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STELLAR MODEL COMPUTATIONS FOR DETERMINING MASS AND AGE OF STARS

BY

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SUMMARY

A condensed version of a recent work on stellar model computation is given here, stress being laid on the computational methods used. The basic program is that of the Goettingen team (R. Kippenhahn et al.) to which a certain number of minor modifications were applied.

RÉSUMÉ

Nous présentons ici une version abrégée d'un travail récent sur le calcul de séquences évolutives de modèles stellaires, en insistant plus spécialement sur la technique des méthodes utilisées. Le programme de base est celui de l'équipe de Goettingen (R. Kippenhahn et al.), auquel ont été apportées quelques modifications de détail.

1. INTRODUCTION

Stars evolve over a very long time scale, spending most of their life consuming some nuclear fuel during quasi static phases of evolution. In the main sequence stage in particular, where hydrogen is transmuted into helium in the innermost region of the star, it is customary to divide the star into three main regions, namely the atmosphere, the outer envelope and the inner core. Owing to different physical properties, each of these three regions has to be described by its own set of basic equilibrium equations together with an appropriate integration method.

The initial data for any model are the overall chemical composition of the star and its total mass M: quantities such as the total radiated power or luminosity L and the radius R will appear here as free parameters of the stellar model. The most directly observable quantities are L and the effective temperature T_{eff} related to R and L by the black-body relation

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4$$

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where σ is Stefan's constant. In practice, many steps have to be overcome when passing from the truly observed magnitudes and colour indices to the quantities L, T_{eff}.

The purpose of building a model star is to get an idea of the internal structure of the star and to ascribe to it a representative point in the colour-magnitude or Herzsprung diagram (L, T_{eff}). However, since the effect of nuclear burning is to gradually change the chemical composition and the internal structure as well, we are therefore faced with a whole evolutionary series of models, to which shall correspond an evolutionary track in the (L, T_{eff}) plane.

Let us now turn to the study of a stellar model, divided into the three aforementioned main regions.

2. MODEL ATMOSPHERE

The atmosphere comprises the outermost layers and is usually very thin, compared to the star's radius R; we may locally describe the atmosphere by a simplified model consisting of a plane-parallel layer of gas subject to a constant gravity

$$g = G\frac{M}{R^2}$$

and pervaded by a constant net outward radiative flux F.

Moreover, a temperature T can be defined at any level as the temperature which would warrant a local thermodynamic equilibrium state within any volume element located on that level. A slight vertical displacement downwards $-dr$ corresponds to an increase

$$d\tau = -\kappa\rho\, dr$$

in optical depth τ, where ρ is the density and κ a mean absorption coefficient, depending on the local physical conditions (ρ, T) and the chemical composition.

The model atmosphere thus defined will then provide us with a temperature-depth law

$$T = T(\tau;\, R, L) \tag{1}$$

for assumed values of the parameters R, L corresponding to a given representative point of the model in the (L, T_{eff})-plane.

Since the pressure P varies rapidly in these outer layers, it is advisable to adopt $\log P$ as the independent variable and consequently to write the hydrostatic equilibrium equation in the form

$$\frac{d\tau}{d\log P} = 2.30258\,\frac{\kappa P}{g} \tag{2}$$

where κ is usually given by opacity tables in terms of ρ, T for a known chemical composition. The equation of state of the atmospheric partially ionized gas mixture

and the law (1) allow us to express κ in function of τ, P and consequently to integrate (2) through the atmosphere.

The integration, necessarily numerical, is carried out by a predictor-corrector type of method from the "outer edge" $\tau = 0$ where

$$T = T(0; R, L), \quad P = \frac{1}{3} a T^4 (0; R, L)$$

down to the "photosphere" $\tau = \tilde{\tau}$ defined by

$$T(\tilde{\tau}; R, L) = T_{\text{eff}}(R, L)$$

and often chosen as the bottom of the atmosphere.

3. STELLAR ENVELOPE

The envelope extends from the photosphere down to a "fitting level" located at a distance $r = r_F$ from the star's centre, such that the mass contained within r_F amounts to a certain fraction q_F of the total mass. We must choose q_F in such a way that the two predominant elements, *H* and *He*, be completely ionized at $r = r_F$ where convection, if present, should have become adiabatic. On the other hand, q_F must not be taken too small, in order to ensure that all sources of radiant energy remain inside the core $r < r_F$. The compromise value adopted by Hofmeister-Kippenhahn-Weigert (1964) * of $q_F = 0.97$ means that only 3% of the mass (but something like 40% of the radius) are contained in the envelope.

The basic equations take here into account: the hydrostatic equilibrium, the mass variation (M_r being the mass inside the sphere of radius r) and the temperature variation. With $\log P$ as independent variable they read

$$\frac{d \log r}{d \log P} = - \frac{r P}{G M_r \rho} \tag{3a}$$

$$\frac{d \log M_r}{d \log P} = - \frac{4 \pi r^4 P}{G M_r^2} \tag{3b}$$

$$\frac{d \log T}{d \log P} = \nabla \tag{3c}$$

where the T-gradient ∇ may take, in a first approximation, the value appropriate either to radiative energy transfer ($\nabla = \nabla_{\text{rad}}$) or to convective energy transfer ($\nabla = \nabla_{\text{conv}}$); both values depend on the opacity κ, in addition, ∇_{conv} contains also the rather uncertain parameters of the so-called mixing length theory.

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* Quoted further as H.K.W.

We notice further that the elements *H* and *He*, usually partially ionized in a large part of the envelope, tend to become fully ionized at the botton $M_r = M_F$ and this is reflected on the equation of state linking P, ρ, T.

Starting from the photospheric conditions

$$r = R, \quad M_r = M, \quad T_{\text{eff}} = T(R, L), \quad P = P(\tau; R, L)$$

we integrate the three ordinary differential equations (3) by a predictor-corrector method to obtain, for any given R, L, the three fit-values (r_F, P_F, T_F) at the fit level $M_r = q_F M$. Further, we still have $L_r = L_F = L$ at $r = r_F$, since all the sources of radiant energy were located below the envelope.

Eliminating formally (R, L) between these 4 fit-values leads to 2 relations between the latter and considering only neighbouring points in the $(\log L, \log T_{\text{eff}})$ plane, these relations may be taken as linear, namely

$$\log r_F = \alpha_1 \log P_F + \beta_1 \log T_F + \gamma_1 \tag{4}$$
$$\log L_F = \alpha_1 \log P_F + \beta_1 \log T_F + \gamma_1$$

and 3 integrations of the envelope will be sufficient to determine the 6 coefficients α, β, γ.

As evolution proceeds, the representative point of the model moves in the $(\log L, \log T_{\text{eff}})$ plane; the technique of (H.K.W. 1964) based on the subdivision of that plane into small triangles having two of their sides parallel each to one of the axes $\log L$ or $\log T_{\text{eff}}$ permits to reduce to a minimum the number of necessary integrations through the envelope.

This is important, because the complicating features presented by the partial ionization of the medium and the non-adiabatic character of the convection, if any, require a long computing time.

4. INNER REGION

In the inner region, the elements *H*, *He* are, in the rule, totally ionized, although it may happen that the gas becomes partially electron-degenerate. Further, if convection is present, it will be considered as adiabatic with $\nabla = \nabla_{\text{ad}}$ throughout the convective core if one neglects, in first approximation, any overshooting at the edge of that core.

The new feature here is that L_r is no more constant, and we now have to envisage the following four basic equations, written with M_r as independent variable.

$$\frac{\bar{\partial} r}{\bar{\partial} M_r} = \frac{1}{4\pi r^2 \rho} \tag{5a}$$

$$\frac{\bar{\partial} P}{\bar{\partial} M_r} = -\frac{G M_r}{4\pi r^4} \tag{5b}$$

$$\frac{\bar{\partial} T}{\bar{\partial} M_r} - = \frac{G M_r T}{4\pi r^4 P} \nabla \tag{5c}$$

$$\frac{\bar{\partial} L_r}{\bar{\partial} M_r} = \varepsilon - c_p \frac{\bar{\partial} T}{\bar{\partial} t} + \frac{\delta}{\rho} \frac{\bar{\partial} P}{\bar{\partial} t} \tag{5d}$$

In (5c) we shall let $\nabla = \nabla_{\text{rad}}$ or $\nabla = \nabla_{\text{ad}}$ according to the dominant mode of energy transfer. In the energy equation (5d), the functions ε (rate of radiated energy), δ (thermal dilatation coefficient), c_p (specific heat at constant pressure) all depend, like ∇_{rad} or ∇_{ad} in (5c), on the physical conditions prevailing in the medium. Moreover, (5d) contains the time explicitly, so the equations (5) build a system of partial differential equations, we are faced with an initial value problem and a boundary value problem as well. As long as we start from an initial main sequence state, where the chemical composition is still uniform throughout the star as the thermonuclear reactions are on the verge of ignition at the centre, matters are greatly simplified: the evolution begins very slowly so that the 2 terms of (5d) containing t explicitly remain still negligibly small and the first time step $t - t_0$ can be chosen fairly large.

At any epoch $t = t^n$, the equations (5) have to be integrated inwards from the fit-level $M_r = M_F$ in the relevant model, where the linear boundary conditions (4) were obtained from the integrations through the atmosphere and envelope at given R, L. To avoid possible divergences, we do not carry the integration right down to the centre, but to a very small fixed mass level M_s where the 4 dependent variables, r, P, T, L_r are determined from the central values (P_c, T_c) with $\rho = \rho_c$, by first order expansions in M_r.

The equations (5) have first to be converted into finite difference equations, both in t and M_r. According to Henyey (1964) one adopts, for reasons of numerical stability against short-period perturbations, an implicit and asymmetrical difference scheme $\Delta t = t^n - t^{n-1}$, index $n - 1$ pertaining to the preceeding formerly computed known model.

On the other hand, we divide the inner model into m concentric shells labelled $j = 1, 2, \ldots m$ where $j = 1$ denotes the outer fit-level and $j = m$ the centre.

The physical quantities r, P, T, L_r undergoing considerable variation inside the model, accuracy reasons lead us to replace them by the respective logarithmic variables x, p, θ, s which vary much more smoothly, and take instead of M_r a dimensionless variable such as $\xi = ln\left(1 - \frac{M_r}{M}\right)$, in order to obtain a finer mass-level distribution near M_F. All these new variables remain finite in the interval

$$0 < M_s \leqslant M_r \leqslant M_F < M$$

and it is to them that we apply now a centered difference scheme, where the values at two consecutive levels $j, j+1$ appear symmetrically.

Symbolically, the equations (4), (5) together with the four central conditions at $j = m$, are of the form

$$\begin{aligned} &B_k(p_1, \theta_1, x_1, s_1) && = 0 \quad k = 1, 2 \\ &G_i(p_j, \theta_j, x_j, s_j, p_{j+1}, \theta_{j+1}, x_{j+1}, s_{j+1}) && = 0 \quad j = 1, 2, ..., m-2; i = 1, ..., 4 \\ &C_i(p_{m-1}, \theta_{m-1}, x_{m-1}, s_{m-1}, p_m, \theta_m) && = 0 \quad i = 1, ..., 4 \end{aligned} \tag{6}$$

Starting from a preliminary approximative model, therefore not satisfying equations (6) exactly, the Henyey relaxation method consists in finding the corrections needed in order to have

$$B_k + \delta B_k = 0, \quad G_i + \delta G_i = 0, \quad C_i + \delta C_i = 0$$

Expanding to first order quantities, these conditions give rise to a linear algebraic system of $4m-2$ equations yielding the $4m-2$ corrections $\delta p_j, \delta \theta_j, \delta x_j, \delta s_j$ ($j = 1, ..., m$; $\delta x_m = \delta s_m = 0$) expressed in terms of the preliminary model quantities. Now, m is usually larger than 100, but the particular echelon-structure of the matrix of the algebraic system simplifies greatly the resolution.

With the new values $p_j + \delta p_j$ etc. at level j, one may repeat the procedure until all corrections remain below a certain preassigned value. Note also that the improvement of the model by successive iterations also reflects itself on the fit-values at $j = 1$, therefore on the initially chosen R, L values.

The model obtained at time $t^n = t^{n-1} + \Delta t$ will serve as preliminary approximate model for the construction of the model at time t^{n+1}, unless one prefers to extrapolate linearly in time from the two preceeding t^n, t^{n-1} models. At each epoch $t = t^n$, the chemical composition is readjusted, from the preceeding value, at epoch t^{n-1}.

5. GRID RESOLUTION CRITERIA

One of the important items in carrying out such a program resides in the accuracy limits to decide upon, in connection with the choice of the time and mass steps. Thus, over a mass step $\Delta \xi_j = \xi_{j+1} - \xi_j$, we should require that, in absolute value, the change in variables $\Delta p_j, \Delta \theta_j, ...$ between two consecutive levels in a model built at time t, have both an upper and a lower bound. Since the step $\Delta \xi_j$ entails a change in all the local variables, we may expect that only certain of these changes might be retained. For example, it happens in practice that the bounds imposed on $\Delta p_j, \Delta s_j$ are indeed quite sufficient to ensure the limited variations of $\Delta \theta_j, \Delta x_j$.

Similarly, the limitations on the abundance changes of hydrogen, helium and carbon usually suffice to keep the overall chemical composition within reasonable limits. As regards $\Delta \xi_j$ itself, one often prefers limiting $\Delta M_r/M_r$ together with $\Delta M_r/M$. All these variation limits are chosen somewhat arbitrarily at first, and may be revised when the model, having evolved, exhibits a markedly different inner structure.

The choice of the time step Δt is related, apart from accuracy requirements, to the convergence of the iterations performed in the Henyey method. Consider the maxima $|\delta p|_{\max}, |\delta \theta|_{\max}, |\delta x|_{\max}, |\delta s|_{\max}$ of the absolute values of the respective corrections $\delta p_j, \delta \theta_j, \delta x_j, \delta s_j$ obtained at each level j after the first iteration in Henyey's method. The largest of these maxima (GKØR) is then tested against a fixed chosen value GKØRM:

If GKØR . GT . GKØRM

one steps out of the HENYEY subroutine to go back to the main program where, in case the model is not the initial one (viz AGE > 0), the time step Δt should be halved. The value of GKØRM is to be chosen the smaller, the faster the evolutionary phase comprising the model considered and in connection with this, it appeared useful to control also the run of the central density.

If GKØR . LT . GKØRM

at the first iteration, we go on performing successive iterations until all corrections are brought down below a preassigned value, such as 10^{-4}; usually in fact, they become less than 10^{-9} after 5 or 6 iterations.

On the other hand, accuracy reasons force us to adjust the time-step Δt so that the relative decrease in hydrogen abundance, due to nuclear burning, is kept small (a few percent).

Once hydrogen has become exhausted in the stellar core and goes on burning in a surrounding shell, the core contracts and the outer envelope expands markedly; consequently, the representative point of the star in the colour-luminosity diagram moves rapidly to the right, towards the region of subgiants, and in order to keep then a sufficiently dense network of evolutionary tracks, it is often advisable to set an upper bound B to the surface-temperature variation $|\Delta \log T_{\text{eff}}|$ when passing from model at time t^{n-1} to model at $t^n = t^{n-1} + \Delta t$. If ever $|\Delta \log T_{\text{eff}}| > B$, then Δt is to be halved.

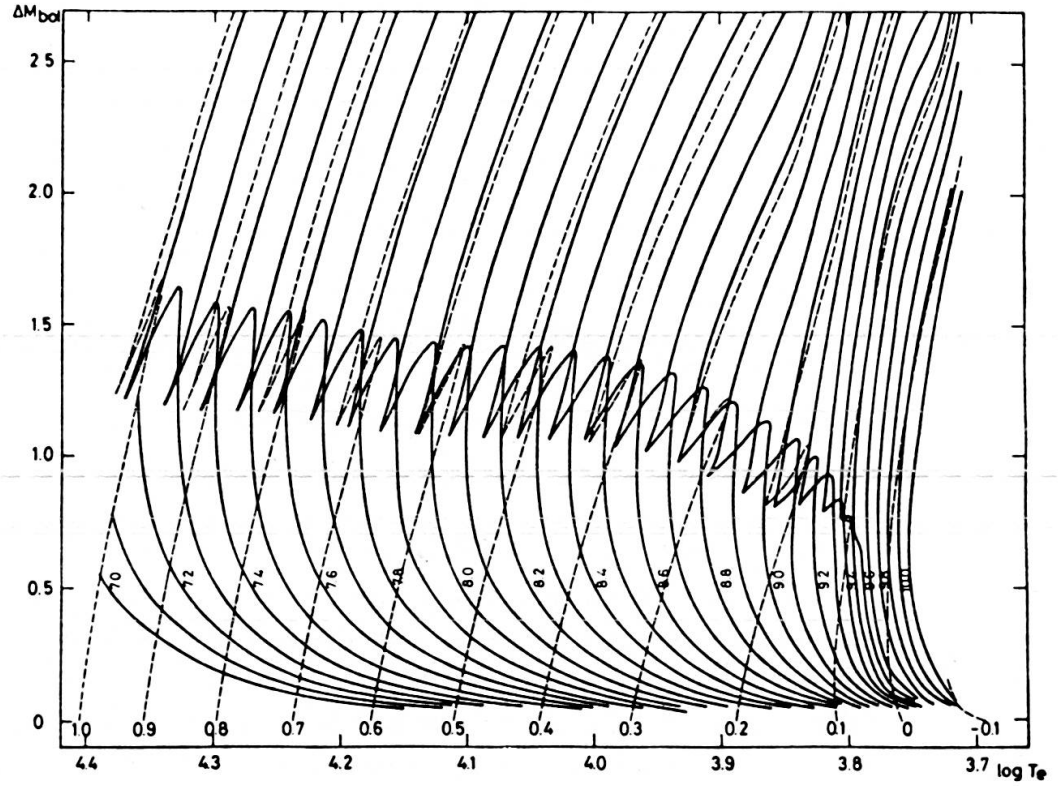
All these various control criteria render the running of the program rather cumbersome.

We produced a dozen evolutionary tracks over a mass range $0.9 \leqslant M \leqslant 10\, M_\odot$ at a given initial overall chemical composition, and also other tracks for a composition with reduced abundance of heavy elements (on the whole 23 tracks). Each track, extending from the zero-age main sequence to the subgiant stage, involves

over a hundred models. For models of low mass, complicating features of the input physics such as partial ionization, ionization pressure, electron-degeneracy, non-adiabatic convection in the envelope, tend to increase the computing time as compared to that of a high mass evolutionary track. As an average figure of the computing rate, we can mention about 20 seconds per high mass model and 30 seconds per low mass model on a CDC 3800 computer, or, roughly speaking, one hour or so of computing time for a single track.

In addition, a separate program enabled us to obtain, from these evolutionary tracks, a family of isochrones or lines of equal age; the construction of an isochrone of given age entails interpolating linearly in log M between two evolutionary tracks and in time along any given track.

This double network or grid of tracks and isochrones covering a large area of the (log L, log T_{eff})-diagram, illustrated by the figure, enables us finally to ascribe a mass and an age to any star of known representative point.



Grid of evolutionary tracks (dashed lines, labelled by log $M/M_{\odot}$) and of isochrones (full lines, labelled by log of age in years). ΔM_{bol} is the difference in bolometric magnitude between the star's magnitude and that on the zero-age main sequence for same T_{eff}.

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