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10. References

Aaltonen, V. T., 1940: Metsämaa. Metsämaatieteen opi-ja kasikirja, Porvoo, Helsinki.

Acton, D. F., J. S. Clayton, J. G. Ellis, E. A. Christiansen, and W. O. Kupsch, 1960: Physiographic divisions of Saskatchewan. Saskatchewan Soil Survey in co-operation with Geol. Div. Sask. Res. Council and Geol. Dept. Univ. of Saskatchewan.

Baker, F. S., 1950: Principles of silviculture. McGraw-Hill Co. Inc., New York.

Bertsch, K., 1959: Moosflora von Südwestdeutschland. E. Ulmer, Stuttgart.

Binz, A., 1961: Schul- und Exkursionsflora für die Schweiz. 10th ed., rewritten by A. Becherer.

Bray, J. R., 1956: A study of mutual occurrence of plant species. Ecology *37*, 21–28.

Bray, J. R., and J. T. Curtis, 1957: An ordination of the upland forest communities of southern Wisconsin. Ecol. Monogr. *27*, 325–349.

Canfield, R. H., 1941: Application of the line interception method in sampling range vegetation. J. For. *39*, 388–394.

Curtis, J.T., 1959: The vegetation of Wisconsin: an ordination of plant communities. Univ.Wisconsin Press.

Dagnelie, P., 1960: Contribution a l'étude de communautés vegetales par l'analyse factorielle. Bull.Service de la Carte Phytogéographique, Sér. B, tome V, fasc. 1, 7–71, and fasc. 2, 93–195.

Delfs, J., 1954: Niederschlagszurückhaltung in verschiedenen alten Fichtenbeständen. Mitt.Arbeitskreis Wald u. Wasser *1.*

Doughty, J.L., 1941: The advantages of a soil paste for routine pH determinations. Sci. Agr. *22*, 135–138.

Ehrendorfer, F., 1954: Gedanken zur Frage der Struktur und Anordnung der Lebensgemeinschaften. Angew.Pflanzensoziol. (Wien), Festschr.E.Aichinger. *1*, 151–167.

Ellenberg, H., 1953: Physiologisches und ökologisches Verhalten derselben Pflanzenarten. Ber.dtsch.bot.Ges. *65*, 350–350.
– 1954: Zur Entwicklung der Vegetationssystematik in Mitteleuropa. Angew.Pflanzensoziol. (Wien), Festschr. E.Aichinger. *1*, 134–143.
– 1956: Aufgaben und Methoden der Vegetationskunde. In: Einführung in die Phytologie. Vol. IV. Grundlagen der Vegetationsgliederung, part 1. E. Ulmer, Stuttgart.
– 1963: Vegetation Mitteleuropas mit den Alpen. In: Einführung in die Phytologie. Vol. IV, Grundlagen der Vegetationsgliederung, part 2. E. Ulmer, Stuttgart.

Fernald, M.L., 1950: Gray's Manual of Botany. 8th ed. American Book Co., New York.

Frehner, H.K., 1963: Waldgesellschaften im westlichen Aargauer Mittelland. Beitr. geobot.Landesaufn. Schweiz *44.*

Goodall, D.W., 1954: Objective methods for the classification of vegetation. III. An essay in the use of factor analysis. Austr.J.Bot. *2*, 304–324.

Greig-Smith, 1952: The use of random and contiguous quadrats in the study of the structure of plant communities. Ann.Bot., London, N.S. *16*, 293–316.
– 1961: Data on pattern within plant communities. II. Ammophila arenaria (L) Link.J. Ecol. *49*, 703–708.
– 1964: Quantitative plant ecology. Butterworth Co., London.

Grout, A.J., 1928–1940: Moss flora of North America north of Mexico. Publ. by the author, Newfane, Vermont.

Harman, H.H., 1960: Modern factor analysis. Univ. Chicago Press.

Hotelling, H., 1933: Analysis of a complex of statistical variables into principal components.J.Educ.Psych. *24.*

Hughes, R.E., 1954: VIII Congress International de Botanique, Paris. Rapports et Communications aux Section 7 et 8, 16–18.

Ihm, P., 1964: Automatic classification in anthropology; in "The use of computers in anthropology", ed. D.Hymer, Mouton & Co., 's-Gravenhage, 373–411.

Kershaw, K.A., 1958: An investigation of the structure of a grassland community. I. The pattern of Agrostis tenuis. J.Ecol. *46*, 571–592.
– 1959: An investigation of the structure of a grassland community. II. The pattern of Dactylis glomerata, Lolium perenne and Trifolium repens. III. Discussion and conclusions. J.Ecol. *47*, 31–53.
– 1961: Association and covariance analysis of plant communities. J.Ecol. *49*, 643–654.

Law, F., 1956: The effect of afforestation upon the yield of water catchment areas. J.Br. Waterworks Ass. *10*, 489–494.

Mahalanobis, P.C., 1936: On the generalised distance in statistics. Proc.Nat.Inst.Sc. (India) *12*, 49.

Mahalanobis, P.C., D.N.Majumbar and C.R.Rao, 1949: Anthropometric survey of the United Provinces (1941): A statistical study. Sankhya (India) *9*, 90–253.

Maycock, P.F., 1963: The phytosociology of the deciduous forests of extreme southern Ontario. Can.J.Bot. *41*, 379–438.

Meyer, P., 1949: Das Mastigobryeto-Piceetum abietosum im Schweizerischen Mittelland und seine forstlich-waldbauliche Bedeutung. Vegetatio *1*, 203–216.

– 1954: Das Piceeto-Abietetum praealpinum Oberdorfer 1950 asperuletosum subass. nova im Schweizerischen Mittelland. Vegetatio *5–6*, 302–308.

Mitchell, J., H.C.Moss, and J.S.Clayton, 1950: Soil Survey of Saskatchewan. Saskatchewan Soil Survey Rep. *13*.

Noirfalise, A., 1959: Sur l'interception de la pluie par le couvert dans quelques forêts belges. Bull.Soc.For.Belg. *66*, 433–439.

Ovington, J.D., 1953: Studies of the development of woodland conditions under different trees. I. Soils pH. J.Ecol. *41*, 13–34.

Pallmann, H., E.Frei, and H.Hamdi, 1943: Die Filtrationsverlagerungen hochdisperser Verwitterungsprodukte im Bodenprofil einiger Glieder der schweizerischen Braunerdenserie. Ber.schweiz.bot.Ges., B. 53 A.

Poore, M.E.D., 1955 a: The use of phytosociological methods in ecological investigations. I. The Braun-Blanquet system. J.Ecol. *43*, 226–245.

– 1955 b: The use of phytosociological methods in ecological investigations. II. Practical issues involved in an attempt to apply the Braun-Blanquet system. J.Ecol. *43*, 245–269.

Rao, C.R., 1952: Advanced statistical methods in biometric research. J. Wiley & Sons, Inc., New York.

Rowe, J.S., 1959: Forest Regions of Canada. Can.Dept.north. Aff., Nat.Res., For. Branch, Bull. *123*.

Rydberg, P.A., 1954: Flora on the Rocky Mountains and adjacent plains. 2nd ed., Hafner Pub. Co., New York.

Schlüter, H., 1957: Ein Beitrag zur Frage ökologischer und soziologischer Artengruppen. Arch.Forstwes. *6*, 44–59.

Schönhar, S., 1953: Die ökologischen Artengruppen. Mitt. Ver. forstl. Standortskartierung *3*, 26.

Stålfelt, M.G., 1937: Der Gasaustausch der Moose. Planta *27*, 30–60.

Steneker, G.A., and J.M.Jarvis, 1963: A preliminary study to assess competition in a white spruce-trembling aspen stand. For.Chron.*39*, 334–336.

Torgerson, W.S., 1952: Multidimensional scaling: I. Theory and method. Psychometrika *17*, 401–419.

van Groenewoud, H., 1956: A root disease complex in Saskatchewan white spruce. For.Chron. *32*, 11–13.

– 1961: The variation of pH and buffering capacity of the organic layer of grey wooded soils. Soil Sci. *92*, 100–105.

– 1963: A simple method for "integrated light" measurements. Ber.geobot.Inst. ETH Stiftg. Rübel, Zürich, *34*, 133–134.

– 1965: An analysis and classification of white spruce communities in relation to certain habitat features. Can. J. Bot. *43*, 1025–1036.

Table 1.

Klassen-, Ordnung-, Verband-, Assoziations-Zugehörigkeit	Systematische Einheit	Orno-Quercetum ilicis pinosum dalmaticae					Erico-Rosmarinetum pinetosum dalmaticae					Genisto-Ericetum verticillatae pinetosum dalmaticae				Seslerio-Ostryetum pinosum dalmaticae						Stipo-Salvietum officinalis pinetosum dalmaticae								Junipero-Pinetum dalmaticae												
	Nr. der Aufnahme	1	2	3	4	5	6	7	8	9	1o	11	12	13	14	15	16	17	18	19	2o	21	22	23	24	25	26	27	28	29	3o	31	32	33	34	35	36	37	38	39	4o	41
	Gebiet	H	H	B	H	B	H	H	B	H	H'	P	P	P	P	B'	B	B	Bi	Bi	Bi	B	B	B	B	B	B	B'	B	Bi	Bi	Bi	Bi	Bi	Bi	Bi	Bi	Bi	Bi	Bi	Bi	Bi
	Höhe über dem Meer	48o	45o	48o	4oo	42o	45o	4oo	42o	5oo	3oo	8oo	8oo	5oo	75o	7oo	46o	64o	56o	9oo	87o	65o	72o	66o	64o	74o	6oo		6oo	115o	118o	128o	123o	11oo	137o	135o	139o	137o	14oo	14oo	13oo	13oo
	Exposition	NW	N	NW	N	S	O	S	O	N	N						O	SO	N	S	W	N	-	-	N	NW	-		-	SW	SW	S	NO	NO	SW	SW	SW	NO	SW	SW	O	SO
	Inklination (∢)	4o	4o	4o	5o	3o	1o	2o	25	2o							5o	4o	25	2o	25	1o	-	-	15	1o	-		-	4o	4o	3o	45	25	3o	3o	35	25	4o	35	25	3o
	Deckungsgrad in %	9o	9o	9o	9o	8o	9o	9o	8o	9o							8o	6o	85	8o	9o	8o	5o	7o	5o	5o	6o		7o	7o	75	8o	65	8o	65	7o	7o	9o	7o	9o	9o	8o
	Grösse der Aufnahme in m^2	3oo	3oo	2oo	3oo	2oo	3oo	2oo	3oo	2oo	3oo	3oo	4oo	15o	2oo		3oo	2oo	3oo	5oo	4oo	2oo	3oo	3oo	3oo	3oo	2oo	3oo	3oo	2oo	2oo	2oo	5oo	4oo	1oo	1oo	1oo	1oo	1oo	2oo	1oo	1oo
VO	Fraxinus ornus (Strauch)		2	1	1	1	1	+	1	+		2	2	2	+	+	3	2	1	+	+					+				+												+
VO	Fraxinus ornus (Baum)			1		1	1	1	+										1	+	+																					
AI	Quercus ilex (Strauch)	1	3	2	2	2	1	1	1	1		+	+		+	+	3	2																								
AI	Quercus ilex (Baum)		2		1	1			+			+	+		+																											
VCB	Brachypodium ramosum	2	2	1	4	1	2	3	1	2	3	+	2	+	+																											
VO	Coronilla emeroides			+	+	1	+	+	1		+			1		+	1		+		+									+						+						
AI	Lonicera implexa			+	+	+			+								1	+																								
AP	Pistacia terebinthus	+	+		+	+			1								+																									
AI	Rhamnus alaternus	+	+		2		+	1		1																																
VCB	Convolvulus elegantissimus	+	+		+		+	+		+																																
VE	Juniperus phoenicea		+					1		+																																
AI	Rosa sempervirens			+		+		+																																		
KI	Arbutus unedo				+			+		+																																
VE	Dorycnium hirsutum				+		+				+																															
KI	Phillyrea media	+	1	+	+								+																													
KQ	Viola silvestris	+		+	+	+																																				
KQ	Festuca heterophylla	1	2		+	+																																				
AI	Carex distachya	3	1	1																																						
AE	Erica multiflora						3	2	4	3	+																															
AE	Rosmarinus officinalis						+	3	2	2	2																															
VE	Erica verticillata						3	2	3	2	3	2	2	3	2																											
	Helianthemum nummularium						+	+	1	+	+	+	+		+																											
VE	Hieracium stupposum						1	+	3		+	1	1	+	+										1																	
OSC	Linum tenuifolium						+	+		+		+																														
AG	Asperula scutellaris											3	1	+	1																											
AG	Genista sericea var. rigida											+	1	+	+																											
AG	Linum elegans											1	+		1																											
VI	Lonicera etrusca											+	+	1																												
AG	Salvia brachyodon											+	+		+																											
AG	Genista silvestris subsp. michelii												+																													
	Micromeria croatica												+																													
ASO	Ostrya carpinifolia			+								1	+	+	+	2	+	1	1	+	+																					
VO	Acer monspessulanum											+		+	+	+	+	+	+	+	+									+							+				+	
VO	Rhamnus rupestris											1	+		+	+	+	+	+	+															+							
VO	Sesleria autumnalis															3	2	1	3	2	3											+								+		
AO	Astragalus illyricus																		+	1	+																					
AP	Rhamnus intermedia															+		+	+				1	1		1	2		+													
	Ornithogallum tenuifolium															+			+		+		+	+		1		+														
KB	Tunica saxifraga																+						+	+		+	+															
VF	Helianthemum nitidum																	+					+	+	+	+																
	Cerastium trichogynum																					2	3	1	2	1	2		+													
KB	Teucrium polium					+	+															+	+	+	+	1	1	+														
OSC	Festuca vallesiaca																					+	1	1	2			4	+													
	Paronychia kapela																					+	+		+	1	+	+														
	Trifolium aureum			+																		2		+	+	+	+															
	Allyssum petraeum																						+	+	+	+			+													
	Rosa rubiginosa																					+		+			+		+													
	Lappula echinata																					+	+	+			+															
	Viola arvensis																					+	+	+		+																
	Carduus micropterus																						+	+			+	+														
	Satureia acinos					+															+	+	+	1	+	+	1	+	+	+	+					+	+		+	+		+
VCS	Euphorbia myrsinites																			+	+		+	+	+	+	+	+	+	+	+	+	+						1			+
OSC	Bromus erectus											+								+					+	+				+	+				2		3	2	2	2	2	2
	Hieracium cymosum var. laxiflorum																		+		+										+		+	+	+	+	1	+	+	+		+
	Myosotis collina																				+	+	+	+								+	+	+		+	+	+	+			+
	Sedum ochroleucum							+															+		+	+	1	+		+				+		+	+		+			+
	Festuca duriuscula																			+	+												+	+	1	2	1	2	2	2	2	
KA	Ceterach officinarum																						+				+			+	+		+	+		+			+			
	Carduus candicans subsp. cylindricus																		+	+										+	+		+	+		+						+
	Arabis muralis																				+				+	+					+	+	+						+			
AC	Campanula portenschlagiana																				+									+		+				+	+		+			+
	Trifolium pratense var. nivale																				+													+	+	+	+	+		+		
VPM	Juniperus sibirica																													+	+	1	2	2	3	3	3	3	3	3	4	4
AJ	Sesleria nitida (robusta)																													+	2	2	1	1	1	1	2	+	1	+	+	1
	Polygala vulgaris																													+	+	+		+	+	+	+	+	+	+	+	+
	Muscari botryoides																														+	+	+		+	+	+	+	+	+	+	+
AC	Cerastium grandiflorum																													+		+	+		+		+	+	+		+	+
AD	Cynanchum adriaticum																													+	+	+	+		+	+	+		+			+
KA	Asplenium ruta muraria																													+	+		+	+	+	+					+	
KA	Asplenium trichomanes																													+	+		+	+				+			+	+
	Arenaria serpyllifolia																																		+	+	+	+	+	+		+
	Festuca rubra subsp. fallax																													+	+	1						2			+	2
VSV	Centaurea triumfetti																													+	+				+		+					
VPM	Juniperus sabina																															3		1		+						
	Senecio rupester																																			+		+	+			
	Luzula campestris subsp. multiflora																																				+	+		+	+	+
	Poa pumila																																				+	2		+	+	
OS	Ranunculus montanus subsp. carinthiacus																																				+	1	1			
	Lathyrus pratensis																																					1		+	+	
OF	Lilium martagon var. cattaniae																																					+		+	+	
	Hieracium psammogenes																																					+		+	+	
AA	Linaria cymbalaria																																					+		+	+	
KT	Poa alpina var. vivipara																																							+	+	+
OQ	Trifolium alpestre																																					1		1	1	
VE	Juniperus oxycedrus (Strauch)	2	2	2	1	2	3	3	1	3	2	3	2	2	1		1	1	+	+	1	3	2	3	4	4	3	3	3				+			+						
VE	Juniperus oxycedrus (Baum)						2																																			
	Genista silvestris subsp. dalmatica	3	1	+	4	1	3	2	2	1	2	+	+		+	+		1	+				3	3	1	3	2	1	2													
VO	Teucrium chamaedrys	1	+		+		+	+			+	1	3		+	+		+	+	+					1	+	1		+	+	+											
OSC	Eryngium amethystinum	+			+		+	+		+		+					+	+	+	+	+		+	1	+	+	+	+	+													
OSC	Koeleria splendens	2	1				1	1	1	1		+							+		+				+	2		+														
	Rubus dalmatinus	+		1	+					+								+				1	+	+	1		+	2	1													
	Echinops ritro		+	+	+	+				+							+	+	1				+	+	+			+														
VCC	Picris hispidissima	+	+			+	+		+	+							1	+					+	+	+		2															
VCS	Edraianthus tenuifolius						+	+		+			+					+								+																
AJ	Pinus nigra subsp. dalmatica s.l. (Baum)	3	3	3	3	3	3	3	2	3	3	4	4	4	4	+	4	3	3	4	3	4	3	3	3	3	3	4	4	4	4	4	3	3	4	4	3	3	2	3	3	3
AJ	Pinus nigra subsp. dalmatica s.l. (Strauch)	+		+		+		+	1	1	1	2	1	3	3	1	+	+	1	3	1	1	1	2	1	3	1	2	+	2	3	2	4	3	3	3	2	2	3	2	3	2
OQ	Galium lucidum	+	+		1	+	+	+		+	+		+				+	+	+	+	+		3	1	+	+	2	+	1	+	+	+	+	+	+	+	+		+	+	+	+
VPA	Bunium montanum	+					+			+	+								+		+	2	1	2	+	1	2	+	1	+	1	+	+	+	+	+	+	+	+	+		+
KB	Sanguisorba muricata			+		+		+		+			+						+	+	+	+	1	1	+	+	1	+	1	+	+	+	+		+	+	+		+			+
ASS	Salvia officinalis	1	+	+	+	+		+		+	+	+	1		+	+		1		+			+	+	4	2	+	2			+	2	3	2			+					
VCS	Aethionema saxatile	+						+	+			+	+		+			+			+			+	+	+		+		+	+	+	+		+	+	+		+	+		+
KB	Lotus corniculatus var. hirsutus			+	+	+					+					+	+		1	+	+	+	+	1	+	+		+		+					+	+	+	+			+	
VO	Prunus mahaleb (Strauch)			+								+	+			+				+	+	1					1								+				+			
VO	Prunus mahaleb (Baum)																			+	+	1																				
VST	Sesleria tenuifolia											2	+		+			1											1	1		+		+		+					1	
VCS	Teucrium montanum	1	+					+		+		+	+		+			+			+									+												
KQ	Hedera helix			+								+	1	2	1				+		+						+			+												
VCS	Carex humilis											4	3		2								1		+						+		+							+		+
	Geranium purpureum		+			+																	+	+	+	+	1		+													
	Cerastium viscosum								+	+																						+				+		+	+	+		+
VCC	Inula candida		+																+		+				+					+			+			+			+			
VCS	Globularia cordifolia											+			+	+					+									+			+	+	+							
VCS	Satureia montana		+				+	+										+																+			+	+				
VCS	Stachys subcrenata					+			+			+	+		+		+																					+				
AC	Moltkea petraea												+		+					+													+		+	+						+
	Cephalaria leucantha				+		+							+												+				+			+									
	Hypericum perforatum				+	+	+	+		+																+																
OSC	Brachypodium pinnatum														2				3	4										2			1	1								
OSC	Asperula longiflora												+		+				3																		+		+			+
	Leontodon crispus																		+		+						+	+			+	+										
	Potentilla hirta	+								+												+			+	+																
KI	Asparagus acutifolius			+	+	+																					+		+													
KB	Helichrysum italicum	+					+	+		+								+																								
	Leucanthemum cinerariaefolium						+		+	+							+	+								+																
	Muscari comosum		+					+		+						+																										
KC	Centaurea calcitrapa	+	+				+			+															+																	
VO	Cnidium silaifolium																		+												+					+	+					+
	Thymus balcanus																		+											+					+	1	+					
	Tragopogon balcanicus																				+										+				+	+			+			
	Hieracium biokovoënse																				+									+	+	+	+									
VF	Helianthemum grandiflorum																			+										+		+		+							+	
OQ	Dorycnium germanicum											1	+	+	+																					+						
VCS	Euphorbia spinosa		+						+	1																											+					
VI	Clematis flammula				+	+			+								+																									
AG	Veronica orbiculata					+										+							+				+															
OSC	Helianthemum italicum											+	+														+	+														
VCS	Onosma javorkae								+									+							+	+																
	Hieracium florentinum		+		+												2									1																
	Stachys italica					+		+												+					+																	
	Polygala nicaeensis	+			+														+						+																	
	Plantago lanceolata			+								+	+																						+							
	Veronica dentata subsp. austriaca																			+	+									+				1								
	Medicago orbicularis																			+	+										+											+
KQ	Clematis vitalba			1																		+					+															
KQ	Prunus spinosa			+																		+			+																	
	Sorbus domestica				+					+								+																								
KE	Fragaria vesca																										+			+			+									
OSC	Thymus longicaulis												+														+	+														
OSC	Hippocrepis comosa										+		+															+														
	Valeriana tuberosa																																+				+					+
	Armeria canescens																																			+	+				+	
	Cerastium campanulatum																				+									+				+								
VSV	Dorycnium herbaceum																													+	+	+										
KT	Rumex scutatus																				+											+						+				
ASH	Thalictrum aquilegifolium																															+	+						+			
VO	Colutea arborescens				+		+																														+					
VO	Rhamnus saxatilis																																		+	+						
KQ	Rhamnus fallax																													+	+											
VSV	Ononis spinosa																			+														+								
KQ	Crataegus oxyacantha																		+		1																					
KQ	Crataegus monogyna																							+												+						
KQ	Geranium robertianum																					1	+																			
VE	Pinus halepensis (Baum)										+																															
VE	Pinus halepensis (Strauch)										+																															
VE	Spartium junceum								+																																	
VE	Cistus salviaefolius										+																															
OF	Fagus silvatica (Baum)																																								+	
	Spiraea ulmifolia																																				+					
OQ	Sorbus aria																													+												

Appendix I

Vegetation table, Swiss data

Differential species-groups		Vegetation unit	I					II					III				
		Plot number	1	2	3	6	12	7	11	13	14	15	4	5	8	9	10
		Tree cover (%)	85	70	75	75	50	55	80	78	90	45	75	80	80	80	85
		Shrub cover (%)	1/10		3	20	2	20	40	0	0	70	1	13	0	0	0
		Herb cover (%)	60	30	35	25	40	40	20	1	1/10	1/10	2	15	1/100	30	3
		Moss cover (%)	1/10	85	60	90	50	55	90	80	95	70	90	95	90	95	80
		Trees															
		Abies alba	4	2	4	2	3	2	+	3	4	2	3	4	4	3	5
		Picea abies		1	1	3	1	3	4	3	2	3	1	3	1	2	
		Fagus silvatica	2	1	1	2	1			+			+			1	
		Quercus robur			+	1			+								
		Shrubs (1 - 5 m)															
		Abies alba	+		+	+	+	2	3			4	+	2			
		Picea abies				2							+				
		Fagus silvatica				1	+	+	+				+				
A		Differential species															
	H	Vaccinium myrtillus		+		1	+	3	2	1	+	+	+	3	+	+	+
		Abies alba (seedlings)	+	+		+	+	+	+	1	+	+	1	2		+	+
		Picea abies (seedlings)		+		2	+	+	+	1	+	+	+	1		+	+
	M	Hylocomium splendens		+		+	+	2	+	1	1	+	2	3	4	+	+
		Rhytidiadelphus triquetrus	+	1	1	2	2	+	+						+	+	
		Polytrichum formosum	+	4	1	5	2	2	4	3	+		3	3	2	2	1
		Thuidium tamariscifolium	+	1	1	1	1	+	+	1	+	4	1	1	3	1	+
B	H	Anemone nemorosa	+		+		+										
		Fragaria vesca	+	+	+		+										
		Hedera helix			+	+	+										
		Lysimachia nemorum	+	+	+	+	+										
		Viola silvatica					+										
		Galium rotundifolium	+			+	+										
	M	Catharina undulata				+	+										
		Mnium undulatum	+		+		+		+								
		Mnium affine			2	+	+		+								
C	H	Athyrium filix-femina	+	+	+	+	+		+	+			+				
		Dryopteris austriaca	+	1	+	1	+		+		+	+		+			
		Luzula pilosa	+	+		+	+	+	+	+	+		+				
		Majanthemum bifolium	+	+	+	+	+	+	1					+			
		Oxalis acetosella	3	2	3	2	3		2	+	+	+	+				
		Rubus spec.	+	+	+	1	+	+	+								
	M	Eurhynchium striatum	+	+	2	1	3		1	2	3	2					
D	M	Pleurozium schreberi				+		2		1				1	+	+	+
		Rhytidiadelphus loreus							2					3	+	1	
		Dicranum scoparium				+		+	+	+		+	1	1	1	+	+
		Hypnum cupressiforme				+	+	+	+	+	+	+	1	1		1	+
		Plagiothecium undulatum								+		+			+		
E	M	Bazzania trilobata											2	3	+	3	3
		Spagnum girgensohnii											2	2		1	2
		Other species															
		Herb layer															
		Melampyrum pratense		+		+		+	+							+	+
		Blechnum spicant						+					+	+			
		Carex brizoides				+			1								
		Carex pilulifera				+											
		Dryopteris filix-mas							+				+				
		Solidago virgaurea			+		+									+	
		Fagus silvatica (2 cm < seedlings)	+	+	+		+	+		+							
		Quercus robur							+								
		Moss layer															
		Plagiochila asplenoides		+	3	1	+		1		+		1		2		
		Leucobryum glaucum	+							+							
		Lophocolea bidentata				+			+		+						
		Lophocolea heterophylla								+	+						
		Calypogeia trichomanis				+			+								
		Dicranella heteromalla				+				+	+			+	+		
		Lepidozia reptans												+			
		Picea abies (seedlings < 2 cm)													+		

Note: Instead of *Spagnum girgensohnii* read: *Sp. quinquefarium*, instead of *Catharina Catharinea*, instead of *asplenoides asplenioides*. Appendix II: Instead of *Actea Actaea*.

Appendix II

Vegetation table, Canadian data

Vegetation unit	I														II																		III.								
Plot number	37	27	31	22	11	35	43	10	36	9	32	14	29	23	39	20	25	40	26	30	4	28	41	33	24	42	34	12	15	2	18	21	17	38	3	1	7	19	13	8	6	5	16
Tree cover (%)	64	50	65	65	45	65	47	57	65	53	65	50	36	60	55	70	50	57	60	70	40	55	72	60	50	50	60	51	60	59	53	70	50	70	65	60	53	62	52	56	45	70	70
Shrub cover (%)	1	5	1	8	2	1	1	1	1	1	1	5	1	1	1	1	2	1	2	1	2	1	3	1	1	8	1	1	1	1	2	1	1	1	1		1	1	1	2	1	1	1
Herb cover (%)	5	45	30	70	75	10	20	20	20	10	5	50	45	20	10	65	65	13	50	10	7	15	80	7	40	90	17	62	25	4	40	40	30	25	1	4	37	50	25	36	60	12	1
Moss cover (%)	80	30	31	1	74	83	70	76	25	98	5	22	15	85	85	7	4	85	7	75	67	75	5	12	4	12	47	78	90	80	40	65	60	35	73	85	75	45	85	30	35	30	95
Trees																																											
Picea glauca	4	3	3	3	3	3	3	3	4	4	3	3	2	4	3	3	3	4	3	4	3	3	2	3	2	3	3	4	4	4	3	3	3	4	4	4	4	3	4	2	3	4	4
Populus tremuloides		1	3	+	+			2	2	+	2	2	2			3	3		2		+	2	3		2	3	2			+				2	+	+	+	2	+	3	+	1	+
Populus balsamifera	+			3		1	1		+		+			+	+	+		+		1	+				+					+	+	+	+	+			+				+	+
Abies balsamea								+							+														1		+		+								+
Picea mariana																									+										+							
Pinus banksiana																												+									+			+		
Salix spec.								+										+																				+			
Betula papyrifera																																									+	
Shrubs																																										
Viburnum trilobum	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+			+	+	+	+	+		+	+	+	+	1	+	+			+		+	+	+	+
Symphoricarpus albus	+	+		+		+	+		+	+	+		+	+	+	+	+	+	+	+	+	+	+		+	1			+	+		+		+	+			+			+		
Cornus stolonifera	+	+		1		+			+	+	+				+	+	+				+					+	+			1	+			+						+	+	
Alnus crispa				+	+							1						+					+					+										+		1	3	+	
Ribes lacustre		+		+		+		+		+			+		+			+			+			+	+				+	+						+					
Rubus idaeus			+	+				+	+				+		+	+							+			+					+			+	+			+			
Ribes americanum										+				+		+		+							+						+								+			
Differential species																																										
S Lonicera involucrata	+	+		+		+	+	+	+	+	+				+	+		+		+	+	+	+	+				+		1		+		+							
Lonicera dioica	+	+	+	+		+	1	+	+				+		+	+			+		+	+	+			+			+	+			+								
Ribes hirtellum	+	+	+	1		+	+	+		+	+				+		+			+	+	+	+	+									+								
Ribes triste		+	+	+	+	+	+	+	+		+						+				+		+	+		+	+					+									
Shepherdia canadensis		+		+			1											+			+	+		+	+	+	+			+											
Amelanchier alnifolia	+				+					+													+			+				+						+					
H Lathyrus orchroleucus	+	+	+	1	+	+		+	+			+	+		+	+	+	+			+		+	+	+	1	+				+	+	+				+	+	+		
Habenaria obtusata	+		+		+					+	+	+		+	+	+					+	+			+	+	+														
Geocaulon lividum	+	+				+	+	+			+	+	+	+	+	+		+		+					+		+			+	+										
Actea rubra	+	+			+		+	+		+				+	+		+	+	+		+		+	+			+														
Galium boreale	+			+	+							+				+									+			+													
M Eurynchium pulchellum		1	+		+				+		+	1	+			+	+		+	+		+		+	+			+	+		+										
H Galium triflorum	+	+		+			+	+			+	+							+	+																						
Elymus innovatus	+		+	+	+				+		+		+				+																									
Equisetum scirpoides	+			+		+	+								+	+																									
Equisetum pratense	+		+	+						+																																
Hieracium canadensis		+		+									+	+																												
Carex capillaris						+								+																												
M Peltigera spec.	+					+		+	+	+								+																								
Other species																																										
Herbs																																										
Linnaea borealis v. americana	+	2	+	+	+	+	+	2	2	2	+	3	2	1	1	2	2	1	2	1	+	1	2	+	1	2	3	2	1	+	1	2	2	1	+	+	2	2	2	1	3	+	+
Cornus canadensis	+	2	2	2	2	+	+	1	+	1	+	2	2	+	+	2	2	+	2	1	+	1	3	1	2	2	+	2	1	+	2	2	1	1	+	2	1	1	2	3	+	+
Rosa acicularis	+	+	+	1	+	+	+	+	+	+	+	1	+	+	+	+	1	+	1	+	+	+	+	+	+	+	+	+	+	+	1	1	1	1	+	+	2	1	1	2	+	+	
Petasites palmatus	+	1	1	2	2	+	+	1	+	1	+	+	2	1	+	1	1	+	1	1	+	+	+	2	+	1	2	+	2	1	+	1	+	1	+			2	1	+	1	+	
Mertensia paniculata	+	2	1	2	2		+	1	+	+	+	+	+	+	+	2	2	+	2	+	+	+	1	+	1	1	+	2	1	+	2	+	1	+	+	+		1	+	+	+	+	
Fragaria vesca	+	1	+	2	+	+	+	1	+	+	1	1	1	+	+	1	1	+	1	+	+	+	+	+	2	1	+	1	+	1	+	1	1	1	+			1	+	1	+	+	
Fragaria virginniana	+	+	+	2	+	+	+	+	+	+	+	+	1	+	+	1	+	+	+	+	+	+	1	+	1	1	+	+			+	+	+	+	+			1	+	+	+	1	+	
Mitella nuda	+	1	+	1	2	+	+	1	+	+	1	+		1	+	1	1	+	+	+	+	+	+	+	+	+	+	2	1	+	1	+	+	+	+	+	+	1	1		+	+	
Pyrola virens	+	+	+		+	+		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	1	+					+	+	+	+	+	
Pyrola secunda	+	+	+	+	+			+	+	+	+	+	1	+	+	+	+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+			+	+	+	+	
Rubus pubescens	+	1	+	2	2	1	+	1		+		1	+	1	+	2	1		1	+	+	+	1	+	1		+	2	1	+	1	1	1			+	+	+	1	+	1	+		
Maianthemum canadense	+	+	+	+	+	+		+	+	+		1	+	+	+	+	+		+			+	2	+	+	2	+	1	+		+	+	+	+	+			+	+	+	+	
Aster puniceus	+	+	+	1	+	+	+	+	+	+	+	+	+	+	+	1	1	+	+	+	+	+	+	+	+		+	+	+	+	+	1	1				+	1	1	+		
Viola renifolia	+	+			+	+	+			+	+	+	+		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		+				1	+			
Pyrola asarifolia	+		+	+	+			+			+	+	+			1	1	+	+		+	+	+	+	1	+	+	+		+	+	1				+	+	+	+	+	+	
Achillea millefolium	+	+	+	+	+	+	+	+	+	+	+		+		+	+	+	+			+				+	+					+	+							+		
Aralia nudicaulis	+			+	+	+	+					+											+		2	2		+	+		1	+	+		+		+	+	+			
Carex disperma	+	+	+	+		1	+	+	+		+			1	+	+	+		+	+	+	+		+			+		+			+					1					
Solidago hispida	+	+	+		+	+	+		+	+			+	+	+			+	+	+			+		+		+	+		+		+						+	+	+	
Trientalis borealis	+		+	+	+	+						+		+	+	+			+					+		+	+		+		+	1	+					+	1	+	+	
Vicia americana		+					+			+	+			+	+		+	+	+		+				+		+							+						+		
Arctostaphylos uva-ursi						+	+	+	+			+	1							1					1		+			+		+										+
Epilobium angustifolium		+			+							+	+				+	+					+		+		+				+				+			+	+			
Taraxacum spec.						+	+				+				+			+	+								+									+			+			
Vaccinium vitis-idaea v. minus					+		+					+														+	+					+							+			
Ledum groenlandicum		+																		+		+	+			+	+											+	+			
Moneses uniflora														+		+				+							+	+		1			+							+		
Rubus acaulis	+			+	+								+					+																			+			+		
Goodyera repens						+				+						+			+					+			+										+	+				
Lycopodium complanatum					+							1	+														+										+			+		
Equisetum arvense								+	+		+										+																			+		
Lycopodium clavatum					+							+																											+	+		
Actea alba			+								+																		+										+	+		
Equisetum silvaticum	+		+														+																									
Viola rugulosa												+				+																										
Habenaria orbiculata																				+						+																
Mosses																																										
Hylocomium splendens	4	2	2	+	4	3	1	3	2	2	+	2	1	4	4	1	+	4	1	3	3	2	+	1	+	1	3	4	4	5	3	3	3	2	3	5	4	2	3	3	2	2	5
Pleurozium schreberi	3	1	1		2	3	+	2	2	4	+	+	1	2	2	+	+	2	+	2	2	2		+		1	2	2	3		1	3	+	3	2	3	1	2	1	+		
Dicranum rugosum	+	+	+		+	1	4	+	+	+			+	+	+		+	+	+	+	+	+	+			+	+	+	+		-	1		+	+		+	+	+	+	+	+	
Hypnum crista-castrensis			+		+			+				+	+	+				+	+		+					+	+	+			+		+			+	+	+			+
Mnium cuspidatum	+			+				+	+	+							+												+		+					+	+			+		
Thuidium recognitum		1			+							1	+		+	+		+							+						1								+			
Rhytidiadelphus triquetrus													+																		+											+
Polytrichum juniperinum																											+													+			

Other species: Alnus rugosa 17(+), 22(+); Corallorhiza maculata 3(+), 1(+); Empetrum nigrum 29(+), 43(+); Smilacina stellata 32(+), 39(+); Hedysarum alpinum 12(+); Habenaria dilatata 42(+); Lycopodium obscurnum 11(+); Deschampsia caespitosa 22(+); Lilium philadelphicum v. andinum 5(+).

Appendix III

Computational scheme for finding clusters (Swiss data)

Plot No.	D^2	Mean D^2	Increase
13–14	18.367907	18.367907	
15–13	24.239020		
15–14	26.371720		
	60.610740 : 2 =	30.305370	11.937463
78–15	25.544942		
8–13	26.386967		
8–14	36.276877		
	88.208786 : 3 =	29.402929	–0.902441
12– 8	50.735104		
12–15	58.855423		
12–13	27.257727		
12–14	45.991048		
	182.839302 : 4 =	45.709826	16.30689
4–12	72.138194		
4– 8	46.041220		
4–15	74.711975		
4–13	42.746613		
4–14	53.580815		
	289.218817 : 5 =	57.843763	12.133937
1– 4	151.77657		
1–12	76.46608		
1– 8	132.04625		
1–15	147.99643		
1–13	116.10162		
1–14	160.93687		
	785.32382 : 6 =	130.887303	73.043540
9–10	47.815447	= 47.815447	
5– 9	72.679807		
5–10	119.59655		
	192.276357 : 2 =	96.138179	48.322732
1– 5	254.41571		
1– 9	147.25925		
1–10	189.28631		
	590.96127 : 3 =	196.987090	100.848911
2–11	61.445963	61.445963	
6– 2	73.735058		
6–11	60.437793		
	134.172851 : 2 =	67.086426	5.640463

Plot No.	D^2	Mean D^2	Increase
7– 6	255.19504		
7– 2	350.06796		
7–11	277.88278		
	883.14578 : 3	= 294.381927	227.295501
1– 3	169.41938	= 169.41938	
7– 1	371.67753		
7– 3	527.89705		
	899.57458 : 2	= 449.787290	280.367910

Appendix IV

Computation of Mahalanobis' generalized distance (D^2)

If the species are not quantitatively correlated the generalized distance, D^2, between the sample plots simply is the sum of the squares of the differences in average percentage cover for the various species. When the species are correlated, however, the cover percentages can be replaced by a set of transformed variates which are linear functions of the original cover percentages and which are mutually uncorrelated (Rao 1952). The following method of transforming correlated into uncorrelated variables is due to Rao (Mahalanobis *et al.* 1941, Appendix 5, p. 251; and Rao 1952).

If $x_1, x_2 \ldots\ldots x_m$ represent the original variables (indices for plots are not given, N measurements for each species), transformed to unit standard deviation and r_{ij} is the correlation between the i^{th} and j^{th} species, a system of new variables $y_1, y_2 \ldots\ldots y_m$ can be defined by

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 - a_{21}y_1 \\ y_3 &= x_3 - a_{32}y_2 - a_{31}y_1 \\ &\vdots \\ y_m &= x_m - a_{mm-1}\, y_{m-1} \ldots\ldots - a_{m1}\, y_1 \end{aligned}$$

in which the constants $a_{21}, \ldots\ldots a_{mm-1}$ are selected in such a manner that the correlation between the y's become zero (the y's are then said to be linearly independant).

In order to find the first coefficient a_{21} the covariance between y_1 and y_2 must be made equal to zero.

$$\text{cov}\,(y_1y_2) = \text{cov}\,(x_1x_2) - a_{21}V\,(y_1) = 0$$

in which (cov y_1y_2) denotes the covariance of y_1 and y_2, V (y_1) denotes the variance of y_1. Since Vx_1, Vx_2, and Vy_1 all have standard deviation = 1.

$$\text{cov}\,(y_1y_2) = r_{21} - a_{21} = 0$$

thus $a_{21} = r_{21}$

and $V(y_2) = 1 - r_{21}^2$

To facilitate the computation of the constants a, it is advantageous to introduce at this stage a new constant b, as will be defined below. The computational steps to obtain the constants a_{31} and a_{32} are now as follows:

$$b_{31} = \text{cov}\,(x_3x_1) = r_{31} \qquad a_{31} = b_{31}$$
$$b_{32} = \text{cov}\,(x_3x_2) - a_{21}b_{31} \qquad a_{32} = b_{32}/V(y_2)$$
$$= r_{32} - a_{21}b_{31},$$
$$V\,(y_3) = 1 - b_{31}a_{31} - b_{32}a_{32}$$

The computational steps to find y_4 are

$$b_{41} = r_{41} \qquad a_{41} = b_{41}$$
$$b_{42} = r_{42} - a_{21}b_{41} \qquad a_{42} = b_{42}/V(y_2)$$
$$b_{43} = r_{43} - a_{32}b_{42} - a_{31}b_{41}, \qquad a_{43} = b_{43}/V(y_3)$$
$$V\,(y_4) = 1 - b_{41}a_{41} - b_{42}a_{42} - b_{43}a_{43}$$

In general if $y_1, y_2 \ldots\ldots y_{i-1}$ are known, the steps to calculate y_i are

$$b_{i1} = r_{i1} \qquad a_{i1} = b_{i1}$$
$$b_{i1} = r_{i2} - a_{21}b_{i1} \qquad a_{i2} = b_{i2}/V(y_2)$$
$$b_{i3} = r_{i3} - a_{32}b_{i2} - a_{31}b_{i1} \qquad a_{i3} = b_{i3}/V(y_3)$$
$$\vdots$$
$$b_{ij} = r_{ij} - \sum_{t=i-1}^{1} a_{jt}b_{it}, \qquad a_{ij} = b_{ij}/V(y_j),\ (j \leqq i-1)$$

$$V\,(y_i) = 1 - \sum_{j=1}^{i-1} a_{ij}b_{ij}$$

Since the calculation of each new constant depends on the constants derived previously it is necessary to keep checking for errors.

At the end, the new variables are transformed to unit standard deviation by deviding each variate by its standard deviation, $\sqrt{V(y)}$.

Now the calculation of the "generalized distances" can progress with very little difficulty by applying the Pythagorean theorem. If $d_1, d_2 \ldots\ldots d_m$ are the differences, in transformed data of the species, between two sample plots, the distance (D^2) is calculated as follows

$$D^2 = d_1^2 + d_2^2 + \ldots\ldots + d_m^2$$

It is clear that when the number of species is large, the computation of the D^2's becomes very time consuming. The use of large electronic computers, however, has made it possible to apply the method to almost any number of species (up to 200).

Appendix V

Principal component analysis

Principal component analysis of the two types of matrices, vector or point representation, is the same.

Because the mathematics of principal component analysis involves the manipulation of matrices, a short review of basic matrix operations is presented here, before proceeding to the explanation of the analysis.

(1) A vector is a directed line segment in test space which is described by a one-dimensional array of ordered numbers, which is arranged either as a column or as a row. The numbers are the coordinates of the end point of the vector. The beginning of all vectors is in the origin zero. Every vector is thus adequately described by the coordinates of its endpoint. The numbers are ordered because a vector with the coordinates (x_1, x_2, x_3) is obviously not the same as the vector with the coordinates (x_2, x_3, x_1) or (x_3, x_2, x_1).

Example:

$$\alpha = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ (a vector in two dimensional test space).

$\beta = (x_1, x_2, x_3)$ (a vector in three dimensional test space).

(2) A matrix is a two-dimensional array of numbers
Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

a_{23} denotes that element of the matrix A which is found in the second row and the third column.

(3) Matrices are added or subtracted by adding or subtracting the corresponding elements of the matrices.

Example:

if $$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$ and $$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

then $$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

and $$A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}$$

(4) In matrix multiplication the sum of the products of the elements from a row of the first matrix and a column of the second matrix are computed for each combination of rows and columns.
Example, using the same matrices A and B:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{pmatrix}$$

In the foregoing example A is called the premultiplier and B the postmultiplier. If A has more rows than B has columns it is not possible to form the product.

(5) To multiply a matrix with a scalar (ordinary algebraic quantity, real or complex number) each element of the matrix is multiplied with the scalar.

(6) Post multiplying of a matrix with m columns by a column vector with m elements produces a column vector with as many elements as the matrix has rows.

(7) The matrix I, in which all the elements of the main diagonal are unity (1) and all the off-diagonal elements are zero, is called the identity matrix.

An example is:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In matrix algebra it plays the same role as unity (the number 1) does in ordinary algebra. It is important in the inversion of matrices.

(8) Matrix division involves an operation called matrix inversion. If a matrix A is divided by a matrix B then this is not denoted by A/B but by $A \cdot B^{-1}$ (B^{-1} is called B inverse). The matrix B^{-1} has the property that $B^{-1} \cdot B = I$ (identity matrix).

(9) The determinant of a square matrix (number of columns = number of rows) is a single number which represents a unique function of the numbers in the matrix. The determinant is calculated from the elements of the matrix. By definition a determinant of the n^{th} order (n rows, n columns) stands for the sum of n! (n factorial) terms ($n! = 1 \times 2 \times 3 \ldots\ldots n$) each of which is the product of the n elements, only one from each column, and only one from each row. The signs attached to these products must satisfy the following rule. If the number of interchanges in each permutation is even, the sign is +, if the number is odd the sign is —.

example $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The six terms entering in the expansion of the determinant are:

	order of rows			number of interchanges	sign
$a_{11} \cdot a_{22} \cdot a_{33}$	1	2	3	0	+
$a_{11} \cdot a_{32} \cdot a_{23}$	1	3	2	1	—
$a_{21} \cdot a_{12} \cdot a_{33}$	2	1	3	1	—
$a_{21} \cdot a_{32} \cdot a_{13}$	2	3	1	2	+
$a_{31} \cdot a_{12} \cdot a_{23}$	3	1	2	2	+
$a_{31} \cdot a_{22} \cdot a_{13}$	3	2	1	3	—

The determinant in expanded form thus is as follows:

$$A = a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{32} \cdot a_{23} - a_{21} \cdot a_{12} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} - a_{31} \cdot a_{22} \cdot a_{13}$$

(10) The characteristic equation of a square matrix is formed by subtracting from the diagonal elements of the matrix some value, which is chosen so that the determinant of the new matrix is equal to zero.

example $$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

In expanded form the determinant may be written as follows:

$$(a_{11} - \lambda)\ (a_{22} - \lambda)\ (a_{33} - \lambda) - (a_{11} - \lambda)\ (a_{32})\ (a_{23}) - (a_{21})\ (a_{12})\ (a_{33} - \lambda) + (a_{21})\ (a_{32})\ (a_{13}) + (a_{31})\ (a_{12})\ (a_{23}) - (a_{31})\ (a_{22} - \lambda)\ (a_{13}) = 0$$

The various non-zero λ's, are usually called roots, eigenvalues or latent roots. In plant ecology the R matrix (matrix of correlation coefficients or covariances between species) is symmetric and in the case of the matrix of correlation coefficients the diagonal elements are all unity (1's) which simplifies the determinant considerably

$$|R| = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix}$$

the characteristic equation now becomes

$$\begin{vmatrix} 1-\lambda & r_{12} & r_{13} \\ r_{12} & 1-\lambda & r_{23} \\ r_{13} & r_{23} & 1-\lambda \end{vmatrix} = 0$$

or $$(1 - \lambda)^3 - (1 - \lambda)\ r_{23}^2 - r_{12}^2\ (1 - \lambda) + 2\ (r_{12})\ (r_{23})\ (r_{13}) - r_{13}^2\ (1 - \lambda) = 0$$

(11) Several methods are available for solving for the various nonzero roots or eigenvalues (λ's) of a polynomial of the third order. Consider the polynomial $\lambda^3 - a\lambda^2 + b\lambda - c$.

a, b, and c are real coefficients. The 3 roots (λ) for which the polynomial equals 0, are desired. First the upper and lower limits of this polynomial are determined.

$$f(\lambda) = \lambda^3 - a\lambda^2 + b\lambda - c = 0$$

$$f^1(\lambda) = 3\lambda^2 - 2a\lambda + b = \lambda^2 - {}^2/_3 a\lambda + {}^b/_3 = 0$$

$$\lambda = \frac{2a \pm \sqrt{(-2a)^2 - 4 \cdot 3 \cdot b}}{2a} = 1 \pm \sqrt{\frac{4a^2 - 12b}{4a^2}} = 1 \pm \sqrt{1 - \frac{3b}{a^2}}$$

thus the equation reaches a maximum value for

$$\lambda_1 = 1 + \sqrt{1 - \frac{3b}{a^2}}$$

and $$\lambda_2 = 1 - \sqrt{1 - \frac{3b}{a^2}}$$

thus λ_1 for which $\lambda^3 - a\lambda^2 + b\lambda - c$ becomes 0, lays between $-\infty$

and $1 - \sqrt{1 - \frac{3b}{a^2}}$

λ_2 lays between $1 - \sqrt{1 - \frac{3b}{a^2}}$ and $1 + \sqrt{1 - \frac{3b}{a^2}}$

and λ_3 between $1 + \sqrt{1 - \frac{3b}{a^2}}$ and $+\infty$.

Now trial values for λ can be inserted in the equation and the roots found by interpolation.

(12) For each eigenvalue found, there will be an associated vector, called the eigenvector, V which satisfies the matrix equation $A \cdot V = \lambda V$. If all the eigenvalues of the matrix A are placed in the diagonal elements of a matrix $\wedge$ in which the off diagonal elements all are zero, and the corresponding set of eigenvectors of the matrix A are placed as columns in a matrix X the following matrix equation holds true $A \cdot X = X \cdot \wedge$

(13) The trace of a matrix is the sum of all the diagonal elements. The trace of matrix A is identical to the trace of the matrix $\wedge$. In other words the sum of the eigenvalues is identical to the trace of the original matrix A.

As mentioned before the basic task of principal component analysis firstly is to find an axis (component, factor) in the original test space along which the variance is maximum, then successively a number of axes along which the remaining variance each time is maximum. This task can be accomplished by solving for a set of equations which are the result of partial differentiation of the function to be maximized, subject to a restriction due to the use of Lagrange multipliers.

This set of equations may be written as follows:

$$\begin{aligned}
&V_1(1-\lambda) + V_2 r_{12} + \ldots\ldots V_n r_{1n} = 0\\
&V_1 r_{21} + V_2(1-\lambda) + \ldots\ldots V_2 r_{2n} = 0\\
&\vdots\\
&V_1 r_{n1} + V_2 r_n + \ldots\ldots V_n(1-\lambda) = 0
\end{aligned}$$

In matrix form these equations can be written as follows:

$$\begin{vmatrix} (1-\lambda) & r_{12} & \cdots & r_{1n} \\ r_{21} & (1-\lambda) & \cdots & r_{2n} \\ \vdots & \vdots & & \\ r_{n1} & r_{n2} & \cdots & (1-\lambda) \end{vmatrix} \cdot \begin{vmatrix} V_{1i} \\ V_{2i} \\ \vdots \\ V_{ni} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{vmatrix}$$

or in matrix notation $(R - \lambda_i I) \cdot V_i = 0$ $(i = 1, 2 \ldots\ldots n)$ in which R is the matrix of the correlation coefficients. There are n nontrivial solutions in which each V_i represents the coefficients (transformation vector) for converting the original data to one of the uncorrelated scores of the new components (factors). For the nontrivial solutions the determinant of the coefficient of V is zero. In matrix notation this can be written as follows: $R - \lambda I = 0$. This is the characteristic equation for which there are n possible solutions for λ (if R is a correlation matrix with n column and n rows).

The vector V_i can be calculated for any corresponding eigenvalue by substituting the eigenvalue in the set of equations mentioned before and solving for V_i. When the vectors are normalized, the variance of each set of component scores is λ_i. The eigenvector V_1 produces the principal component scores with the maximum variance, this variance is the value of the largest eigenvalue. The sum of all the eigenvalues is equal to the sum of the diagonal elements of R matrix, which is called the trace. Because the trace of the R matrix is the total variance to be accounted for by the principal components (factors) the sum of the eigenvalues, associated with the principal components retained, divided by the trace of the matrix R is the proportion of the variance accounted for. Numerical example of the principal component analysis of the matrix of correlation coefficients (hypothetical).

Let the matrix of correlation coefficients be

$$R = \begin{pmatrix} 1 & 0.10 & 0.20 \\ 0.10 & 1 & 0.90 \\ 0.20 & 0.90 & 1 \end{pmatrix}$$

The task is then to find the eigenvalues and the associated eigenvectors. The eigenvalues are the roots of the characteristic equation which may be written in the form of a determinant as follows.

$$\begin{vmatrix} 1-\lambda & 0.10 & 0.20 \\ 0.10 & 1-\lambda & 0.90 \\ 0.20 & 0.90 & 1-\lambda \end{vmatrix} = 0$$

or in expanded form (a polynomial of the third order)

$$(1-\lambda)^3 - (1-\lambda)\,(0.90^2) - (0.10^2)\,(1-\lambda) + (0.10)\,(0.90)\,(0.20) + (0.20)\,(0.10)\,(0.90) - (0.20)\,(1-\lambda)\,(0.20) = 0$$

$$1 - 3\lambda + 3\lambda^2 - \lambda^3 - 0.81\,(1-\lambda) - 0.01\,(1-\lambda) + 0.018 + 0.018 - 0.04\,(1-\lambda) = -\lambda^3 + 3\lambda^2 - 2.14\lambda + .176 = \lambda^3 - 3\lambda^2 + 2.14\lambda - .176 = 0$$

We now determine the upper and lower limit of the roots

$$f(\lambda) = \lambda^3 - 3\lambda^2 + 2.14\lambda - .176$$

$$f'(\lambda) = 3\lambda^2 - 6\lambda + 2.14 = \lambda^2 - 2\lambda + .7133$$

$$\lambda_{12} = 1 \pm \sqrt{1 - .7133} = 1 \pm .535$$

$$\lambda_1 = .465$$

$$\lambda_2 = 1.535$$

Inserting these values in the characteristic equation we find that $\lambda^3 - 3\lambda^2 + 2.14\lambda - .176$ reaches a positive maximum for $\lambda = .465$ and a negative maximum for $\lambda' = 1.535$

Also for $\lambda = - \ldots, \lambda^3 - 3\lambda^2 + 2.14\lambda - .176$ is negative
and for $\lambda = + \ldots, \lambda^3 - 3\lambda^2 + 2.14\lambda - .176$ is positive

By inserting trial values and interpolating we now find:

$$\lambda^1 = 1.948$$
$$\lambda^2 = .958$$
$$\lambda_3 = .0943$$

The trace of the R-matrix is 3 and the sum of the calculated eigenvalues is 3.0003. The accuracy of the computations is thus very satisfactory. The first principal component accounts for $1.948/3 \times 100\% = 64.93\%$; the second component for $0.958/3 \times 100\% = 31.93\%$ and the third component for $.0943/3 \times 100\% = 3.14\%$ of the total variance. The first and second components together thus account for 96.86% of the total variance. The third component has such a low eigenvalue that for practical purpose it does not need to be considered.

To find the eigenvectors, corresponding to these eigenvalues, the eigenvalues are substituted in the set of equations.

$$V_1 (1 - \lambda) + V_2 r_{12} + V_3 r_{13} = 0$$
$$V_1 r_{12} + V_2 (1 - \lambda) + V_3 r_{23} = 0$$
$$V_1 r_{13} + V_2 r_{23} + V_3 (1 - \lambda) = 0$$

If λ_1 is substituted, this results in the following:

$$V_1 (1 - 1.948) + V_2 (0.1) + V_3 (0.2) = 9$$
$$V_1 (0.1) + V_2 (1 - 1.948) + V_3 (0.3) = 0$$
$$V_1 (0.2) + V_2 (0.3) + V_3 (1 - 1.948) = 0$$

These equations can be solved for V_1, V_2, and V_3 with the use of determinants.

$$\frac{V_1}{\begin{vmatrix} 0.1 & 0.2 \\ (1 - 1.948) & 0.3 \end{vmatrix}} = \frac{V_2}{\begin{vmatrix} 0.2 & (1 - 1.948) \\ 0.3 & 0.1 \end{vmatrix}} = \frac{V_3}{\begin{vmatrix} (1 - 1.948) & 0.1 \\ 0.1 & (1 - 1.948) \end{vmatrix}}$$

Thus $\lambda_1 = 1.948$ has the following eigenvectors

$$V_1 = \{(0.1)\ (0.3) - (-0.948)\ (0.2)\}\ \varkappa = .1926\ \varkappa$$
For $\varkappa = 1$, $V_1 = 0.1926$ and
$$V_2 = \{(0.1)\ (0.2) - (0.3)\ (-0.948)\}\ \varkappa = .3044\ \varkappa$$
For $\varkappa = 1$ $V_2 = 0.3044$
$$V_3 = \{(-0.948)\ (-0.948) - (0.1)\ (0.1)\}\ \varkappa = 0.889\ \varkappa$$
For $\varkappa = 1$ $V_3 = 0.889$

If λ_2 is substituted the corresponding eigenvectors are:

$$\frac{V_1}{\begin{vmatrix} 0.1 & 0.2 \\ 0.042 & 0.3 \end{vmatrix}} = \frac{V_2}{\begin{vmatrix} 0.2 & 0.042 \\ 0.3 & 0.1 \end{vmatrix}} = \frac{V_3}{\begin{vmatrix} 0.042 & 0.1 \\ 0.1 & 0.042 \end{vmatrix}}$$

$$V_1 = (0.03 - 0.0084)\ \varkappa = 0.0216\ \varkappa$$
For $\varkappa = 10$, $V_1 = 0.216$
$$V_2 = (0.02 - 0.0126)\ \varkappa = 0.0074\ \varkappa$$
For $\varkappa = 10$, $V_2 = 0.074$
$$V_3 = (0.001764 - 0.01)\ \varkappa = -0.0082\ \varkappa$$
For $\varkappa = 10$, $V_3 = -.0082$

With this the actual principal component analysis is complete.

It is easy to see that, with more than three variables, it is difficult to perform the analysis with a desk calculator and it is very time consuming. With the help of an electronic computer of sufficient capacity, however, the complete analysis takes only minutes. The time required for an analysis depends on the size of the computer used, the number of variables, and the accuracy required. In many cases, large institutions make computertime available without cost for research purposes.