

# 3. Applications

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$$(*) \quad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A.$$

For  $f \in \mathcal{O}(\tilde{X})$ ,  $a \in \tilde{X}$  and  $b \in p^{-1}(p(a))$ , it is now possible to conclude that  $f(a) = f(b)$  is true. Let  $x := \pi(a)$ . Due to (\*), functions  $g \in S_x$  and  $h \in A$  exist with  $f = h/g \circ \pi$ . Since  $a$  and  $b$  are equivalent with respect to the equivalence relation  $R$ ,  $f(a) = f(b)$  follows, and a continuous function  $F: Y \rightarrow \mathbf{C}$  exists with  $F \circ p = f$ .

Since the Stein complex structure on  $Y$  is not in general the canonical ringed quotient structure, it is still necessary to verify that  $F$  is holomorphic in order to prove the density of  $A$  in  $\mathcal{O}(\tilde{X})$ . To that end, let  $H \in \mathcal{O}(Y)$  and  $G \in \mathcal{O}(Y)$  have the property that  $H \circ p = h$  and  $G \circ p = g \circ \pi$ . Such functions exist because  $p^*(\mathcal{O}(Y)) = \bar{A}$  holds. Then  $F = H/G$  follows, and the germ  $F_{p(a)}$  is the germ of a holomorphic function at  $p(a)$ , since the germ  $G_{p(a)}$  of  $G$  at  $p(a)$  is a unit. The surjectivity of  $p$  implies that  $F$  is holomorphic on  $Y$ , completing the proof of the theorem.

Note that the topology induced by  $\mathcal{O}(\tilde{X})$  on any subalgebra  $A$  of  $\mathcal{O}(\tilde{X})$  is the metrizable topology of uniform convergence on compact subsets of  $X$ . Because the closure  $\bar{A}$  of  $A$  in  $\mathcal{O}(\tilde{X})$  is its completion,  $\bar{A}$  can be obtained without referring directly to  $\mathcal{O}(\tilde{X})$ . Thus the Main Theorem can be stated as follows:

*If  $\tilde{X}$  denotes the normalization of an irreducible Stein space  $X$ , then  $\mathcal{O}(\tilde{X})$  is the completion of the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$ .*

### 3. APPLICATIONS

In this section  $X$  will denote an irreducible Stein space with normalization  $\pi: \tilde{X} \rightarrow X$ ,  $\widetilde{\mathcal{O}(X)}$  will be the integral closure of the holomorphic functions  $\mathcal{O}(X)$  on  $X$ ,  $\tilde{\mathcal{O}}(X)$  the Fréchet algebra of weakly holomorphic functions on  $X$  (or equivalently, the Fréchet algebra of holomorphic functions  $\mathcal{O}(\tilde{X})$  on  $\tilde{X}$ ), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \quad \text{for } x \in X.$$

Although the example given in the first section shows that the algebras  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  are not always equal, the inclusion (\*) in the proof of the Main Theorem implies that they are locally equal in the following sense.

**THEOREM 2.** *For every  $x \in X$ , the localizations of  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  with respect to  $S_x$  coincide.*

The next theorem implies an algebraic description of the topological closure of  $\widetilde{\mathcal{O}(X)}$  in  $\widetilde{\mathcal{O}(X)}$ .

**THEOREM 3.**  $\mathcal{O}(\widetilde{X}) = \bigcap_{x \in X} S_x^{-1} \widetilde{\mathcal{O}(X)}$ .

*Proof.* Let  $f \in M(\widetilde{X}) = M(X)$  be such that for every  $x \in X$  there is a  $g \in S_x$  and an  $h \in \widetilde{\mathcal{O}(X)}$ , with  $f = h/g \circ \pi$ . Then the germ  $f_a$  of  $f$  at an arbitrary point  $a \in \widetilde{X}$  is holomorphic, because the germ of  $g \circ \pi$  at  $a$  is a unit. Hence  $f \in \mathcal{O}(\widetilde{X})$ , and the assertion is proved.

**COROLLARY 2.** The topological closure of  $\widetilde{\mathcal{O}(X)}$  in  $\widetilde{\mathcal{O}(X)}$  is the intersection of the localizations of  $\widetilde{\mathcal{O}(X)}$  with respect to  $S_x$  for all  $x \in X$ .

The next result characterizes the weakly holomorphic functions on  $X$  as being exactly those meromorphic functions on  $X$  which are almost integral over  $\mathcal{O}(X)$ .

**COROLLARY 3.**  $\mathcal{O}(\widetilde{X})$  is completely normal.

*Proof.* Let  $f \in M(\widetilde{X})$  be almost integral over  $\mathcal{O}(\widetilde{X})$ . Then  $f$  is almost integral over  $\mathcal{O}(X)$  and therefore over  $S_x^{-1} \widetilde{\mathcal{O}(X)}$  for every  $x \in X$  which has been shown to be completely normal in the proof of the Main Theorem. An application of Theorem 3 yields  $f \in \mathcal{O}(\widetilde{X})$  and hence the assertion.

Using the classical Oka-Weil-Cartan Theorem [1, Anhang zu VI, § 4], an immediate consequence of the Main Theorem is

**THEOREM 4.**  $\widetilde{X}$  is  $\widetilde{\mathcal{O}(X)}$ -convex,  $\widetilde{\mathcal{O}(X)}$ -separable and has local coordinates by functions in  $\widetilde{\mathcal{O}(X)}$ .

A property which ensures that the holomorphic functions on  $\widetilde{X}$  are integral over the holomorphic functions on  $X$  is that  $\mathcal{O}(\widetilde{X})$  is a finite  $\mathcal{O}(X)$ -module.

**THEOREM 5.** Let  $u \in \mathcal{O}(X)$  be any global universal denominator for  $X$ . Then  $\mathcal{O}(\widetilde{X})$  is isomorphic to the closed ideal  $u\mathcal{O}(\widetilde{X})$  in  $\mathcal{O}(X)$ , and  $\mathcal{O}(\widetilde{X})$  is a finite  $\mathcal{O}(X)$ -module if and only if this ideal is finitely generated.

*Proof.* Recall that a global universal denominator  $u$  for  $X$  always exists [10, E.73a]. The multiplication map

$$\mathcal{O}(\widetilde{X}) \rightarrow \mathcal{O}(X), f \mapsto uf,$$

defines an injective  $\mathcal{O}(X)$ -module homomorphism. Thus,  $\mathcal{O}(\tilde{X})$  is isomorphic to the ideal  $u\mathcal{O}(\tilde{X})$  in  $\mathcal{O}(X)$  which will now be denoted by  $I$ . Consider the transporter ideal  $J := \tilde{\mathcal{O}} : \frac{1}{u} \mathcal{O}$  of  $\frac{1}{u} \mathcal{O}$  into  $\tilde{\mathcal{O}}$  which is a coherent sheaf of ideals in  $\tilde{\mathcal{O}}$ . The global sections  $J(X)$  form a closed ideal of  $\mathcal{O}(X)$  by a theorem of Cartan [4, 5], due again to the fact that  $X$  is Stein. Because  $J(X) = I$  holds, the assertion follows.

COROLLARY 4. *If  $\mathcal{O}(\tilde{X})$  does not coincide with  $\widetilde{\mathcal{O}(X)}$ , the closed ideal  $u\mathcal{O}(\tilde{X})$  in  $\mathcal{O}(X)$  is not finitely generated.*

In a Stein algebra  $\mathcal{O}(X)$ , every finitely generated ideal is closed, as Cartan [4, 5] showed. If  $X$  is at least two-dimensional, Forster [6] gave examples of closed ideals in  $\mathcal{O}(X)$  which are not finitely generated. According to Corollary 4, the space constructed in § 1 gives a one-dimensional example.

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