KALUZA-KLEIN APPROACH TO HYPERBOLIC THREE-MANIFOLDS

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A KALUZA-KLEIN APPROACH TO HYPERBOLIC THREE-MANIFOLDS

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§ 1. INTRODUCTION

In the recent past Thurston has caused a revolution in three-dimensional topology with the creed: "Every 3-manifold is essentially geometric". In particular a large class of 3-manifolds with boundary can be supplied with a hyperbolic structure. This situation is much the same as that for two-dimensional surfaces, which can also be given hyperbolic structures.

Another even more recent revolution in mathematics came about when mathematicians started paying close attention to the methods employed in theoretical physics. In particular S. K. Donaldson found deep applications of Yang-Mills theory to four-dimensional topology.

On three-dimensional manifolds there exists a set of partial differential equations, the Bogomol'nyi equations, which describe magnetic monopoles in M. This equation is closely related to the Yang-Mills equation in dimension four, and can only be formulated in presence of a Riemannian metric and orientation on the 3-manifold. In the last three sections of this paper we shall study some aspects of this equation on hyperbolic 3-manifolds.

Kaluza-Klein theory, another favorite of the physicists, leads to a natural way to study these equations, thereby circumventing a large amount of analysis associated with more direct approaches. Basically Kaluza-Klein theory amounts to studying space through the geometry of a fibre bundle over space. In our case this fibre bundle over a hyperbolic 3-manifold is simply the product of the manifold with the circle. The analytical problems alluded to above are largely due to the fact that a 3-manifold with boundary, supplied with a hyperbolic metric, is very non-compact as a metric space. Although this is not changed by taking the product with a circle, it turns out that this 4-manifold has a natural conformal compactification (yet another popular ingredient in physical theories).

The upshot is (§ 2) that we canonically associate a conformally flat, compact 4-manifold (without boundary) with a circle action, to a hyperbolic

3-manifold (provided some conditions are satisfied see § 2). This provides a link between conformal geometry in dimension 4, and hyperbolic geometry in dimension 3. It is very similar to Poincaré's observation in 1883 that hyperbolic geometry in dimension 3 is related to conformal geometry in dimension 2, by considering the boundary surfaces of a hyperbolic 3-manifold.

In going over to the 4-manifold, no information is lost. This allows one to deduce precise facts concerning the 3-manifold from known facts about conformally flat 4-manifolds; therefore, before we start studying the Bogomol'nyi equation, we study some global differential geometric questions concerning hyperbolic 3-manifolds in the light of the conformal compactifications.

In particular we can exploit recent work of Schoen and Yau to classify a family of hyperbolic 3-manifolds (§ 3), namely those which are geometrically finite without cusps and have a limit set of Hausdorff dimension $\leqslant 1$.

On the analytical side, knowledge about conformally invariant differential operators in dimension 4 can be exploited to obtain a Hodge theory for hyperbolic 3-manifolds (§ 4). This answers a question posed by Thurston. We prove that the L^2-cohomology in dimension 1 of the 3-manifold is equal to the de Rham cohomology with compact supports. On the universal cover H^3, Poisson transformation gives an identification between closed and co-closed one forms on H^3 and closed hyperfunction one forms on δH^3. Our L^2 harmonic forms now correspond to closed, invariant currents on δH^3 with support in the limit set. Additionally this theory gives an invariant of the hyperbolic structure, of a type familiar from algebraic geometry.

After these digressions we start studying magnetic monopoles on the hyperbolic 3-manifolds by relating them to S^1-invariant instantons on the 4-manifolds. Relevant definitions and background can be found in § 5.

The twistor spaces associated to the conformally flat 4-manifolds are studied in § 6. Not only do these provide a way to study monopoles, they also encode a wealth of geometrical information belonging to the 3-manifold such as the entire geodesic flow. Finally in § 7, we use the twistor theory to construct some explicit formulas for monopoles on handlebodies. Here we naturally encounter the Eisenstein series associated to the hyperbolic 3-manifold.

We end this introduction by briefly indicating what kind of future developments can be expected. The compact 4-manifolds should allow for easy study of many natural differential operators on the 3-manifold; in § 4 it is indicated how. Using generalizations of Poisson transformation to fields of higher spin, it seems very likely that a wealth of hyperfunction objects with support in the limit set can be obtained. The twistor spaces may provide a natural environment to study theorems about the 3-manifold which rely on properties of the geodesic flow. In particular one could try to prove Mostow's theorem (and Thurston's generalisation of it) along the lines outlined in § 6.

From an analytical study of monopoles it is known that monopoles exist under reasonable conditions. This shows that there are interesting holomorphic bundles on twistor space. Understanding the structure of these will almost certainly reveal a large amount of geometry and analysis associated to the hyperbolic manifold. Finally, properties of the moduli spaces of monopoles which are independent of the metric on the 3-manifold are topological invariants of the 3-manifold. This is related to the work of Donaldson and Casson.

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§ 2. Conformal compactifications and their topology

Let $\bar{M}$ be an oriented, irreducible, atoroidal, compact, three-dimensional manifold with non-empty boundary $\delta\bar{M}$. *Atoroidal* means that every map $T^2 \to \bar{M}$ has a kernel on the level of fundamental groups. For simplicity we shall avoid cusps and thus we assume that:

2.1 either no component of $\delta\bar{M}$ is of genus 1 or $\bar{M} = \bar{D}^2 \times S^1$.

Thurston's uniformization theorem (see Morgan [29]) asserts that there is a complete, geometrically finite, hyperbolic structure on $M = \bar{M} - \delta\bar{M}$. This means that M can be realised as follows (see Bers [7], Maskit [27], Morgan [29], Beardon [6] for background).

Recall that $PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\{\pm 1\}$ is the isometry group of hyperbolic 3-space H^3, and that the right action of an isometry on $H^3 = SU(2)\backslash SL(2, \mathbf{C})$ extends over the boundary $S^2 \cong \delta H^3$ as an action by a fractional linear transformation of S^2. A *Kleinian group* Γ without cusps is a discrete subgroup of $PSL(2, \mathbf{C})$ all elements of which are loxodromic (i.e. have exactly two fixed points in $\bar{H}^3 = H^3 \cup S^2$), and which acts freely and properly on a non-empty open set $\Omega \subset S^2$ (Felix Klein, the man of the dicrete groups, and Oscar Klein, of the Kaluza-Klein theories mentioned in the introduction, are not the same). Proper means that the map $\Omega \times \Gamma \to \Omega \times \Omega : (x, \gamma) \to (x\gamma, x)$ is proper. Proper actions are well behaved, and a proper free action has a smooth quotient, see Gleason [14].

There is a preferred region $\Omega(\Gamma)$, in which Γ acts properly. Define the *limit set* $\Lambda(\Gamma)$ of the group Γ to be the set of all $y \in S^2$ such that there is a sequence of different elements $\gamma_j \in \Gamma$ and an $x \in S^2$ with $\gamma_j \cdot x \to y$. The *region of discontinuity* $\Omega(\Gamma)$ is the complement $S^2 - \Lambda(\Gamma)$, and Γ acts properly on $\Omega(\Gamma)$. The limit set may be quite wild and has Hausdorff dimension $\dim_H \Lambda(\Gamma) \in [0, 2]$. If no confusion is possible we shall denote $\Omega(\Gamma)$ by Ω and $\Lambda(\Gamma)$ by Λ.

The number of components of Ω is 1, 2 or infinite, and Ω/Γ is a collection of N Riemann surfaces $S_1, ..., S_N$, where N is the number of Γ-orbits in the set of components of Ω (N can be infinite). It is well known that the Γ-action on H^3 is proper and that it extends to a proper action on $\bar{H}^3 - \Lambda$; therefore $(\bar{H}^3 - \Lambda)/\Gamma$ is a smooth manifold with boundary $\Omega/\Gamma = \cup_j S_j$.

In order to ensure that $(\bar{H}^3 - \Lambda)/\Gamma$ is compact we introduce another notion. The group Γ is said to be *geometrically finite* iff there is a finitely sided fundamental polyhedron (Maskit [27]) for the Γ-action on H^3. In this case the quotient $M = H^3/\Gamma$ is the interior of a compact, smooth manifold $\bar{M} = (\bar{H}^3 - \Lambda)/\Gamma$ which has boundary $\delta M = \Omega/\Gamma$, now equal to a finite collection of compact Riemann surfaces without boundary. In this case the hyperbolic structure on M is said to be geometrically finite. If $\Gamma = \{e\}$ we have $N = 1$, $S_1 = S^2$, and if Γ is cyclic then $N = 1$, $S_1 = T^2$; in both of these cases Ω is connected. In all other cases every S_j is a surface of genus $\geqslant 2$.

The conjugacy class of Γ in $PSL(2, \mathbf{C})$ is not uniquely determined by M as a smooth manifold; in fact continuous deformations of the complete hyperbolic structure on M can be realized by deforming the embedding $\Gamma \to PSL(2, \mathbf{C})$. Thus the situation is much the same as that for Riemann surfaces, which also admit families of hyperbolic structure (or equivalently complex structures).

As a metric space, M endowed with such a hyperbolic structure is highly non-compact, and the boundary surfaces lie at infinity, i.e. they are the celestial surfaces in M. Following the physical idea of a Kaluza-Klein theory we shall study the fibre bundle $M \times S^1$ over M instead of M itself. Another popular notion in physics is that of a *conformal compactification*: $M \times S^1$ has a natural conformal compactification X without boundary (or X_Γ if we want to indicate the dependence on Γ), i.e. there is an injective conformal immersion $M \times S^1 \to X$ onto a dense subset. To get X we spin $\bar{M}$ around $\delta\bar{M}$, see figure 1, i.e. X is $\bar{M} \times S^1$ with the circles over δM identified to a point. This gives a compact 4-manifold X with an S^1-action. The action is free away from the fixed point set, which is isomorphic to the boundary $\delta\bar{M} = \cup_{j=1,N} S_j$. The normal bundles of the S_j are trivial and of S^1-weight 1. For example take $M \cong S \times \mathbf{R}$ with S a surface. Then X is the compactification of $S \times \mathbf{R} \times S^1 \cong S \times \mathbf{C}^*$, that is $X \cong S \times S^2$, where S^1 acts on S^2 by earth rotation and has two fixed surfaces $S \times \{0, \infty\}$ in X.

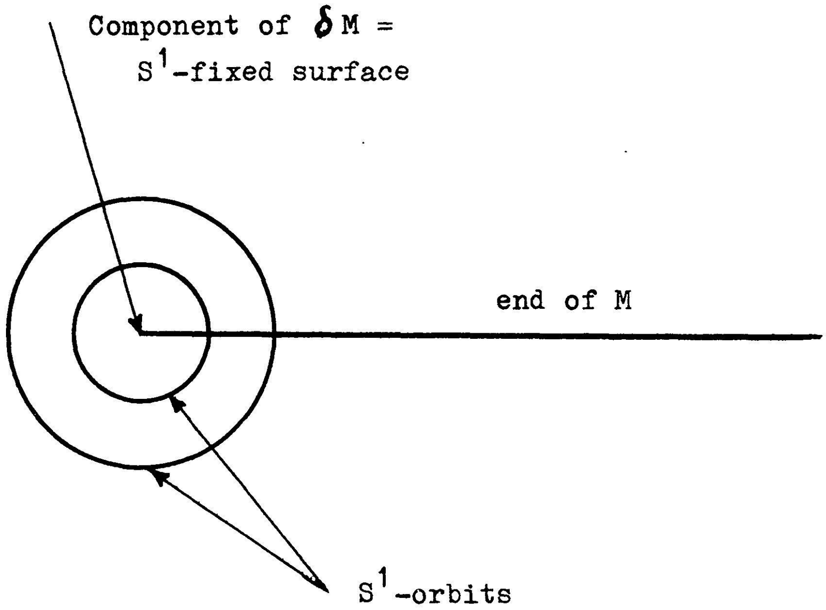

FIGURE 1.

In order to relate the hyperbolic structure on M to a conformal structure on X we proceed more formally. Recall that $H^3 = \{(x, y, t) \in \mathbf{R}^3; t > 0\}$ with metric $ds^2 = (dx^2+dy^2+dt^2)/t^2$. It follows that:

2.2 $$i: H^3 \times S^1 \to (\mathbf{R}^2 \oplus \mathbf{R}^2) - (\mathbf{R}^2 \oplus 0) \cong \mathbf{R}^4 - \mathbf{R}^2 \cong S^4 - S^2: \quad ((x, y, t), \vartheta) \to (x, y, t\cos\vartheta, t\sin\vartheta)$$

is an orientation preserving, conformal diffeomorphism. The map i intertwines the S^1-action on $H^3 \times S^1$ with rotations in the second summand of $\mathbf{R}^2 \oplus \mathbf{R}^2$. The S^1-action extends to S^4 with fixed point set $S^2 = (\mathbf{R}^2 \oplus 0) \cup \{\infty\} \subset S^4$. This fixed point set corresponds to $\delta H^3 \times S^1$ under:

2.3 $$i' : \bar{H}^3 \times S^1 \to S^4 ,$$

the continuous extension of i. To get further we shall show that the compactification S^4 of $H^3 \times S^1$ is natural enough to transfer group actions from H^3 to S^4. The maps i and i' are equivariant with respect to the group $S^1 \times PSL(2, \mathbf{C})$, which will act on the right on S^4 by *conformal transformations*. To see this, recall that the $PSL(2, \mathbf{C})$-action on S^4, which is the quaternionic projective line $\mathbf{H}P^1 = \mathbf{H}^* \backslash (\mathbf{H}^2 - \{0\})$ (i.e. divide out the left action of multiplication by invertible quaternions), is by fractional linear transformations:

2.4 $$\left([x, y] , \begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) \to [xa + yb, xc + yd]$$

As a result a geometrically finite Kleinian group Γ acts on S^4. The limit set Λ' of the Γ-action on S^4 equals $i'(\Lambda \times S^1)$, so it is contained in the S^1-fixed point set $S^2 \subset S^4$. Clearly Λ' is isomorphic to Λ, and we shall simply identity Λ and Λ'. The restriction:

$$i' : (\bar{H}^3 - \Lambda) \times S^1 \to S^4 - \Lambda$$

is proper, equivariant and surjective. This implies immediately that the Γ-action on $S^4 - \Lambda$ is proper. Since Γ is geometrically finite the quotient $X = (S^4 - \Lambda)/\Gamma$ is compact and without boundary. Finally, the fact that the Γ-action is free ensures that X is smooth and inherits a smooth S^1-action.

The S^1-action is free away from the fixed surfaces S_j, which correspond as conformal surfaces to $\Omega/\Gamma = (\delta H^3 - \Lambda)/\Gamma \cong i'((\delta H^3 - \Lambda) \times S^1)/\Gamma$. It is useful to realise that i and i' induce maps $i : M \times S^1 \to X$ and $i' : \bar{M} \times S^1 \to X$. Summarizing we have proved:

THEOREM 2.1. *Let $\bar{M}$ be an oriented, geometrically finite, complete hyperbolic 3-manifold with non-empty boundary $\delta\bar{M} = \cup S_j$ satisfying* 2.1. *Then $M \times S^1$ has an oriented, smooth conformal compactification X (without boundary) upon which S^1 acts. X is conformally flat and the S^1-action is free away from its fixed surfaces $S_j (j = 1, ..., N)$ which correspond as conformal surfaces to the boundary surfaces of $\bar{M}$. The normal bundles N_j of S_j in X are topologically trivial and of S^1-weight* 1. *The hyperbolic structure on M can be reconstructed from X by giving $X - (\cup S_j)$ that metric in the conformal class for which the S^1-orbits have length* 2π. *Then M is the Riemannian quotient of $X - (\cup_j S_j)$ by S^1.* □

Remark. It is worth pointing out that if one chooses an equatorial embedding of S^n in S^{n+1} then any conformal transformation of S^n extends uniquely to a conformal transformation of S^{n+1} leaving invariant the components $S^{n+1} - S^n$. Thus if Γ is a group acting on S^n then it also acts on S^{n+1}. A Kleinian group can be thought of as a group acting on S^3 with limit set in an equatorial S^2. Theorem 2.1 now says that if Γ is geometrically finite and purely loxodromic then in S^4 we have $\Lambda(\Gamma) \subset S^2 \subset S^4$ and $\Omega(\Gamma)/\Gamma$ is a compact 4-manifold.

The existence of a conformal compactification is not automatic. It is easy to see that $\mathbf{R}^3 \times S^1$ cannot be compactified by adding an S^2 at infinity.

The topology of X is easily described:

PROPOSITION 2.2.

a) $\pi_1(X, m) \cong \pi_1(\bar{M}, m)$ *for* $m \in S_1$ *(*S_1 *a fixed or boundary surface).*

b) There are natural isomorphisms $H_2(\bar{M}, \delta\bar{M}; \mathbf{Z}) \to H_3(X; \mathbf{Z})$ *and* $H_2(\bar{M}; \mathbf{Z}) \oplus H_1(\bar{M}, \delta\bar{M}; \mathbf{Z}) \to H_2(X; \mathbf{Z})$

The two summands of $H_2(X; \mathbf{Z})$ *(modulo torsion) are isotropic and dual to each other under the intersection form* Q *on* $H_2(X; \mathbf{Z})$*; consequently the signature* $\sigma(X) = 0$, *and* $Q = n$ *times* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *with* $n = \operatorname{rk} H_2(\bar{M}; \mathbf{Z})$.

c) $\chi(X) = \Sigma_j \chi(S_j)$ *with* χ *denoting the Euler characteristic.*

d) Spin structures on X *exist and the double cover of* S^1 *acts naturally and effectively on any spin structure.*

Proof. a) Of course this is what one expects to be true: $\pi_1(\bar{M} \times S^1, m) \cong \pi_1(\bar{M}, m) \times \mathbf{Z}$, but the $\mathbf{Z}$ factor is killed by shrinking the circles to a point. Formally, remark that a tubular neighbourhood of $\cup S_j$ looks like $(\cup S_j) \times D^2$, and apply the Seifert-van Kampen theorem.

b) Define $j: \bar{M} \to X$ by $j(m) = i'(m, 1)$. Up to S^1-rotation j is defined uniquely by the conformal structure of X. This induces a homomorphism $j_*: H_2(\bar{M}; \mathbf{Z}) \to H_2(X; \mathbf{Z})$. Next remark that if c is a chain in $C_j(\bar{M}, \delta\bar{M}; \mathbf{Z})$ then $i'_*(c \times S^1)$ is a chain in $C_{j+1}(X; \mathbf{Z})$, because the circle shrinking to a point enforces $\delta i'_*(c \times S^1) = 0$. Taking a careful look at the Mayer-Vietoris sequence applied to $(\cup S_j) \times D^2$ and $M \times S^1$ shows that this gives natural isomorphisms as indicated in the proposition. The properties of the intersection form Q follow from the intersection pairing: $H_2(\bar{M}; \mathbf{Z}) \times H_1(\bar{M}, \delta\bar{M}; \mathbf{Z}) \to \mathbf{Z}$.

c) This is easy, using either a and b, or equivariant Lefschetz formulas.

d) Every orientable 3-manifold admits a spin structure, see Stiefel [35]. Give S^1 the spin structure corresponding to the connected double cover, which extends to the disc in $\mathbf{R}^2$; therefore a product spin structure on $M \times S^1$ extends to X. Clearly every spin structure on X arises in this way. The double cover of S^1 is needed to define an action on the spin structure of the orbits in X. □

The spin bundle of H^3 is the Spin(3) $\cong SU(2)$ bundle $SL(2, \mathbf{C}) \to H^3 = SU(2)\backslash SL(2, \mathbf{C})$; thus a spin structure on X is in fact nothing else but a lift of the homomorphism $r: \Gamma \to PSL(2, \mathbf{C})$ which defines M, to a homomorphism $r': \Gamma \to SL(2, \mathbf{C})$.

If N denotes the number of boundary components of $\bar{M}$ (as before) then it follows from the exact sequence of the pair $(M, \delta M)$ that:

2.5 $$\operatorname{rk} H_2(X; \mathbf{Z}) = 2 \cdot \operatorname{rk} H_1(\bar{M}, \delta\bar{M}; \mathbf{Z}) \geqslant 2 \cdot (N-1)$$

Another useful fact to keep in in mind is:

2.6 $$\operatorname{rk}\{\ker(H_1(\delta\bar{M}; \mathbf{Z}) \to H_1(\bar{M}; \mathbf{Z}))\} = \tfrac{1}{2} \cdot \operatorname{rk} H_1(\delta\bar{M}; \mathbf{Z}),$$

which can easily be deduced from Alexander duality and the exact sequence of the pair $(\bar{M}, \delta\bar{M})$.

Examples 2.3. 1) If Γ is the cyclic group generated by $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $\lambda \in \mathbf{C}^*$ then the limit set equals $\{0, \infty\}$ in the coordinates on δH^3 supplied by the upper half space model. It is easy to see that $M = H^3/\Gamma = D^2 \times S^1$. To find X, it is easiest to divide out the Γ-action on $S^4 - \Lambda = \mathbf{C}^2 - \{0\}$ which is given by $(z_0, z_1) \to (\lambda^2 z_0, |\lambda|^2 z_1)$. As a result X is a *Hopf surface* diffeomorphic to $S^3 \times S^1$. The S^1-action is given by $(z_0, z_1) \to (z_0, e^{i\vartheta} z_1)$, so the fixed surface is the two-torus $\mathbf{C}^*/<\lambda^{2k}>$.

2) If Γ is *Fuchsian*, i.e. $\Gamma \subset PSL(2, \mathbf{R})$, then H^2/Γ is a compact Riemann surface without boundary S of genus $\geqslant 2$ with metric ds^2. The 3-manifold M is diffeomorphic to $\mathbf{R} \times S$ with metric given by $dl^2 + \cosh^2 l \cdot ds^2$. Clearly it follows that X must be diffeomorphic to $S^2 \times S$. A little computation shows that X is even conformally equivalent to $S^2 \times S$. Thus X is conformally equivalent to the Kähler surface $\mathbf{CP}^1 \times S$.

From the point of view of Kleinian groups, we remark that Ω is the union of two round discs which are both invariant under Γ. The limit set is a smooth circle.

3) A Kleinian group Γ which is not itself Fuchsian, but which contains a Fuchsian subgroup Γ_0 of index two is said to be an *extended Fuchsian*

group. For details see Maskit [28]. The limit sets $\Lambda(\Gamma_0)$ and $\Lambda(\Gamma)$ are equal, and any $\gamma \in \Gamma - \Gamma_0$ swaps the two components of Ω. Such an element γ also gives rise to a fixed-point-free, orientation reversing involution σ of S (compare 2), and one deduces from this that $M \cong H^3/\Gamma$ is a nontrivial $\mathbf{R}$-bundle over S/σ. Remark that $\delta\bar{M} \cong S$.

A standard way to get more interesting 3-manifolds is through the *Klein-Maskit combination theorems* (Maskit [27], Morgan [29]). We shall explain how the simplest of these relates to the 4-manifolds involved. Let Γ_0 and Γ_1 be geometrically finite groups without cusps and $M_j = H^3/\Gamma_j$. Every pair of points $x_j \in \delta\bar{M}_j$ has neighbourhoods K_j in M_j isometric to a hyperbolic half space i.e. to a component of $H^3 - H^2$. The $H_j^2 = \delta K_j - \delta\bar{M}_j$ are embedded in M_j and $\delta H_j^2 \cap \bar{M}_j = H_j^2 \cap \delta\bar{M}_j$ are circles which bound discs in $\delta\bar{M}_j$. Define $M = M_0 \# M_1$ to be $M_0 \backslash K_0 \cup_\rho M_1 \backslash K_1$, where ρ is an isometry $\delta K_0 \to \delta K_1$. The metric structure of $M = M_0 \# M_1$ depends on ρ, the choice of x_j and the choice of the half spaces K_j. M is called a *boundary connected sum* of M_0 and M_1.

The *first combination theorem* expresses the fact that $M = H^3/\Gamma$ with Γ a Kleinian group which is isomorphic to the free product of Γ_0 and Γ_1. In $PSL(2, \mathbf{C})$ the group Γ is generated by Γ_0 and $g\Gamma_1 g^{-1}$ for a suitable $g \in PSL(2, \mathbf{C})$. It is easy to verify this.

Reverting to the 4-manifolds, we see that we are identifying, by S^1-equivariant conformal maps, balls B_j around the points x_j in the fixed surfaces. Thus X_Γ equals $X_{\Gamma_0} \# X_{\Gamma_1}$ with $\#$ now denoting a conformal connected sum. Summarizing we get:

PROPOSITION 2.4. *If Γ is the Kleinian group corresponding to a boundary connected sum of H^3/Γ_0 and H^3/Γ_1 then Γ is a Kleinian group such that X_Γ is the S^1-equivariant conformal connected sum of X_{Γ_0} and X_{Γ_1} at points in the fixed surfaces.*

Example 2.5. A *classical Schottky group* Γ of genus g is a free product of g cyclic groups (compare example 2.3 (1)), formed as in the gluing construction described before proposition 2.4. The 3-manifold M_Γ is a handlebody of genus g, and by proposition 2.4, X_Γ equals the connected sum $(S^3 \times S^1)^{\# g}$. In fact if Γ is any geometrically finite free Kleinian group without cusps, then H^3/Γ is a handlebody; this follows from standard results in 3-manifold topology (see Hempel [17]). We shall refer to such free groups as Schottky groups.

§ 3. Classification of Γ with $\dim_H \Lambda(\Gamma) \leqslant 1$

In the previous section we constructed a compact, oriented, conformally flat 4-manifold X starting from a suitable (see § 2) hyperbolic 3-manifold. By Schoen's solution of the Yamabe problem [33] there is a metric in the conformal class of X, for which the scalar curvature is a constant. The sign of this constant $-$, 0 or $+$ is called the *type* of X. A lot is known about X of non-negative type, and we shall classify 3-manifolds M which give rise to X of non-negative type.

In a different direction Schoen and Yau [34] proved that if X is the quotient of $S^4 - \Lambda$ by a discrete group of conformal transformations, then X is of type $+, 0, -$ if and only if the *Hausdorff dimension* of Λ satisfies that $\dim_H \Lambda - 1$ is negative, zero, positive respectively. Hence our classification is that of M for which $\dim_H \Lambda \leqslant 1$. The classification for $\dim_H \Lambda = 1$ seems to be new, for < 1 the result was known.

Up to now, the only Kleinian groups to have been classified are the so called *function groups*, those Kleinian groups which leave a component of $\Omega = S^2 - \Lambda$ invariant. This has been done by Maskit. A special case of this, which we shall use repeatedly below, occurs when Ω is connected. In this case the Kleinian group is Schottky (see example 2.5).

Theorem 3.1.

a) If the type of X is $+$ then M is a handlebody equal to H^3/Γ with Γ a Schottky group.

b) If the type of X is 0 then one of the following holds:

1) M equals $\mathbf{R} \times S = H^3/\Gamma$ with Γ Fuchsian and S a compact surface.

2) M equals H^3/Γ with Γ extended Fuchsian (see example 2.3 (3)).

3) M is a handlebody as in a).

Proof. a) $R > 0$ implies $\dim_H \Lambda < 1$, see the proof of proposition 3.3 of Schoen and Yau [34]. This implies that $\Omega(\Gamma)$ is connected, because a set of Hausdorff dimension smaller than 1 cannot disconnect S^2. By Maskit's classification theorems (see Maskit [27]) it follows that Γ is Schottky.

b) First assume $H^2(X, \mathbf{R}) \neq 0$ and give X a metric of zero scalar curvature in the conformal class. From proposition 2.2 we see that the intersection form is indefinite, so there is a self-dual harmonic 2-form ω

on X. A Weitzenbock formula asserts that on 2-forms $(d+d^*)^2 = \nabla^*\nabla$ with ∇ the total covariant derivative. It follows that ω is covariantly constant, and a multiple of ω serves as Kähler form for an integrable complex structure on X: compare LeBrun [24]. LeBrun proceeds to classify these as (1) a K3 surface, (2) a four dimensional torus modulo a finite group and (3) a flat $\mathbf{CP}^1$ bundle with the local product metric, over a Riemann surface S which carries a metric of curvature -1. From proposition 2.2 we see that only (3) is possible in our case because (1) has the wrong Euler characteristic, and (2) with Euler characteristic 0 should have had $H_2(X;\mathbf{R}) = 0$.

The Kähler form of X is the unique self-dual harmonic 2-form on X. This is preserved by the conformal S^1-action, thus the action is a holomorphic action on X. As a result the vector field v induced by the S^1-action on X is holomorphic. The fibration $\pi: X \to \mathbf{CP}^1$ helps us further: we get a map $\pi_*: TX \to \pi^*TS$ and $\pi_* v$ is a section of π^*TS. Such a section is constant on fibres, so it is a pull-back of a section of TS. The only holomorphic section of TS is 0; so v is a vertical vector field.

From theorem 2.1 we see that zeroes of v must be simple, hence two per fibre. One of these is a sink, the other a source of $i \cdot v$ so we get two sections $S \to X$. This proves that X is the projectivization of a direct sum of holomorphic line bundles, say $X = P(L_0 \oplus L_1)$. The next step is to remember that the circle bundle $X - [P(L_0) \cup P(L_1)]$ over H^3/Γ may have no monodromy. Infinitesimally this implies that $L_0 \otimes L_1^*$ is a trivial line bundle. So $X = S \times \mathbf{CP}^1$ and consequently Γ must be Fuchsian.

Next we come to the case $H^2(X, \mathbf{R}) = 0$. If $S^2 - \Lambda$ has only one component then we can apply Maskits classification theorem as in a), and conclude that Γ is Schottky; therefore we shall concentrate on the case that Ω has at least two components.

If Ω_0 is one of these components then the stabilizer $\Gamma_0 \subset \Gamma$ of Ω_0 is a geometrically finite Kleinian group, and has Ω_0 as a component, see Marden [26] corollary 6.5 (it should be remarked that subgroups are not automatically geometrically finite). As $S^2 - \Omega_0$ is Γ-invariant and has non-empty interior, it follows that H^3/Γ_0 must have at least two ends. By formula 2.5 and the fact that $\dim_H \Lambda(\Gamma_0) \leqslant \dim_H \Lambda(\Gamma) \leqslant 1$, the above implies that Γ_0 is Fuchsian. Thus every component of Ω is a round disc.

Before we proceed let us briefly recall what effect a conformal rescaling of the metric has on the scalar curvature. If on the 4-manifold X one has $g_1 = u^2 \cdot g_0$ then $\frac{1}{6} \cdot u^3 \cdot R(g_1) = \left(d^*du + \frac{1}{6} R(g_0)u\right)$, where d^* is taken with respect to g_0. Since here metrics of zero scalar curvature are involved, this equation loses its nonlinear character. An immediate consequence is that metrics of zero scalar curvature are unique up to constants multiples and hence S^1-invariant.

We have the hyperbolic covering $H^3/\Gamma_0 \to H^3/\Gamma$, and on the 4-manifolds corresponding to each of these there exists an S^1-invariant metric of zero scalar curvature. Denote these by g_0 and g, and denote the hyperbolic metric on the 3-manifolds by g_h. Then we have positive functions $u_0\colon H^3/\Gamma_0 \to \mathbf{R}_{>0}$ and $u\colon H^3/\Gamma \to \mathbf{R}_{>0}$ such that $g_0 = u_0^2 \cdot g_h$ and $g = u^2 \cdot g_h$. By the above u_0 and u are in the kernel of (d^*d-1) on H^3/Γ_0 and H^3/Γ respectively (here d^* is w.r.t. the hyperbolic metric).

Results of Sullivan [36] imply that positive solutions of $d^*d - 1$ on H^3/Γ_0 are unique (up to positive scalar factors) as $\dim_H \Lambda(\Gamma_0) = 1$. Therefore the pullback of u equals u_0, and hence the cover $X_{\Gamma_0} - (S_1 \cup S_2) \to X_\Gamma - S_1$ is an isometry (S_i are the fixed surfaces). The map can readily be extended to an isometry $X_{\Gamma_0} - S_2 \to X_\Gamma$ and then extends to a double cover $X_{\Gamma_0} \to X_\Gamma$. It follows that Γ is extended Fuchsian as claimed. □

Reformulating in terms of Kleinian groups gives:

COROLLARY 3.2. *Let* Γ *be a geometrically finite Kleinian group without cusps. If* $\dim_H \Lambda(\Gamma) < 1$, *then* Γ *is Schottky. If* $\dim_H \Lambda(\Gamma) = 1$ *then* Γ *is Schottky, Fuchsian or extended Fuchsian.*

Proof. We shall see in section 7 that $\dim_H \Lambda(\Gamma) < 1$ implies that the type of X is $+$, which is essentially an old observation due to Poincaré. Together with the results of Schoen and Yau mentioned in the proof above, the corollary is now obvious. □

Remark. 1) Existence of Schottky groups with limit set of any dimension smaller than 2 has been proved (Thurston [37]).

2) In Schoen & Yau [38] and Gromov & Lawson [15] the conclusion is drawn that for so-called classical Schottky groups Γ the manifold X_Γ admits a metric of positive constant scalar curvature.

4) R. Bowen [9] has proved that any quasifuchsian group with $\dim_H \Lambda = 1$ is Fuchsian. Of course this is a special case of theorem 3.1.

It will be interesting to see if further developments in the theory of compact, 4-dimensional, conformally flat manifolds are going to have similar applications to Kleinian groups. On the other hand it seems likely that a purely 3-dimensional proof of theorem 3.1 could be found as well. The crucial element seems to be to exploit the existence of a harmonic two form, in the way Lebrun did. LeBrun arrives at his flat $\mathbf{CP}^1$ bundle through a foliation argument which presumably can be mimicked in the 3-manifold.

§ 4. Hodge theory for hyperbolic 3-manifolds

Apart from the topological and geometrical applications which we discussed in § 3, our Kaluza-Klein approach also has some more analytical applications.

Recall that the Hodge-star $*: \Omega^n(Y) \to \Omega^n(Y)$, on a $2n$-dimensional oriented Riemannian manifold Y, depends only on the conformal structure underlying the metric. This has two consequences:

1) The L^2-norm $\| \omega \|^2 = \int \omega \wedge *\omega$, of $\omega \in \Omega^n(Y)$, is conformally invariant.

2) The harmonic n-forms, i.e. the $\omega \in \Omega^n(Y)$ s.t. $d\omega = d^*\omega = 0$, depend only on the conformal structure of Y.

Of course conformal rescaling lies at the heart of our construction in § 2, and we shall now show how the above applies to this situation. Let X be the conformal compactification of $M \times S^1$ as in § 2. Harmonic 2-forms on X are automatically S^1-invariant because they are in one-one correspondence with the elements of $H^2(X;\mathbf{R})$ $(=H^2(M;\mathbf{R}) \oplus H^1(M, \delta M;\mathbf{R})$, see § 2). By restriction to the open subset $M \times S^1 \subset X$ and a conformal rescaling of the metric on $M \times S^1$, 2) above implies that we get S^1-invariant harmonic 2-forms on $M \times S^1$ with respect to the product metric.

An S^1-invariant form can be written as $\omega = \rho^*\alpha + \rho^*\beta \wedge d\theta$, with $\alpha \in \Omega^2(M)$, $\beta \in \Omega^1(M)$ and $\rho: M \times S^1 \to M$ the projection. A short computation shows that such S^1-invariant forms ω are harmonic iff α and β are harmonic on M. If ω is a harmonic 2-form on $M \times S^1$ arising from a form on X then it follows from proposition 2.2 that $\alpha \in \Omega^2(M)$ and $\beta \in \Omega^1(M)$ are harmonic representatives for the class $\omega \in H^2(M;\mathbf{R}) \oplus H^1(M, \delta M;\mathbf{R})$. The forms α and β have finite L^2-norm on M by 1) above.

Conversely any S^1-invariant, harmonic 2-form $\tilde{\omega}$ on $M \times S^1$ with finite L^2-norm arises in this way. By 1) above one can always consider $\tilde{\omega}$ to be an L^2-form ω on X because $\cup S_j = X \backslash M \times S^1$ has measure 0. Applying the first order elliptic operator $d \oplus d^*$ to ω gives a distributional form in $L^2_{-1}(\Lambda^*(X))$ of distributional order $\leqslant 1$, which has support in the codimension 2 manifold $\cup_j S_j \subset X$. The following lemma shows that this implies that $(d \oplus d^*)\omega = 0$, which proves that ω is a smooth harmonic form on X, as we claimed.

LEMMA 4.1. *Let* μ *be a distribution of order* $\leqslant 1$ *in* $L^2_{-1}(\mathbf{R}^n)$. *If* $\operatorname{supp}\mu$ *is contained in* $\mathbf{R}^{n-2}$ *then* $\mu = 0$.

Proof. Without loss of generality assume that μ is compactly supported. The structure theorem for distributions carried by submanifolds (see Hörmander [21] theorem 2.3.5) asserts that μ is a finite linear combination of distributions ν of the form $<\nu, f> = <\eta, \delta_{\mathbf{R}^{n-2}} \cdot D^k_{\mathbf{n}} \cdot f>$, where η is a compactly supported distribution on $\mathbf{R}^{n-2}$, $\delta_{\mathbf{R}^{n-2}}$ is restriction to $\mathbf{R}^{n-2}$ and $D^k_{\mathbf{n}}$ is a k-th derivative $(0 \leqslant k \leqslant 1)$ in a direction $\mathbf{n}$ normal to $\mathbf{R}^{n-2}$.

The Fourier transform $\hat{\mu}(u, x, y)$ is a smooth function on $\mathbf{R}^{n-2} \oplus \mathbf{R} \oplus \mathbf{R}$ of the form $f_0(u) + f_1(u) \cdot x + f_2(u) \cdot y$. It is easy to see from this that the L^2_{-1}-norm cannot be finite, unless $\mu = 0$. □

Denote by $\mathscr{H}^i(M)$ the vectorspace of harmonic (i.e. closed and coclosed) i-forms on M with finite L^2-norm. Summarizing the above we have proved:

THEOREM 4.2. *The natural maps* $\mathscr{H}^1(M) \to H^1(M, \delta M; \mathbf{R})$ *and* $\mathscr{H}^2(M) \to H^2(M; \mathbf{R})$ *are isomorphisms.* □

On the universal cover, *Poisson transformation* gives a one-one correspondence between closed and co-closed 1-forms on H^3 and exact one forms with hyperfunction coefficients on δH^3, and this is what we shall exploit next. If the hyperfunction one form is continuous then it is the boundary value of the one form on H^3 in the classical sense, this is special for hyperbolic space. Thus in this case Poisson transformation is solving a Dirichlet boundary value problem on $(H^3, \delta H^3)$. The Poisson transform $\mathscr{P}(\phi)$ of a continuous function ϕ on δH^3 is defined as (see e.g. Gaillard [13]):

$$\mathscr{P}(\phi)(h) = \int_{S^2} P(h, b) \cdot \phi(b) \quad \text{with} \quad P(h, b) = \pi^{-1}(h_3/|h-b|^2)^2 db_1 \Lambda db_2,$$

where $h = (h_1, h_2, h_3) \in \mathbf{R}^3_+ \cong H^3$, $h_3 > 0$ and $b = (b_1, b_2, 0) \in \mathbf{R}^2 \subset \delta H^3$. For exact one-forms $\alpha = d\phi$ we define $\mathscr{P}(\alpha) = d\mathscr{P}(\phi)$. As $\mathscr{P}(\phi)$ is harmonic, $\mathscr{P}(\alpha)$ is closed and co-closed. Using this, we can identify our L^2 cohomology as follows:

THEOREM 4.3. *Poisson transformation induces an isomorphism from* Γ-*invariant closed one-forms with hyperfunction coefficients on* δH^3 *with support* *in the limit set to closed and co-closed one forms on* H^3/Γ *with finite* L^2 *norm. Such hyperfunction one-forms are one-currents.*

Proof. An L^2 harmonic 1-form on M lifts to an invariant 1-form ω on H^3. From Gaillard [13] we know that ω is the Poisson transform of a unique closed 1-form α on $S^2 = \delta H^3$ with hyperfunction coefficients. From theorem 4.2 it follows that ω is bounded on a fundamental domain, so it is of slow growth and therefore α is a current. Now write $\alpha = d\phi, \omega = d\psi$ for a distribution ϕ and a function ψ. It follows that ψ is the Poisson transform of ϕ (after adding a constant). From theorem 4.2 it follows that the one form ω extends smoothly to a one form on $(H^3 \cup \delta H^3) - \Lambda$, zero on the boundary $\delta H^3 - \Lambda$. This implies that ψ is smooth on $(H^3 \cup \delta H^3) - \Lambda$. In Schlichtkrull [32], chapter 4, it is proved that under these conditions ψ converges uniformly to ϕ. But then ϕ must be constant on components of $\delta H^3 - \Lambda$ and therefore the support of α is contained in Λ.

Conversely let α be a closed 1-form with hyperfunction coefficients in S^2 with support in Λ, and let ω be its Poisson transform. We shall prove that ω, which is automatically closed and co-closed, has finite L^2 norm. As above let $\omega = d\psi$ and $\alpha = d\phi$, then ϕ is constant on components of $\delta H^3 - \Lambda$. Apart from the boundary value ϕ there is another "boundary value" ϕ', just as in the classical case there is the von Neumann boundary value problem next to the Dirichlet boundary value problem. In further analogy with the classical case the global boundary value ϕ' can be obtained from ϕ by applying a pseudo-differential operator on S^2 to it, which has a real analytic integral kernel, see Schiffmann [31]. So, ϕ and ϕ' are real analytic in $\delta H^3 - \Lambda$.

Oshima [30] theorem 5.3 shows then that locally in $\delta H^3 - \Lambda$ we have:

$$\psi(h_1, h_2, h_3) = c_1(h_1, h_2, h_3) + c_2(h_1, h_2, h_3) \cdot h_3^2 \cdot q(\log h_3),$$

with (h_1, h_2, h_3) upper half space coordinates, q a polynomial in one variable and $c_1(h_1, h_2, 0) = \phi(h_1, h_2)$, $c_2(h_1, h_2, 0) = \phi'(h_1, h_2)$. From this it follows that ω has an expansion locally bounded by $cst \cdot h_3 \cdot q(\log h_3)$.

Recall that a fundamental domain for the Γ-action on H^3 intersects δH^3 in a compact fundamental domain for the Γ-action in $\delta H^3 - \Lambda$. This together with our estimate implies readily that the L^2 norm of ω restricted to a fundamental domain is finite. □

A few remarks are in order. First of all it should be possible to give an effective bound on the distributional order of the currents α on S^2, and also if $\alpha = d\phi$ it should be possible to determine if the function ϕ (constant on components of $\delta H^3 - \Lambda$) is locally integrable. Also it should be noted that $\omega \Lambda d\theta$ is a solution on X of a p.d.e with real analytic coefficients, i.e. it is real analytic. This shows inmedeately that ω has an expansion as in the proof of theorem 4.3, without logaritmic terms.

Next we can use the above to define a simple invariant of the hyperbolic structure on M. The Hodge star of the hyperbolic 3-manifold M gives an isomorphism $*_3 : \mathscr{H}^1(M) \to \mathscr{H}^2(M)$. Both $\mathscr{H}^1(M)$ and $\mathscr{H}^2(M)$ contain an integral lattice of maximal rank coming from integral cohomology. These lattices do not generally coincide under $*_3$; in fact their intersection is empty unless the 4-manifold carries a self-dual harmonic form which represents an integral cohomology class. The relative position of the two lattices in $H^2(M; \mathbf{R})$ is described by:

4.1 $$h(M) \in GL(H^2(M; \mathbf{R}))/GL(H^2(M; \mathbf{Z}) \otimes \mathbf{Z}),$$

which is an invariant of the hyperbolic structure of M. Similar invariants are very popular in algebraic geometry. There discrete lattices in a complex vector space give rise to invariants associated to the complex structure of manifolds.

We proceed to sketch how the above theory relating solutions of elliptic p.d.e. on M to invariant solutions on X generalizes. Suppose $D : \Gamma(E) \to \Gamma(F)$ is a conformally invariant first order (possibly over-determined) elliptic operator acting on sections of the vector bundle E over X. This class of operators was studied in detail by Hitchin [18], and comprises, among others, Dirac and twistor operators on X and the operator $d + d^*$ on 2-forms which we studied above. Again restriction of S^1-invariant solutions on X to $M \times S^1$ gives solutions to a closely related geometric p.d.e. on M.

Conversely we can start with a solution on M and require that it has a finite L^2-norm on $X \backslash (\cup S_j)$. In general this is not the same as having a finite L^2-norm on M, but it is the same as having a finite weighted L^2-norm on M. The weighting function is a suitable power of the function on M which conformally rescales the hyperbolic metric on M to a metric on X. Such a function is determined up to multiplication by functions $\phi : M \to \mathbf{R}_{>0}$ which are bounded above and below. The exact value of the power needed is an inhomogeneous linear function of the conformal weight of E. The extension over the fixed surfaces S_j goes now as in lemma 4.1. We shall not make use of this in the sequel and therefore leave the details to the reader.

Remarks. 1) It would be interesting to see what kind of harmonic representatives for classes in $H^1(M; \mathbf{R})$ can be found.

2) Theorem 4.2 generalizes to identify elements of $H^j(M, \delta M; \mathbf{R})$ with L^2 harmonic forms for any oriented n-dimensional Riemannian manifold M for which a conformal compactification of $M \times S^k$ exists, for all k, provided $j < n/2$.

§ 5. Monopoles and Instantons

Our goal is now to exploit the compactification X of $M \times S^1$ (see § 2) to get monopoles on M from S^1-invariant instantons on X. We shall also relate the instanton number on X to various topological invariants of the monopoles on M. General background for this section can be found in Freed-Uhlenbeck [12] and Jaffe-Taubes [22]. More specifically our approach here is very similar to the one taken in Atiyah [2].

Let P be a principal $SU(2)$-bundle over X, with $c_2(P) = k \geqslant 0$. Recall that X comes naturally with a conformal structure. This enables us to talk about *instantons or anti-self-dual connections* A on P. These are defined to be the solutions of the *anti-self-duality equation*:

5.1 $$F^A = - *_4 F^A \quad (*_4 \text{ the Hodge star on } \Lambda^2(X)) .$$

Here F^A is the curvature of A, a section of $\Lambda^2(X) \otimes g_P$ with $g_P = P \times_{Ad} su(2)$. The instantons are the absolute minima of the *Yang-Mills functional*:

5.2 $$YM(A) = (16\pi^2)^{-1} \int_X <F^A \wedge *F^A>$$

where $<\alpha, \beta> = - 2 \cdot tr(\alpha\beta)$ is an invariant inner product on $su(2)$. For an instanton $YM(A) = k$.

Next assume that the double cover $\tilde{S}^1$ of S^1 acts on P by bundle automorphisms, covering the action on X; the double cover will be needed in order to include the spin bundles of X. Our interest will now lie in $\tilde{S}$-invariant instantons on P. To relate these to objects on M introduce the map:

$$j: M \to X : m \to i'(m, 1) \quad (\text{compare } 2.2),$$

which is a diffeomorphism onto its image. Let v be the vectorfield on P induced by the $\tilde{S}^1$-action. If we interprete an $\tilde{S}^1$-invariant connection A as a 1-form on P, then define the Higgs-field Φ to be the $su(2)$-valued function $j^*A(\frac{1}{2}v)$ on j^*P. It is easy to see that Φ is a section of j^*g_P.

Further $A_3 = j^*A$ defines a connection on the bundle j^*P over M. A little computation shows that the $\tilde{S}^1$-invariant connection A is anti-self-dual iff (A_3, Φ) satisfy the so called *Bogomol'nyi equation* on M:

5.3 $$d^{A_3}\Phi = - *_3 F^{A_3}.$$

As 5.3 is the standard equation describing *magnetic monopoles* on three dimensional manifolds, this leads to the definition.

Definition 5.1. A monopole on P is an $\tilde{S}^1$-invariant instanton on P.

Normally one defines a monopole by imposing certain asymptotic conditions rather than requiring it to extend over a compact manifold. In Braam [10] it is explained that results of the Sibners imply that this amounts to the same. We shall see below that the boundary data are the same.

If $GA(P)$ denotes the group of $\tilde{S}$-invariant gauge transformations on P, then $GA(P)$ leaves the set of monopoles invariant. Just as for instantons one can therefore define a *monopole moduli space*, equal to:

5.4 $$\{\text{solutions of } 5.3\}/GA(P)$$

In Braam [10] is shown that under some assumptions these moduli spaces are non-empty finite dimensional manifolds.

We shall now return to our $\tilde{S}^1$-equivariant bundle P and relate topological invariants of the action to asymptotic invariants of (A_3, Φ) on M. Restricted to one of the fixed surfaces S_j, $\tilde{S}^1$ acts by gauge transformations on P. The fibres of $E = P \times_{SU(2)} \mathbf{C}^2$ over S_j decompose into eigenspaces for the $\tilde{S}^1$ action. Denote by $m_j \in \mathbf{Z}_{\geqslant 0}$ the $\tilde{S}^1$-weight which is non-negative.

If $m_j > 0$ then:

5.5 $$E_{|S_j} \cong L_j \oplus L_j^*$$

where L_j is the complex line bundle in E of weight m_j and L_j^* that of weight $- m_j$; because $c_1(E_{|S_j}) = 0$, L_j^* is also the dual of L_j. In order to define the first Chern classes of L_j it is convenient to have an orientation of S_j. Recall that X is oriented and that a neighbourhood of S_j in X looks like $S_j \times \mathbf{R}^2$. The $\mathbf{R}^2$ is oriented by the S^1-action, and this induces an orientation of S_j. Now write $c_1(L_j) = - k_j \cdot x_j$ with $k_j \in \mathbf{Z}$ and x_j the positive generator of $H^2(S_j; \mathbf{Z})$. If $m_j = 0$ then $E_{|S_j}$ is trivial as an $\tilde{S}^1$-equivariant vector bundle. We shall leave k_j undefined in this case.

There is one important constraint on the m_j. This becomes clear by remarking that $- 1 \in \tilde{S}^1$ acts as a gauge transformation on all of E, i.e. as

$+1$ or as -1. This implies that either all m_j are even or they are all odd. In Braam [10] we have shown that any set of invariants (m_j, k_j) satisfying this constraint arises from a suitable $\tilde{S}^1$-equivariant bundle, and that the $\tilde{S}^1$-isomorphism class is determined by (m_j, k_j).

Definition 5.2. The moduli space of monopoles on a principal $SU(2)$-bundle P with invariants (m_j, k_j) will be denoted by $\mathcal{M}(m_j, k_j)$.

Having defined the relevant invariants of P, the question now arises what they amount to in terms of asymptotic conditions for a pair (A_3, Φ) on M. The vector field v on P turns vertical over S_j. This shows that:

5.6 $$|\Phi(y)| \to m_j \quad \text{if} \quad y \to S_j \subset \delta M .$$

This is the Prasad-Sommerfeld boundary condition used in physics and the numbers m_j are called the *masses* of the monopole.

The solutions of the Bogomol'nyi equation 5.3 are minima of the *energy functional*:

5.7 $$E(A_3, \Phi) = (8\pi)^{-1} \int_M |F^{A_3}|^2 + |d_{A_3}\Phi|^2 \, dV_3 .$$

If the pair (A_3, Φ) arises from an invariant connection A on P then $E(A_3, \Phi) = YM(A)$. If we assume that (A_3, Φ) satisfies 5.4, then:

$$|d_{A_3}\Phi|^2 \, dV_3 = |F^{A_3}|^2 \, dV_3 = \langle F^{A_3} \wedge d_{A_3}\Phi \rangle = d\langle F^{A_3} \cdot \Phi \rangle ,$$

by the Bianchi identity. It follows that:

$$E(A_3, \Phi) = -2 \textstyle\sum_j (8\pi)^{-1} \cdot \int_{S_j} \langle F^{A_3} \cdot \Phi \rangle .$$

The minus sign appears because the boundary orientation of S_j does not agree with orientation we have given it above. A moments reflection shows that $2 \cdot (8\pi)^{-1} \cdot \int_{S_j} \langle F^{A_3} \cdot \Phi \rangle = -m_j \cdot k_j$. Putting things together we get:

5.7 $$\textstyle\sum m_j \cdot k_j = E(A_3, \Phi) = YM(A) = k .$$

This is essentially the localization formula in equivariant cohomology applied to the equivariant $c_2(P)$, see Atiyah [2].

Exactly what the physical symmetry breaking would lead one to expect does indeed happen: far away in M, that is near an S_j with $m_j \neq 0$, the connection almost becomes a $U(1)$-connection on L_j, the bundle of eigenvectors of Φ of eigenvalue $\frac{1}{2} \cdot m_j$. The *charges* k_j appear as first Chern classes of these line bundles on the boundary surfaces. This is of course nothing but the quantized charge of a $U(1)$-monopole, a so called Dirac monopole, on L_j. Dirac monopoles have singularities, but the genuine non-

Abelian character of $SU(2)$-monopoles in the core of M allows for non-singular solutions.

From 5.7 we see that $\sum m_j \cdot k_j \geqslant 0$ is necessary for the existence of monopoles, however this is by no means sufficient as we shall see below (also compare Braam [10]).

We shall end this section by giving some simple examples of monopoles.

Examples 5.3. 1) Monopoles with all $m_j = 0$. For these monopoles $YM(A) = 0$, so we are dealing with flat connections. The Higgs field Φ vanishes, this follows from the Bogomol'nyi equation. It is not hard to see that the moduli space $\mathcal{M}(0, 0)$ equals the space of all representations $\pi_1(X) \to SU(2)$ modulo conjugacy: one assign to a flat connection its holonomy representation. This space can be very non-trivial; e.g. if $M = H^3/$ Fuchsian group $\cong S \times \mathbf{R}$, with S a surface, then $\mathcal{M}(0, 0)$ is the space of representations of $\pi_1(S) \to SU(2)$ modulo conjugacy. By the theorem of Narasimham-Seshadri this is the same as the moduli space of semi-stable $SL(2, \mathbf{C})$-bundles on S, for any complex structure on S. The topology of this $\mathcal{M}(0, 0)$ was investigated by Atiyah-Bott [4].

2) Next keep $k_j = 0$ but take at least one m_j to be nonzero. The connections are still flat so Φ is covariantly constant. This shows that $\mathcal{M}(m_j, 0) = \emptyset$ unless all m_j are equal. Further

$$\begin{aligned}\mathcal{M}(m, o) &\cong \operatorname{Repr}\left(\pi_1(M), S^1\right) \cong \operatorname{Repr}\left(H_1(M;\mathbf{Z}), S^1\right)\\ &\cong H_1(X;\mathbf{Z})_{tor} \times \{H_1(X;\mathbf{R})/H_1(X;\mathbf{Z})\}.\end{aligned}$$

3) For $M \cong H^3$ all monopoles were determined by Atiyah [2]. The moduli space $\mathcal{M}(m, k)$ equals $\{\phi: S^2 \to S^2;\ \phi \text{ rational, degree } \phi = k, \phi(\infty) = 0\}$, modulo multiplication by complex scalars of length 1. The monopole associated to the rational function $\sum_j \exp(i\alpha_j) \cdot \lambda_j/(z - a_j)$ with $\lambda_j \in \mathbf{R}_{>0}$, $a_j \in \mathbf{C}$, represents k lumps, centered at approximately $(a_j, \lambda_j) \in \mathbf{R}^3_+ \cong H^3$, with relative phase factors $\exp\left(i(\alpha_{j_1} - \alpha_{j_2})\right)$.

4) Monopoles arising from Riemannian curvature. If X is a oriented Riemannian 4-manifold then one can write the curvature tensor $R: \Lambda^2 \to \Lambda^2$ as $\begin{bmatrix} W_+ + (R_{sc}/3) & B \\ B^* & W_- + (R_{sc}/3) \end{bmatrix}$ relative to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, in which B equals the Ricci curvature and $W_\pm$ the Weyl tensor. If X is a conformally flat spin manifold with a metric of zero scalar curvature then this curvature tensor equals $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$. It follows that the connection

on the spin bundle S_+ is anti-self-dual. Recall (see § 3) that for Γ Fuchsian, extended Fuchsian or a suitable Schottky group X_Γ admits such a metric. The connection on S_+ is a monopole because the metrics are S^1-invariant. The mass(es) is (are) 1 by proposition 2.2, and the charges k_j equal $g - 1$, where g is the genus of the fixed surface(s). Choosing a different spin structure amounts to tensoring the bundle with a 2-torsion element in $\mathrm{Repr}(\pi_1(M), S^1)$, compare 2).

In section 7 we shall come to grips with explicit formulae for nontrivial monopoles on certain handlebodies. In Braam-Hurtubise [11] the moduli spaces of monopoles on a solid torus are investigated in considerable detail. A general existence theory for monopoles on hyperbolic manifolds has been developed in Braam [10].

§ 6. Twistor spaces

To a conformally flat oriented 4-manifold X there are naturally associated two complex manifolds Z_+ and Z_-, the *twistor spaces* of X. Applying our construction of § 2 we thus get twistor spaces for hyperbolic 3-manifolds. It will be shown here that these carry a lot of geometric information associated to the 3-manifold M, such as the complete geodesic flow. Also they allow for a description of monopoles through holomorphic geometry. For the rest of this section let X be the conformal compactification of $M \times S^1$, with M a hyperbolic 3-manifold H^3/Γ as in § 2. We shall state those properties of $Z_\pm$ that we will need, and refer to Atiyah [1] and Atiyah-Hitchin-Singer [5] for proofs and more details. The general line of thought in this section is very similar to that of Hitchin [20] and Atiyah [2].

If $S_+(S_-)$ is the spin bundle of positive (negative) chirality on X, then $Z_+(Z_-)$ can be realised as the $\mathbf{CP}^1$-bundles over X:

$$P(S_+) \to X \qquad (P(S_-) \to X)\,,$$

where $P(\ )$ denotes projectivization of vectorbundles. A remarkable fact is that Z_+ and Z_- are *complex manifolds* with a complex structure encoded in the conformal structure of X. However, the twistor spaces are only Kähler if $X \cong S^4$ or $X \cong \mathbf{CP}^2$, which in our case results in $\Gamma = \{e\}$ (see Hitchin [19]). There is an orientation reversing isometry of X arising from conjugation of the circles. This interchanges the two spin bundles and makes

Z_+ holomorphically equivalent to Z_-. Henceforth we shall only consider Z_+ and denote it by Z.

Z carries an *anti-holomorphic involution*:

$$\sigma: Z \to Z, \quad \sigma^2 = 1.$$

This involution is a bundle map, inducing the identity on the base X, and is equal to the antipodal map upon restriction to the fibres. The complex structure on Z is such that (orientation preserving) conformal transformations on X lift to holomorphic transformations of Z. So our S^1-action on X lifts to an action on Z by holomorphic transformations and complexifies to a holomorphic $\mathbf{C}^*$-action on Z. We shall show that this $\mathbf{C}^*$-action is essentially the *geodesic flow* in H^3/Γ (as one would expect from Hitchin [20]).

The naturality with respect to conformal transformations has one further important application.

Recall (see Atiyah [1]) that the twistor space of S^4 is $\mathbf{CP}^3$ with projection and real structure:

$$\pi: \mathbf{CP}^3 \to S^4 = \mathbf{HP}^1: [z_0, z_1, z_2, z_3] \to [z_0 + z_1 \cdot j, z_2 + z_3 \cdot j]$$
$$\sigma: \mathbf{CP}^3 \to \mathbf{CP}^3: [z_0, z_1, z_2, z_3] \to [-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2]$$

As $X = (S^4 - \Lambda)/\Gamma$ it follows that the twistor space of X is the quotient:

$$Z = [\mathbf{CP}^3 - \pi^{-1}(\Lambda)]/\Gamma.$$

To study Z it will be useful to know how $\mathbf{C}^*$ and $PSL(2, \mathbf{C})$ act on $\mathbf{CP}^3$. The $\mathbf{C}^*$ action is described by $[z_0, z_1, z_2, z_3] \to [z_0, \lambda \cdot z_1, z_2, \lambda \cdot z_3]$, and the right $PSL(2, \mathbf{C})$-action by mapping $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to $\begin{bmatrix} a & 0 & c & 0 \\ 0 & \bar{a} & 0 & \bar{c} \\ b & 0 & d & 0 \\ 0 & \bar{b} & 0 & \bar{d} \end{bmatrix} \in PSL(4, \mathbf{C})$

which acts naturally on $\mathbf{CP}^3$, compare 2.3. Clearly the S^1-action fixes precisely two lines in $\mathbf{CP}^3$ namely:

6.1
$$P_1^+ = \{[z_0, 0, z_2, 0] \in \mathbf{CP}^3\} \quad \text{and}$$
$$P_1^- = \{[0, z_1, 0, z_3] \in \mathbf{CP}^3\}$$

These lines are also invariant under the hyperbolic isometries. The projections to the fixed point set $S^2 \subset S^4$ are the orientation preserving map $P_1^+ \to S^2: [z_0, z_2] \to [z_0, z_2]$ and the orientation reversing map

$P_1^- \to S^2: [z_1, z_3] \to [\bar{z}_1, \bar{z}_3]$ respectively. Here we have used homogeneous quaternionic coordinates on $S^4 = \mathbf{HP}^1$. The real structure maps P_1^+ to P_1^- and vice versa.

Non-trivial $\mathbf{C}^*$-orbits in $\mathbf{CP}^3$ are in one-one correspondence with a pair of begin- and end-points $(z, w) \in P_1^+ \times P_1^-$. Upon projecting the orbit $\mathcal{O}$ corresponding to (z, w) down to H^3:

$$\mathcal{O} \subset \mathbf{CP}^3 \to \pi(\mathcal{O}) \subset S^4 = \overline{H^3 \times S^1} \to g(\mathcal{O}) \subset H^3$$

one easily sees that $g(\mathcal{O})$ is an oriented geodesic in H^3 from $z \in S^2 = \delta H^3$ to $\bar{w} \in S^2$. The constant geodesics at infinity are included. Further for $p \in \mathcal{O} \subset \mathbf{CP}^3$ and $\lambda \in \mathbf{C}^*$ we have that the distance of $\pi(p)$ and $\pi(\lambda p)$ on $g(\mathcal{O})$ equals $\log|\lambda|$. As the $\mathbf{C}^*$-action commutes with the Γ-action, this shows that the $\mathbf{C}^*$-action is essentially geodesic flow in M. More precisely consider a copy of $M = i(M \times \{1\})$ in X. Then $Z_{|M}$ is the projectivized spin bundle of M which is canonically isomorphic to the unit tangent sphere bundle of M. Further the action of $\mathbf{R}_{>0} \subset \mathbf{C}^*$ preserves $Z_{|M}$ and is exactly the geodesic flow.

It is now possible to describe Z in detail. First of all the fixed points of the $\mathbf{C}^*$-action on Z are surfaces S_j^+, S_j^-, which project down to $S_j \subset X$. The surfaces S_j^+, S_j^- equal the components of $[P_1^+ - \Lambda]/\Gamma$ and $[P_1^- - \Lambda]/\Gamma$ respectively. The real structure maps S_j^+ to S_j^-.

The nontrivial $\mathbf{C}^*$-orbits in Z come in three types. *Good orbits* emanate from a plus surface, say S_j^+, and end on a minus surface, say S_k^-. The closure of one of these orbits in Z is a $\mathbf{CP}^1$. Note that these orbits are not determined by their two "endpoints". This corresponds precisely to the fact that two geodesics in M may have the same two endpoints, but in between one of them may run through different loops than the other. Denote by $\Omega_j^+ (\Omega_j^-)$ the pre-image in $P_1^+ (P_1^-)$ of $S_j^+ (S_j^-)$ under the quotient map. From the above we get the following

PROPOSITION 6.1. *The good orbits from S_j^+ to S_k^- are in one-one correspondence with oriented geodesics in $M \cong H^3/\Gamma$, which go from S_j to S_k. These have the complex analytic parameter space $[\Omega_j^+ \times \Omega_k^-]/\Gamma$, which is a holomorphic Ω_k^- bundle over S_j^+ or equivalently an Ω_j^+ bundle over S_k^-.*

Considering all good orbits emanating from S_j^+ and ending on some S_k^-, one gets that these are holomorphically parametrized by a $\bigcup_k \Omega_k^- = P_1^- - \Lambda$ bundle over S_j^+. Indeed, all orbits emanating from S_j^+ have a nice algebraic parameter space, which is equal to the projectivized holomorphic normal bundle $P(N_j^+)$ of S_j in Z. This is a $\mathbf{CP}^1$-bundle over S_j^+. The *bad orbits* correspond to geodesics in M which, in the universal cover, start in Ω_j and end in Λ. Of course similar statements hold concerning arriving geodesics and the projectivized normal bundle of S_j^-. Concerning the normal bundles we have the following

PROPOSITION 6.2. *There are injective, open, locally biholomorphic maps $\psi_j^\pm : N_j^\pm \to Z$, where $N_j^\pm$ is the holomorphic normal bundle of $S_j^\pm$ in Z. The $\mathbf{C}^*$-multiplication on the bundle N_j^+ is intertwined with the $\mathbf{C}^*$-action on Z by ψ_j^+, whereas ψ_j^- intertwines multiplication by the inverse with the $\mathbf{C}^*$-action on Z. The projectivized normal bundles $P(N_j^+)\,(P(N_j^-))$ are an algebraic parameter space for all geodesics in M going out from (arriving at) S_j.*

Proof. This is easy for the normal bundles of P_1^+ and P_1^- in $\mathbf{CP}^3$. Because the Γ action is linear and commutes with the $\mathbf{C}^*$-action the result also holds in Z. □

Remark 6.3. 1) The relation of the normal bundles with Eichler's modules. If $\mathscr{H} \to \mathbf{CP}^1$ is the positive Hopf bundle, then $H^0(\mathbf{CP}^1, \mathscr{H}^n) = \Pi_n$ is an $SL(2, \mathbf{C})$-module, called an *Eichler module*, see Bers [7]. Hence after choice of a spin structure $\Gamma \to SL(2, \mathbf{C})$ a Γ-module (compare the discussion after proposition 2.2). A short computation shows that the normal bundle of S_j^+ in Z is isomorphic to:

$$N_j^+ = (\Omega_j^+ \times_\Gamma \bar{\Pi}_1) \otimes V_{+,j},$$

where $V_{+,j}$ is the positive spin bundle of S_j^+.

2) In general for complex submanifolds $V \subset W$ there are obstructions for locally embedding the normal bundle in a holomorphic way, see Kodaira [23].

3) It may be possible to derive the geometry of the ends of the hyperbolic manifold M from the holomorphic structure of a normal bundle of a fixed surface. It would be interesting to have a formula for the metric on an end, giving the end as a foliation by surfaces such that the foliation is invariant under geodesic flow.

Finally there are *very bad orbits*, corresponding to geodesics going from Λ to Λ in the universal cover. In M they keep spiralling around, and never find and endpoint in either direction. For example closed geodesics are among these, in fact points in non-trivial orbits have a non-trivial stabilizer iff the orbit corresponds to a closed geodesic. The $\mathbf{C}^*$-orbits in Z corresponding to closed geodesics are compact holomorphically embedded *elliptic curves* in Z. The set of very bad orbits is closed in Z, is disjoint from the S_j, and lies in the closure of the set of very good orbits. In figure 2 we have sketched the orbit situation.

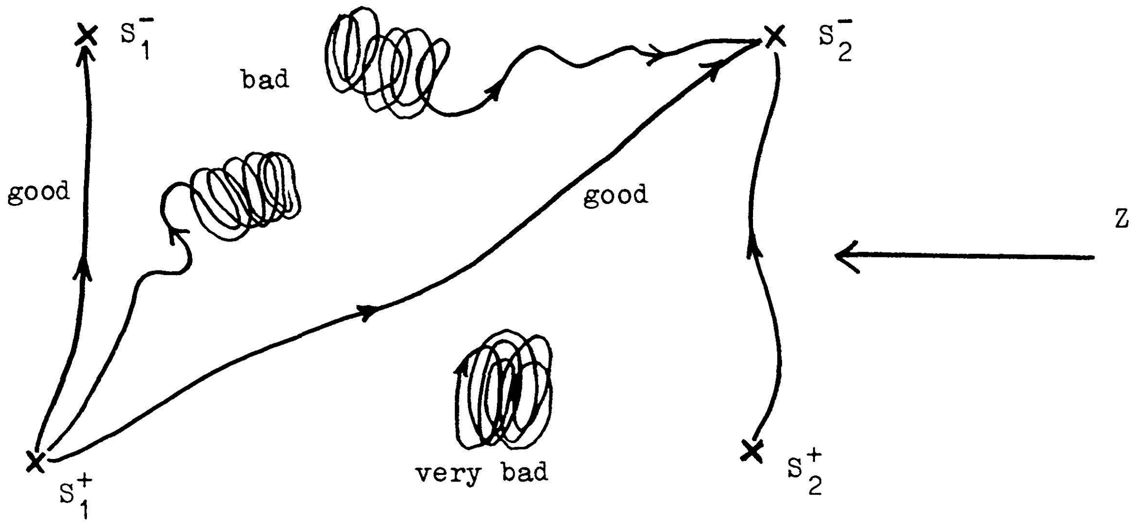

FIGURE 2.

The next objective of this section is to give a holomorphic description of monopoles. The relation between twistor spaces and anti-self-dual connections lies in the Atiyah-Ward correspondence (see Atiyah-Hitchin-Singer [5], for the instanton case):

THEOREM 6.4. *Let* $P \to X$ *be an* $\tilde{S}^1$*-equivariant* $SU(2)$*-bundle, and* A *a monopole on* P. *Put* $E = P \times_{SU(2)} \mathbf{C}^2$. *Then* $\pi^* A$ *induces a* $\tilde{\mathbf{C}}^*$*-invariant holomorphic structure on* $F = \pi^* E$ *such that:*

1) F *is trivial on the fibres of* π.

2) The natural antiholomorphic antilinear bundle map $\sigma: F \to \bar{F}^*$, *covering* σ *on* Z, *induces an* S^1*-invariant Hermitian metric on the vector spaces* $H^0(\pi^{-1}(x), F)$.

3) $\Lambda^2 F$ *is holomorphically trivial.*

Conversely a $\tilde{\mathbf{C}}^*$*-invariant holomorphic* $\mathbf{C}^2$*-bundle* F *over* Z, *with a real structure* $\sigma: F \to \bar{F}^*$ *satisfying 1, 2 and 3 arises from a unique monopole on* $P \to X$. □

Real structures on indecomposable holomorphic bundles F over twistor space are unique. Hence all the information is encoded in the holomorphic structure. However, existence of real structures is not automatic. The gauge equivalence relation for monopoles on $P \to X$ is the same as holomorphic $\tilde{\mathbf{C}}^*$-equivariant equivalence, preserving real structures, for the holomorphic bundles F on Z.

Let A be a monopole on $P \to X$, with all $m_j \neq 0$ and even, for simplicity. In this case we need not consider double coverings of groups and we shall denote the weights of S^1 by $p_j = \frac{1}{2} \cdot m_j$. Denote by $F = \pi^*(P \times_{SU(2)} \mathbf{C}^2)$ the holomorphic bundle over Z, with real structure σ. By theorem 6.4 the holomorphic structure on F is $\mathbf{C}^*$-invariant. An important aspect of monopole geometry of $\mathbf{R}^3$ and H^3 is to consider the quotient bundle $\mathscr{F} = F/\mathbf{C}^*$ on $Z/\mathbf{C}^*$ as far as this makes sense. On $Z/\mathbf{C}^*$, $\mathscr{F}$ will be an extension of certain standard line bundles, and this has been put to constructive use in the $\mathbf{R}^3$ case, see Hitchin [20]. It will be shown that a more complicated but essentially similar picture persists in our more general case. As yet, the constructive power seems to be rather limited.

Restricting F to S_j^+ it splits holomorphically, since the $\mathbf{C}^*$ action is fibre-wise, with nonzero weights $\pm p_j$:

6.2
$$F_{|S_j^+} = L_j^+ \oplus (L_j^+)^*$$
$$F_{|S_j^-} = L_j^- \oplus (L_j^-)^*$$

Here L_j^+ has $\mathbf{C}^*$-weight p_j and $c_1(L_j^+) = -k_j$, as in § 5. For L_j^- we have $\mathbf{C}^*$-weight $-p_j$ and $c_1(L_j^-) = -k_j$. The real structure gives an anti-linear isomorphism $L_j^+ \to L_j^-$.

PROPOSITION 6.5. *On $N_j^+ \subset Z(N_j^- \subset Z)$ there are line bundles $K_j^+(K_j^-)$, extending the $L_j^\pm$ of 6.2 (which were defined on the zero sections $S_j^\pm$ of $N_j^\pm$), such that on the $N_j^\pm$ the bundle F is an extension:*

$$0 \to K_j^+ \to F_{|N_j^+} \to (K_j^+)^* \to 0$$
$$0 \to K_j^- \to F_{|N_j^-} \to (K_j^-)^* \to 0$$

The real structure interchanges these two extensions.

Proof. Recall that sections of $P(F)$ correspond to line sub-bundles of F. We shall look at the $\mathbf{C}^*$-action on $P(F)$ restricted to the fibres $(N_j^+)_z$ with $z \in S_j^+$. Over $(N_j^+)_z$ we have two fixed points in $P(F)$ namely $[(L_j^+)_z]$ and $[(L_j^+)_z^*]$, lying in the fibre above $0 \in (N_j^+)_z$. At $f = [(L_j^+)_z]$ the weights of the infinitesimal $\mathbf{C}^*$-action on $T_f P(F)$ are $(+1, +1, -p_j)$. This means that most of the $\mathbf{C}^*$-orbits will actually flow to $[(L_j^+)_z^*]$, compare figure 3.

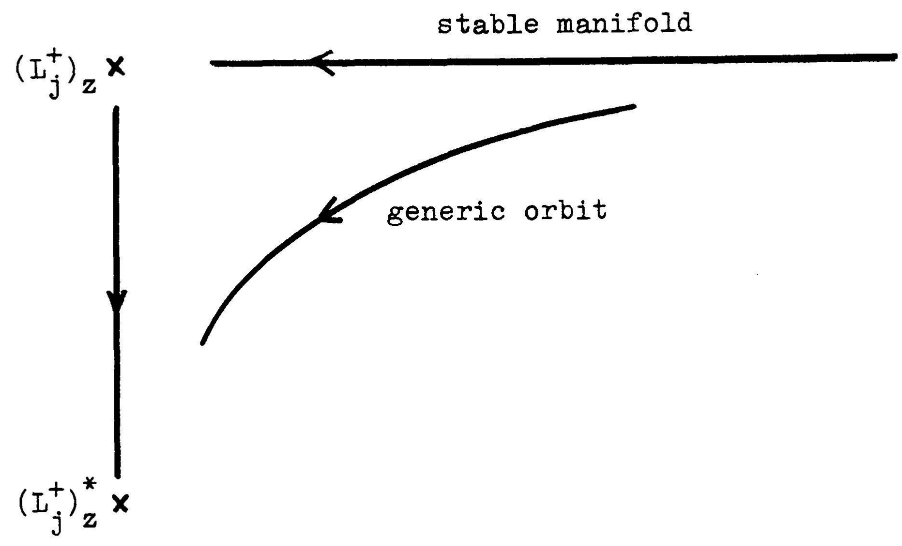

FIGURE 3.

By the stable manifold theorem with holomorphic parameter $z \in S_j^+$, we get a $\mathbf{C}^*$-invariant, $\operatorname{codim}_{\mathbf{C}}$ 1, complex submanifold $[L_j^+]$ of $P(F)$, consisting of precisely those orbits that flow into L_j^+. For the stable manifold theorem see Hadamard [16]. On N_j^- the situation is of course similar. □

In the case of monopoles on H^3 these extensions extend as bundle maps from $N_j^+ = \mathbf{CP}^3 - P_1^-$ to $\mathbf{CP}^3$ (also for N_j^-) but in our more general situations there can be obstructions to this.

The extensions of proposition 6.5 descend to the quotient $P(N_j^\pm)$, and we proceed by identifying them there. Holomorphic line bundles on the ruled surfaces are of the form:

$$\rho^* L \otimes O(n)$$

where $\rho\colon P(N_j^\pm) \to S_j^\pm$ is the projection, L a line bundle on $S_j^\pm$, and $O(n)$ the n-th power of the positive Hopf bundle on $P(N_j^\pm)$, which has fibre $(\mathbf{C}v)^*$ at the point $[v] \in P(N_j^\pm)$. On the fibres of $N_j^\pm$ the structure of the bundle follows from:

LEMMA 6.6. *Let* $\mathbf{C}^*$ *act on* $\mathbf{C}^2$ *by scalar multiplication. A* $\mathbf{C}^*$-*equivariant* $\mathbf{C}^2$-*bundle* $E \to \mathbf{C}^2$ *is equivariantly isomorphic to* $E_0 \times \mathbf{C}^2$ *with* E_0 *the representation of* $\mathbf{C}^*$ *on the fibre over* $0 \in \mathbf{C}^2$.

Proof (see Atiyah [2]). On $\mathbf{C}^2\backslash\{0\}$ a $\mathbf{C}^*$-equivariant bundle is the same as a bundle on $\mathbf{CP}^1$, i.e. a sum of powers of the Hopf bundle. This establishes the given isomorphism on $\mathbf{C}^2\backslash\{0\}$. By Hartog's theorem it extends to $\mathbf{C}^2$. $\square$

The point of the lemma is that it identifies $K_j^{\pm}$ as the pull back of $L_j^{\pm}$ under the projection $N_j^{\pm} \to S_j^{\pm}$, with $\mathbf{C}^*$ acting on it by a character of weight $\pm p_j$. Now one concludes readily that the extension on $P(N_j^+)$ reads:

6.3
$$0 \to \mathscr{L}_j^+ \to \mathscr{F} \to (\mathscr{L}_j^+)^* \to 0 \quad \text{with}$$
$$\mathscr{L}_j^+ = \rho^* L_j^+ \otimes O(p_j) \quad \text{and} \quad \mathscr{F} = [F_{|N_j^+\backslash\{0\}}]/\mathbf{C}^* .$$

Similarly on $P(N_j^-)$ we get:

6.4
$$0 \to \mathscr{L}_j^- \to \mathscr{F} \to (\mathscr{L}_j^-)^* \to 0 \quad \text{with}$$
$$\mathscr{L}_j^- = \rho^* L_j^- \otimes O(p_j) \quad \text{and} \quad \mathscr{F} = [F_{|N_j^-\backslash\{0\}}]/\mathbf{C}^* .$$

This results in:

THEOREM 6.7. *The monopole A defines extensions of $\mathscr{F}$ on $P(N_j^+)$ and $P(N_j^-)$ for $j = 1, \ldots, N$ as in 6.3 and 6.4. These extensions are interchanged by the real structure.* $\square$

In the case of monopoles on H^3 these restrictions are essentially all the data one obtains about the quotient bundles and the monopole is determined by the extensions and the real structure: see Atiyah [2]. In our case the intersection of N_i^+ with N_j^- will generally have many components and we get extra data in the form of a set of invariant holomorphic identifications:

6.5
$$g_{ij} \colon N_i^+ \cap N_j^- \to \mathrm{Hom}(F_{|N_i^+}, F_{|N_j^-}) .$$

Conjecture. Under general conditions on the hyperbolic structure on M bundles F arising from irreducible monopoles are determined by the extensions 6.3, 6.4 and the real structure on these. $\square$

One can almost certainly prove that if F_0 and F_1 are two holomorphic bundles on Z such that upon restriction to $\cup_i(N_i^+ \cup N_i^-)$ they become isomorphic, then they are isomorphic on Z. In order to prove the conjecture it remains to show that for irreducible monopoles no information is contained in the g_{ij}. Evidence for this conjecture comes from Thurston's version of Mostow's theorem (see Morgan [29]). This theorem implies that the flat $PSL(2, \mathbf{C})$-bundles encoding the holonomy of the hyperbolic structure are determined by their restriction to the fixed surfaces, despite the fact that the fundamental group of Z is not necessarily generated by that of the fixed surfaces. In fact one may hope to reverse this procedure: a proof of the conjecture would be a good first step towards a proof of Mostow's theorem.

It might be a good point to stress that although Z is not Kähler, suddenly algebraic objects such as elements of Picard groups and ruled surfaces have appeared. This makes algebraic geometry enter the picture, perhaps somewhat unexpectedly.

Next we shall consider spectral curves, of which we shall obtain a whole bunch instead of just a single one, as obtained in the case of $\mathbf{R}^3$ and H^3 (see Hitchin [20] and Atiyah [2]). Just as in the $\mathbf{R}^3$ and H^3 case we should compare two extensions. On $P(N_j^+ \cap N_k^-)$ we have:

6.6
$$0 \to \mathscr{L}_j^+ \to \mathscr{F} \to (\mathscr{L}_j^+)^* \to 0 \quad \text{and}$$
$$0 \to \mathscr{L}_k^- \to \mathscr{F} \to (\mathscr{L}_k^-)^* \to 0 \,.$$

Definition 6.8. The spectral curve

$$C_{jk} \subset P(N_j^+ \cap N_k^-) = (\Omega_j^+ \times \Omega_k^-)/\Gamma \qquad j, k = 1, \ldots, n$$

is the zero set of the canonical map

$$\mathscr{L}_j^+ \to (\mathscr{L}_k^-)^*$$

arising from 6.6. □

Hence for a manifold with N ends, we get N^2 spectral curves. However, the real structure clearly interchanges C_{jk} with C_{kj}, so effectively we are left with $(N^2+N)/2$ spectral curves, N of which, namely the C_{jj}, have to satisfy reality constraints. The curves can be interpreted geometrically as follows:

PROPOSITION 6.9. *The following three are equivalent:*

1) A $\mathbf{C}^*$ *orbit* $\mathcal{O} \in (\Omega_j^+ \times \Omega_k^-)/\Gamma$ *lies in* C_{jk}.

2) The bundle F *restricted to* $\bar{\mathcal{O}} \cong P_1 \subset Z$ *is isomorphic to* $\mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$. *(For other good orbits it will be isomorphic to* $\mathcal{O}(p_j-p_k) \oplus \mathcal{O}(-p_j+p_k)$.*)*

3) The Hitchin equation (compare Hitchin [20]*):*

$$\frac{\partial s}{\partial l} + A_1 \cdot s + i\Phi \cdot s = 0\,, \qquad s\colon g(\mathcal{O}) \to \mathbf{C}^2$$

on the corresponding geodesic $g(\mathcal{O}) \subset H^3/\Gamma$ *has a bounded solution.*

Proof. To see the equivalence of 1) and 2) we first digress on bundles on $\mathbf{CP}^1$. The result of lemma 6.6 also holds if one replaces $\mathbf{C}^2$ by $\mathbf{C}$; this follows by using an arbitrary projection $\mathbf{C}^2 \to \mathbf{C}$ and pulling back. Thus $E_{|\bar{\mathcal{O}}}$ trivializes in a $\mathbf{C}^*$-equivariant way as:

$$\begin{array}{lll} L_j^+ \oplus (L_j^+)^* & \text{on} & \bar{\mathcal{O}} - \{\infty\} \\ L_j^- \oplus (L_j^-)^* & \text{on} & \bar{\mathcal{O}} - \{0\} . \end{array}$$

The $\mathbf{C}^*$-equivariant automorphisms of $E_{|\bar{\mathcal{O}}-\{\infty\}}$ are easily seen to be of the form $\begin{bmatrix} a & b \cdot z^{2p_j} \\ 0 & c \end{bmatrix}$, and thus form a Borel subgroup of $GL(2, \mathbf{C})$. The situation is the same at infinity, and from this it follows that isomorphism classes of $\mathbf{C}^*$-equivariant holomorphic bundles on $\mathbf{CP}^1$ are given by the set of two elements $B\backslash GL(2, \mathbf{C})/B$. The exceptional case is that in which the transition function maps L_j^+ to L_j^-, i.e. $\mathcal{O} \in C_{jk}$. Then $F_{|\bar{\mathcal{O}}}$ equals $\mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$, otherwise it is isomorphic to $\mathcal{O}(p_j-p_k) \oplus \mathcal{O}(p_k-p_j)$.

To prove the equivalence of 2) and 3), we first remark that $F_{|\bar{\mathcal{O}}}$ has a bounded $\mathbf{C}^*$-invariant holomorphic nonzero section, iff $F_{|\bar{\mathcal{O}}} \cong \mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$. This follows from the standard description of sections of line bundles over $\mathbf{CP}^1$ as homogeneous polynomials and from the fact that the weights of the action are is p_j at 0 and $-p_k$ at ∞. The Hitchin equation is nothing but the Cauchy-Riemann equation for invariant sections, see Hitchin [20]. Therefore the proposition follows. □

Remark 6.10. 1) One expects that the spectral curves will generally not be compact and more or less resemble a curve of infinite genus. This is because on the universal cover H^3 we are dealing with a monopole of infinite charge.

2) It should also be remarked that the complex manifolds $(\Omega_j^+ \times \Omega_k^-)/\Gamma$ in which the spectral curves lie are far from nice generally. In the case of cyclic groups they are a $\mathbf{C}^*$-bundle over a torus and for quasi-Fuchsian groups they are disc bundles over a Riemann surface of genus $\geqslant 2$. Generally they will be Ω_j^+ bundles over S_k^- and the fibre will have infinitely many components; see § 2 where we discussed Kleinian groups.

As remarked in the introduction, it should be very interesting to find constructions for monopole bundles on these twistor spaces. It seems however that methods previously employed for $\mathbf{CP}^3$ fail, mainly due to the fact that the twistor spaces are not Kähler.

§ 7. Atiyah-Ward ansatzes, summing 't Hooft solutions and Eisenstein series

In this section we shall derive some explicit formulae for monopoles on handlebodies, using the complex geometry of their twistor spaces. A detailed study of the moduli spaces of monopoles on a solid torus has been made in Braam-Hurtubise [11].

From the description of Z as $P(S_+)$, it follows that on Z there exists a *tautological line bundle* L, which upon restriction to the fibre over $x \in X$, equals the negative Hopf bundle on $P(S_{+,x})$. It turns out that L is naturally holomorphic, and to tie in with the $(\mathbf{CP}^3, S^4)$ case we shall denote the $(-q)$-th power of L by $\mathcal{O}(q)$.

If $F \to \mathbf{CP}^3$ is an instanton bundle on the twistor space of S^4 then *Atiyah-Ward ansatzes*, that is an explicit formula for the instanton on S^4, arise from a suitable description of F as holomorphic bundle. Let s be a section of $F \otimes \mathcal{O}(q) = F(q)$. Generically s will be nonzero away from a *complex curve* $C_s \subset Z$ and give rise to an extension class $e_s \in H^1(Z - C_s, \mathcal{O}(-2q))$. Elements of such sheaf cohomology groups correspond to solutions ϕ_s of linear p.d.e. on open sets of S^4: this is the celebrated *Penrose correspondence*. Explicit formulas for the instanton, such as those of 't Hooft, can be constructed in terms of this ϕ_s. Every instanton on S^4 can theoretically be computed in this way. For background see Atiyah [1].

We shall see that on our manifolds $X = (S^4 - \Lambda)/\Gamma$, for $\Gamma \neq \{e\}$, the situation is rather different, but that nevertheless in some cases explicit constructions can be made again. As before attention will only be paid to $\tilde{S}^1$-invariant instantons, i.e. monopoles. In those cases which we treat in detail, it will appear that we are essentially summing together a monopole, much in the same way as automorphic forms are constructed by summing kernels. It is however quite remarkable that "summing" of solutions is possible for the non-linear anti-self-duality equations, and may be these summation procedures are best thought of as a kind of Backlund transformations.

Recall from § 2 and § 3, that X comes with a natural conformal structure, and that X can be given a metric in the conformal class with constant scalar curvature R_X. We proved that the majority of X's give rise to negative R_X. Assume a spin structure on X has been fixed, then the line bundle $\mathcal{O}(q)$ above is well defined.

PROPOSITION 7.1. *If* $R_X < 0$, *then no monopole on* X *arises from an Atiyah-Ward construction, since* $H^0(Z, F(q)) = 0$ *for all* $q \in \mathbf{Z}\backslash\{0\}$.

Proof. For $q < 0$ any section would vanish on the fibres $\pi^{-1}(x)$, and hence be zero; this is independent of the sign of R_X. For $q > 0$, we know from Hitchin [18], that elements of $H^0(Z, F(q))$ are in one-one correspondence with solutions of the twistor equation on X with coefficients in $E = P \times_{SU(2)} \mathbf{C}^2$:

$$\bar{D}_q s = 0$$
$$\bar{D}_q = \mathscr{P} \circ \nabla_A : \Gamma(S^q(S_+) \otimes E) \to \Gamma(S^{q+1}(S_+) \otimes S_- \otimes E),$$

with S^q the q-th symmetric product, $\mathscr{P}: \Lambda^1 \otimes S^q(S_+) \to S^{q+1}(S_+) \otimes S_-$ the projection, and A the anti-self-dual $SU(2)$-connection on $E \to X$. For these equations we have a vanishing theorem of Weizenbock type in the case of negative scalar curvature, see Besse [8]. □

Hence attention here needs only be paid to the $R_X \geqslant 0$ manifolds, which were classified in theorem 3.1. But even here there is a very fundamental difference between the case $X = S^4$, i.e. $\Gamma = \{e\}$, and the cases of non-trivial Γ.

On $X = S^4$, $Z = \mathbf{CP}^3$, the dimensions of $H^0(Z, \mathcal{O}(q))$ (and also of the invariant part $H^0(Z, \mathcal{O}(q))^{S^1}$) increase with q. Tracing through the (equivariant) Riemann-Roch formula (as in Hitchin [19]), one learns that the increasing character is due to the fact that for the fixed point sets $S^+ = P_1^+$, $S^- = P_1^- \subset Z = \mathbf{CP}^3$ we have $\chi(S^\pm) > 0$. For $\Gamma \neq \{e\}$ these Euler characteristics satisfy $\chi(S^\pm) \leqslant 0$. This leads one to suspect that it may not always be possible to find sections of $F(q)$, which would be needed to obtain Atiyah-Ward ansatzes in general.

After all these negative remarks, let us proceed to show that, at least in some cases, the construction works satisfactorily. To simplify things even further, we shall assume that X is a manifold with $R_X > 0$; by theorem 3.1, X arises from a Schottky group. Consider on X the *conformally invariant Laplacian* D_0 acting on densities of conformal weight 1, with values in densities of weight 3, which equals

$$D_0 = d^*d + \frac{1}{6} \cdot R_X .$$

Since $R_X > 0$, we get $\ker D_0 = 0$, and hence unique fundamental solutions ϕ_x exist satisfying

$$D_0 \cdot \phi_x = \delta_x \qquad x \in X .$$

Through the twistor correspondence (see Atiyah [3], [1], and Hitchin [18]) ϕ_x corresponds to a *cohomology class*:

$$\varphi_x \in H^1(Z - \pi^{-1}(x), \mathcal{O}(-2)),$$

and hence ϕ_x gives rise to a vector bundle F on $Z - \pi^{-1}(x)$, which is an extension:

$$0 \to \mathcal{O}(-1) \to F \to \mathcal{O}(1) \to 0 .$$

In fact one can show (Atiyah [3]) that the bundle F extends to a bundle F on Z, such that $F(1)$ has a holomorphic section vanishing precisely on $\pi^{-1}(x)$. The maximum principle applied to D_0 ensures that $\phi_x(y) > 0$, for all $y \in X$, and this implies that F is trivial on the real lines $\pi^{-1}(x)$. Since ϕ_x is real, F gets a real structure. Thus F is an instanton bundle.

To get a monopole rather than just an instanton we have to assume $x \in S_1$, the fixed surface in X. The weight m_1 of a monopole constructed in this way equals 1, because the Hopf bundle $\mathcal{O}(1)$ is of weight 1. The charge also equals 1.

Obviously the process can be generalized by using a positive linear combination of k fundamental solutions:

$$\varphi = \Sigma \lambda_j \varphi_{x_j} \qquad \lambda_j > 0\,, \quad j = 1, \dots, k\,,$$

which is called an *'t Hooft potential.* If the x_j lie in $S_1 \subset X$, then the 't Hooft potential will be invariant, and it follows that we have created a monopole of mass 1 and charge k. All positive scalar multiples of ϕ give the same instanton, so the number of parameters in the solutions is $3k - 1$: we have 2 for every $x_j \in S_1$, and 1 for every λ_j. These solutions therefore don't give an open set in the $4k - \frac{1}{2} \cdot \chi(S)$ dimensional moduli space.

We proceed to identify these potentials ϕ. In the course of this, explicit formulas for the connection A will also be given. Besides, a slight generalization of the Atiyah-Ward construction will emerge.

Pulling back ϕ_x to $S^4 - \Lambda$, under the quotient map, one gets a generalized function $\tilde{\phi}_x$ on $S^4 - \Lambda$ satisfying:

$$D_0 \tilde{\varphi}_x = \sum_{\gamma \in \Gamma} \delta_{\gamma y}$$

with $y \in S^4 - \Lambda$ mapping to x. Of course the next step is to try to reverse this and to put:

7.1 $$\tilde{\varphi}_x = \sum_{\gamma \in \Gamma} \psi_{\gamma y}$$

where ψ_y is a fundamental solution on S^4 of D_0 at y. In the flat metric on $\mathbf{R}^4 \subset S^4$, fundamental solutions are equal to:

7.2 $$\psi_y(r) = (2\pi\|y-r\|)^{-2}.$$

Since the flat metric is not Γ-invariant, conformal weight factors will occur in 7.1. It is easier to see what happens if one uses the Γ-invariant metric on $H^3 \times S^1$:

$$t^{-2}(dx_1^2+dx_2^2+dt^2) + d\theta^2 \qquad (x_1, x_2, t, \theta) \in H^3 \times S^1$$

Under conformal rescaling, 7.2 transforms to the θ-independent *summation kernel of the Eisenstein series* on H^3 (compare Mandouvalos [25]):

$$E(y, h) = t/[(x_1-y_1)^2+(x_2-y_2)^2+t^2] \quad y \in \mathbf{R}^2 \subset S^2, \quad h = (x_1, x_2, t) \in H^3$$

Summing, we get for 7.1:

7.3 $$E_\Gamma(y, h) = \sum_{\gamma \in \Gamma} E(y, \gamma h),$$

which is the *Eisenstein series* for Γ, see Mandouvalos [25]. As settled by Poincaré already, 7.3 is convergent if $\delta(\Gamma) < 1$, where $\delta(\Gamma)$ is the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of Γ. The groups Γ for which this holds are the cyclic groups and classical Schottky groups (with their defining circles wide apart, compare Bers [7]). In passing by we note that $\delta(\Gamma) < 1$ implies that X is of positive type because the Eisenstein series is a strictly positive Green's function for $d^*d + \frac{1}{6} \cdot R_{sc}$: the maximum principle implies $R_{sc} > 0$.

To compute the gauge potentials, it is easiest to go back to the flat metric on $\mathbf{R}^4$. The more general potentials there look like

7.4 $$\phi(h, \theta) = \sum_{i=1}^{k} \lambda_i \cdot t^{-1} \cdot E_\Gamma(x_i, h),$$

and the formulas of 't Hooft give for the connection (see Atiyah-Hitchin-Singer [5])

$$A = \sum_i P_+(-1/2\, d \log \varphi \Lambda e_i) \otimes e_i \in \Gamma(\mathbf{R}^4, \Lambda^2_+ \otimes \Lambda^1),$$

with e_i an orthonormal, covariantly constant framing of $T^*\mathbf{R}^4$ and Λ^2_+ identified with $su(2)$. To see what this looks like, assume that Γ is cyclic, generated by $\begin{bmatrix} \lambda^{\frac{1}{2}} & \\ & \lambda^{-\frac{1}{2}} \end{bmatrix}$, $\lambda \in \mathbf{R}_{>0}$. Then

$$\varphi(r) = \sum_{n=1}^{\infty} \left[\sum_i \left\{ \frac{\lambda^n \lambda_i}{\| \lambda^n r - y_i \|^2} + \frac{\lambda^{-n} \lambda_i}{\| r - \lambda^{-n} y_i \|^2} \right\} \right] + \sum_i \frac{\lambda_i}{\| r - y_i \|^2} \tag{7.5}$$

$$\text{with } y_i \in \mathbf{R}^2 \subset S^2 \quad \text{and} \quad r \in S^4 \backslash \Lambda = \mathbf{R}^4 \backslash \{0\}.$$

So we see that for $\lambda \gg \lambda_i$ and $1 \leqslant \| r \|$, $\| y \| \leqslant \lambda$, the second term dominates strongly and the monopole will look much like a "grafted S^4-monopole". On making λ smaller, nearby nonlinear interaction makes the monopole look more complicated.

Finally we discuss a modification of this construction which supplies a few more solutions. Suppose we put $k = 1$ and consider the harmonic function:

$$\phi_\alpha(r) = \sum_{n \in \mathbf{Z}} \lambda^{(1+\alpha) \cdot n} \cdot \| \lambda^n r - y \|^{-2},$$

which converges for $-1 < \alpha < 1$. Then $\phi_\alpha(\lambda r) = \lambda^{-\alpha - 1} \cdot \phi_\alpha(r)$, so the instanton is invariant. This results in a 3-parameter family of monopoles.

Now ϕ_α describes a fundamental solution of the Laplacian acting on sections of a flat real line bundle with monodromy λ^α along the non-trivial loop in H^3/Γ, so we have constructed a bundle F on twistor space, which is an extension of $L(1)$ by $L^*(-1)$, where L is a real flat line bundle in the Picard group of Z with monodromy $\lambda^{\frac{1}{2} \cdot \alpha}$.

The same procedure can be used for Schottky groups Γ of genus g, by twisting the sum with a character $\Gamma \to \mathbf{R}_{>0}$ close to 1. This gives a $3k - \frac{1}{2} \cdot \chi(S)$ parameter family of monopoles. This too doesn't give an open set in the moduli spaces and it appears that the construction of the general solution is not yet clear, even in these simple cases.

Possibly this can be remedied by going over to the next Atiyah-Ward ansatz, which exploits the self-dual Maxwell equations on X. Here the vanishing sets could be choosen to be elliptic curves corresponding to closed geodesics in M.

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