0. Introduction

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ISOCLINIC n-PLANES IN R^{2n} AND THE HOPF-STEENROD SPHERE BUNDLES $S^{2n-1} \to S^n$, $n = 2, 4, 8$

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0. Introduction

The construction of the sphere bundles $S^{2n-1} \to S^n$, $n = 2, 4, 8$, by N. Steenrod was accomplished in an ingenious but rather roundabout way, using the famous Hopf maps and the systems of complex numbers, quaternions and Cayley numbers (cf. Hopf [2], Steenrod [5, pp. 105-110] and Hilton [1, pp. 51-55]). In this paper, we show how the theory of mutually isoclinic n-planes in a real Euclidean $2n$-space R^{2n} as developed by Wong in [8, 9] enables us to reconstruct these sphere bundles in a more natural manner by working strictly within the field of real numbers and giving the three cases $n = 2, 4, 8$ a more unified treatment. In addition, we prove that the bundle group $O(8)$ of the Hopf-Steenrod sphere bundle $S^{15} \to S^8$ can be replaced by $SO(8)$ but not by any subgroup of $SO(8)$.

In § 1, we recall certain results on maximal sets of mutually isoclinic n-planes in R^{2n} that motivated our investigation. In § 2, we confine ourselves to the cases $n = 2, 4, 8$, and prove some results that will be used later. In § 3, we construct three sphere bundles by using maximal sets of mutually isoclinic n-planes in R^{2n}. In § 4, we give a unified and explicit formulation of the three Hopf-Steenrod sphere bundles, using as Steenrod did the Hopf maps and systems of complex numbers, quaternions and Cayley numbers. In § 5, we prove that the Hopf maps and maximal sets of mutually isoclinic n-planes in R^{2n}, $n = 2, 4, 8$, are equivalent concepts, and that the reformulated Hopf-Steenrod sphere bundles described in § 4 are topologically essentially the same as the sphere bundles constructed in § 3. The paper ends with two appendices in which we explain the operations of Cayley numbers, and give a direct proof that for $n = 2, 4$, or 8, the n-planes in R^{2n} containing the Hopf fibers of S^{2n-1} are mutually isoclinic n-planes.

In a continuation of this paper being prepared, we shall show that the image of the Hopf fibers of S^{2n-1}, $n = 2, 4$, or 8, under an inversion in R^{2n} has some very interesting properties which include those recently found by J. B. Wilker [7] for the case $n = 2$.

We wish to thank Prof. Wilker for letting us have a preprint of his paper, and Prof. Kee-Yuen Lam for some helpful discussions.

1. Some results on isoclinic n-planes in R^{2n}

By a Euclidean (vector) m-space R^m, where m is a positive integer, we mean an m-dimensional vector space provided with a positive definite inner product. An r-plane $(1 \leqslant r \leqslant m-1)$ in R^m is an r-dimensional vector subspace of R^m provided with the induced inner product. In R^m, length of a vector, angle between two vectors, orthogonality between a k-plane and an r-plane, (orthogonal) projection of a vector on an r-plane, orthonormal bases and rectangular coordinates are defined in the usual way.

In an R^{2n}, let $\mathbf{A}$, $\mathbf{B}$ be any two n-planes. Then we say that $\mathbf{A}$ is *isoclinic* with $\mathbf{B}$ at angle θ if the angle between every nonzero vector in $\mathbf{A}$ and its projection on $\mathbf{B}$ is always equal to θ. It turns out that if $\mathbf{A}$ is isoclinic with $\mathbf{B}$ at angle θ, then $\mathbf{B}$ is isoclinic with $\mathbf{A}$ at the same angle θ. Therefore, in this case, we shall say that $\mathbf{A}$ and $\mathbf{B}$ are isoclinic at angle θ, or simply, $\mathbf{A}$ and $\mathbf{B}$ are isoclinic.

A set Φ of n-planes in R^{2n} is said to be a *maximal set of mutually isoclinic n-planes* if every pair of n-planes in Φ are isoclinic and Φ is not contained in a larger set of mutually isoclinic n-planes. It is easy to see from definition that if $\mathbf{A}$ is isoclinic with $\mathbf{B}$ at angle θ, then its orthogonal complement $\mathbf{A}^\perp$ is isoclinic with $\mathbf{B}$ at angle $\frac{\pi}{2} - \theta$. Consequently, if Φ is any maximal set of mutually isoclinic n-planes in R^{2n} and $\mathbf{A} \in \Phi$, then $\mathbf{A}^\perp \in \Phi$.

In his memoir [8] Wong determined, for each n, the dimensions of the maximal sets of mutually isoclinic n-planes in R^{2n}, the number of non-congruent maximal sets of a given dimension, and explicit equations of the n-planes in any maximal set of mutually isoclinic n-planes containing a given n-plane.

In the following, we summarize some of his results related to the problem studied in this paper.

Theorem 1.1. (Wong [8, pp. 25-26]). *In R^{2n} provided with a rectangular coordinate system $(x, y) \equiv ([x_1 \dots x_n], [x_{n+1} \dots x_{2n}])$, any maximal set Φ of mutually isoclinic n-planes containing the n-plane $\mathbf{O}: y = 0$ (and consequently, also the n-plane $\mathbf{O}^\perp: x=0$) is congruent to the set of n-planes with equations*