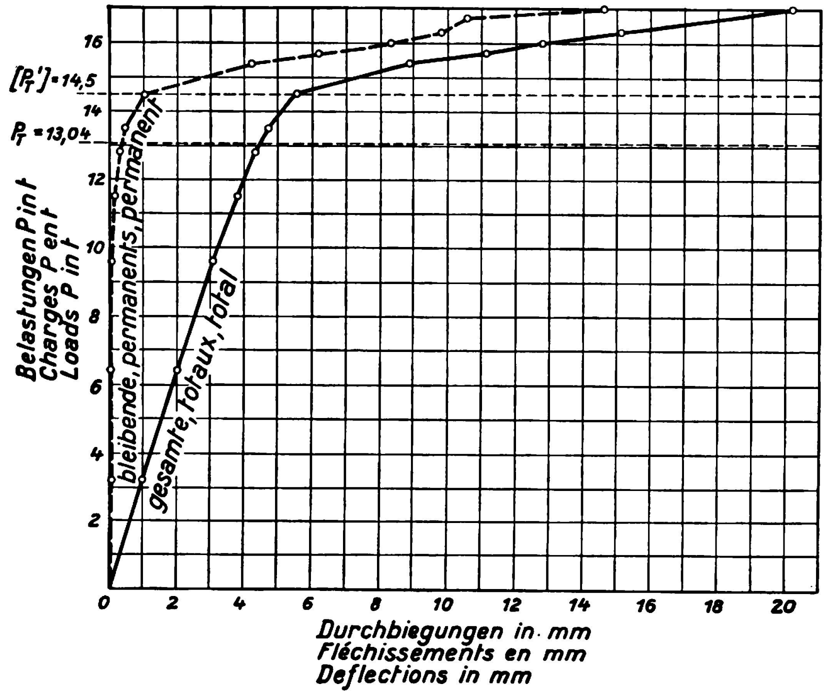


Fig. 2.
Beam 11 a, mean of deflections for points 3, 4 in mid-span.

permanent deformations and deflections would occur. The corresponding bending moment is in this case of the following value:

$$M_s = W \cdot \sigma_s = 214 \cdot 2.437 \text{ cmt},$$

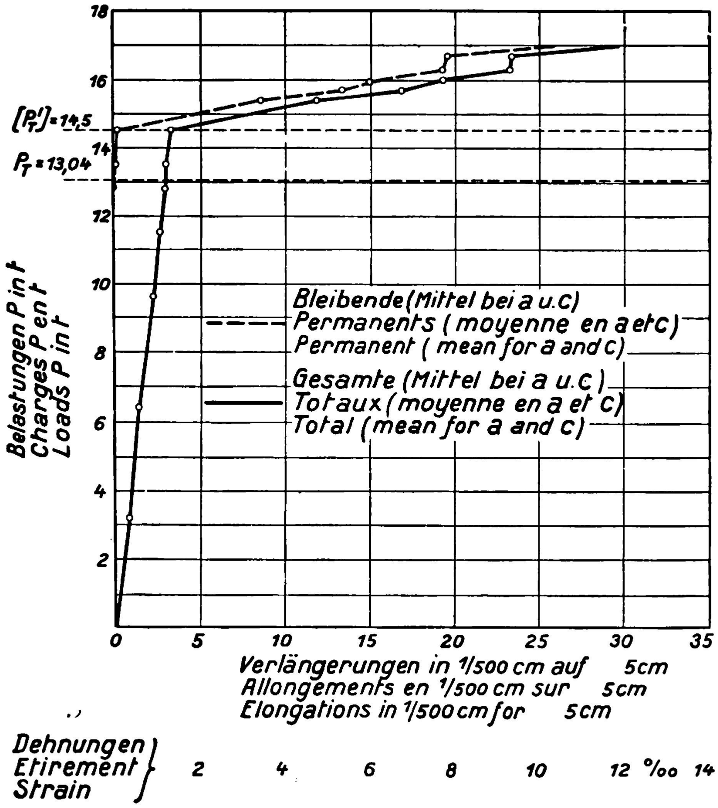


Fig. 3.
Beam 11 a.

giving a carrying capacity of

$$[P_T] = \frac{4 \cdot W \cdot \sigma_s}{l} = 13.04\,t.$$

From Fig. 3 it can be seen that permanent straining really starts with this load.

Attempts have often been made to arrive analytically at the value of M'_s, of the *carrying moment* which ultimate value cannot be exceeded and which corresponds to the internal moment of the stresses after yielding and for complete plasticity of the whole cross section. (See, e.g. [2], [3], [4], [5], further references to publications are given in [5] concerning the analytical method of *Prager* and *Kuntze*). As shown in the report by *Fritsche* [5], numerical conformity between experiment and calculation cannot be expected (the cross sections do not remain even as plasticity increases; the values of yield stresses vary for different points along the flange; hardening of the material due to yield).

From Fig. 3 and also from Fig. 2 relating to the tests under consideration, the value $[P'_T] = 14{,}5$ t, corresponding to M'_s, can be measured distinctly, after which permanent deformations and deflections develop at the bottom flange at the point of attack of the load and increase almost unrestrictedly. The corresponding *carrying moment* for this load has the value:

$$M'_s = \frac{[P'_T] \cdot l}{4} = 580 \text{ cmt. (See also section D a)}$$ [2]

The ratio $\frac{[P'_T]}{[P_T]} = \frac{14.5}{13.04} = 1.10$

applies for the present case.

The tests published in [1] for loading according to Fig. 1 give ratios as under:

I-beam Burbach $\frac{152}{127}$: $\frac{[P'_T]}{[P_T]} = \frac{14.7}{12.66} = 1.16,$

I-beam 16: $\frac{[P'_T]}{[P_T]} = \frac{8.3}{7.61} = 1.09$

In any case, as already shown by *Grüning* in [2] it does not appear justifiable to consider the quantities P_v and P_w (in the treatise of *Stüssi-Kollbrunner* [6]) as being decisive for the carrying capacity, or even to regard these values as the carrying capacity of beams.

For a clear conception (qualitative interpretation), concerning the tests described in sections B to E for continuous and fixed beams, and for the purpose of establishing a simple mode of calculation for such beams, it appears important to understand fully that the relation between the *permanent* deflection f of a beam AB (and similarly the residual angle φ of the deflection curve) and the load P or the bending moment $\frac{P \cdot l}{4}$ (see Fig. 4a) can be expressed by the curve OCDE in Fig. 4b (identical with Fig. 5 in [1]). To render the above

[2] If a curve were to replace the straight lines in Fig. 2 and 3 the value of $[P'_T]$ would be read as 15 t in which case $M'_s = 600$ cmt.

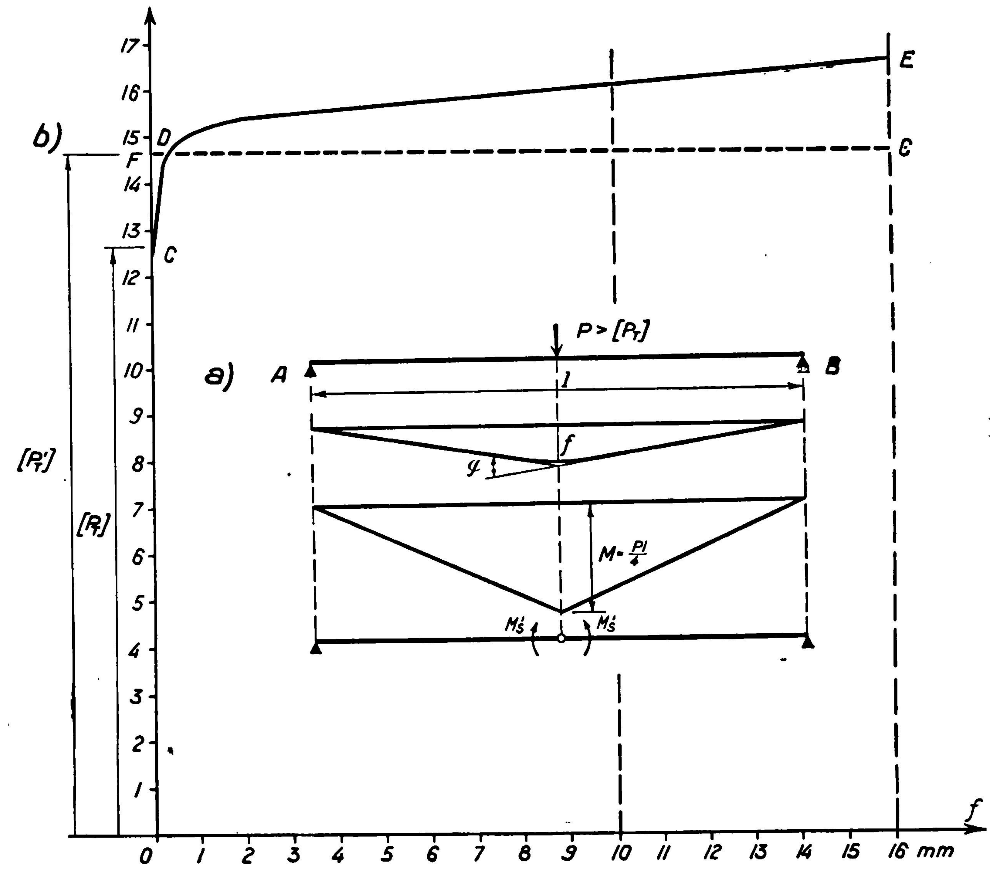


Fig. 4.

still easier to understand, the curve CDE can be replaced by a horizontal line FG and the permanently bent beam can be considered to have a sharp kink at the middle of the span (Fig. 4a). In this case we get

$$\varphi = \frac{2 \cdot f}{\frac{l}{2}} \quad \text{or} \quad f = \frac{\varphi \cdot l}{4}$$

The simplified conception of the problem leads to the assumption that the beam would not be capable of withstanding a higher moment than $M'_s = \frac{[P'_T] \cdot l}{4}$ (in this in particular case $M'_s = 580$ cmt) and that under this moment the beam is bent with a sharp kink at the load point. This is equivalent to the conception of a beam loaded with $[P'_T]$ being hinged at the point of attack of the load, the hinge being equipped with an internal moment maintaining equilibrium with the bending moment M'_s.

Further tests with simply supported beams can be found in [7], [8], [9], [10], [15].

B. Continuous beams of two spans and equal loading for both spans.

Fig. 5 shows the results of tests carried out by the Author with two compound I-beams with a depth of 16 cm, described in [11]. Four different cases were studied and the values P_{zul} of the permissible loads are given, for which

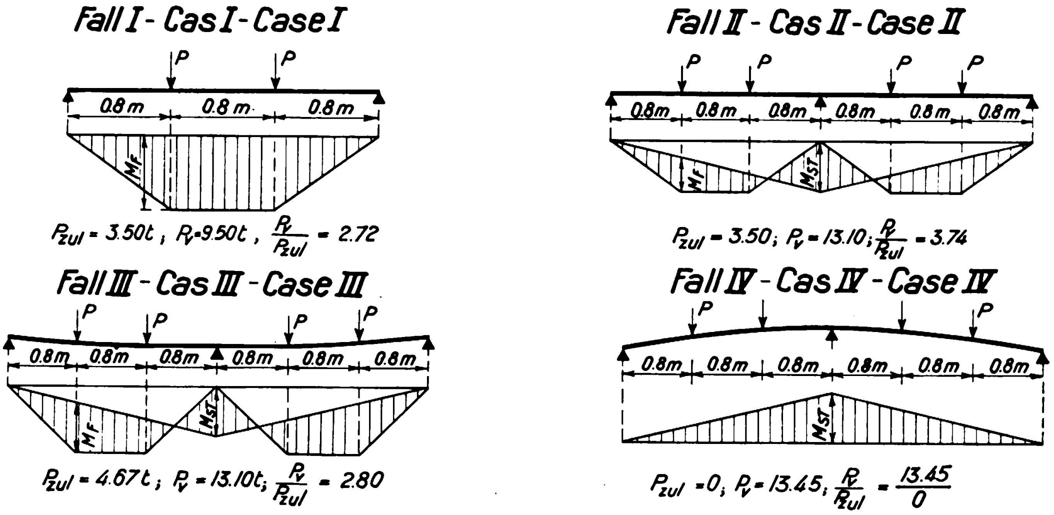


Fig. 5.

according to the usual theory of elasticity the permissible stress $\sigma_{zul} = 1.2$ t/cm^2 is attained.

Case I is that of a simply supported beam over one span, Case II a continuous beam over two spans with bearings all at the same level. Case III treats a continuous beam over two spans with the intermediate support lowered in such a way as to produce a bending moment over the central support equal in value to the maximum moment in the span, but being equal to the permissible bending moment: $W \cdot \sigma_{zul}$. Case IV has the following characteristics: continuous beam over two spans, but with the end supports at a lower level than for the intermediate support, in such a way as to establish a moment over the middle support equal to the permissible moment $W \cdot \sigma_{zul}$.

Failure of all beams occurred for the loads P_v after the production of unreasonably high deformations, by lateral buckling in the vicinity of the outer

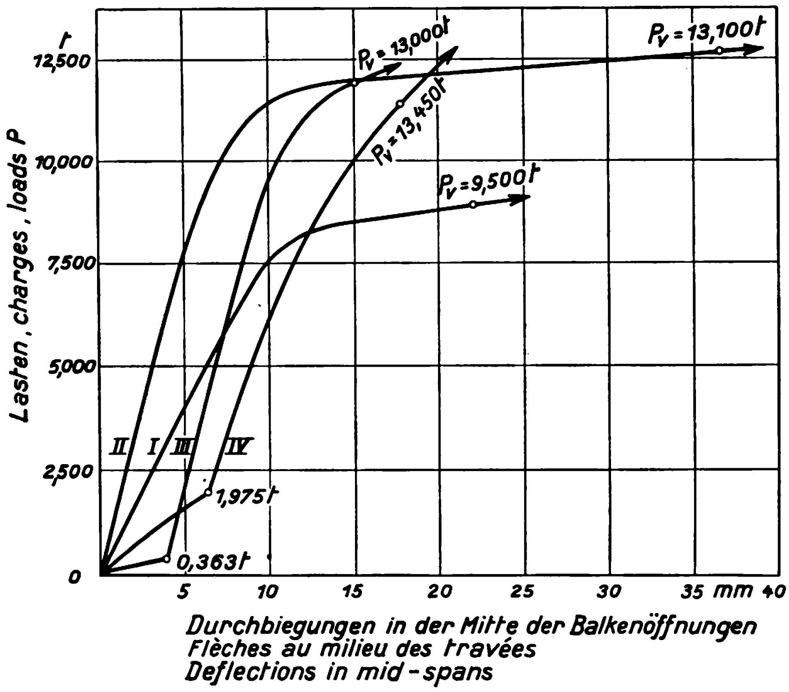


Fig. 6.

forces. The deflections registered for the middle of the spans are given in Fig. 6.

In case I the first traces of yielding were observed at the tension flange under the point of attack of the load for $[P'_T] = 7.5$ t, giving a stress of 2.70 t/cm^2, which is less than the average yield point stress of 2.94 t/cm^2 as determined by tensile tests. Based on observations made with these tests, it can be assumed that $[P'_T] = 8.5$ t,[3] giving a corresponding carrying moment of $M'_s = 8.5 \cdot 80 = 680$ cmt. The modulus of section for the two compound I-beams is $W = 222 \text{ cm}^3$. With this value and for $\sigma_s = 2.70 \text{ t/cm}^2$ we receive a moment of $M_s = W \cdot \sigma_s = 222 \cdot 2.70 = 600$ cmt or correspondingly for $\sigma_s = 2.94 \text{ t/cm}^2$ a moment of $M_s = 222 \cdot 2.94 = 665$ cmt. For $\frac{[P'_T]}{[P_T]} = 1.14$ the following ratios are received:

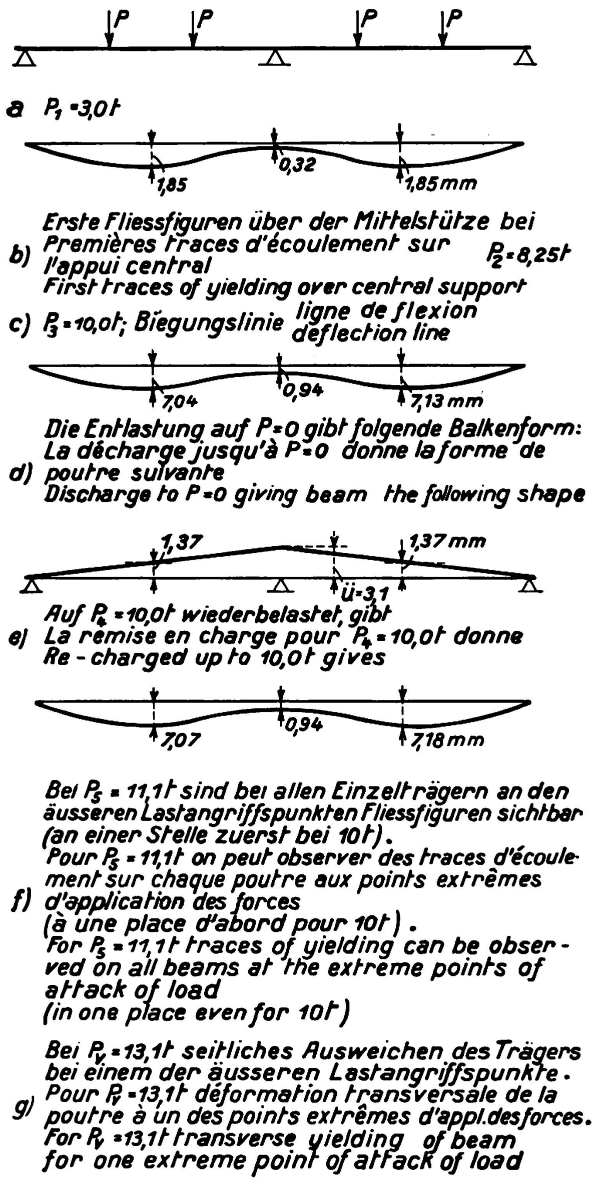


Fig. 7.

Diagram showing principal test II.

[3] The values for carrying capacity of simply supported beams are marked thus: [ ]; $[P_T]$ and $[P'_T]$ refer throughout to the simply supported single span reference beam.

$\frac{M'_s}{M_s} = \frac{680}{600} = 1.14$ in one case and $\frac{680}{665} = 1.02$ in the other case. These results show that it is impossible to deduce the bending moment M'_s from the yield stress received by tensile tests.

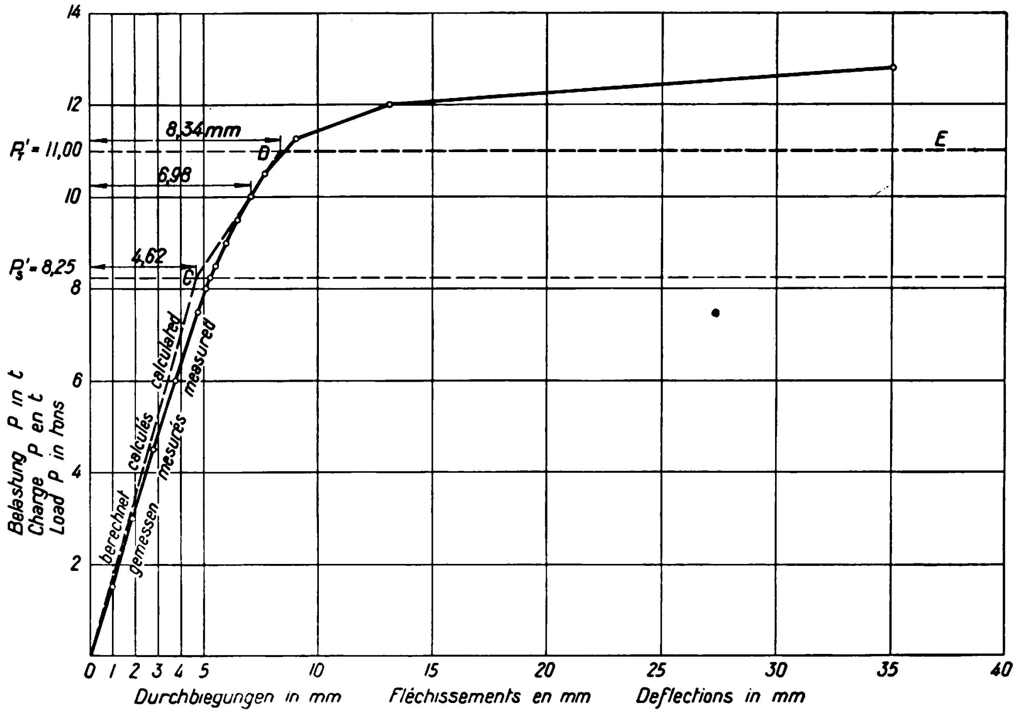


Fig. 8.

Case II.

The most important readings taken from this experiment are shown in Fig. 7. The mean deflection values are plotted on Fig. 8, while Fig. 9 shows the elonga-

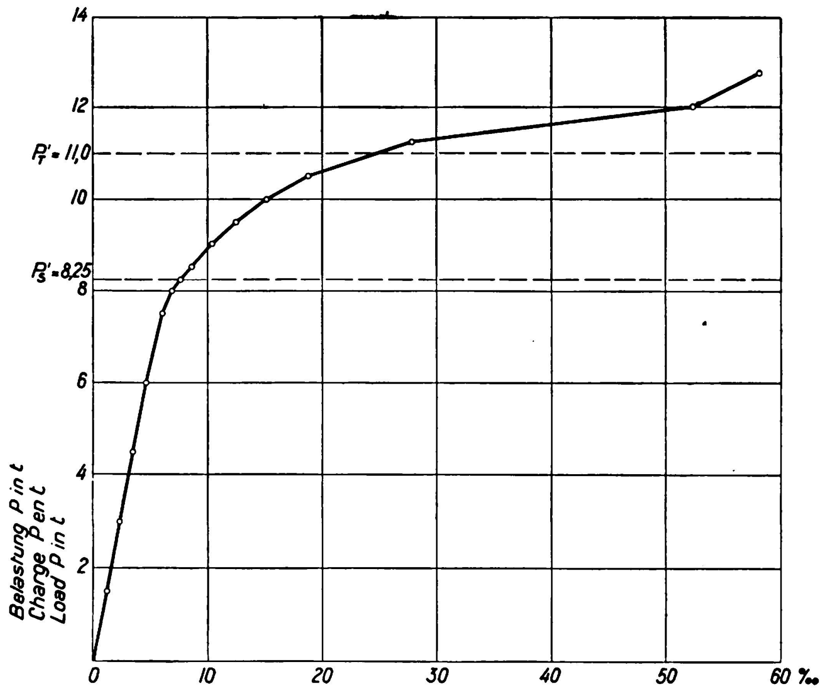
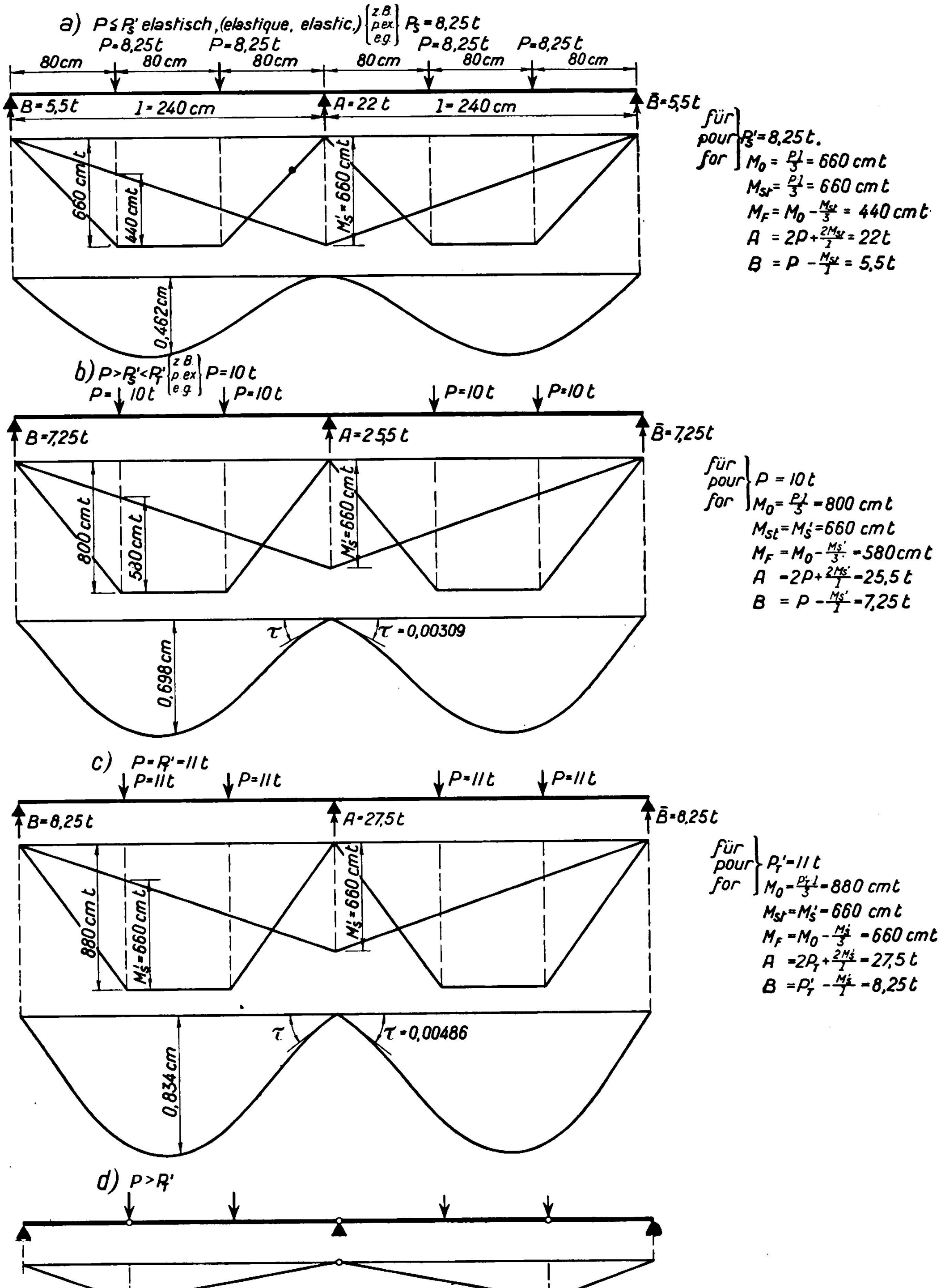


Fig. 9.

tions of a measured stretch of 100 mm on the upper flange over the central support. The properties of the compound beams based on actual measurements are: Sectional area F = 43 cm²; Moment of inertia J = 1727 cm⁴; Modulus of section W = 211 cm³ and an average yield stress of 2.51 t/cm² was found by tensile tests. If this latter value could be employed for calculating M'_s using the same notations as in section A, this bending moment would be $M_s = 211 \cdot 2.51 = \sim 530$ cmt. According to *M. Grüning* [2], M'_s would be $232 \cdot 2.51 = 585$ cmt



Fig, 10.

and according to *Fritsche* [3] p. 854, it would be $1.16 \cdot 530 = 620$ cmt. During the execution of the test the first traces of yielding were observed at the junction from web to the upper flange over the central support, for one joist under a load of $P = 8.25$ t (and for the other under a load of $P = 8.50$ t). The corresponding moment for this condition is $M'_s = \frac{P l}{3} = 8.25 \cdot 80 = 660$ cmt and $8.50 \cdot 80 = 680$ cmt respectively. It can be seen clearly from Fig. 8 that from $P'_s = \sim 8.25$ t onwards the deflections start to increase more.

To interpret this experiment by the simplified hypothesis given at the end of section A, th evalue $M'_s = 660$ cmt is used, and the results are shown on Fig. 10.

The quantity P'_s represents the value of P — assuming fully elastic conditions — for which the moment M_{st} over the support is equal to M'_s (Fig. 10a). This relation yields

$P'_s = \frac{M'_s \cdot 3}{l} = \frac{660 \cdot 3}{240} = 8.25$ t. If P should assume higher values than P'_s, the moments over the support will not increase above the value of M'_s (Fig. 10b). The carrying capacity is reached if a load P'_T produces under the point of attack of the outer load a moment $M_F = \frac{P'_T \cdot l}{3} - \frac{M'_s}{3}$ equal to M'_s, (Fig. 10c). Hence $P'_T = \frac{4\,M'_s \cdot 3}{3 \cdot l} = \frac{4 \cdot 660}{240} = 11$ t. From Fig. 8 can be seen, that indeed the deflections grow more rapidly with increasing loads above $P = 11{,}25$ t, thus rendering the beam useless for practical purposes. (If, as previously mentioned, the calculation had been based on $M'_s = 680$ cmt, we should have received $P'_s = 8.5$ t and $P'_T = 11.66$ t, which values correspond still better with the actual conditions).

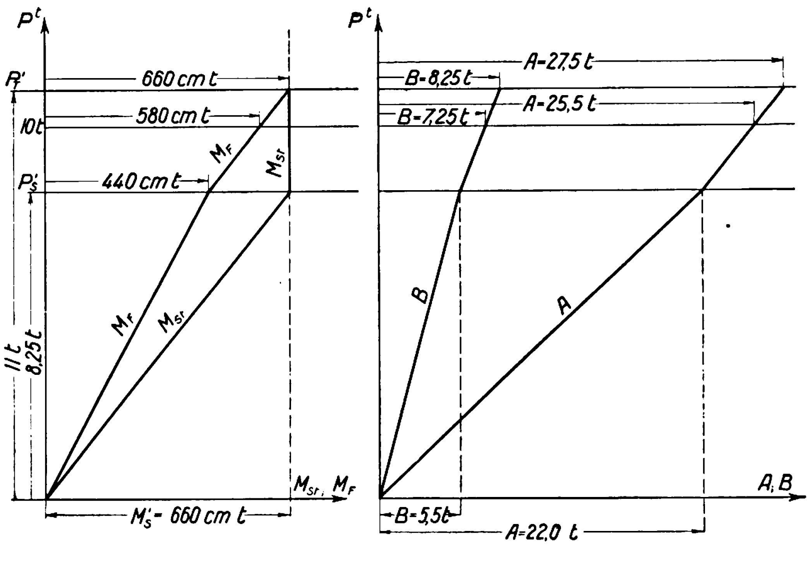


Fig. 11.

The values given in connection with Fig. 10 are plotted in Fig. 11.

In Fig. 10 and 12 are shown diagrammatically the deformations based on the simplified interpreting hypothesis with $J = 1727$ cm^4, $E = 2100$ t/cm^2

but without considering the influence of shear. The deflections for the middle of the span 4.62 mm, 6.98 mm and 8.34 mm calculated with $P = 8.25$ t, $P = 10$ t and $P = 11$ t are shown in Fig. 8. To the basis of calculation corresponds the line OCDE of the deflections and further, that the deflections grow unrestrictedly when the value P'_T is reached.

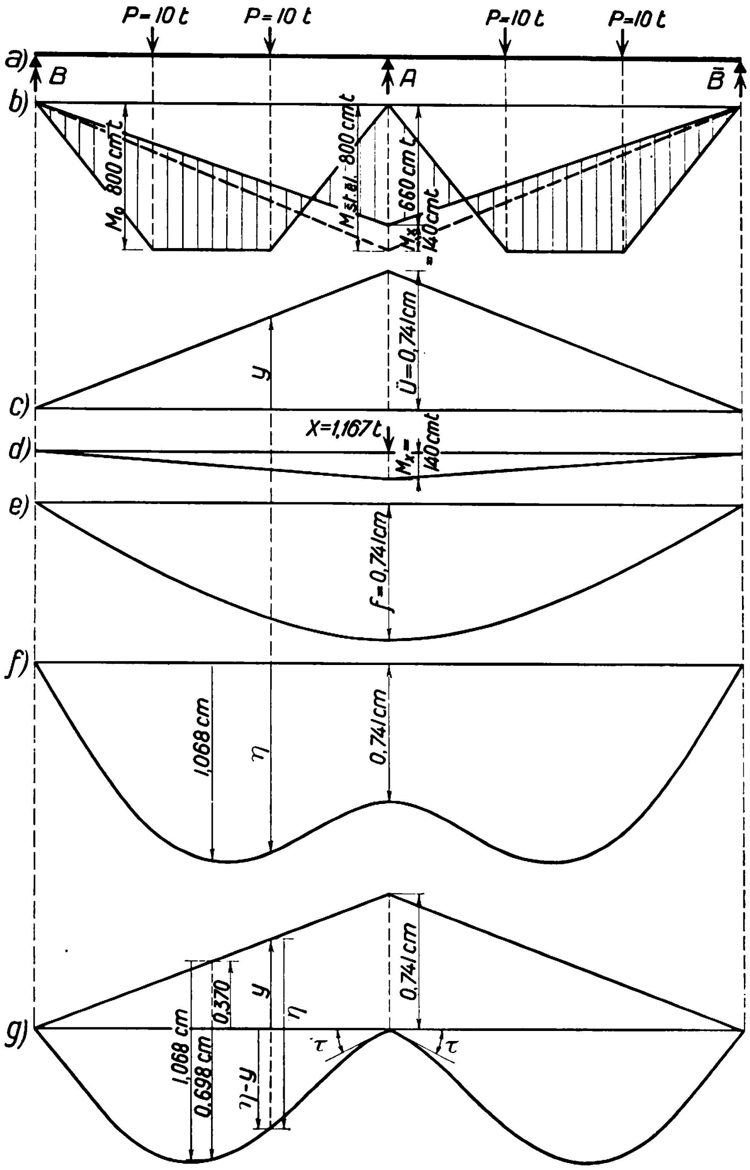


Fig. 12.

The simplified hypothesis makes it quite clear how in the range $P > P'_s < P'_T$ the equalization between the bending moment M_{st} over the middle support and the moment M_F in span takes place. The deflection line of the beam $BA\overline{B}$ (Fig. 10b) shows that at the point A which might be considered as a hinge, there is an abrupt change of direction of $\varphi = 2 \cdot 0.00309$. If the beam $B\,A\,\overline{B}$ is released from its load, it will assume the shape indicated by Fig. 12c with a camber $ü = 0.00309 \cdot 240 = 0.741$ cm. On reloading the beam the camber ü must first disappear by a load $x = \dfrac{48\,E\,J\,ü}{l} = 1.167$ t

(Fig. 12e). This load corresponds to a positive bending moment at the support of $M_x = \frac{1.167 \cdot 480}{4} = 140$ cmt. When the beam now is loaded again with 10 t there is a total bending moment at the support of $M_{st\,.\,el} - M_x = \frac{10 \cdot 240}{3}$ $- 140 = 660$ cmt (Fig. 12b). Since it is assumed that up to this bending moment of 660 cmt the strains are elastic, reloading with loads up to 10 t takes place within the elastic range. Finally the hatched area of the bending moment diagram given in Fig. 12b again appears. The whole procedure can also be
diagram given in Fig. 12b again appears. The whole procedure can also be
expressed in the following manner: If the beam is restrained from lifting from the support, firstly a restraining force X is acting and the corresponding moment $M_x = 140$ cmt. Due to the loads of 10 t the bending moment over the support now continues to increase quite elastically up to the value

$$M_{st\,.\,el} - M_x = \frac{Pl}{3} - 140 = 10 \cdot 80 - 140 = 660 \text{ cmt.}$$

The deflections η of the reloaded beam B A $\overline{B}$ with the initial camber ü are received in the form of ordinates of a bending moment diagram for a beam B$\overline{B}$, loaded by $\frac{1}{EJ}$ — times the values of the hatched moment area given in Fig. 12b. (Mohr's method applied to cambered beam.) The deflection line, computed this way, is shown in Fig. 12f. The deflection a point A must check with camber ü. If the ordinates of the deflection curve η are plotted from the axis line of the cambered beam as a base (ordinates y) the shape of the axis of the reloaded beam is obtained. (Ordinates $\eta - y$.) This shape (Fig. 12g) is identical with the sape of the axis after the first loading with 10 t (Fig. 10b).

Comparing the preceding qualitative interpretation of the loading, unloading and reloading due to loads $P = 10$ t with the results received by experiments (Fig. 7) we find general agreement in two points. Firstly, after loading and reloading with 10 t a camber ü appears over the middle support and secondly, the reloading is perfectly elastic. Full numerical agreement cannot be expected for the following reasons:

a) the simplifying assumptions of the interpreting hypothesis,
b) differences in the geometrical properties of the beams,
c) differences in the behaviour of the beams under load (influence of shear on the strain, variations in modulus E and yield point stress σ_s),
d) the practical impossibility of keeping the supports exactly at the same level.

The experiments, however, show clearly that the moment over the support does not increase above the value of M'_s for loads above the value P'_s, although an increase of moments in the span takes place. The reason for this is that the beam automatically undergoes a deformation producing a camber ü as if established by cold bending in a machine. This phenomenon of producing a camber ü, reduces the bending moment over the middle support in exactly the same way as if the middle support were lowered artificially by the same amount. According to calculation we receive for $P = 10$ t and $P = 11$ t the values ü = 7.41 mm and 11.65 mm respectively. They are quite small for a beam of 480 cm in length. The values obtained by experiment are smaller.

The summary of this would be:

1) The interpretation and their simplified assumptions give a sufficiently accurate basis to judge the carrying capacity of the continuous beam under consideration. It is quite safe to introduce for a simply supported beam the carrying moment $M_S = W \cdot \sigma_S$ and to calculate with this moment according to Fig. 11 the values of P_s and P_T for the continuous beam under consideration.

2) The following definitions correspond to the simplified hypothesis of interpretation:

a) P'_∞ represents an ultimate load for which loading, unloading and reloading of the beam still produce only fully elastic deformations.

b) It is essential for $P > P'_S$ but less in value than the maximum load P'_T, that only limited and practically harmless deformations similar to cold bending of the beam are produced near the cross section where, according to the theory of elasticity, the maximum bending moment occurs. The unloading and recharging of the beam creates fully elastic deformations only.

c) P'_T represents an ultimate load for which, if exceeded, the beam enters into a condition of instability. (See Fig. 10d.)

3) The actual behaviour of the continuous beam is as follows:

a) P_S creates the first permanent although small deformations.

b) The deformations increase considerably with $P > P'_S$, though without impairing the practical usefulness of the beam.

c) For $P > P'_S < P'_T$ unloading and recharging of the beam takes place under fully elastic conditions.

d) With $P > P'_T$ the deformations grow to such an extent as to render the beam useless for practical purposes.

e) At P_V the beam fails completely.

The interpretation of the test results of cases III and IV need not be gone into here. In comparison with case II, the load P'_S entered later into account for case III, and earlier for case IV, but P'_T was found to have almost the same value in all three cases.

An article published in [12] describes tests proposed by *J. H. Schaim*, with simply supported single span and continuous double span beams, spanning 4 m, and loaded with four equal point loads in each span. (See Fig. 13 and also [13].)

It can be assumed that $M'_s = \sim 614$ cmt $(= \sim 234 \cdot \sigma_s = 234 \,.\, 2.62)$ as for test I for a simply supported beam loaded with four equal point loads P, with $\sigma_S = 2.645$ t/cm², the beam failed at $P_V = \frac{10}{4}$ t corresponding to a bending moment of $240 \cdot \frac{10}{4} = 600$ cmt. For test 4 with $\sigma_S = 2.895$ t/cm² the deformations grow rapidly for a load of $P = \frac{10}{4}$ t. The load P_V of this simply supported beam was found to be $\frac{12.03}{4} = 3.01$ t.

For the above mentioned value $M'_s = 614$ cmt the load P'_s assumes the value

$P'_s = \frac{614}{240} = 2.56$ t. It can be seen from Fig. 13c (total and elastic elongations in span and over support) that permanent deformations developed already at $P = 2$ t. The probable explanation for this is that as stated [12], p. 14 'a relative lifting-off of the middle support has taken place, thus causing increased stressing over the intermediate support'. Therefore M_{st} become equal to M'_s

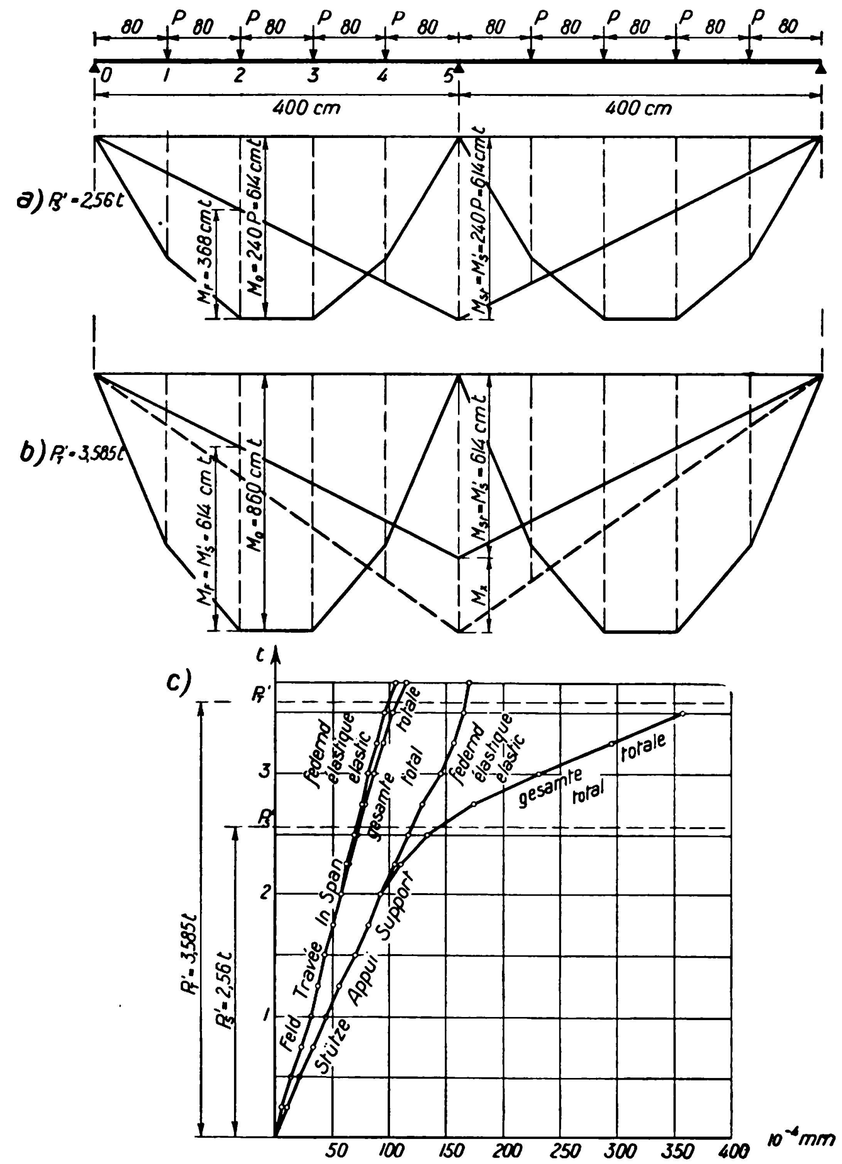


Fig. 13.

even at $P < P'_s$. The load P'_T is reached since for the cross section 2 the moment $M_F = M_O - \frac{2}{5} \cdot M_{st} = M'_s$, i. e. $240\,P'_T - \frac{2}{5}\,M'_s = M'_s$, hence $P'_T = 3.585$ t. The beam fails at $P_V = \frac{19}{4} = 4.75$ t.

The foregoing and the following remarks are concerned with loads increasing from o to P'_T (to P_V) whilst the tests of *O. Graf*, published in [14], deal with the investigation of the fatigue strength of simply supported single span and continuous double span beams of steel St. 37.

C. Continuous I-beams over two spans, with only one span loaded.

If a continuous beam (Fig. 14) having two equal spans AB and BC is loaded with a point load P at the distance a from the left outer bearing, all deformations remain fully elastic until P reaches the value P'_s or in other words as long as the moment M_F in the span remains smaller than M'_s. The closing line

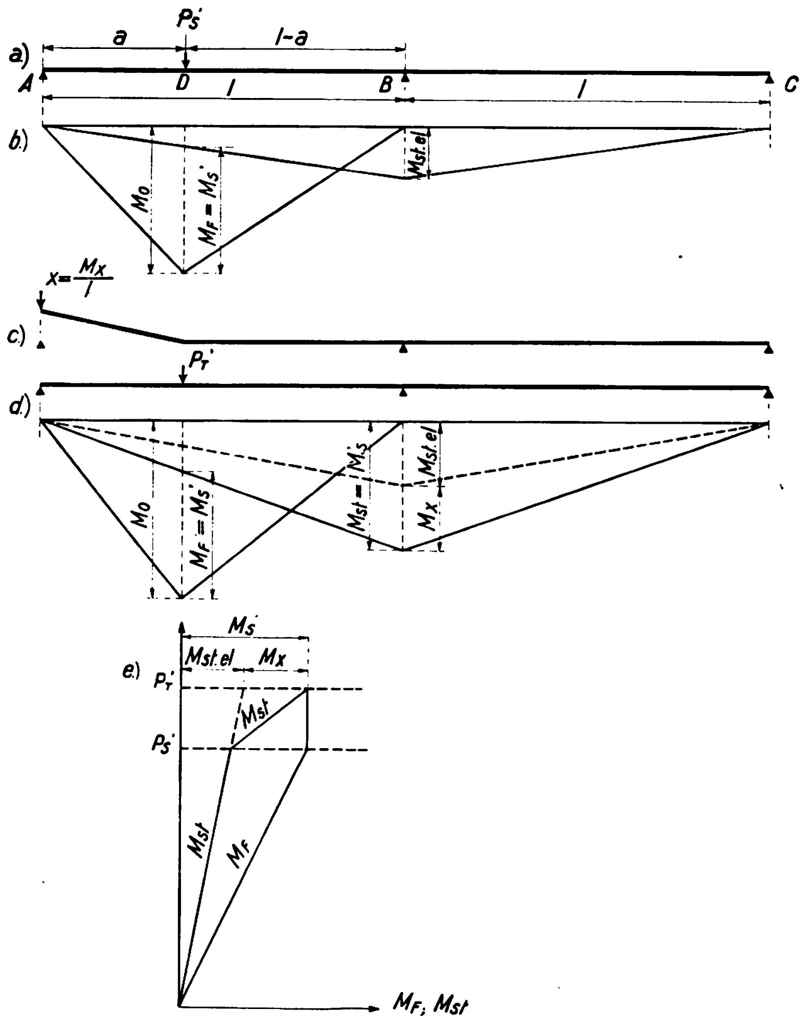


Fig. 14.

of the bending moment diagram (Fig. 14b) is determined by $M_{st\ el}$. For loads $P > P'_s$ the bending moment diagram corresponds to the M_O-diagram and a closing line determined by the stipulation that the moment M_F in the span shall not exceed M'_s. The ultimate value of the load P'_T is attained at $M_{st} = M'_s$ (Fig. 14d) according to the following condition:

$$\frac{l - a}{l} \cdot a \cdot P'_T - \frac{a}{l} \cdot M'_s = M'_s$$

hence

$$P'_T = \frac{M'_s \left(1 + \frac{a}{l}\right) l}{a\,(l - a)}$$

The relations M_F (P) and M_{st} (P) are shown in Fig. 14e.

For $P > P'_s < P'_T$ and for $P = P'_T$ (Fig. 14d) and in accordance with the explanations given in section B, the moment M_{st} over the central support can be considered as composed of two parts: $M_{st \cdot el}$ and $M_x = M_{st} - M_{st \cdot el}$. A visible lift away from the support (cold bending at the attack of the load) (Fig. 14c) corresponds to the bending moment M_x in such a way that on reloading with the force $X = \frac{M_x}{l}$ the end of the beam can be brought back again to the support. Afterwards, for loads increasing from o to P all deformations are again fully elastic, and the elastic bending moment $M_{st \cdot el}$ produced by P has to be added to the moment M_x due to X. The shape of the bent beam (Fig. 14c) could also be derived from the deflection line of the span AB, similarly as shown in section B the test carried out by F. Hartmann and described in [4] is based on a beam charged with two point loads in the left span. The beam used was an I-beam with a depth of 12 cm with the following properties: $J = 328$ cm^4; $W = 54.7$ cm^3; $\sigma_s = 2.51$ t/cm^2; $M_s = 54.7 \cdot 2.51 = 137.5$ cmt. If we choose $M'_s = 160.9$ cmt $(= \sim 1.16 \cdot M_s)$, according to data given in [4] p. 79, we receive following the simplified hypothesis of interpretation the conditions shown in Fig. 15 and 16. As long as $P < P'_{s1}$ the distribution of

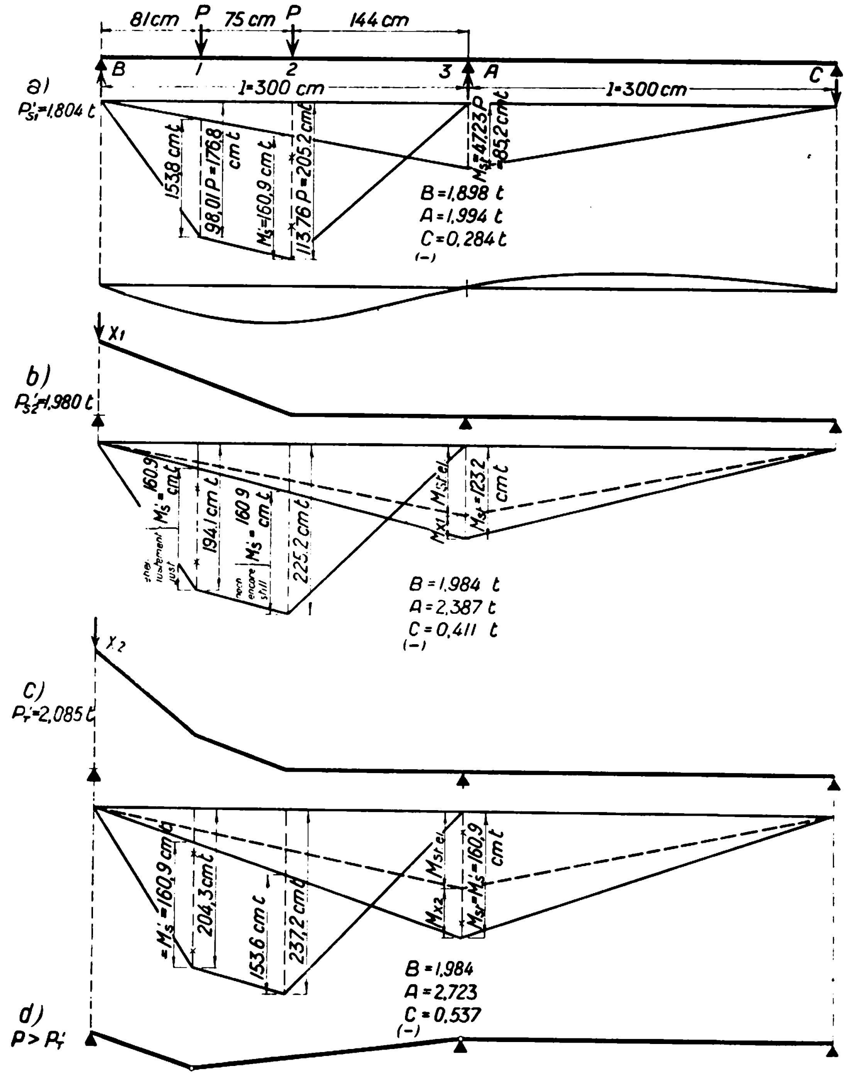


Fig. 15.

moments is established by the M_O — diagram and the closing line, fixed by the moment $M_{st\ el}$. The moment in the cross section 2 of the beam (Fig. 15a) attains M'_s for a load $P'_{s1} = 1.804$ t. For the loads $P = P'_{s1}$ and subsequent unloading the beam undergoes permanent bending in point 2 (cold bending). The same values M'_s are obtained for the bending moments in point 1 and 2 if $P = P'_{s2} = 1.98$ t (Fig. 15b); but for $P > P'_{s2}$ the bending moment under point 2 is less than M'_s. If the beam is released of its loads we observe a further bending-up of the beam at point 1. The moment at point 1 and the moment over the support become equal to M'_s for an ultimate load $P'_T = 2.085$ t (Fig. 15c). The moment over the support e. g. of Fig. 15c can again be considered as consisting of two parts: M_{x2} and $M_{st \cdot el}$, as previously explained. To the part-moment M_{x2} belongs a force X_2 which for recharging of the beam causes the previous permanent deformation to disappear at the support A, allowing in consequence the beam to undergo fully elastic deformations again. The relations between M_1, M_2, M_{st} and P are shown diagrammatically in Fig. 16; in addition to this are shown the strain values ε_1, ε_2 and ε_3 received by experiment from various stages of loading taken from [4] (Fig. 13). It can be seen that the elongations ε_2 are greater than ε_1 provided the bending moment at point 1 has not reached the value M'_s. For $P = 1.94$ t, a value close to $P'_{s2} = 1.98$ t, the two lines cross each other, i. e. for $P > 1.94$ t the elongations ε_1 are greater than the elongations ε_2. The test was stopped at $P = 2.2$ t.

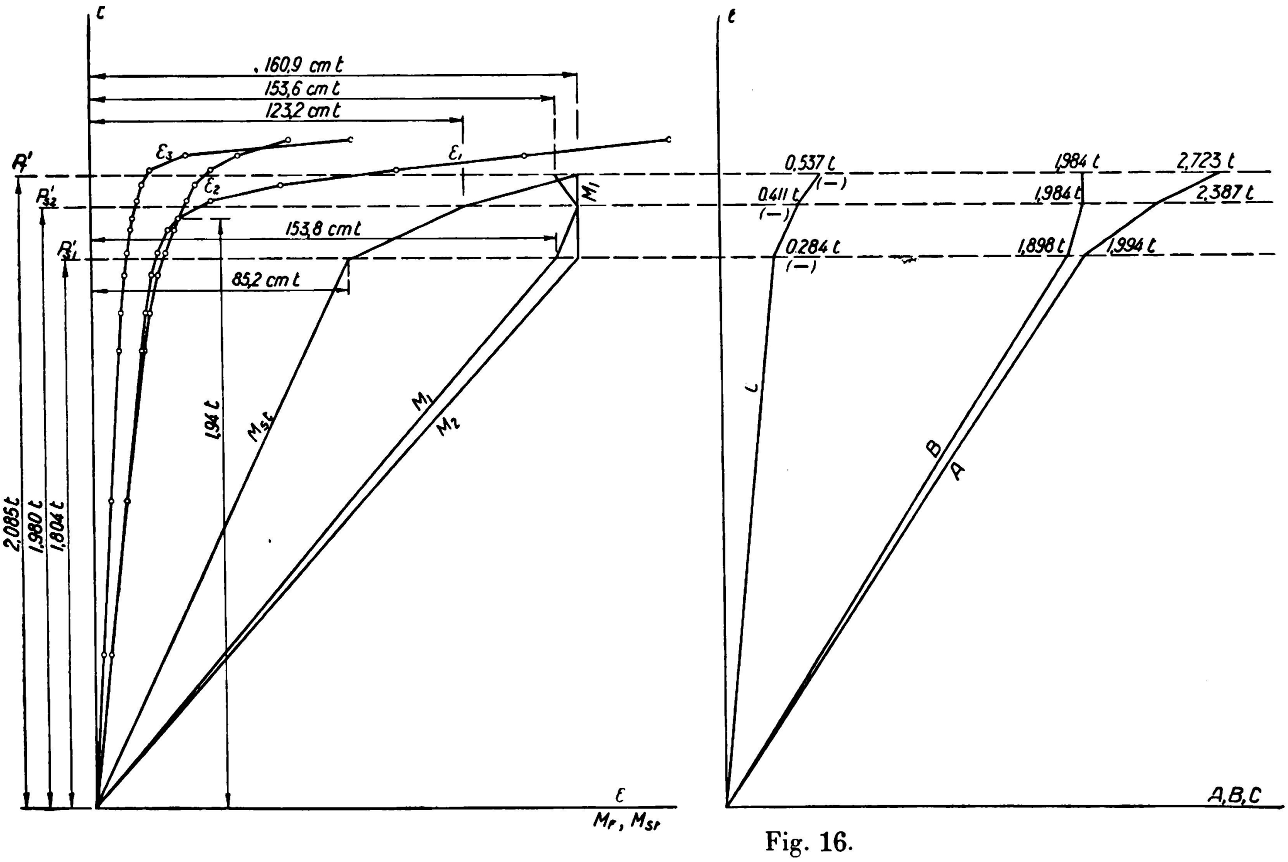


Fig. 16.

D. I-beams fixed at both ends.

a) Tests carried out in such a manner, that the cross sections at the supports remain vertical.

From the tests carried out by the Author and publshed in [1] only the tests for beam No. 11 shall receive consideration in the following.

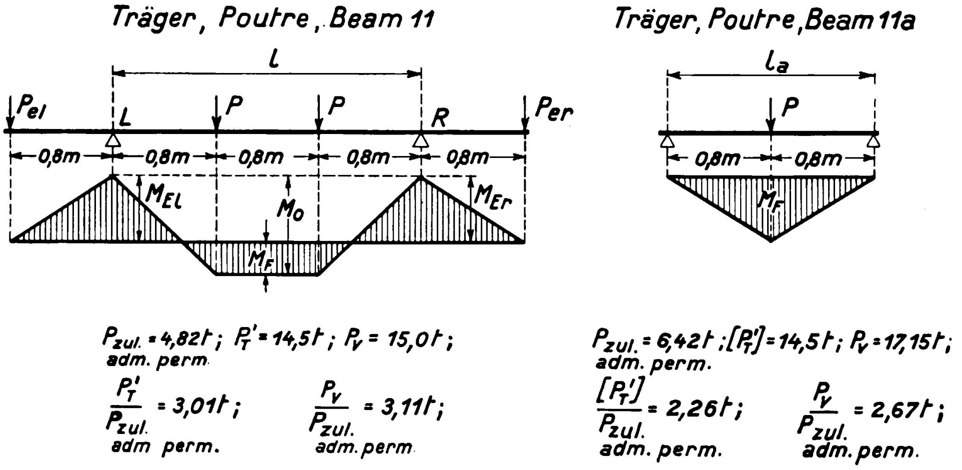


Fig. 17.

In Fig. 17 are shown side by side as previously in Fig. 5 of section B the values of P_{zul}, P'_T and P_V for a fixed beam (beam 11) and a simply supported beam (beam 11a); the behaviour of this simply supported beam has already been explained in section A. Both beams No. 11 and 11a were cut from the same I-beam 40 cm · 40 cm. The forces P_e shown in Fig. 17 were determined during the test and act in such a way as to keep, for all ranges of loading the cross sections of the beam at L and R perfectly and permanently vertical, with the purpose of establishing as nearly as possible the proper conditions required for the calculation of beams with fixed ends.

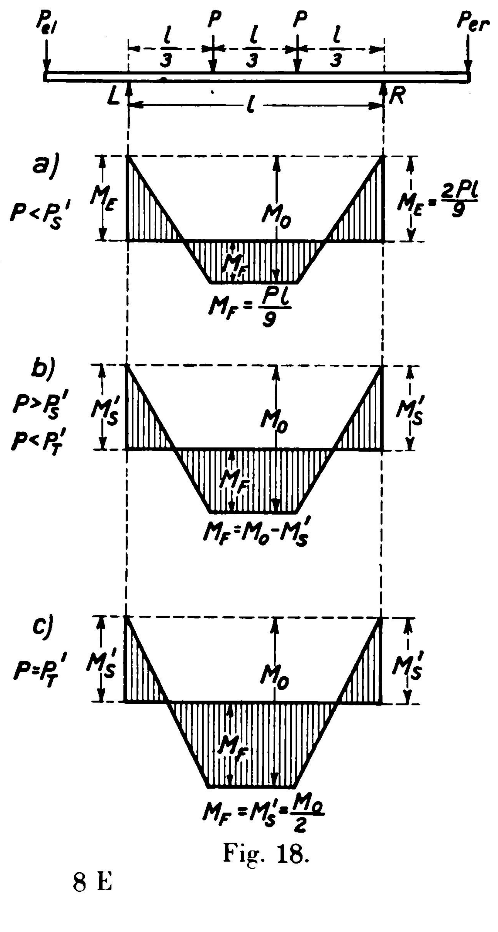


Fig. 18.

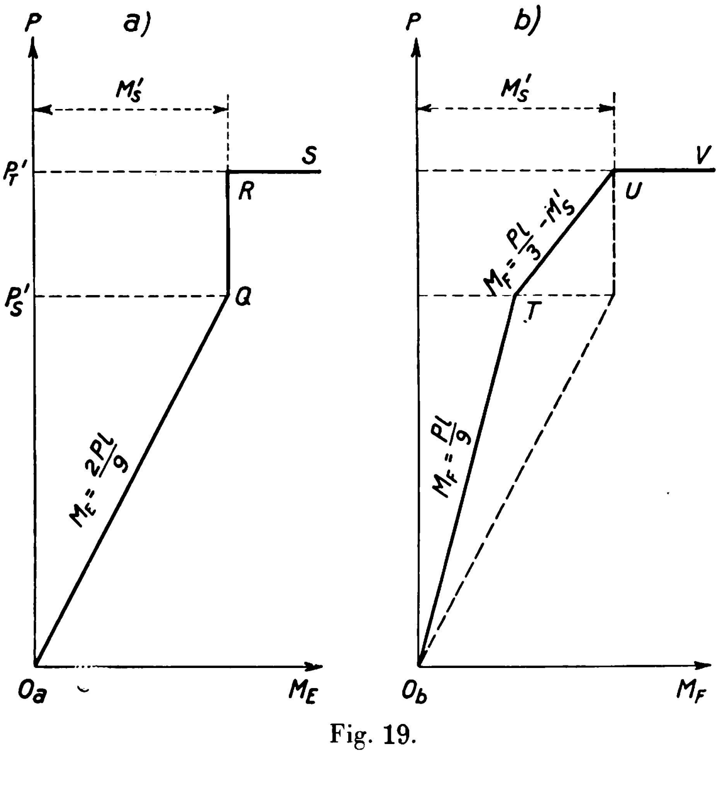


Fig. 19.

In accordance with the explanations given in the sections B and C and with the simplified hypothesis of interpretation the following various stages of loading of a beam with fixed ends are shown in Fig. 18:

a) For a load $P < P'_s$ (Fig. 18a) all deformations are of an elastic nature. Accordingly we receive:

$$M_O = \frac{P\,l}{3}, \quad M_E = \frac{2\,P\,l}{9}, \quad M_F = M_O - M_E = \frac{P\,l}{9}.$$

b) Due to a load $P'_s = \frac{9 \cdot M'_s}{2\,l}$ the moment over the support reaches the value of the carrying moment M'_s. For $P > P'_s$ the moments M_E cannot go much beyond the value of M'_s (Fig. 18b). The following relations apply:

$$M_E = M'_s; \qquad M_F = M_O - M'_s.$$

At this stage a change of the angle between the tangent of the deflection curve and the previously horizontal axis of the beam takes place. For elastic deformations only this tangent is horizontal. After the beam is released from its loads, a permanent deformation (cold bending) is observed in such a way that the ends of the beam at the left of L and at the right of R point downwards. To fulfil the assumptions on which the calculations of the beam with fixed ends are based it is necessary, before recharging of the beam takes place, to apply forces P_{el} and P_{er} (each equal to X, Fig. 20b), acting upwards, bringing the cross section of the beam at L and R into vertical position again. The moments M_s produced by this action, can be found in a similar way as explained under B and C:

$$M_x = M_{E \cdot el} - M'_s.$$

c) Due to P'_T the moment at the place of attack becomes $= M'_s$, i. e. it is expressed by the following term:

$$M_F = \frac{P'_T \cdot l}{3} - M'_s = M'_s; \text{ hence } P'_T = \frac{6\,M'_s}{l}.$$

Should the loads still increase, above the value of P'_T, then the beam enters into a state of instability.

The relations M_E (P), M_F (P) are plotted in Fig. 19, from which diagram the values for P'_s and P'_T for a given value of M'_s can readily be taken.

The simply supported reference beam 11a of Fig. 17 treated in section A furnishes the following relation:

$$M'_s = \frac{[P'_T] \cdot l_a}{4} = \frac{14.5 \cdot 160}{4} = 580 \text{ cmt.}$$

Accordingly we receive from the formula given above under b):

$$P'_s = \frac{9.580}{2.240} = 10.87 \text{ t},$$

and from the formula laid down under c):

$$P'_T = \frac{6 \cdot 580}{240} = 14.50 \text{ t}.$$

Under observance of the simplified mode of calculation of the sections A and B and with $J = 1525\ cm^2$, $E = 2100\ t/cm^2$, the deformations due to the ultimate loads $P'_s = 10.87$ t and $P'_T = 14.5$ t respectively, are shown in Fig. 20, in combination with the relations f (P) and τ (P).

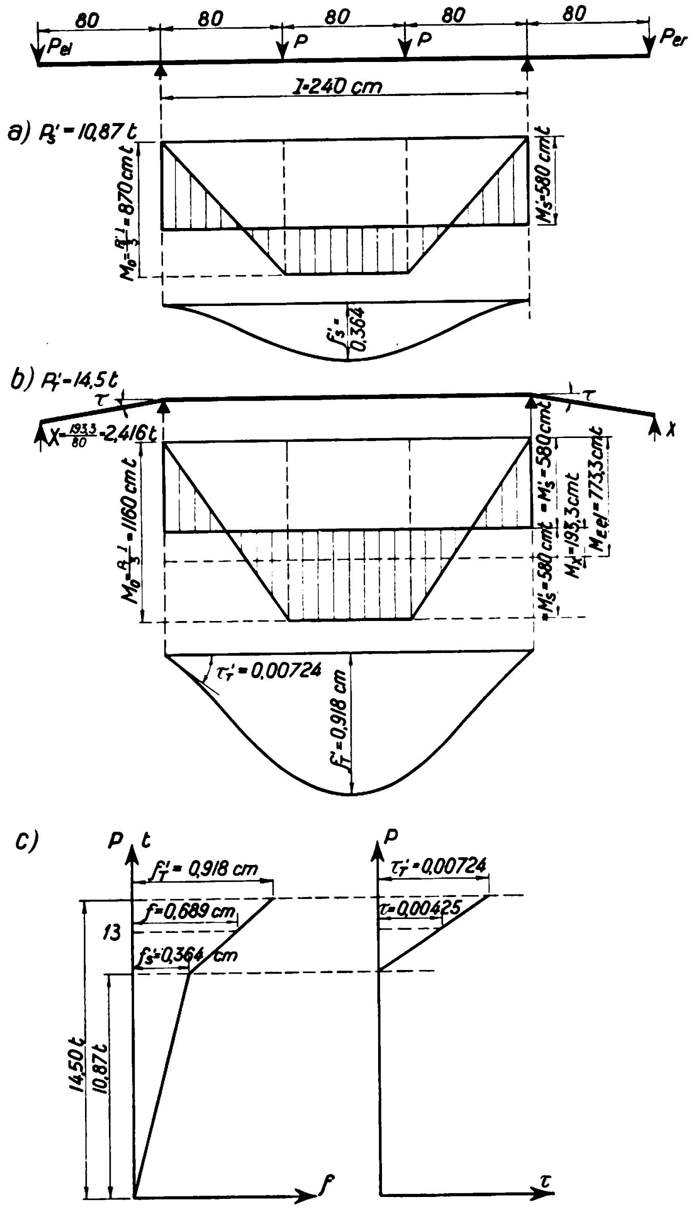
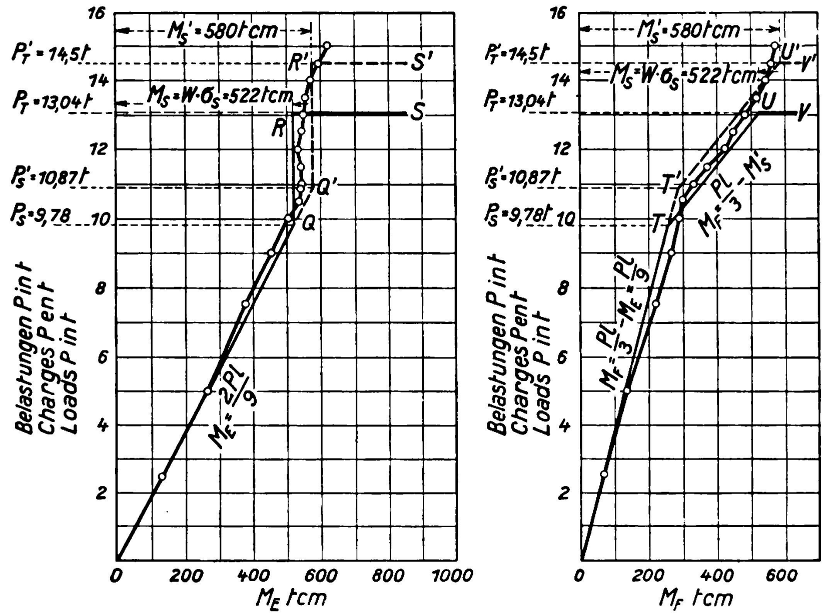


Fig. 20.

As mentioned, during the execution of the tests the auxiliary forces P_{el} and P_{er} were measured and in consequence also the restrained moments M_E || $80 \cdot P_e$ together with the moments $M_F = M_O - M_E$. These values, in analogy to Fig. 19, were entered in Fig. 21. This Fig. shows also the lines OQRS and OTUV for M_E and M_F, according to the values of P_s and P_T, calculated by means of the relation $M_s = W\ \sigma_s$. This procedure may be recommended if it is desired to be on the safe side when preparing the analytical calculation for the actual carrying capacity of a beam. For the purpose of comparison the values f of deflections measured in mid-span are shown in Fig. 22, while Fig. 20 illustrates the corresponding calculated quantities. The measured values become higher for $P < P_s$ compared with the calculated deflections. The reason for this may be, as shown in Fig. 21, that the actual bending moments in the span produced by the test are higher than the calculated values.



Fig, 21.
Beam 11.

Of particular interest are the permanent deflections of the beam ends shown in Fig. 23. The deflections are proportional to the angles τ (Fig. 20) if the assumptions made for calculation are observed. Fig. 24 gives the measured unit tensile elongations at the supports and in mid-span.

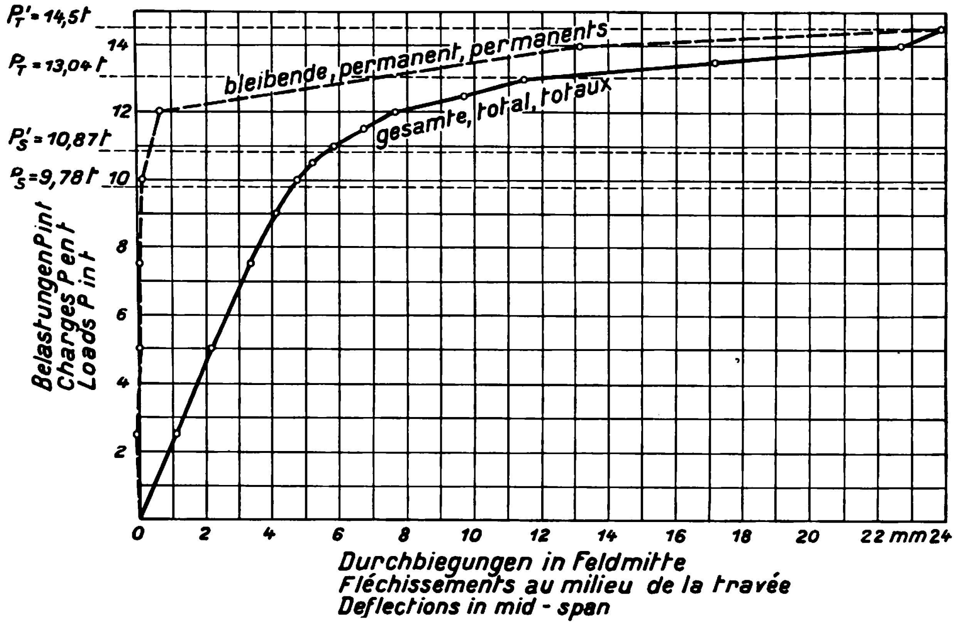


Fig. 22

d) Test with I-beams, ends fixed in masonry.

G. v. Kazinczy published in [15] tests which he carried out with I-beams with a depth of 160 mm, having clear spans of 5.60 and 6.00 m[4]. The size

[4] The Author heard of the first mentioned tests at the end of 1928 and of the second tests only in 1930.

of the beams was based on a bending moment of $M = \frac{p\,l^2}{24}$. The deficient carrying capacity at the supports was supplied by providing top reinforcement of round bars and a concrete slab acting as compression flange. Based on

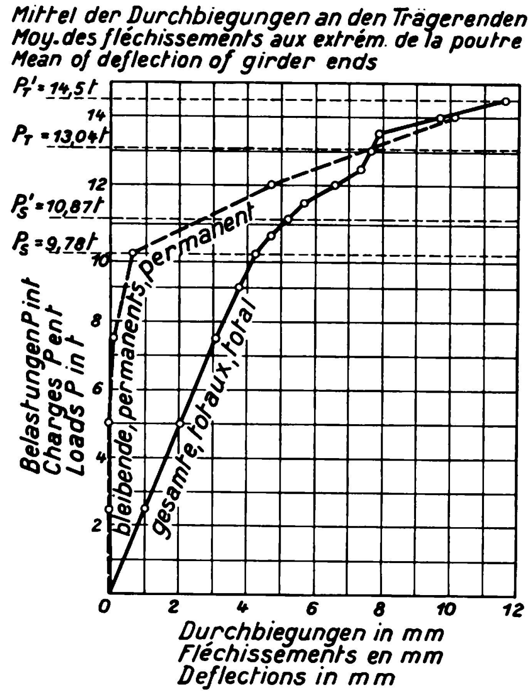


Fig. 23.

these tests, *G. v. Kazinczy* proposed that a beam with fixed ends not encased in concrete should be designed for a moment of $\frac{p\,l^2}{16}$, irrespective whether the

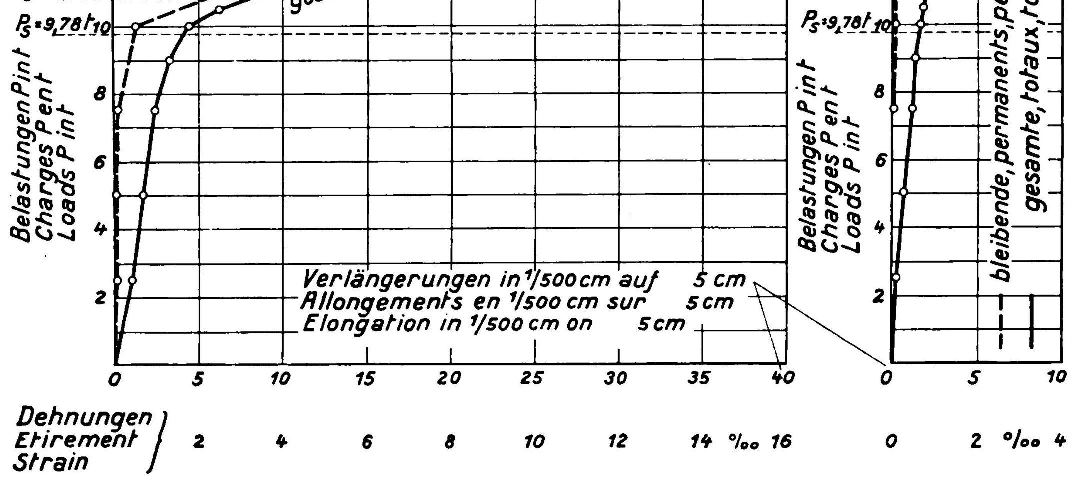


Fig. 24. Beam 11.

beam be fully or only partially fixed. He also points out that such cross sections having reached the yield point stresses may be regarded as acting like a hinge, equipped with a permanent bending moment.

F. v. Emperger reports in [16] on experiments carried out with one simply supported beam and six beams with ends fixed in different kinds of masonry. The beams used were I-beams No. 15 with a clear span of 4 m[4]. In his report

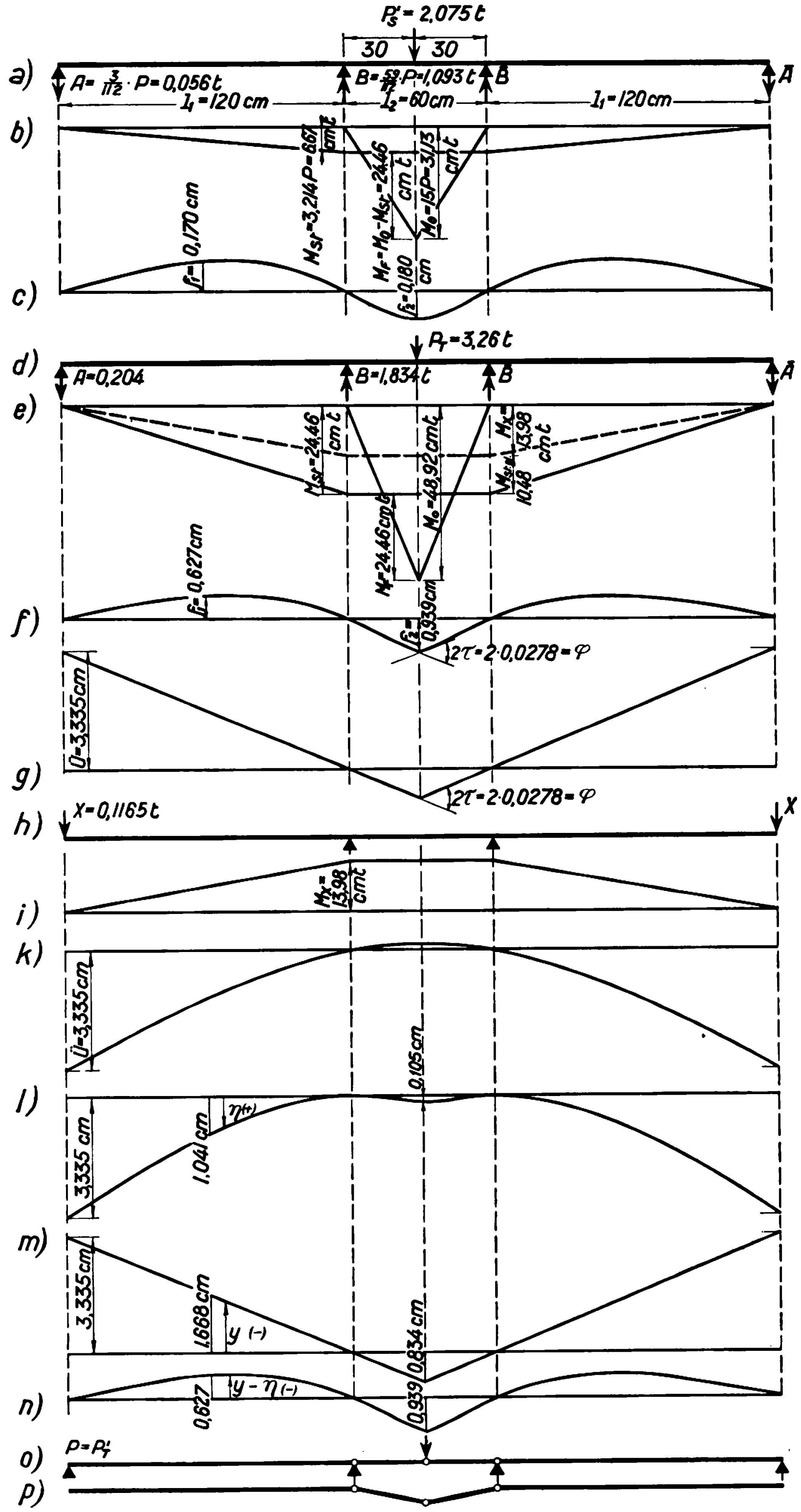


Fig. 25.

F. v. Emperger says 'that provided certain conditions for fixing in masonry are observed, the carrying capacity of steel beams can be expected to be such as almost to reach the attainable maximum bending moment of $\frac{\mathrm{Pl}}{16}$.'

E. Continuous I-beams over three spans, with loaded central span.

Such beams were studied and tested by *F. Stüssi* and *C. F. Kollbrunner* as published in [6] (especially tests 532/6 and 534/8). The behaviour of a

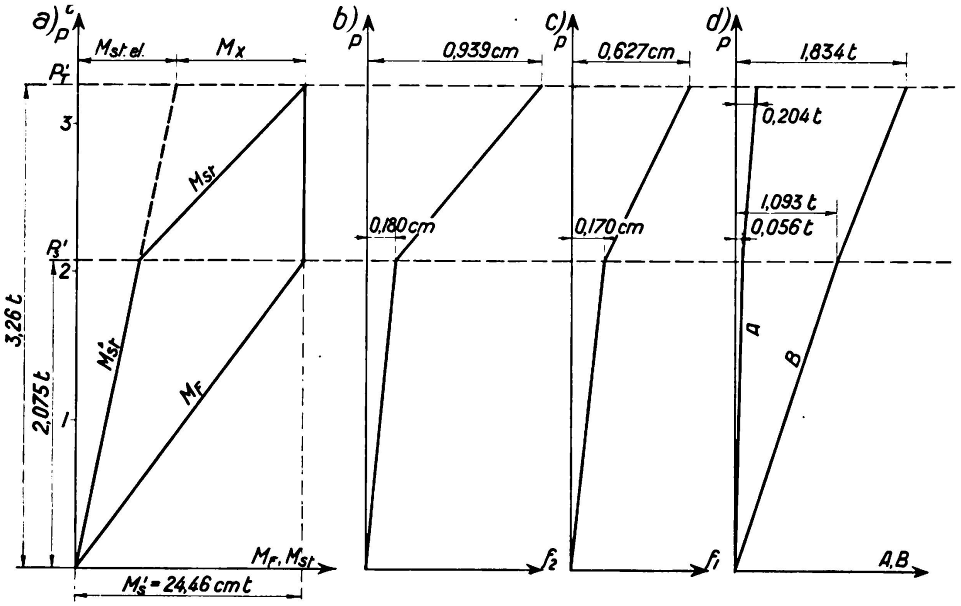


Fig. 26.

beam corresponding to these tests for loads ranging from o to P'_T under consideration of the simplified hypothesis of interpretation, is shown in Fig. 25 and 26. For the beam I $\frac{46}{35}$ is $J = 16.73$ cm^4 and $W = 7.28$ cm^3. Specimens cut from the flanges gave an average yield point stress of $\sigma_s = 3.36$ t/cm^2 with margins of $\pm 10\,\%$. The results of the tests 532/6 and 534/8 (see Fig. 14 and 15 [6] which are identical to Fig. 27 of this report) allow to

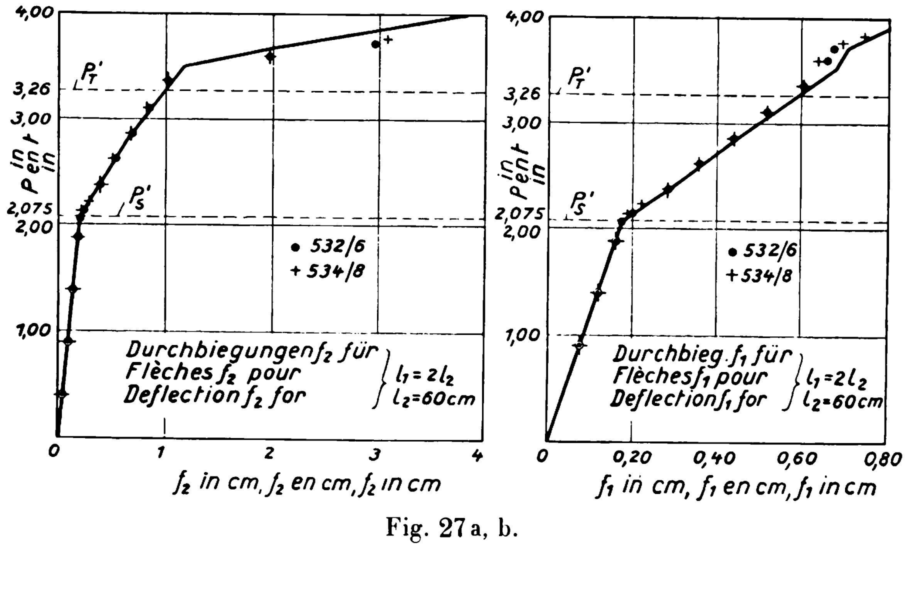


Fig. 27 a, b.

deduce that only incidentally $M'_s = 7.28 \cdot 3.36 = 24.46$ cmt.[5] This, since for the range of elastic deformations, under the sole consideration of deformations due to bending moments only, is

$M_{st} = 3.214$ P and $M_F = (15 - 3.214)$ P $= 11.786$ P, hence we receive $P'_s = \frac{24.46}{11.786} = 2.075$ t. According to Fig. 26a and for $P > P'_s$ the bending moment over the support is equal to $M_O - M_F = 15\ P - M'_s$; therefore with $15\ P'_T - M'_s = M'_s$ we receive $P'_T = \frac{2 \cdot 24.46}{15} = 3.26$ t. In agreement with the procedure adopted under A), B) and C), Fig. 25b shows the deflection curve for the ultimate load $P'_s = 2.075$ t under consideration of $E = 2100$ t/cm^2 and deformations due to bending moments only. In the bending moment diagram of Fig. 25e produced by the ultimate load $P'_T = 3.26$ t is $M_O = 15\ P = 48.92$ cmt, and $M_{st} = 48.92 - 24.46 = 24.46$ cmt. For the conditions of Fig. 25e the deflection lines (shape of beam axis) passing from A to B and $\bar{A}$ to $\bar{B}$ are designed in Fig. 25f. At the centre of l_2 an angle of $\varphi = 2\ \tau = 2 \cdot 0.0278$ can be observed. After release the beam assumes the shape shown in Fig. 25g, with lifting up at A and $\bar{A}$ to the extent of $ü = 0.0278 \cdot 120 = 3.335$ cm. Whilst reloading the elevation ü has first to be brought back again by introducing the forces $X = 0.1165$ t, to which corresponds the bending moment diagram of Fig. 25i and the deflection curve of Fig. 25k showing deflections at A and $\bar{A}$ identical with $ü = 3.335$ cm. The reloading of the beam with $P'_T = 3.26$ t causes elastic deformations only. The same bending moment diagram as given by Fig. 25e is reproduced through the combination of $M_x = 0.1165 \cdot 120 = 13.98$ cmt and the moment over the support $M_{st\,.\,el} = 3.214 \cdot 3.26 = 10.48$ cmt. The reaction at A is composed of two parts: the force X and the force due to merely elastic action of the beam: $\frac{3}{112} \cdot 3.26 = 0.0873$ t, both together $= 0.204$ t. The deflection line (Fig. 25l ordinates η) of the beam deformed as in Fig. 25g was calculated, starting with the supports $B\bar{B}$, according to the bending moment diagram given by Fig. 25e. The shape of the axis of the beam after reloading with $P'_T = 3.26$ t (Fig. 25n) is found by forming algebraically the quantities $\eta + y$, giving for instance in mid-span of the central bay a deflection $f_2 = 0.105 + 0.834 = 0.939$ cm and in mid-span of the end spans a deflection $f_1 = 1.041 - 1.668 = -0.627$ cm.

In the same way as in the sections B) and C) and according to Fig. 25c and f the relations between P and 1) the deflection f_2 in the intermediate span, 2) the deflection f_1 in the outer span, 3) the reaction A, 4) the reaction B are shown in Fig. 26b to 26d.

For the purpose of comparison are given in 27a and b the measured de-

[5] According to a letter received from Mr. F. Stüssi, the curve of deflections for a simply supported reference beam of $l = 60$ cm loaded in mid-span gives us: $[P'_T] = 1.71$ t corresponding to $[P_T] = 1.63$ t. In place of $M'_s = 24.46$ cmt the value $M'_s = 1.71 \cdot 15 = 25.65$ cmt had possibly been introduced for the interpretation of the test. But with $M'_s = 24.46$ cmt one is safe in calculating the value P'_T.

flections; the values for P'_s and P'_T are also entered in the same figures. It can be seen quite clearly that the quantity P'_T actually defines the actual carrying capacity.

It is not advisable to call the test result $P_w = 3.902$ t the carrying capacity, this has been explained already in the sections A) and B). For $P > P'_T$ the deflections increase much quicker than for $P < P'_T$. The measure results of the deflections agree well enough with the results received by the calculation based on the simplified hypothesis of interpretation.

To study the flow of moments due to $P > P'_s < P'_T$ more closely, the Author carried out in May—June 1936, two tests (see [17]) with loading as in Fig. 25 using an I-beam 10 · 10 cm with spans of $l_1 = 2.40$ m and $l_2 = 1.20$ m.

From a comparison test carried out with a simply supported single span beam loaded in mid-span was found $M'_s = 262$ cmt. Under elimination of the dead weight and the deformation due to shear we get: $P'_s = 11.12$ t and $P'_T = 17.47$ t.

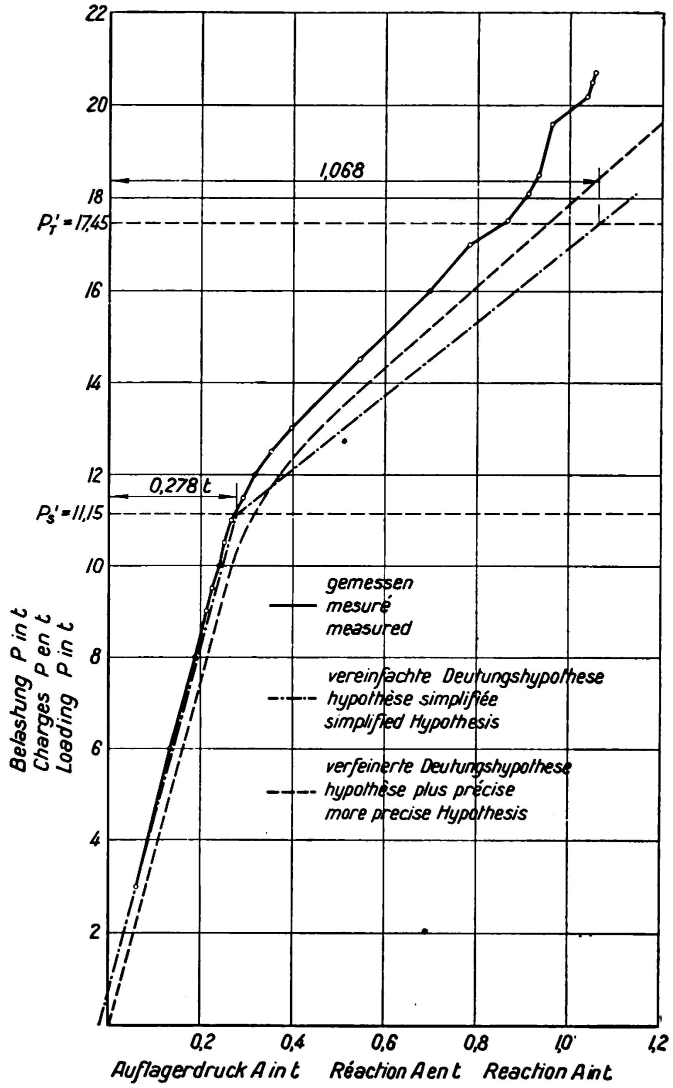


Fig. 28 a.

In Fig. 28 a are plotted the average values (received by test) for the reactions at A and $\overline{A}$, in Fig. 28 b the bending moments M_F and M_{st}, calculated with the help of the measured values A and $\bar{A}$ under consideration of the dead weight. The influence of dead weight has also been considered in connection with $P'_s = 11.15$ t and $P'_T = 17.45$ t. The measured deflections in the centre of the intermediate span are given in Fig. 28 c.

The details of Fig. 28b show that the actual moments M_F and M_{st} deviate quite considerably from the values (shown in dash-dotted line) worked out

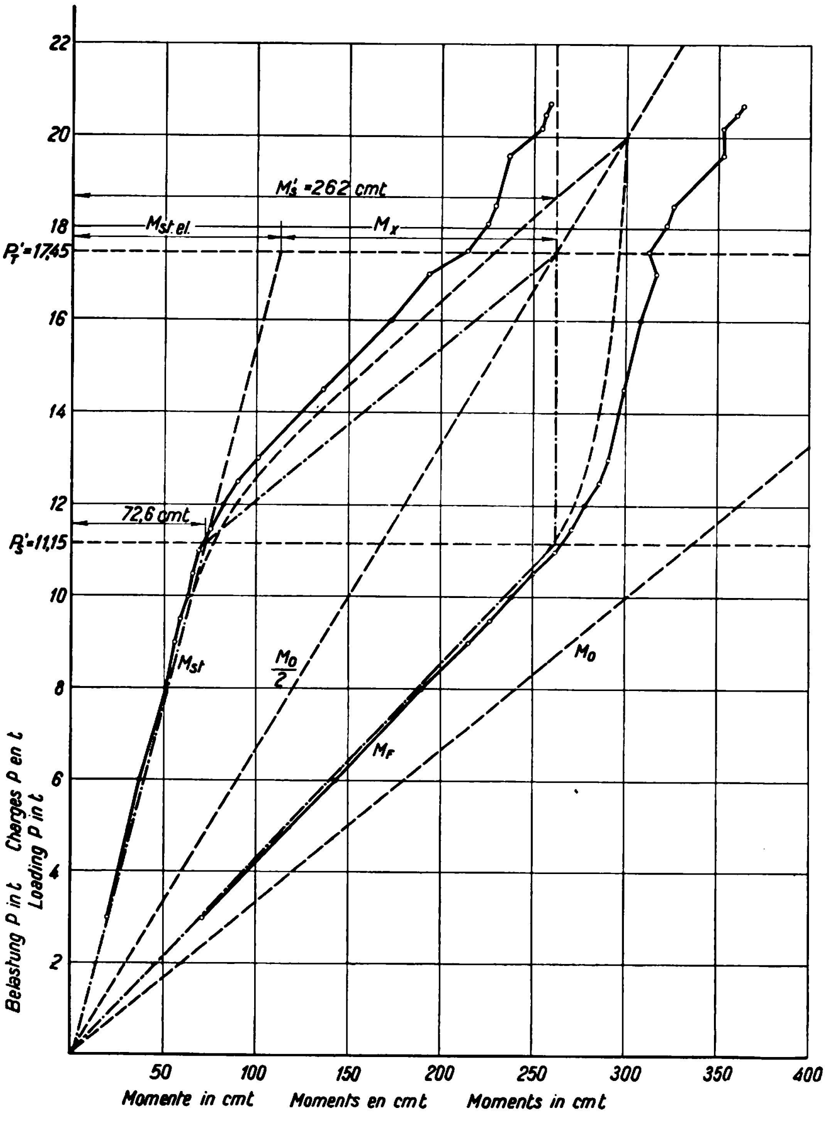


Fig. 28 b.
The dotted lines for M_F and M_{st} naturally hold only so long as $M_{st} < M_s$. The load P' corresponds to this limiting case. When this load is exceeded the beam buckles not only in the middle but also at the points of support.

according to the simplified hypothesis of interpretation. It is obvious that the quantities P'_s and P'_T are characteristic for the actual behaviour of the beam. (See also Fig. 28c, from which it will be noticed that the deflections f_2 increase only very considerably for values greater than P'_T.) The bending moments M_F exceed the value of M'_s if $P > P'_s$. The practical limit for the carrying capacity of the beam is only attained if permanent deformations develop also over the supports, or differently expressed, if M_{st} reaches the value of M'_s.

In any case, from the fact that both moments M_F and M_{st} are not equal for P'_T, it cannot be deduced that the practical limit of the carrying capacity of a continuous girder will be less than for P'_T. The deviations in the actual values of M_{st} and M_F can be explained by the choice of a straight horizontal line F G in Fig. 4 to replace the curve C D E, in accordance with the simplified hypothesis of interpretation. Based on the line CDE established by test results and under elimination of the influences of dead weight and deformation due to shear we find the bending moment M_F and M_{st} as shown dotted in Fig. 28 b in conformity with what we said in [17].

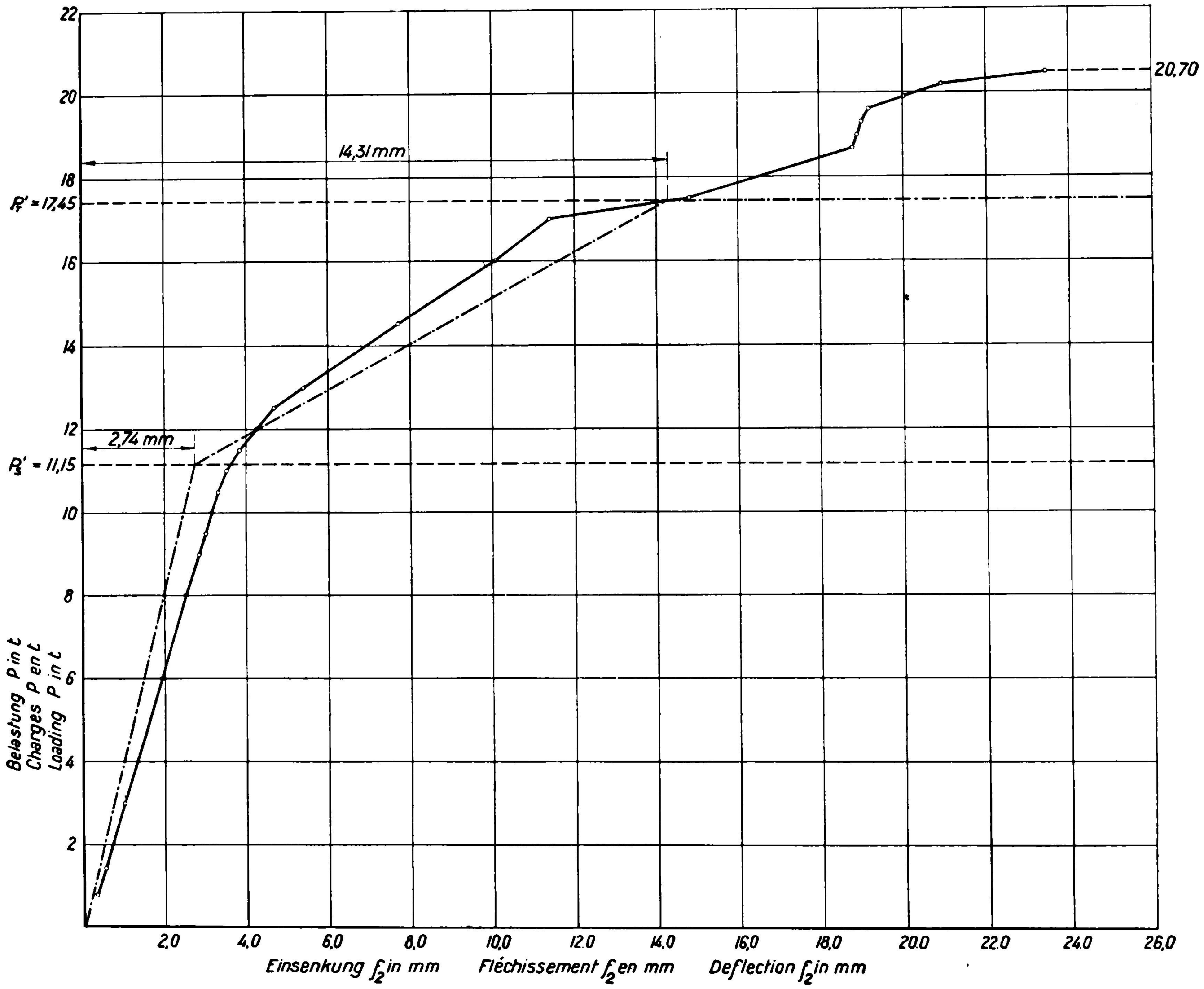


Fig. 28 c.

F. Continuous latticed girders with parallel chords, over three spans.

The tests started by *M. Grüning* and completed by *G. Grüning* and *E. Kohl* were based on the type of girder shown in Fig. 29 a (see [18]). The bars above the intermediate supports B $\overline{B}$ in the top chord and the three bars in the centre of the lower chord are removable I-bars. The panel point No. 17 was subsequently fitted with a truss-pin. The quantities of the reactions at A and $\overline{A}$ were determined by tests.

In one of the first experiments the members O_{10}, U_{17} and $\overline{O}_{10}$ were all of the same section, namely 2.88 cm². The yield point stress for these bars was $\sigma_s = 2.68$ t/cm² and the carring moment $M_s = 2.88 \cdot 50 \cdot 2.68 = 386$ cmt. Making no allowance for the dead weigth of the structure for 4 equal concentrated loads acting in point 14, 16, $\overline{16}$, $\overline{14}$ respectively the purely elastic reactions at the outer supports are equal to 0.418 P

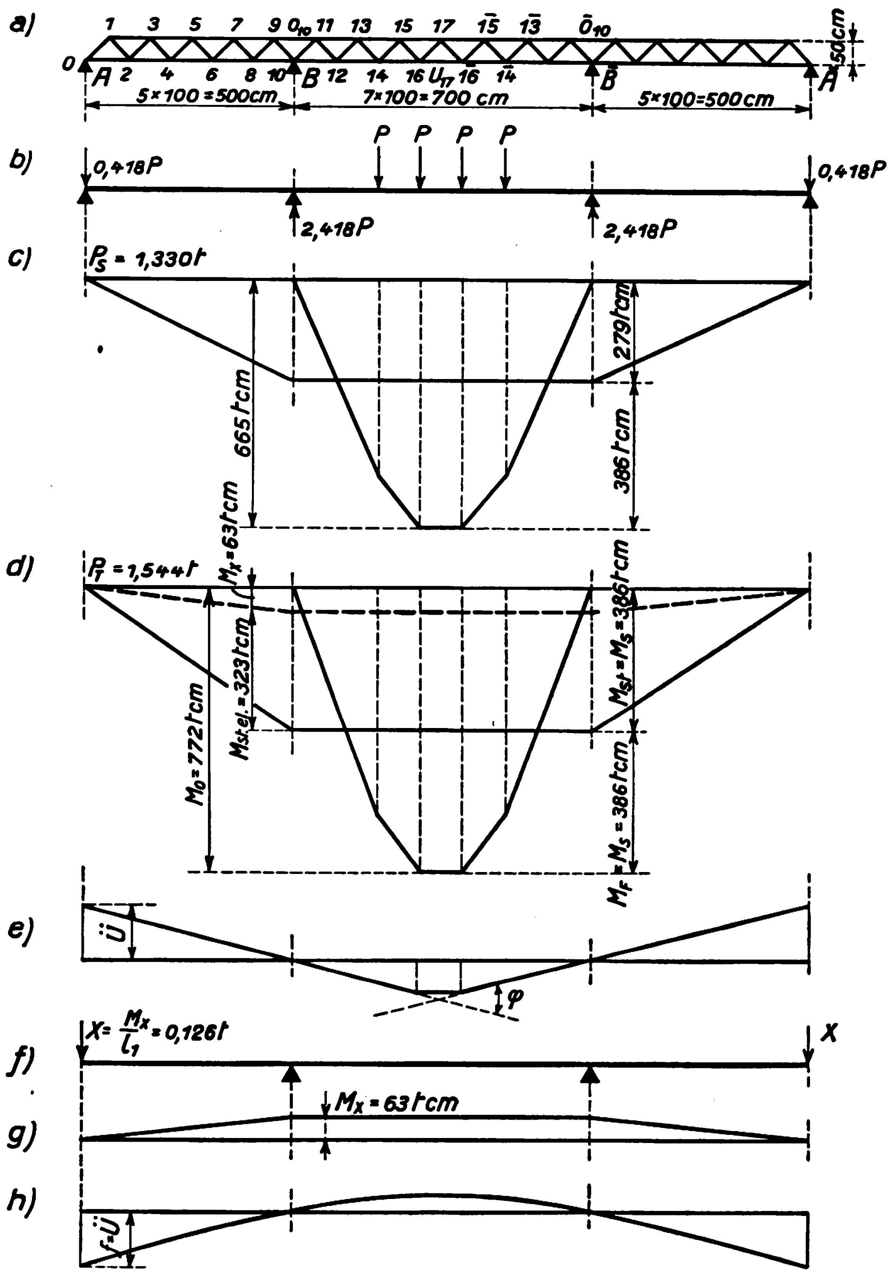


Fig. 29.

and hence those at the intermediate supports equal to 2.418 P. Accordingly the moments at those supports are $M_{st} = 0.418 \cdot 500$ P $= 209$ P and the maximum moment in the central span $M_F = (500 - 209)$ P $= 291$ P. The loads P_s causing the member U_{17} to begin to yield, are deduced from 291 $P_s = M_s = 386$ cmt to $P_s = 1.33$ t, see Fig. 29c. The loads P_T for which the members O_{10} and $\bar{O}_{10}$ begin also to yield are found to be $P_T = 1.544$ t based (Fig. 29 d) on $M_{st} = M_O - M_s = M_s$ i. e. $500 \cdot P_T = 2 \cdot 386$ cmt. The relations between M_{st}, M_F, P_s and P_T are given in Fig. 30a. If the girder is released of its loads P_T we find permanent deformations in

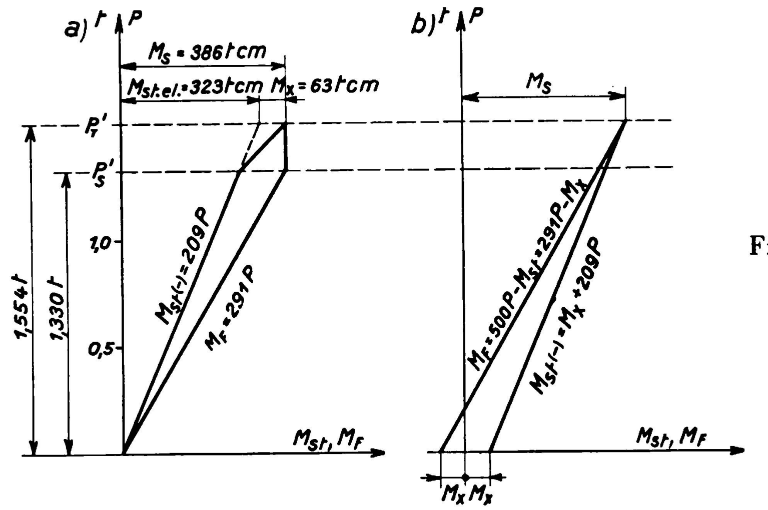


Fig. 30.

the lower chord as given in Fig. 29 e. The deflection line shows an angle φ under point 17, causing an elevation ü. Before actual reloading takes place, the ends of the girder have to be brought back first on to the bearings through the introduction of the forces X (Fig. 29 f). These forces create bending mo-

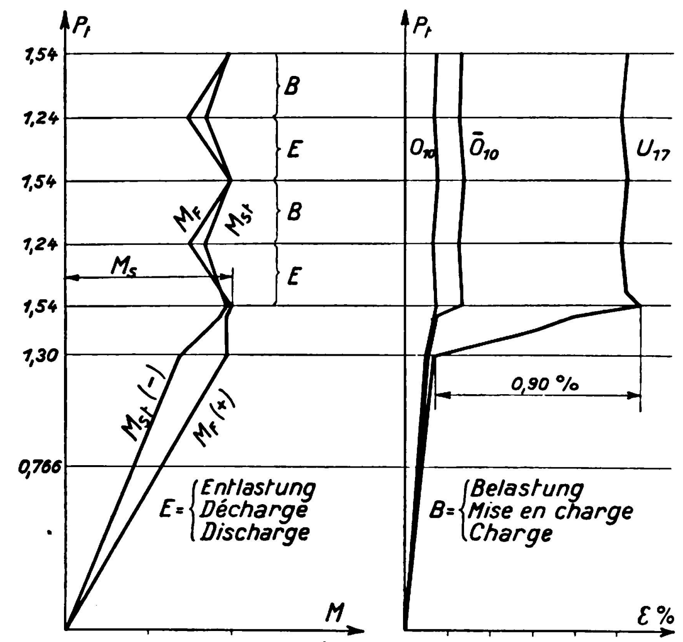


Fig. 31.

ments over the supports of $M_x = X \cdot l_1$, to which have to be added the moments $M_{st \cdot el} = 209\ P_T = 323$ cmt produced by loading of the girder with the forces P_T. From $M_x + M_{st \cdot el} = M_s$ follows that (Fig. 29 d) $M_x = 63$ cmt and $X = 0.126$ t. The deflection f at the girder end due to X must be equal to ü; with this condition we receive φ and the permanent deformation

Δ s of the bar U_{17}, due to the loads P_T [Δs could alternatively be obtained from the deformations of the line A, B, 16, 17, $\overline{16}$, $\overline{B}$, $\overline{A}$]. A reloading of the girder produces the conditions laid down in Fig. 30b. The test results given in Fig. 7 of [18] are produced in Fig. 31. These values show good agreement with the values deduced from the interpretation of the tests (29 and 30), making no allowance, however, for the dead weight of the girder and assuming a frictionsless hinge in point 17.

Further tests described in [18] were for cases with I-bars having reduced cross sections for certain portions of these bars and for cases charged with heavier external loads than P_T. Further, the influence of subsidence of the supports on the carrying capacity was studied. A single lifting of the outer

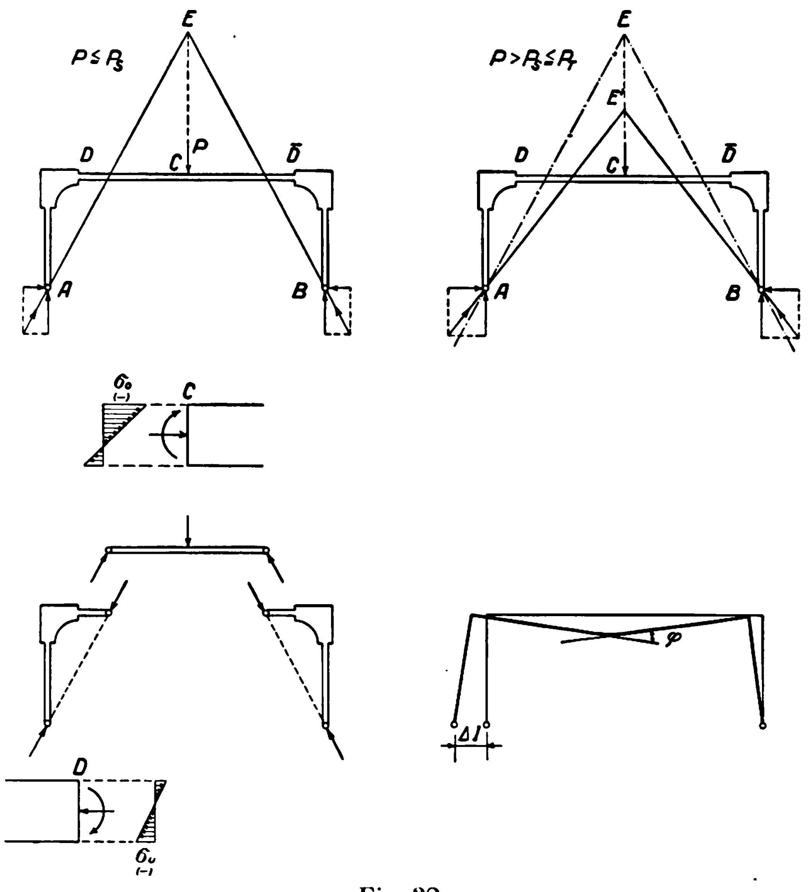


Fig. 32.

bearings did not impair the carrying capacity; but repeated lowering and lifting of the supports of the girder loaded with P_T will naturally be detrimental.

As regards the conclusions to be drawn from the experiments which still require to be continued, a reference is made to [18] p. 72.

It would be of particular interest to obtain some knowledge about the actual carrying capacity of continuous latticed girders whose critical members are weakened, e. g. in consequence of rivet holes.

G. Rectangular portal frame.

K. Girkmann reported in [19] on a test which he carried out and to which refers Fig. 32.

The frame structure AB with corners purposely overdimensioned received an increasing load P in mid-span of the brace D $\overline{D}$. First a pressure line AEB is established by the load P, giving a stress distribution in cross sections C and D as shown in Fig. 32. The maximum stress takes place in the upper extreme fibres of the cross section C (compression).

This stress reaches the yield point stress for a load P_s. For increasing loads, $P > P_s$, the elements of the brace near C will be deformed in such a way that on replacing one of the fixed bearings of the frame by a movable bearing a displacement of Δl will take place. This means that for loads $P > P_s$ a greater horizontal thrust will be established than expected according to the usual

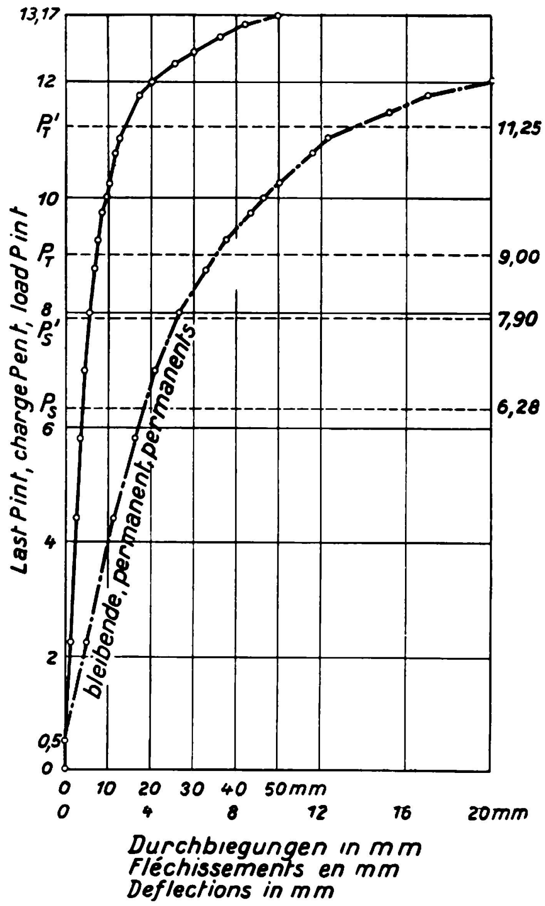


Fig. 33.

theory. The pressure line will be less steep. The frame starts to fail definitely if yield point stresses are obtained at the bottom of the cross section D. The carrying capacity can be calculated, with sufficient accuracy for practical purposes, by assuming that the pressure line passes in the centre between C and D; or more exactly between the core-points which are decisive for σ_o in C and σ_u in D. Introducing the respective cross sections into the calculation without making deductions for rivet holes we receive:

$P_{adm} = 2.88$ t for $\sigma_{zul} = 1.2$ t/cm^2; $P_s = 6.28$ t with stressing at one place only of $\sigma_s = 2.62$ t/cm^2 (on top of cross section C); $P_T = 9.00$ t the practical carrying capacity. The assumption is hereby made that cross section acts as a hinge immediately after attaining the yield stress at the topmost fibres.

The customary theory of elasticity allows for the frame under consideration a carrying capacity of 6.28 t only. The test has proved (see Fig. 33) that only from a load of P = 11.25 t upwards do the deformations in mid-span of the brace $D\overline{D}$ grow rapidly; therefore $P'_T = 11.25$ t and accordingly $P'_s = \sim P'_T \cdot \frac{P_s}{P_T} = \sim 11.25 \cdot \frac{6.28}{9.00} = 7.9$ t. Fracture occured, starting from a rivet hole in mid-span, at $P_v = 13.17$ t. For other observations made attention is drawn to details given in [19].

List

of important treatises on the influence of ductility (plasticity) of building steel on the dimensioning of steel structures particularly such publications which refer to tests.

F. Bleich: La ductilité de l'acier. Son application au dimensionnement des systèmes hyperstatiques. *L'Ossature Métallique 3 (1934), p. 93.*

F. Bleich: La théorie et la recherche expérimentale en construction métallique. *L'Ossature Métallique 3 (1934), p. 627.*

H. Bleich: Über die Bemessung statisch unbestimmter Stahltragwerke unter Berücksichtigung des elastisch-plastischen Verhaltens des Baustoffes. *Bauingenieur 13 (1932), p. 261.*

[8] *F. Bohny:* Über die Verwendung von Nickelstahl im Brückenbau (see page 90 test with I-beam No 30). *Der Eisenbau 2 (1911), p. 79.*

[9] *J. E. Brik:* Über den Knickwiderstand der Druckgurte vollwandiger Balkenträger (contains informations about tests with simple beams). *Der Eisenbau 3 (1912), p. 351 und 4 (1913), p. 189, 192, 353.*

Eisenmann: Theorie und Statik plastischer Träger des Stahlbaues. *Stahlbau 6 (1933), p. 25. Erwiderung 6 (1933), p. 54.*

[16] *F. v. Emperger:* Versuche mit eingespannten Balken in Heft 5 der Mitteilungen über Versuche ausgeführt vom Eisenbetonausschuß des österreichischen Ingenieur- und Architekten-Vereins. *F. Deuticke, Leipzig und Wien 1917.*

[3] *J. Fritsche:* Die Tragfähigkeit von Balken aus Stahl mit Berücksichtigung des plastischen Verformungsvermögens. *Bauingenieur 11 (1930), p. 851, 873, 888.*

J. Fritsche: Die Tragfähigkeit von Balken aus Baustahl bei beliebig oft wiederholter Belastung. *Bauingenieur 12 (1931), p. 827.*

[5] *J. Fritsche:* Grundsätzliches zur Plastizitätstheorie. *Stahlbau 9 (1936), p. 65.*

[19] *K. Girkmann:* Über die Auswirkung der „Selbsthilfe" des Baustahls in rahmenartigen Stabwerken. *Stahlbau 5 (1932), p. 121.*

[14] *O. Graf:* Über Dauerversuche mit I-Trägern aus St. 37. *Stahlbau 7 (1934), p. 169.*

[18] *G. Grüning & E. Kohl:* Tragfähigkeitsversuche an einem durchlaufenden Fachwerkbalken aus Stahl. *Bauingenieur 14 (1933), p. 67.*

Martin Grüning: Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl bei beliebig häufig wiederholter Belastung. *Broschüre Berlin 1936.*

[2] *M. Grüning-Kulka* und andere: Zuschriften an die Schriftleitung betreffend Plastizität *Bautechnik 6 (1928), p. 274.*

[4] *F. Hartmann:* Die Formänderungen einfacher und durchlaufender Stahlträger. Mit einem Versuch.
Schweiz. Bauzeitung 101 (1933), p. 75.

F. Hartmann: Über die Erhöhung der zulässigen Inanspruchnahme von stählernen Brücken.
Zeitschrift des Österr. Ingenieur- und Architekten-Vereins 1935, Heft 21/22 und 23/24.

A. S. Joukoff: Principes d'application de la plasticité au calcul des constructions métalliques hyperstatiques.
L'Ossature Métallique 3 (1934), p. 379.

F. Kann: Der Momentenausgleich durchlaufender Traggebilde im Stahlbau.
Broschüre, Verlag de Gruyter & Co. 1932.

F. Kann: Rechnerische Untersuchungen über die Größe des Fließbereiches in stählernen Durchlaufbalken unter Berücksichtigung des Momentenausgleichs.
Stahlbau 5 (1932), p. 105. Berichtigung (1932), p. 160.

[15] *G. v. Kazinczy:* Kisérletek befalozott tartékkal (Tests with fixed-end beams.)
Betonszemle 2 (1914), Nr. 5, 6, 7.

G. v. Kazinczy: Bemessung von statisch unbestimmten Konstruktionen unter Berücksichtigung der bleibenden Formänderungen.
Internationaler Kongreß für Eisenbau, Liége September 1930.

[13] *G. v. Kazinczy:* Statisch unbestimmte Tragwerke unter Berücksichtigung der Plastizität.
Stahlbau 4 (1931), p. 58.

G. v. Kazinczy: Die Weiterentwicklung der Plastizitätslehre.
Sonderdruck aus dem 12. Jahrgang der „Technica", Budapest 1931.

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2. Band der Abhandlungen der Internationalen Vereinigung für Brückenbau und Hochbau, Zürich 1934.

N. C. Kist: Die Zähigkeit des Materials als Grundlage für die Berechnung von Brücken, Hochbauten und ähnlichen Konstruktionen aus Flußeisen.
Der Eisenbau 11 (1920), p. 425.

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Science et Industrie. Edition: Construction et Travaux Publics. März 1933.

O. Knackstedt: Beitrag zur Ermittlung der wirklichen Tragfähigkeit statisch unbestimmter Vollwandträger aus Stahl unter Berücksichtigung der plastischen Verformung.
Dissertation, Braunschweig 1932.

O. Lienau: Der Einfluß der Streckung auf die Tragfähigkeit durchlaufender Eisenträger.
Zeitschrift Der Schiffbau 29 (1928), p. 532.

[11] *Maier-Leibnitz:* Beitrag zur Frage der tatsächlichen Tragfähigkeit einfacher und durchlaufender Balkenträger aus Baustahl St. 37 und aus Holz.
Bautechnik 6 (1928), p. 11, 27.

Maier-Leibnitz: Diskussionsbeitrag im Bericht über die 2. Internationale Tagung für Brückenbau und Hochbau.
Wien 1928, p. 257.

[1] *Maier-Leibnitz:* Versuche mit eingespannten und einfachen Balken von I-Form aus St. 37.
Bautechnik 7 (1929), p. 313. Berichtigung (1929), p. 366.

Maier-Leibnitz: Artikel: „Zulässige Beanspruchung" in Luegers Lexikon der Gesamten Technik und ihrer Hilfswissenschaften.
Luegers Lexikon der gesamten Technik ca. 1930.

[17] *Maier-Leibnitz:* Frage der tatsächlichen Tragfähigkeit durchlaufender vollwandiger Balken aus Baustahl mit drei Feldern.
Stahlbau 9 (1936).

[7] *H. F. Moore:* The strength of I-beams in flexure (consains references to other publications about tests with simple beams).
Bulletin Nr. 68 der University of Illinois, Engineering Experiment Station vom 1. September 1913.

[10] *K. A. Müllenhoff:* Versuche an schmiedeeisernen Blechträgern (references to tests of F. Mc. Kibben der Lehigh University on simple beams).
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[12] *J. H. Schaim:* Der durchlaufende Träger unter Berücksichtigung der Plastizität.
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E. Siebel und *H. F. Vieregge:* Über die Abhängigkeit des Fließbeginns von Spannungsverteilung und Werkstoff.
Mitteilungen aus dem Kaiser-Wilhelm-Institut für Eisenforschung 1934, Band XVI, Lieferung 21.

[6] *F. Stüssi* und *C. F. Kollbrunner:* Beitrag zum Traglastverfahren.
Bautechnik 13 (1935), p. 264.

F. Stüssi: Sicherheit statisch unbestimmter Fachwerke bei Veränderung einzelner Stabquerschnitte. Beitrag zur wirtschaftlichen Verstärkung statisch unbestimmter Fachwerkträger mit geringsten Eingriffen.
Bautechnik 9 (1931), p. 603.

F. Stüssi: Aktuelle baustatische Probleme der Konstruktionspraxis.
Schweiz. Bauzeitung 1935, Sonderdruck.

Summary.

The paper gives a summary of loading tests carried out with different types of structural elements. With some of these tests the yield limit is exceeded. It can be demonstrated in every case that the load P_s is not decisive for the carrying capacity, simply because on application of the theory of elasticity, the yield limit is reached somewhere in the structure. In fact a higher value P_T is reached for the carrying capacity. To determine the actual carrying capacity, a procedure is given (simplified hypothesis of interpretation), which supplies sufficiently accurate values for practical purposes. The procedure also serves to interpret the results of the tests. If necessary, the interpretation of these results and the mode of calculation for the determination of P_T can still be refined.

I 8

Calculation of Statically Indeterminate Systems based on the Theory of Plasticity.

Bemessung statisch unbestimmter Systeme nach der Plastizitätstheorie (Traglastverfahren).

Dimensionnement des systèmes hyperstatiques d'après la théorie de la plasticité.

Baurat Dr. Ing. F. Bleich,
Zivilingenieur, Wien.

1) *Introduction.*

This paper has been written for the purpose of discussing the practical use of the theory of plastic equilibrium for the design of statically indeterminate structures composed of members stiff against bending, of defining the limits of its application, and of giving some examples of its use.

The usual method of designing steel structures proceeds from the assumptions that, under the influence of dead weight and live load, stresses develop which follow *Hooke*'s law, and that these stresses shall at no point exceed a certain limited proportion of the working strength of the material, which proportion is known as the permissible stress. The knowledge that in statically indeterminate structures the elastic limits can be exceeded locally without necessarily reducing the loadcarrying capacity and consequently the factor of safety of the structure, as the overloaded sections can be relieved by those less highly stressed, gave rise to another conception of safety in the design of such structures. The new definition of the safety factor should make it possible to take advantage of the properties of tenacity in the steel in the more economic design of statically indeterminate structures.

This definition, then, is as follows:

The factor of savety ν *is the ratio of the ultimate load to the useful load. By ultimate load is meant that limit of load up to which the load can be increased without causing in the structure inadmissible deformations due to frequent repetition of loading and release from load. If various cases of loading are possible, a factor of safety must exist even for any case of changing loading for any sequence of loading whatever.*

The dead weight of the structure is also included in the safe load.

In the method of designing steel structures up to now the factor of safety was expressed by the permissible stress through the ratio

$$\nu = \frac{\sigma_A}{\sigma_S} \tag{1}$$

wherein σ_A is the working stress of the material, the upper limit of which, however, is considered as being fixed by the elastic limit in order that greater permanent deformation may be safely avoided.

The same value ν as expressed in equation (1) is also chosen as the safety factor for the design of statically indeterminate structures according to the new conception of safety. Employing then the same safety factor, the dimensioning of statically indeterminate structures according to the above conception would give exactly similar results to the dimensioning of statically determinate structures, which always proceeds on the basis of the proportionality between stress and strain.

The determination of dimensions of statically indeterminate structures based on this new conception of safety is called *design on the theory of plastic equilibrium*.

The theory of plastic equilibrium is based on two assumptions:

1) The material must show a stress-strain line in accordance with Fig. 1, with a completely elastic relationship up to the yield point σ_s, the material becoming plastic when this limit is reached. So that the method of plastic equilibrium demands a steel with a sufficiently high elongation[1].

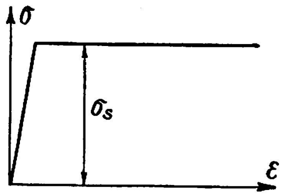

Fig. 1.

2) The material cannot be brought to fracture, in all those cases where the deformations are limited, in other words when a certain upper limit cannot be exceeded even for any number of loading repetitions, irrespective of the difference between the lower and upper limit of stressing. So the method of plastic equilibrium does not deal with the possibility of fatigue failure which can occur without appreciable deformation at the point of fracture. *Strictly speaking, therefore, the use of this method of design can only be permitted where the permanent strength does not come into account, that is to say where the number of loading repetitions during the life of the structure is limited, as is the case in the construction of roofs and floors.*

We must, however, compare the following facts with these observations: For many years now an enormous number of steel bridges have been undergoing in their riveted joints, local stresses in the form of what are called secondary

[1] In soft steels the total permanent deformation at the point of permanent hardening amounts to 10—20% of the measured length.

stresses, which in a considerable number of cases certainly reach the elastic limit. It is well known that in the case of these secondary stresses there are prominent cases of repeated loads producing limited deformation. Fatigue fractures are scarcely known in spite of the fact that in particular cases the number of load fluctuations runs into millions. On the other hand, however, laboratory tests have been carried out on drilled plates, which have shown clearly that in contrast with results obtained with bridges, fatigue strength depends largely on the magnitude of the stress differences and that the fatigue strength is mostly less than the yield point stress, although even with drilled plates there would occur, over the cross sections of the holes, local increases in stress in excess of the average, which are only looked upon as secondary stresses. The conditions have not yet been explained and until they have, precautions should be taken. That is why the line has been taken in the foregoing that dimensioning according to the plastic equilibrium method is limited to those cases where there is no question of fatigue strength. The exploitation of the properties of tenacity of the steel made possible by dimensioning according to this method is of especial significance from the economic point of view in those structures composed of a number of single members stiff against bending, and showing within a definite span or over a series of spans, a constant cross sectional area. If one part of the girder, in which the maximum stress has been reached, fails owing to the occurrence of permanent deformations, then a part of the girder which has up to that time not been completely utilised, will be loaded more heavily owing to the new distribution of the stresses. Nevertheless, in using this new method of dimensioning it is often expedient to carry out local strengthening of the girder.

The plastic equilibrium method, however, offers no further economic advantages in those instances where the cross section of the member is well adapted to the magnitude of the internal forces, as for example is the case in riveted or welded girders, the heights or flange thicknesses of which are designed to suit the bending moment. There is just as little advantage to be gained from the application of the plastic equilibrium method to the dimensioning of statically indeterminate lattice girders as the cross sections of the members of such structures are, as a rule, fitted fairly exactly to the forces in the members. But there is another very important reason why the dimensioning of statically indeterminate lattice structures according to this method is ill advised. The compression members of lattice structures show no tendency to plastic yielding on account of the tendency to buckle, but they fail suddenly, as in the case of tension members which are constructed from brittle material[2]. In this connection lattice work must not be regarded as consisting of members whose material is in every part in an equally elastic-plastic condition. Even in individual members the standard elastic properties as determined by the plastic equilibrium method may vary with changes in compression and tension. The bars may behave like tension bars in the elastic-plastic condition, but if the same beam is also subject to compression it may behave like brittle material.

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[2] This particularly applies to a member which fractures beyond the range of elastic deformations (after the elastic limit is passed).

2) The basic principles for calculation on the theory of equilibrium.

If the main stress at one extreme fibre is plotted to each cross section of a structure composed of members stiff against bending, the line, continuous or continuous in parts, which results is called a *stress line of the structure.* The separate ordinates of this line can naturally belong to different cases of loading of the structure. By selecting for each cross section that particular loading arrangement which creates for the extreme fibre the maximum tension or compression stress, we receive in this way two stress diagrams which are called max. σ and min. σ lines respectively. In asymmetric cross sections or in cases where there are longitudinal forces as well as bending moments, the stresses on the extreme fibres of the member are usually different. In such cases a line max. σ and a line min. σ belongs to each of the extreme fibres.

In each statically indeterminate structure as opposed to a statically determinate structure, stresses and reaction forces can occur, even if there are no external loads. Such a condition is called the *condition of self-stress.* The stresses appertaining to them are called *self-stresses* σ and the stress line, the *self-stress line.* In structures which are composed of straight members of invariable cross sections, this self-stress line is a composition of straight line.

The following general principle now becomes operative[3]:

According to the plastic equilibrium method, a statically indeterminate structure is capable of carrying loads when by appropriate selection of the statically indeterminate quantities it is possible to define a self-stress condition in such a way that at no point does the algebraic sum of the self-stress $\bar{\sigma}$ on the one hand, and the stress limit value max. σ or min. σ, which is calculated according to the usual theory in statically indeterminate structures, on the other, exceed the yield stress σ_s.

Thus, in general the following formula becomes operative:

$$|\max \sigma + \bar{\sigma}| \leqq \sigma_s \text{ and } |\min \sigma + \bar{\sigma}| \leqq \sigma_s \qquad (2)$$

The object of *E. Melan*'s paper is the general proof of this principle. With the aid of the stress lines the above principle can be made easy to understand. In Fig. 2 the lines max. σ and min. σ are drawn for a girder on three supports with two equal spans l. The girder is only loaded with a movable load P. If it is now possible to plot a self-stress line, which has at no place a greater distance than σ_s from the lines max. σ and min. σ, then the structure is capable of being loaded with the movable load P. The load can infinitely often pass over the girder or change its position, without causing the permanent deformation of the girder to exceed a certain limit. If the ν-th part of the load P is taken as the safe load (the dead weight of the girder has been taken as nil for the sake of simplicity) then the girder has ν-fold safety.

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[3] *H. Bleich.* The dimensioning of statically indeterminate structures taking the elastic-plastic behaviour of the constructional material into consideration. Der Bauingenieur, 1932, p. 261. On the condition that with a variable load other laws of dimensioning come into the question than when the same load changes between an upper and lower limit value, *G. v. Kazinczy* was, as far as I know, the first to give an example (three supporting girders). The further development of the plasticity member. Technika, 1931, Budapest.

If the conditions of the structure are observed, the self-stress line can be plotted within the limits which are given in the principles discussed above. Consequently the economic and constructive points of view can be considered when the choice of the plotted line of self-stress is made.

The self-stresses determined by the self-stress line can be regarded as *artificial pre-stressing* in the unloaded structure[4]. Then dimensioning according to the method of plastic equilibrium means nothing more than the superposition of such a system of artificial pre-stressing over the elastic stresses, produced by loading of a statically indeterminate system, that the maximum values of stresses at particular places of the structures are reduced accordingly.

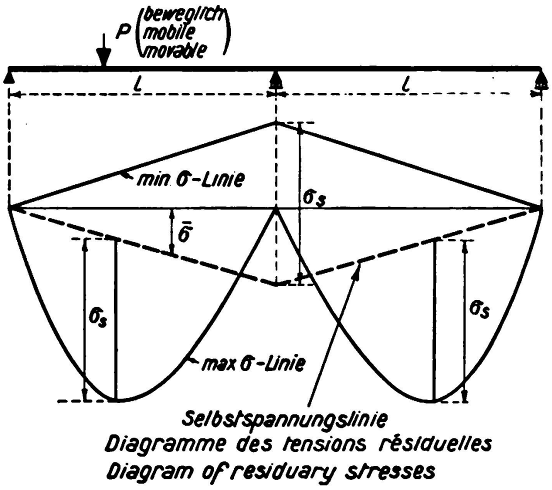

Fig. 2.

Such a condition of pre-stressing does actually occur if the structure is fully loaded and afterwards unloaded.

It is naturally not necessary in the practical application of the plastic equilibrium method to use the maximum loads and the max. σ or min. σ on the one hand and the yield limits σ_s on the other; it is more advantageous to determine the lines max. σ and min. σ corresponding to the actual loads and so to choose the self-stress line, that at no point does the interval between them and the external lines exceed the value of the permissible stress σ_{zul}. The factor of safety is then as required

$$\nu = \frac{\sigma_s}{\sigma_{zul}}$$

If the upper or lower extreme fibres have, in the more usual cases, different stresses in one cross section, then a special stress diagram (Fig. 2) holds for each extreme fibre. But the self-stress lines of both diagrams are no longer dependent upon each other, as they belong to the same condition of stressing.

The use of the stress lines in the graphic determination of the self-stress lines is naturally only possible when the cross sections of the members are already known, so that extreme fibre stresses can be calculated. But if the cross

[4] In the example under consideration these initial stresses can be advantageously produced by raising the middle support to a certain extent.

sections require to be determined first, it is more advantageous to proceed from the lines max. M and min. M, which, reversed, are nothing but the stress lines multiplied by the modulus of section W. The problem can be carried through by suitable plotting of the line of self-stress moments M, so that the conditions

$$|\max M + \overline{M}| \leqq W_{s\,zul} \text{ and } |\min M + \overline{M}| \leqq W\sigma_{zul}$$

are fulfilled for each point of the structure. $W\sigma_{zul}$ is called the *permissible moment*[5]. The line of self-stress moments is always a straight line in the case of straight members. If normal stresses as well as moments are to be considered, then M and $\overline{M}$ represent the moments round the middle thirds of the section. It is also possible to proceed so that in the preliminary dimensioning the longitudinal forces N are considered in such a way that the value σ_{zul} is temporarily reduced by the estimated amount $\sigma = \frac{N}{F}$ (F = cross section of the member) and the permissible moment is expressed by the formula $W\left(\sigma_{zul} - \frac{N}{F}\right)$.

It is important to note that *singular overstressing* which can occur through a slight *subsidence of the supports* has no influence on the safety of the statically indeterminate structure, as in the most unfavourable cases it will lead to limited permanent deformations. On the other hand, temperature influences must be taken into consideration in designing as they are subject to unlimited repetition. This can easily be effected by adding the temperature stresses to the ultimate stress-lines for max. σ and min. σ.

In those cases where there is only a single kind of loading which fluctuates between a lower and upper limit, but where reversions of the load are excluded, the lines max. M and min. M coincide with the bending moment lines for this given case of loading. It is only in this case that the calculation no longer requires to be for a statically indeterminate system. The statically indeterminate quantities are chosen arbitrarily as long as the permissible moment $W\sigma_{zul}$ is not exceeded at any point.

The process of calculation according to the plastic equilibrium method can be briefly summarised as follows:

a) *Dimensioning.*

1. Calculation of the lines max. M and min. M, as well as the longitudinal forces N of the members in the statically indeterminate structure on the usual theory of statically indeterminate structures. This condition of stress is called *Condition I.*

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[5] The introduction of $W\sigma_{zul}$ as permissible moment means that even with bent girders the connection between stress and deflection according to Fig. 1 will be taken as simplified. The actual carrying capacity moment of an I-girder is, for example, as is well known, about 16 — 18% greater than $W\sigma_{zul}$. It is apparently incorrect to call on this last reserve in dimensioning statically indeterminate structures.

2. Determination of the condition of self-stress by means of determining the lines of self-stress moment $\overline{M}$ in such a way that the moment at no point exceeds W σ_{zul}.

   Calculation of the statically supernumerary quantities X_1, X_2, ... which belong to the condition of self-stress chosen, and which we shall designate as *Condition II.*

b) *Stress Calculation.*

1. Determination of the important extreme fibre stresses for Condition I.
2. Determination of the extreme fibre stresses for Condition II. These extreme fibre stresses must be less than σ_{zul}.
3. Forming the sum of the stress from I and II. This total shall not exceed σ_{zul}.

The total stress as determined according to 3 should be regarded as theoretical, since it represents only a sort of measure for the safety. The actual stress is given in each individual case by the stress as calculated according to 1, since, as a rule with the plastic equilibrium method of dimensioning. stresses due to safe load (dead weight + live load) also lie below the yield point stress. In such cases where the deflection due to safe load should be determined, these deflections should be calculated without consideration of the influences of self-stress conditions on to the stresses.

Deflections can therefore be determined in the usual way. Care must be exercised to ensure that no buckling of the flanges of girders in compression takes place.

3) *Application of the plastic equilibrium theory to the calculation of continuous girders.*

1st *Example.*

First of all we will deal with the simplest case, in which one kind of loading fluctuates between a lower and an upper limit, so that in the calculation it will

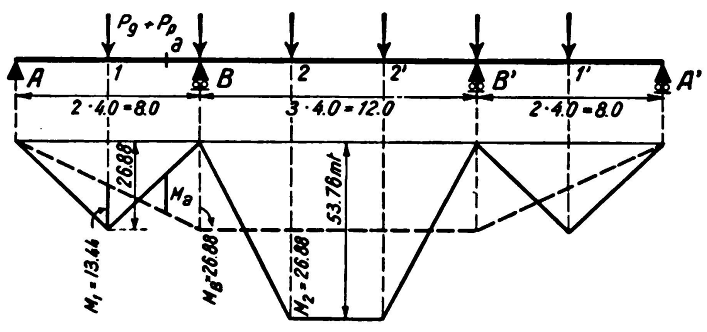

Fig. 3.

be possible to make use of the simplification mentioned on p. The roof joist, a continuous girder with a span 8 + 12 + 8 m, shown in Fig. 3 a, will be taken as an. example. The girder is loaded with a point load from the roof trusses at intervals of 4 m. The dead weight amounts to 180 kg/m², the live load (snow + wind) to $100 \cdot kg/m^2$. The loading area for this joist is 12 m wide so that the following concentrated loads result:

from dead weight: $P_g = 12 \cdot 4 \cdot 0.18 = 8.64$ t.
from snow and wind: $P_p = 12 \cdot 4 \cdot 0.10 = 4.80$ t.

As all the loads in this case represent full loads there is but one kind of loading, whereby the loads fluctuate between $P_g = 4.80$ t and $P_g + P_p = 13.44$ t. The maximum values of the moments for statically determinate fundamental system composed of three single girders are shown in Fig. 3 b. Loading with the concentrated loads $P_g + P_p$. We now choose as statically indeterminate quantities the moments over the supports M_B and $M_{B'}$, in such a way that in the middle span the moment of support and the moment in mid-span are equal to each other. By this means the dotted line in Fig. 3 b is determined. The calculation is the following:

$$M_B = M_2 = \frac{53.76}{2} = 26.88 \text{ mt}; \quad M_1 = 13.44 \text{ mt}.$$

Based on $\sigma_{zul} = 1600$ kg/cm² we receive for the outer spans a joist I 34 and the middle span a joist I $42^1/_2$. The joint between I 34 and I $42^1/_2$ is placed in the outer spans; at point a in Fig. 3 b. The position of a must be plotted in such a way that $W_n \cdot \sigma_{zul} \geqq M_a$. W_n is the modulus of section of I 34 under consideration of rivet hole deductions for the position of the joint.

2nd *Example.*

A floor beam passing over 4 spans, Fig. 4, is loaded in one thirds in the points of the span with point loads from dead weight loads $P_g = 4$ t and the live loads $P_p = 8$ t. The permissible stress is taken at $\sigma_{zul} = 1400$ kg/cm². The maximum and minimum values of moments which are obtained according to the usual theory of continuous beams are given in Fig. 4 a. In calculating the limit values of moments it was assumed that the individual spans of the beams were either completely loaded with the working load or were without load.

Dimensioning.

1st *Solution, Fig. 4 b.* The lines of moment of self-stress $\overline{M}$ are plotted in such a way that $M_1 = - M_B = 20.15$ mt and $M_2 = - M_C = 15.74$ mt. Thus a rolled girder I 40 becomes necessary for the end spans and a girder I 38 for both the middle spans. With the moment of 17.70 tm, which a girder I 38 when $\sigma_{zul} = 1400$ kg/cm² is just capable of supporting, the theoretical situation of the welded joint next to support B is determined. The following values of statically supernumerary magnitudes are obtained for the condition of self-stress:

$$\overline{M}_B = \overline{M}_{B'} = +2.14 \text{ mt}, \qquad \overline{M}_C = +2.55 \text{ mt}.$$

2nd *Solution, Fig. 4 c.* The line of moment is plotted in such a way that the absolute values

$$- M_B = M_2 = - M_C = 16.47 \text{ mt},$$

so that the maximum moment in the end span increases to = 21.34 mt. The end spans require a joist I $42^1/_2$ and for the middle spans a I 38 joist; the welded joint can, however, be placed immediately over the support B.

The condition of self-stress is denoted by the following values of the supernumeraries:

$$\overline{M}_B = \overline{M}_{B'} = +5.81 \text{ mt}, \qquad \overline{M}_C = +1.82 \text{ mt}.$$

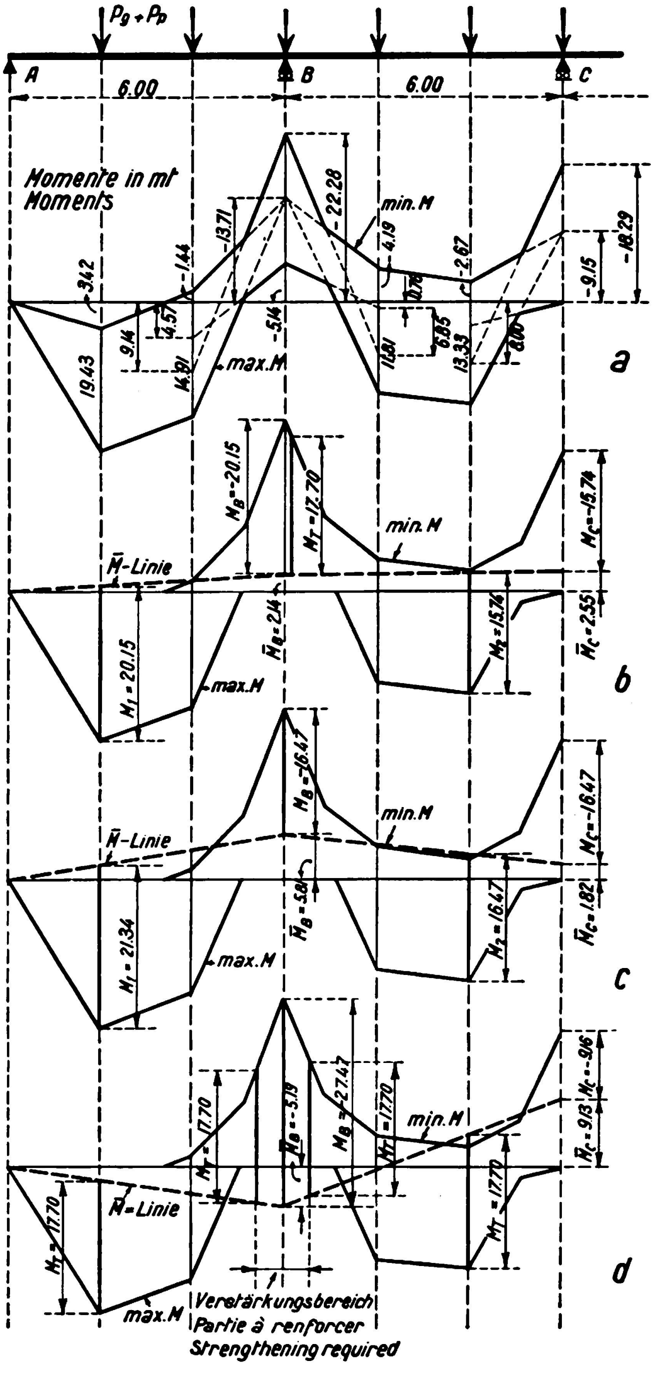

Fig. 4.

3[rd] *Solution, Fig. 4 d.* In the first and second bay at the points of largest positive moments are plotted the largest moments to which a joist I 38 is suited for $\sigma_{zul} = 1400$ kg/cm^2, in this case 17.70 mt.

These two points determine the self-stress bending moment line which produces a moment over the support B of 27.47 mt. This moment can be carried by strengthening the through girder I 38 with welded-on flange-plates of 160 · 12 in the region of the support.

The following values of the supernumerary quantities correspond to the condition of self-stress: —

$$\overline{M}_B = \overline{M}_{B'} = -5.19 \text{ mt}, \qquad \overline{M}_C = +9.13 \text{ mt}.$$

3rd *Example.*

The continuous beam for uniform distributed load plays a prominent part as floor girder in steel structures. Consequently in the following, simple rules for dimensioning such girders, which may have any desired number of equal spans, will be laid down. For this purpose a girder over three spans loaded with a permanent load g and a live load p, the latter always applied at the most unfavourable places, will be taken as an example. In Fig. 5 the lines

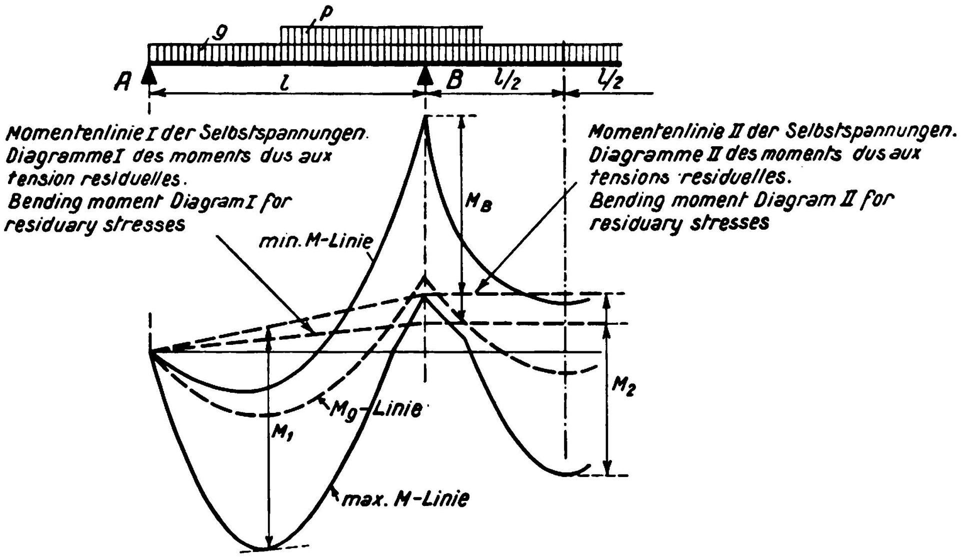

Fig. 5.

max. $M = M_g +$ max. M_p and min. $M = M_g +$ min. M_p are plotted beside the line M_g.

Two solutions of the problem will be considered: The line of bending moment for self-stresses I was plotted so that $M_1 = M_B$. The cross sections are determined in the end bays by the moment M_1, and in the middle span by M_2. It should be observed here that the stronger girder of the end bays must be projected into the middle span to a certain extent (by about 1/10) as M_B is greater than M_2, so that the joint of the girder is to the right of support B.

If the joint of the girder is to lie directly over support B, then the bending moment line for self-stresses should be plotted according to line 11 in Fig. 5. In this case $M_B = M_2$. The moment M_1 is greater than M_B, but the stronger girder of the end bay can now be jointed at B with the weaker girder of the middle span. The bending moment diagrams plotted in Fig. 5 were determined first under the assumption that the cross section of the girder was constant for all spans. So if exact calculations were desired, the bending moment diagrams would have to be re-calculated under consideration of the various cross sections. It becomes clear, however, that in calculating with the plastic equilibrium method the influence of difference in the cross section of the various spans is very slight, so that a single calculation nearly always suffices (assuming constant cross section). In order to understand the incidental correctness of this assertion it suffices to remember Ex. 1, which also represents a strict solution of the problem stipulated by the plastic equilibrium method. In that case the solution is entirely independent of the magnitudes of the moment of inertia of the cross sections in the individual spans.

Fig. 5 also shows that the maximum values of the moments M_g or M_r stipulated by the bending moment line for self-stresses occur either exactly at the same point or very close together. It is therefore possible to calculate the maximum values of the moments for dead weight and live load separately and then add them. The moments are then as follows:

for the end spans: $\max. M = c_1\, gl^2 + d_1\, pl^2$
for the middle span: $\max. M = c_m\, gl^2 + d_m\, pl^2$.

The coefficients c and d depend only, assuming equal spans, on the number of spans and can be calculated in advance. In Table I the coefficient for both solutions are grouped for the lines of self-stress I and II respectively. Since in the middle bays the maximum moments differ only slightly from one another, the maximum value of c and d, which holds good for all middle spans, was given for each case.

Table I. Moments for dimensioning continuous beams with equal spans.

End spans: $\max M = c_1\, gl^2 + d_1\, pl^2$					Intermediate spans $\max M = c_m\, gl^2 + d_m\, pl^2$				
(a) The stronger girder of the end spansprojects by about 1/10 into the second span					(b) The stronger girder of the end goes only as far as the first intermediary support				
No. of spans	End spans		Middle spans		No. of spans	End spans		Middle spans	
	c_1	d_1	c_m	d_m		c_1	d_1	c_m	d_m
2	0.0858	0.1048	—	—	2	0.0858	0.1048	—	—
3	0.0858	0.1061	0.0392	0.0858	3	0.0957	0.1109	0.0625	0.0957
4	0.0858	0.1061	0.0511	0.0942	4	0.0957	0.1104	0.0625	0.0971
>4	0.0858	0.1061	0.0625	0.0950	>4	0.0957	0.1098	0.0625	0.0972

4) *Application of the theory of plastic equilibrium to the calculation of frames.*

4th *Example.*

A frame with encastré ends, with a span of 16 m and uprights of 10 m, being a threefold statically indeterminate frame, shall be so designed that the uprights and brace could be made with rolled girders sections without local strengthening. The following loads are taken into account:

Permanent load $p = 0.72$ t/m,
Snow load $s = 0.45$ t/m, and
Wind pressure on the uprights $w = 0.60$ t/m.

The flow of the moments calculated in the usual way, for the three types of loading mentioned above, is shown in Figs. 6 b to 6 d. As there is a possibility of wind pressure from right and left, the lines max. M and min. M are symmetrical to the vertical axis of the frames. These lines are shown in Fig. 6 e.

We now determine a condition of self-stress which is given by the quantities $\overline{M}_A$, $\overline{M}_B$ and $\overline{H}$ and which fulfils the following conditions: It shall be $\overline{M}_A = \overline{M}_B$ and further that the encastré moment in A, the corner moment in C and the moment in F (in mid-span of brace) are each equal as regards numerical value.

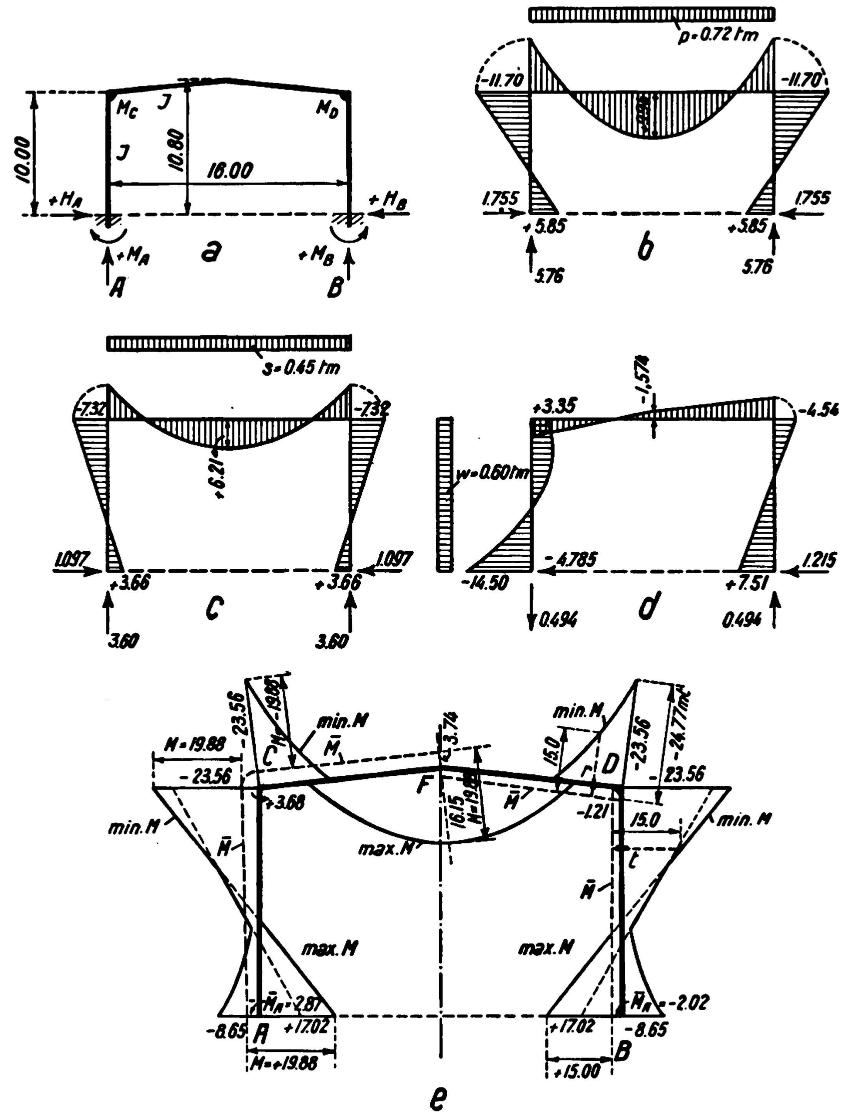

Fig. 6.

If the values for the limit moments as given in Fig. 6 e are considered, the following equations must exist:

$$\text{at A} \quad 17.02 + \overline{M}_A = M,$$

$$\text{at C} \quad -23.56 + \overline{M}_A - 10\,\overline{H} = -M,$$

$$\text{at F} \quad 6.15 + \overline{M}_A - 10.8\,H = M.$$

The unknowns $\overline{M}_A$, $\overline{H}$ and M can be determined from these three equations. The solution gives us

$$\overline{H} = -0.081 \text{ t}; \quad \overline{M}_A = \overline{M}_B = +2.87 \text{ mt}; \quad M = 19.88 \text{ t}.$$

The line of moments for self-stresses $\overline{M}$ is plotted in the left half of Fig. 6e with the values so found. The maximum moment, on which the dimensioning is based, amounts to $M = 19.88$ mt. If $\sigma_{zul} = 1600$ kg/cm² an I 40 girder will suffice[6]. One stipulation in this connection is that the security of the flanges under pressure has been provided for.

If, however, the possibility of local strengthening is considered, then it is possible to proceed in the following way:

As main section an I 36 girder is chosen whose permissible moment when $\sigma_{zul} = 1600$ kg/cm² amounts to $M_T = 1400 \cdot 1089$ kg/cm $= 15.00$ mt, with a reserve of 200 kg/cm² to cover stressing due to normal forces[7].

The condition of self-stress should be so chosen that the maximum moments at the point of fixing A, and in the middle of the brace F, do not exceed the value of 15.00 mt. Hence we have the conditional equations:

$$\text{for A:} \quad 17.02 + \overline{M}_A = 15.00$$
$$\text{for E:} \quad 16.15 + \overline{M}_A - 10.8\,\overline{H} = 15.00$$

From the solution of these two equations we see that $\overline{H} = -0.081$ t and $\overline{M}_A = -2.02$ mt. At points C and D a maximum moment $M_C = M_D = -23.56 - 2.02 + 0.81 = 24.77$ mt occurs.

The appertaining line of moment for self-stress condition is plotted in the right half of Fig. 6 e. With the permissible moment $M_r = 15.00$ the points r and t in Fig. 6 e, which define the limits of the range requiring to be strengthened by welding-on of flange-plates, are obtained, so that this range is rendered capable of withstanding a maximum moment of 24.77 mt.

We have used for the calculation of the second case the same lines max. M and min. M as in the first, although these lines should show a somewhat different course owing to local strengthening of the structure. The error thereby made is not so great, as it is shown again and again that the influence of conditions of rigidity on the moments which are decisive for dimensioning is comparatively small. A statically indeterminate structure designed after to the plastic equilibrium method behaves in accordance with the influence of rigidity of individual parts on the decisive quantities for dimensioning, similarly to a statically determinate structure. For the initial dimensioning it is therefore nearly always sufficient to take as basis of calculation a system with an approximate distribution of the moments of inertia only. For the actual stress calculation it is, however, recommended, in order that all conditions may be correctly conceived, that the correct lines max. M and min. M be used.

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[6] The normal forces in the uprights and brace must also be considered.

[7] The maximum normal force, for example, amounts in the upright to about 10 t.

Summary.

The paper deals with the practical application of the plastic equilibrium method to the dimensioning of statically indeterminate structures composed of members stiff against bending. Firstly the conception of the factor of safety in connection with the plastic equilibrium method is defined and in that connection the stipulations for its application, under consideration of the properties of the material, are discussed. It is concluded that the plastic equilibrium method should be limited above all to structures in which the fatigue strength of the material does not have to be considered. Its application to lattice structures is also inadvisable. Discussion of *H. Bleich's* principles based on the method and discussion of the method of calculation. Several examples are given of the practical application of the method, viz. in three different cases of continuous beams and one frame structure.