Appendix 1. The Cayley numbers

Objekttyp: **Appendix**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

correspond to the coordinate transformations $t \to tB(\lambda)/N(\lambda)^{1/2}$ in $\mathscr{I}_4$, where $B(\lambda)$ are the matrices given in (1.7) in Theorem 1.6. By Theorem 2.5, the elements $B(\lambda)/N(\lambda)^{1/2}$ of $SO(4)$ form a subgroup isomorphic with S^3. Therefore, the bundle group $O(4)$ in $\mathscr{H}\mathscr{S}_4$ can be replaced by S^3. Similarly, the bundle group $O(2)$ in $\mathscr{H}\mathscr{S}_2$ can be replaced by S^1. With these observations, we can now prove the following theorem by proceeding as in the proof of Theorem 5.3.

THEOREM 5.4. *The representative coordinate bundles constructed in* § 4 *for the sphere bundles* $\mathscr{H}\mathscr{S}_2$ *and* $\mathscr{H}\mathscr{S}_4$, *with bundle groups* S^1 *and* S^3 *respectively, are topologically the same as the representative coordinate bundles constructed in* § 3 *for the sphere bundles* $\mathscr{I}_2$ *and* $\mathscr{I}_4$, *respectively.*

Finally, we remark that representative coordinate bundles of the bundles $\mathscr{S}\mathscr{L}_n$ in Theorem 4.2 are topologically essentially the same as the representative coordinate bundles of the bundles $\mathscr{I}\mathscr{L}_n$ in Theorem 3.2.

APPENDIX 1. THE CAYLEY NUMBERS

The Cayley numbers, denoted by X, Y, Z, W, etc. are ordered pairs (q_1, q_2) of quaternions subject to the rules and having the properties listed below. The set of all Cayley numbers, therefore, forms a (non-commutative and non-associative) real division algebra. No proof of the properties will be given as they can all be checked by direct computations.

(1) The *addition* is defined by

$$(q_1, q_2) + (q'_1, q'_2) = (q_1+q'_1, q_2+q'_2).$$

The *zero* is $O = (O, O)$.

(2) The *multiplication* is defined by

$$(q_1, q_2)(q'_1, q'_2) = (q_1q'_1 - q'^*_2 q_2, q'_2 q_1 + q_2 q'^*_1),$$

where q'^*_1, q'^*_2 are respectively the conjugates of (the quaternions) q'_1, q'_2. The (two-sided) *unit* is $1 \equiv (1, 0)$.

(3) Multiplication is

(i) distributive with respect to addition, i.e.,

$$(X+Y)W = XW + YW, \quad W(X+Y) = WX + WY;$$

(ii) not commutative, i.e., generally, $XY \neq YX$ (but see (4) (iv) below);

(iii) not associative, i.e., generally, $(XY)W \neq X(YW)$ (but see (7) below).

(4) The *real part* of $X \equiv (q_1, q_2)$ is $\text{Re } X = (\text{Re } q_1, O) \equiv \text{Re } q_1$. X is said to be *real* if $X = \text{Re } X$; i.e., (q_1, q_2) is real iff q_1 is real and $q_2 = 0$.

(i) $\text{Re}(X+Y) = \text{Re}(X) + \text{Re}(Y)$.

(ii) $\text{Re}(XY) = \text{Re}(YX)$.

(iii) $\text{Re}(CX) = 0$ for all X implies that $C = 0$.

(iv) $CX = XC$ for all X iff C is real. In this case, $C = (c_1, 0)$, where $c_1 =$ real, and $CX = (c_1 q_1, c_1 q_2) = XC$.

(5) The *conjugate* of $X \equiv (q_1, q_2)$ is $X^* = (q_1^*, -q_2)$.

(i) $(X+Y)^* = X^* + Y^*$,

(ii) $(XY)^* = Y^* X^*$.

(iii) $X^* = X$ iff X is real.

(6) The *norm* of X is the non-negative real number $N(X) \equiv XX^*$, which is also equal to X^*X. The *length* of X is the non-negative real number $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$.

(i) $N(X) = 0$ iff $X = 0$.

(ii) If $X \neq 0$, then $X^{-1} \equiv X^*/N(X)$ is a right and left inverse of X.

(iii) $N(XY) = N(X)N(Y)$. It follows from this that $XY = 0$ iff $X = 0$ or $Y = 0$.

(7) Though multiplication is generally non-associative,

(i) $(XY)Y^* = X(YY^*)$.

(ii) If $Y \neq 0$, then $(XY)Y^{-1} = X = Y^{-1}(YX)$.

(iii) $\text{Re}((XY)W) = \text{Re}(X(YW))$.

APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of S^{2n-1} by S^{n-1} over S^n, $n = 2, 4$, or 8, by intersecting the unit sphere S^{2n-1} in $R^{2n} = Q_n \times Q_n$ with the Q_n-lines $Y = CX$ and $X = 0$. In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic n-planes in R^{2n} are equivalent concepts. Here we prove, directly, the