1. Geometry of the unit tangent bundle

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

The paper is organized into the following sections:

1. *Geometry of the unit tangent bundle.* We describe the metric in two ways, and when the base space is a round sphere, we see that geodesics in its unit tangent bundle project to spherical helices on the sphere.

2. *Geodesics in* US^2. Some of the phenomena show up in this case.

3. *Helices in* S^3. Frenet equations, curvature, torsion and writhe.

4. *Sasaki's equations.* A general calculus for geodesics in the unit tangent bundle UM of any Riemannian manifold M.

5. *Proof of the Fundamental Constraint.* A blend of the Sasaki and Frenet equations.

I am grateful to Sharon Pedersen for a detailed reading of the manuscript, and for a number of improvements. Thanks also to Dennis DeTurck for reading the manuscript, and to Wolfgang Ziller for telling me about Sasaki's work. Finally, thanks to the National Science Foundation for their support.

1. Geometry of the unit tangent bundle

Let M be an n-dimensional Riemannian manifold, and $(p(t), v(t))$ a path in its unit tangent bundle UM. It is customary to give UM the Riemannian metric in which arc length $s(t)$ along this path is given by the formula

$$s'(t)^2 = |p'(t)|^2 + |v'(t)|^2 ,$$

where

$$p'(t) = \text{tangent vector to the curve } p(t) \text{ in } M ,$$

$$v'(t) = \text{covariant derivative of } v(t) \text{ along } p(t) \text{ in } M ,$$

and the norms of these vectors are measured in the given Riemannian metric on M.

When M is flat, and hence parallel translation is independent of path, the above metric on UM is simply the product metric of $M \times S^{n-1}$. So the constant speed geodesics in UM, for example, are just the paths $(p(t), v(t))$ for which $p(t)$ and $v(t)$ are themselves constant speed geodesics in their respective spaces. In particular, each geodesic in UM certainly projects to a geodesic in M.

But when M is curved, the story is quite different. A geodesic in the unit tangent bundle UM need not project to a geodesic in M. We can already see this when M is a round two-sphere.

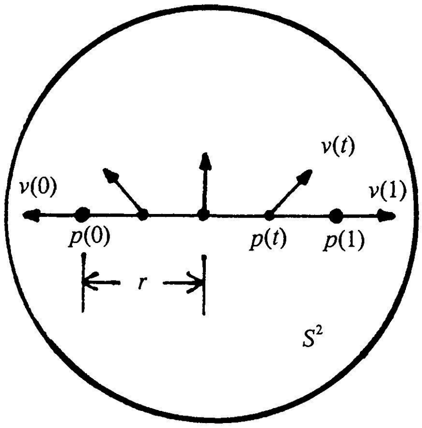

FIGURE 1

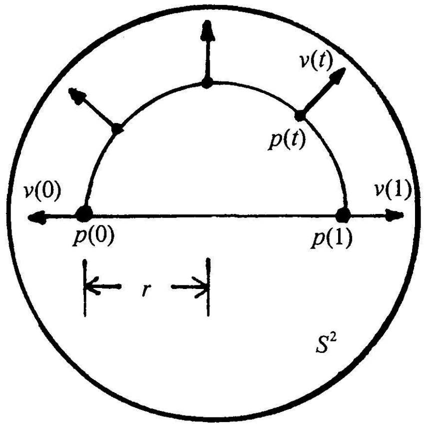

FIGURE 2

In each of Figures 1 and 2, we depict a path $(p(t), v(t))$ in the unit tangent bundle US^2 of a round two-sphere S^2 of radius 1. Though the paths are different, their initial points are the same and their terminal points are the same.

In the first path, the point $p(t)$ travels at constant speed along a geodesic of length $2r$ on S^2. At the same time the tangent vector $v(t)$ rotates at constant speed with respect to a parallel coordinate frame, turning through a total angle π from beginning to end. The length of this path $(p(t), v(t))$ is

$$\sqrt{\pi^2 + 4r^2}\,.$$

If the base space were R^2 instead of S^2, this path in the unit tangent bundle would be a geodesic, indeed a shortest connection between its endpoints.

In the second path, the point $p(t)$ travels at constant speed along a semicircle of length $\pi \sin r$. At the same time the tangent vector $v(t)$ rotates at constant speed with respect to a parallel coordinate frame, turning through a total angle somewhat less than π because of the curvature in the base space S^2. The savings is half of the area $2\pi(1-\cos r)$ inside the small circle. Hence the total angle that $v(t)$ turns through is $\pi \cos r$. It follows that the length of this second path $(p(t), v(t))$ is π.

So the second path is shorter than the first. Indeed, it is a minimizing geodesic in US^2 between its endpoints, whose distance apart is therefore π.

Yet its projection on the base space S^2 is a small circle, not a geodesic.

Immediately one asks: *which curves on* S^n *are projections of geodesics in* US^n*?*

In answering this, we use another approach to the geometry of US^n, viewing it as the homogeneous space $SO(n+1)/SO(n-1)$. Here, the special orthogonal group $SO(n+1)$ is given the usual bi-invariant Riemannian metric, and then the inner products in directions orthogonal to the cosets of $SO(n-1)$ are transfered to the coset space $SO(n+1)/SO(n-1)$. This makes the projection map from $SO(n+1)$ to US^n a Riemannian submersion. We leave it as an exercise to show that this Riemannian metric on US^n coincides with the one described earlier.

A geodesic in $SO(n+1)$ which starts out orthogonal to one of the cosets of $SO(n-1)$ remains orthogonal to all the cosets, and projects to a geodesic in $SO(n+1)/SO(n-1) = US^n$. Furthermore, all the geodesics in US^n are obtained this way.

Suppose, for example, that $n = 3$. If $(p(t), v(t))$ is a geodesic in US^3, then by the above, there must be a geodesic $h(t)$ through the identity in $SO(4)$ such that

$$h(t)\,(p(0)) = p(t) \quad \text{and} \quad h(t)\,(v(0)) = v(t)\,.$$

But every such geodesic $h(t)$ in $SO(4)$ consists of independent, constant speed rotations in a pair of orthogonal two-planes in four-space. Hence $p(t)$ travels along a spiral on an invariant torus, that is, along a spherical helix.

Notice that the isometry $h(t)$ which takes $p(0)$ to $p(t)$ and $v(0)$ to $v(t)$, also takes the entire helix $\{p(t)\}$ to itself. Hence it takes the Frenet frame of the helix at $p(0)$ to the Frenet frame at $p(t)$. It follows that

$$v(t) = aT(t) + bN(t) + cB(t)$$

has constant coeffients with respect to this Frenet frame.

Beyond S^3, nothing new happens for geodesics: it is easy to see that every geodesic in US^n lies inside a totally geodesic submanifold US^3. Indeed, if (p, v) and (q, w) are nearby points on the geodesic, then the vectors p, v, q and w determine the corresponding S^3.

When it comes to proving the Fundamental Constraint, we will capitalize on this observation by restricting our attention to S^3.

We conclude: *the only curves on* S^n *which can be projections of geodesics on* US^n *are spherical helixes (allowing great and small circles and points as special cases) which lie on great 3-spheres. All such spherical helixes will appear in this way.*