0. Introduction

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ABOUT THE PROOFS OF CALABI'S CONJECTURES ON COMPACT KÄHLER MANIFOLDS

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ABSTRACT

The main part in the proof of Calabi's conjectures consists in *a priori* estimates of order zero, two, three. We explain how a reduction to these estimates may be performed in the framework of C^∞ functions and how higher order estimates may be derived without Schauder's elliptic theory. The main tool is an "elliptic" inverse function theorem [22] [11].

0. INTRODUCTION

T. Aubin [1, 2, 3] and S. T. Yau [23, 24] have brought positive answers to the so-called Calabi's conjectures [6], namely,

THEOREM 0.1. (Aubin, Yau). *On a compact (connected) Kähler manifold with negative first Chern class, there exists a unique Kähler-Einstein metric* g' *satisfying:* $\mathrm{Ricci}(g') = -g'$.

THEOREM 0.2. (Yau). *On any compact (connected) Kähler manifold, given a cohomology class* $c \in H^2(X, \mathbf{R})$ *which contains a Kähler form, every 2-form in the first Chern class is the Ricci form of some Kähler form of* c.

Mathematicians from several fields are concerned with these results, whose main consequences are listed in [23] and in [5] sections 2 and 3. Unfortunately, the proofs are quite technical, they involve rather "irregular" mathematical objects such as elliptic equations with non smooth coefficients, and they make a decisive use of Schauder theory. The aim of the present

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note is to analyze how far these tools are necessary for the proof. It turns out that it is possible to reduce the contribution of elliptic theory mainly to a suitable local inverse function theorem for nonlinear elliptic operators acting on smooth functions [22] [11].

The proof presented below deals only with the reduction to the crucial estimates of order zero, two and three, already obtained by Aubin and Yau. Although it is not so clear in [21] [24] these estimates were performed essentially through coordinate free tensor calculus. We show how higher order estimates may be obtained in the same way.

The whole approach applies as well to the corresponding *real* elliptic Monge-Ampère equation on compact Riemannian manifolds [9] and to various generalizations of it. We shall freely use arguments of Calabi [6, 7, 8], Aubin [1, 2, 3], Yau [23, 24], Bourguignon *et al.* [5] [21].

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1. THE MONGE-AMPÈRE EQUATION

Let X be a compact connected finite-dimensional Kähler manifold. ω denotes the original C^∞ Kähler form, g the corresponding Kähler metric, $\varphi \in C^\infty(X)$ denotes a C^∞ real-valued function on X, and we set

$$\omega' = \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi$$

where ∂ and $\bar{\partial}$ are the usual first order differential operators. Let g' denotes the Kähler metric corresponding to ω'. In the sequel, "smooth" means C^∞.

If g and g' are viewed as morphisms from the antiholomorphic tangent bundle into the holomorphic cotangent bundle T^*, then $(g'g^{-1})$ is an endomorphism of T^* the determinant of which, $\det(g'g^{-1})$ is a smooth function on X. The function φ is said to be *admissible* if and only if $\det(g'g^{-1})$ is strictly positive on X. One proves easily that if φ is admissible, then g' is again a (positive definite Kähler) metric e.g. [2], p. 119.

Let $\lambda \in [0, +\infty)$. It is convenient to denote by A_λ the subset of $C^\infty(X)$ consisting in all admissible real-valued smooth functions φ on X, satisfying, in case $\lambda = 0$ the further zero average condition

$$\int_X \varphi\, dX_g = 0,$$