§3. Classification of with $dim_H\Lambda(\Gamma)\leq 1$

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§ 3. Classification of Γ with $\dim_H \Lambda(\Gamma) \leqslant 1$

In the previous section we constructed a compact, oriented, conformally flat 4-manifold X starting from a suitable (see § 2) hyperbolic 3-manifold. By Schoen's solution of the Yamabe problem [33] there is a metric in the conformal class of X, for which the scalar curvature is a constant. The sign of this constant $-$, 0 or $+$ is called the *type* of X. A lot is known about X of non-negative type, and we shall classify 3-manifolds M which give rise to X of non-negative type.

In a different direction Schoen and Yau [34] proved that if X is the quotient of $S^4 - \Lambda$ by a discrete group of conformal transformations, then X is of type $+, 0, -$ if and only if the *Hausdorff dimension* of Λ satisfies that $\dim_H \Lambda - 1$ is negative, zero, positive respectively. Hence our classification is that of M for which $\dim_H \Lambda \leqslant 1$. The classification for $\dim_H \Lambda = 1$ seems to be new, for < 1 the result was known.

Up to now, the only Kleinian groups to have been classified are the so called *function groups*, those Kleinian groups which leave a component of $\Omega = S^2 - \Lambda$ invariant. This has been done by Maskit. A special case of this, which we shall use repeatedly below, occurs when Ω is connected. In this case the Kleinian group is Schottky (see example 2.5).

THEOREM 3.1.

a) *If the type of X is $+$ then M is a handlebody equal to H^3/Γ with Γ a Schottky group.*

b) *If the type of X is 0 then one of the following holds:*

   1) *M equals $\mathbf{R} \times S = H^3/\Gamma$ with Γ Fuchsian and S a compact surface.*

   2) *M equals H^3/Γ with Γ extended Fuchsian (see example 2.3 (3)).*

   3) *M is a handlebody as in a).*

Proof. a) $R > 0$ implies $\dim_H \Lambda < 1$, see the proof of proposition 3.3 of Schoen and Yau [34]. This implies that $\Omega(\Gamma)$ is connected, because a set of Hausdorff dimension smaller than 1 cannot disconnect S^2. By Maskit's classification theorems (see Maskit [27]) it follows that Γ is Schottky.

b) First assume $H^2(X, \mathbf{R}) \neq 0$ and give X a metric of zero scalar curvature in the conformal class. From proposition 2.2 we see that the intersection form is indefinite, so there is a self-dual harmonic 2-form ω

on X. A Weitzenbock formula asserts that on 2-forms $(d+d^*)^2 = \nabla^*\nabla$ with ∇ the total covariant derivative. It follows that ω is covariantly constant, and a multiple of ω serves as Kähler form for an integrable complex structure on X: compare LeBrun [24]. LeBrun proceeds to classify these as (1) a K3 surface, (2) a four dimensional torus modulo a finite group and (3) a flat $\mathbf{CP}^1$ bundle with the local product metric, over a Riemann surface S which carries a metric of curvature -1. From proposition 2.2 we see that only (3) is possible in our case because (1) has the wrong Euler characteristic, and (2) with Euler characteristic 0 should have had $H_2(X;\mathbf{R}) = 0$.

The Kähler form of X is the unique self-dual harmonic 2-form on X. This is preserved by the conformal S^1-action, thus the action is a holomorphic action on X. As a result the vector field v induced by the S^1-action on X is holomorphic. The fibration $\pi: X \to \mathbf{CP}^1$ helps us further: we get a map $\pi_*: TX \to \pi^*TS$ and $\pi_* v$ is a section of π^*TS. Such a section is constant on fibres, so it is a pull-back of a section of TS. The only holomorphic section of TS is 0; so v is a vertical vector field.

From theorem 2.1 we see that zeroes of v must be simple, hence two per fibre. One of these is a sink, the other a source of $i \cdot v$ so we get two sections $S \to X$. This proves that X is the projectivization of a direct sum of holomorphic line bundles, say $X = P(L_0 \oplus L_1)$. The next step is to remember that the circle bundle $X - [P(L_0) \cup P(L_1)]$ over H^3/Γ may have no monodromy. Infinitesimally this implies that $L_0 \otimes L_1^*$ is a trivial line bundle. So $X = S \times \mathbf{CP}^1$ and consequently Γ must be Fuchsian.

Next we come to the case $H^2(X, \mathbf{R}) = 0$. If $S^2 - \Lambda$ has only one component then we can apply Maskits classification theorem as in a), and conclude that Γ is Schottky; therefore we shall concentrate on the case that Ω has at least two components.

If Ω_0 is one of these components then the stabilizer $\Gamma_0 \subset \Gamma$ of Ω_0 is a geometrically finite Kleinian group, and has Ω_0 as a component, see Marden [26] corollary 6.5 (it should be remarked that subgroups are not automatically geometrically finite). As $S^2 - \Omega_0$ is Γ-invariant and has non-empty interior, it follows that H^3/Γ_0 must have at least two ends. By formula 2.5 and the fact that $\dim_H \Lambda(\Gamma_0) \leqslant \dim_H \Lambda(\Gamma) \leqslant 1$, the above implies that Γ_0 is Fuchsian. Thus every component of Ω is a round disc.

Before we proceed let us briefly recall what effect a conformal rescaling of the metric has on the scalar curvature. If on the 4-manifold X one has $g_1 = u^2 \cdot g_0$ then $\frac{1}{6} \cdot u^3 \cdot R(g_1) = \left(d^*du + \frac{1}{6} R(g_0)u\right)$, where d^* is taken with respect to g_0. Since here metrics of zero scalar curvature are involved, this equation loses its nonlinear character. An immediate consequence is that metrics of zero scalar curvature are unique up to constants multiples and hence S^1-invariant.

We have the hyperbolic covering $H^3/\Gamma_0 \to H^3/\Gamma$, and on the 4-manifolds corresponding to each of these there exists an S^1-invariant metric of zero scalar curvature. Denote these by g_0 and g, and denote the hyperbolic metric on the 3-manifolds by g_h. Then we have positive functions $u_0: H^3/\Gamma_0 \to \mathbf{R}_{>0}$ and $u: H^3/\Gamma \to \mathbf{R}_{>0}$ such that $g_0 = u_0^2 \cdot g_h$ and $g = u^2 \cdot g_h$. By the above u_0 and u are in the kernel of (d^*d-1) on H^3/Γ_0 and H^3/Γ respectively (here d^* is w.r.t. the hyperbolic metric).

Results of Sullivan [36] imply that positive solutions of $d^*d - 1$ on H^3/Γ_0 are unique (up to positive scalar factors) as $\dim_H \Lambda(\Gamma_0) = 1$. Therefore the pullback of u equals u_0, and hence the cover $X_{\Gamma_0} - (S_1 \cup S_2) \to X_\Gamma - S_1$ is an isometry (S_i are the fixed surfaces). The map can readily be extended to an isometry $X_{\Gamma_0} - S_2 \to X_\Gamma$ and then extends to a double cover $X_{\Gamma_0} \to X_\Gamma$. It follows that Γ is extended Fuchsian as claimed. □

Reformulating in terms of Kleinian groups gives:

COROLLARY 3.2. *Let* Γ *be a geometrically finite Kleinian group without cusps. If* $\dim_H \Lambda(\Gamma) < 1$, *then* Γ *is Schottky. If* $\dim_H \Lambda(\Gamma) = 1$ *then* Γ *is Schottky, Fuchsian or extended Fuchsian.*

Proof. We shall see in section 7 that $\dim_H \Lambda(\Gamma) < 1$ implies that the type of X is $+$, which is essentially an old observation due to Poincaré. Together with the results of Schoen and Yau mentioned in the proof above, the corollary is now obvious. □

Remark. 1) Existence of Schottky groups with limit set of any dimension smaller than 2 has been proved (Thurston [37]).

2) In Schoen & Yau [38] and Gromov & Lawson [15] the conclusion is drawn that for so-called classical Schottky groups Γ the manifold X_Γ admits a metric of positive constant scalar curvature.

4) R. Bowen [9] has proved that any quasifuchsian group with $\dim_H \Lambda = 1$ is Fuchsian. Of course this is a special case of theorem 3.1.

It will be interesting to see if further developments in the theory of compact, 4-dimensional, conformally flat manifolds are going to have similar applications to Kleinian groups. On the other hand it seems likely that a purely 3-dimensional proof of theorem 3.1 could be found as well. The crucial element seems to be to exploit the existence of a harmonic two form, in the way Lebrun did. LeBrun arrives at his flat $\mathbf{CP}^1$ bundle through a foliation argument which presumably can be mimicked in the 3-manifold.

§ 4. Hodge theory for hyperbolic 3-manifolds

Apart from the topological and geometrical applications which we discussed in § 3, our Kaluza-Klein approach also has some more analytical applications.

Recall that the Hodge-star $* : \Omega^n(Y) \to \Omega^n(Y)$, on a $2n$-dimensional oriented Riemannian manifold Y, depends only on the conformal structure underlying the metric. This has two consequences:

1) The L^2-norm $\| \omega \|^2 = \int \omega \wedge *\omega$, of $\omega \in \Omega^n(Y)$, is conformally invariant.

2) The harmonic n-forms, i.e. the $\omega \in \Omega^n(Y)$ s.t. $d\omega = d^*\omega = 0$, depend only on the conformal structure of Y.

Of course conformal rescaling lies at the heart of our construction in § 2, and we shall now show how the above applies to this situation. Let X be the conformal compactification of $M \times S^1$ as in § 2. Harmonic 2-forms on X are automatically S^1-invariant because they are in one-one correspondence with the elements of $H^2(X; \mathbf{R})$ $(= H^2(M; \mathbf{R}) \oplus H^1(M, \delta M; \mathbf{R})$, see § 2). By restriction to the open subset $M \times S^1 \subset X$ and a conformal rescaling of the metric on $M \times S^1$, 2) above implies that we get S^1-invariant harmonic 2-forms on $M \times S^1$ with respect to the product metric.

An S^1-invariant form can be written as $\omega = \rho^*\alpha + \rho^*\beta \wedge d\theta$, with $\alpha \in \Omega^2(M)$, $\beta \in \Omega^1(M)$ and $\rho : M \times S^1 \to M$ the projection. A short computation shows that such S^1-invariant forms ω are harmonic iff α and β are harmonic on M. If ω is a harmonic 2-form on $M \times S^1$ arising from a form on X then it follows from proposition 2.2 that $\alpha \in \Omega^2(M)$ and $\beta \in \Omega^1(M)$ are harmonic representatives for the class $\omega \in H^2(M; \mathbf{R}) \oplus H^1(M, \delta M; \mathbf{R})$. The forms α and β have finite L^2-norm on M by 1) above.

Conversely any S^1-invariant, harmonic 2-form $\tilde{\omega}$ on $M \times S^1$ with finite L^2-norm arises in this way. By 1) above one can always consider $\tilde{\omega}$ to be an L^2-form ω on X because $\cup S_j = X \backslash M \times S^1$ has measure 0. Applying the first order elliptic operator $d \oplus d^*$ to ω gives a distributional form in $L^2_{-1}(\Lambda^*(X))$ of distributional order $\leqslant 1$, which has support in the co-dimension 2 manifold $\cup_j S_j \subset X$. The following lemma shows that this