

0. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

ISOCLINIC n -PLANES IN R^{2n} AND THE HOPF-STEENROD
SPHERE BUNDLES $S^{2n-1} \rightarrow S^n$, $n = 2, 4, 8$

by Yung-Chow WONG and Kam-Ping MOK

0. INTRODUCTION

The construction of the sphere bundles $S^{2n-1} \rightarrow S^n$, $n = 2, 4, 8$, by N. Steenrod was accomplished in an ingenious but rather roundabout way, using the famous Hopf maps and the systems of complex numbers, quaternions and Cayley numbers (cf. Hopf [2], Steenrod [5, pp. 105-110] and Hilton [1, pp. 51-55]). In this paper, we show how the theory of mutually isoclinic n -planes in a real Euclidean $2n$ -space R^{2n} as developed by Wong in [8, 9] enables us to reconstruct these sphere bundles in a more natural manner by working strictly within the field of real numbers and giving the three cases $n = 2, 4, 8$ a more unified treatment. In addition, we prove that the bundle group $O(8)$ of the Hopf-Steenrod sphere bundle $S^{15} \rightarrow S^8$ can be replaced by $SO(8)$ but not by any subgroup of $SO(8)$.

In § 1, we recall certain results on maximal sets of mutually isoclinic n -planes in R^{2n} that motivated our investigation. In § 2, we confine ourselves to the cases $n = 2, 4, 8$, and prove some results that will be used later. In § 3, we construct three sphere bundles by using maximal sets of mutually isoclinic n -planes in R^{2n} . In § 4, we give a unified and explicit formulation of the three Hopf-Steenrod sphere bundles, using as Steenrod did the Hopf maps and systems of complex numbers, quaternions and Cayley numbers. In § 5, we prove that the Hopf maps and maximal sets of mutually isoclinic n -planes in R^{2n} , $n = 2, 4, 8$, are equivalent concepts, and that the reformulated Hopf-Steenrod sphere bundles described in § 4 are topologically essentially the same as the sphere bundles constructed in § 3. The paper ends with two appendices in which we explain the operations of Cayley numbers, and give a direct proof that for $n = 2, 4$, or 8 , the n -planes in R^{2n} containing the Hopf fibers of S^{2n-1} are mutually isoclinic n -planes.

In a continuation of this paper being prepared, we shall show that the image of the Hopf fibers of S^{2n-1} , $n = 2, 4$, or 8 , under an inversion in R^{2n} has some very interesting properties which include those recently found by J. B. Wilker [7] for the case $n = 2$.

We wish to thank Prof. Wilker for letting us have a preprint of his paper, and Prof. Kee-Yuen Lam for some helpful discussions.

1. SOME RESULTS ON ISOCLINIC n -PLANES IN R^{2n}

By a Euclidean (vector) m -space R^m , where m is a positive integer, we mean an m -dimensional vector space provided with a positive definite inner product. An r -plane ($1 \leq r \leq m-1$) in R^m is an r -dimensional vector subspace of R^m provided with the induced inner product. In R^m , length of a vector, angle between two vectors, orthogonality between a k -plane and an r -plane, (orthogonal) projection of a vector on an r -plane, orthonormal bases and rectangular coordinates are defined in the usual way.

In an R^{2n} , let \mathbf{A} , \mathbf{B} be any two n -planes. Then we say that \mathbf{A} is *isoclinic* with \mathbf{B} at angle θ if the angle between every nonzero vector in \mathbf{A} and its projection on \mathbf{B} is always equal to θ . It turns out that if \mathbf{A} is isoclinic with \mathbf{B} at angle θ , then \mathbf{B} is isoclinic with \mathbf{A} at the same angle θ . Therefore, in this case, we shall say that \mathbf{A} and \mathbf{B} are isoclinic at angle θ , or simply, \mathbf{A} and \mathbf{B} are isoclinic.

A set Φ of n -planes in R^{2n} is said to be a *maximal set of mutually isoclinic n -planes* if every pair of n -planes in Φ are isoclinic and Φ is not contained in a larger set of mutually isoclinic n -planes. It is easy to see from definition that if \mathbf{A} is isoclinic with \mathbf{B} at angle θ , then its orthogonal complement \mathbf{A}^\perp is isoclinic with \mathbf{B} at angle $\frac{\pi}{2} - \theta$. Consequently, if Φ is any maximal set of mutually isoclinic n -planes in R^{2n} and $\mathbf{A} \in \Phi$, then $\mathbf{A}^\perp \in \Phi$.

In his memoir [8] Wong determined, for each n , the dimensions of the maximal sets of mutually isoclinic n -planes in R^{2n} , the number of non-congruent maximal sets of a given dimension, and explicit equations of the n -planes in any maximal set of mutually isoclinic n -planes containing a given n -plane.

In the following, we summarize some of his results related to the problem studied in this paper.

THEOREM 1.1. (Wong [8, pp. 25-26]). *In R^{2n} provided with a rectangular coordinate system $(x, y) \equiv ([x_1 \dots x_n], [x_{n+1} \dots x_{2n}])$, any maximal set Φ of mutually isoclinic n -planes containing the n -plane $\mathbf{O}: y = 0$ (and consequently, also the n -plane $\mathbf{O}^\perp: x = 0$) is congruent to the set of n -planes with equations*