

# 3. Construction of Gröbner Bases

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(2)  $\Rightarrow$  (3): trivial.

(3)  $\Rightarrow$  (1): By (3) we have  $\text{in}(Q) \in \langle \text{in}(F) \rangle$  for every  $Q \in J - \{0\}$ . Hence  $\langle \text{in}(J) \rangle = \langle \text{in}(F) \rangle$ .

2.6. COROLLARY. Let  $F$  be a Gröbner basis of an ideal  $J \leq R[X]$ .

1)  $F$  generates  $J$ .

2) Let  $Q \in R[X]$ . Then  $Q \in J$  iff a rest of  $Q$  after dividing by  $F$  is zero.

*Proof.* Obvious.

2.7. Another characterisation of Gröbner bases can be given as follows:

We shall say that a set  $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$  of admissible combinations of  $F$  (with pairwise different degrees) is an " $F$ -admissible set", if for all  $\alpha$  we have  $\deg(L_\alpha) = \alpha$  and  $\text{lc}(L_\alpha)$  generates the ideal

$${}_R \langle \text{lc}(P) \mid P \in \langle \text{in}(F) \rangle, \deg(P) = \alpha \rangle.$$

Any  $F$ -admissible set is  $R$ -linearly independent.

If  $R$  is a field the condition on  $\text{lc}(L_\alpha)$  is superfluous.

PROPOSITION. Let  $J$  be an ideal in  $R[X]$  containing  $F$ . Then the following conditions are equivalent:

- (1)  $F$  is a Gröbner basis of  $J$ .
- (2) There is an  $F$ -admissible set which is a  $R$ -basis of  $J$ .
- (3) Every  $F$ -admissible set is a  $R$ -basis of  $J$ .

*Proof.* Let  $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$  be a  $F$ -admissible set.

(1)  $\Rightarrow$  (3): Let  $Q$  be an element of  $J - \{0\}$ . Division of  $Q$  by  $\{L_{\deg(Q)}\}$ , of its rest  $\bar{Q}$  by  $\{L_{\deg(\bar{Q})}\}$ , ... yields in a finite number of steps an expression of  $Q$  as  $R$ -linear combination of  $L_\alpha$ 's.

(3)  $\Rightarrow$  (2): trivial.

(2)  $\Rightarrow$  (1): Suppose that  $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$  is a  $R$ -basis of  $J$ . For every  $Q \in J - \{0\}$  the initial term of  $L_{\deg(Q)}$  divides  $\text{in}(Q)$ , hence  $\text{in}(Q) \in \langle \text{in}(F) \rangle$ .

### 3. CONSTRUCTION OF GRÖBNER BASES

3.1. Definition. Let  $P, Q$  be elements of  $R[X]$ , let  $\alpha, \beta \in \mathbb{N}^n$  and let  $a, b \in R$ . Then the polynomial

$$S(P, Q) := aX^\alpha P - bX^\beta Q$$

is called a "S(ubtraction)-polynomial of  $P, Q$ " iff

$$\alpha + \deg(P) = \beta + \deg(Q) = \min(\mathcal{D}(\{P\}) \cap \mathcal{D}(\{Q\}))$$

and  $\text{lc}(P) \cdot a = \text{lc}(Q) \cdot b =$  a least common multiple of  $\text{lc}(P)$  and  $\text{lc}(Q)$ .

3.2. *Example.* Consider the graded lexicographic ordering on  $\mathbb{N}^2$  and

$$P := 6X_1^3X_2 + 1, \quad Q := 8X_1X_2^2 + 3X_1X_2 + X_2 \in \mathbb{Z}[X_1, X_2].$$

Then

$$4X_2P - 3X_1^2Q = -9X_1^3X_2 - 3X_1^2X_2 + 4X_2 \quad \text{and} \quad -4X_2P + 3X_1^2Q$$

are S-polynomials of  $P, Q$ .

See figure 5.

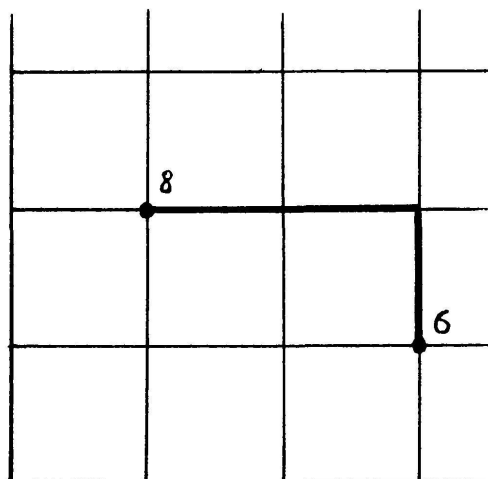


FIGURE 5.

3.3 *Remark.* For  $P, Q \in R[X]$ ,  $S(P, Q)$  as defined above is unique up to multiplication by an invertible element of  $R$ . Therefore we shall call it "the" S-polynomial of  $P, Q$ .

3.4. *LEMMA.* Let  $P_1, \dots, P_k \in R[X]$ ,  $c_1, \dots, c_k \in R$  such that  $\deg(P_1) = \dots = \deg(P_k) =: \delta$  but  $\deg(\sum_{i=1}^k c_i P_i) \neq \delta$ .

Then  $\sum_{i=1}^k c_i P_i$  is a  $R$ -linear combination of the S-polynomials  $S(P_i, P_j)$ ,  $1 \leq i, j \leq k$ .

*Proof.* By induction on  $k$ .

Let  $l_i := \text{lc}(P_i)$ ,  $1 \leq i \leq k$ . Then  $\sum_{i=1}^k c_i l_i = 0$ .

It is sufficient to prove the existence of  $a_{ij}, b_{ij} \in R$  such that

$$\sum_{i=1}^k c_i P_i = \sum_{1 \leq i, j \leq k} (a_{ij} P_i - b_{ij} P_j) \quad \text{and} \quad a_{ij} l_i = b_{ij} l_j, \quad 1 \leq i, j \leq n.$$

For  $k = 2$  we have  $c_1 P_1 + c_2 P_2 = c_1 P_1 - (-c_2) P_2$  and  $c_1 l_1 = (-c_2) l_2$ .  
 $k = 3$ : Let  $l$  be a greatest common divisor of  $l_1, l_2, l_3$ . Since  $c_2 l_2 = -c_1 l_1 - c_3 l_3$ , a greatest common divisor of  $l_1$  and  $l_3$  divides  $c_2 l$ . Hence there are elements  $x_2, x_3 \in R$  such that  $c_2 l = x_1 l_1 + x_3 l_3$ .

Then  $d_1 := (-x_1 l_2 - c_1 l)/l$ ,  $d_2 := (-x_1 l_1)/l$ ,  $d_3 := (x_3 l_2)/l$  are elements of  $R$ . Furthermore, we have

$$\begin{aligned} (c_1 + d_1) l_1 &= d_2 l_2 \\ (c_2 + d_2) l_2 &= d_3 l_3 \\ (c_3 + d_3) l_3 &= d_1 l_1 \quad \text{and} \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^3 c_i P_i &= [(c_1 + d_1) P_1 - d_2 P_2] + [(c_2 + d_2) P_2 - d_3 P_3] \\ &\quad + [(c_3 + d_3) P_3 - d_1 P_1]. \end{aligned}$$

$$k > 3: \text{ Let } Q := \sum_{i=3}^k c_i P_i \quad \text{and} \quad m := \sum_{i=3}^k c_i l_i.$$

If  $m = 0$ , we can apply the induction hypothesis to  $Q$ .

If  $m \neq 0$ , by the  $k = 3$  case there are  $d_1, d_2, d_3 \in R$  such that

$$\begin{aligned} c_1 P_1 + c_2 P_2 + Q &= [(c_1 + d_1) P_1 - d_2 P_2] + [(c_2 + d_2) P_2 - d_3 Q] \\ &\quad + [(1 + d_3) Q - d_1 P_1] \end{aligned}$$

$$\text{and} \quad (c_1 + d_1) l_1 = d_2 l_2, \quad (c_2 + d_2) l_2 = d_3 m, \quad (1 + d_3) m = d_1 l_1.$$

Therefore, we can apply the induction hypothesis to  $(c_2 + d_2) P_2 - \sum_{i=3}^k d_3 c_i P_i$

and to  $-d_1 P_1 + \sum_{i=3}^k (1 + d_3) c_i P_i$  and thus terminate the proof.

*Remark.* If  $R$  is a field, the proof is trivial: Let  $l_i := \text{lc}(P_i)$  and

$$\begin{aligned} P'_i &:= (P_i / l_i), \quad 1 \leq i \leq k, \quad \text{then} \quad \sum_{i=1}^k c_i P_i = c_1 l_1 (P'_1 - P'_2) \\ &\quad + (c_1 l_1 + c_2 l_2) (P'_2 - P'_3) + \dots + \left( \sum_{i=1}^{k-1} c_i l_i \right) (P'_{k-1} - P'_k). \end{aligned}$$

3.5. THEOREM. Let  $J$  be an ideal of  $R[X]$  generated by a finite subset  $F \subseteq R[X] - \{0\}$ .

Then the following assertions are equivalent:

- (1)  $F$  is a Gröbner basis of  $J$ .
- (2) For all  $P, Q \in F$  a rest of  $S(P, Q)$  after division by  $F$  is zero.

*Proof.*

(1)  $\Rightarrow$  (2): Let  $P, Q \in F$ . Then  $S(P, Q)$  and its rest after division by  $F$  are elements of  $J$ . Therefore, this implication is a special case of proposition 2.5., (1)  $\Rightarrow$  (2).

(2)  $\Rightarrow$  (1): Let  $A \in J - \{0\}$ . We have to show that  $\text{in}(A) \in \langle \text{in}(F) \rangle$ . Since  $J$  is generated by  $F$ , there are elements  $c(\gamma, P) \in R$  such that  $A = \sum_{P \in F, \gamma \in \mathbb{N}^n} c(\gamma, P) X^\gamma P$ .

Let  $\delta := \max \{ \gamma + \deg(P) \mid c(\gamma, P) \neq 0 \}$  and  $L := \sum_{\substack{\gamma, P \\ \gamma + \deg(P) = \delta}} c(\gamma, P) X^\gamma P$ .

By lemma 1.3. we may assume that  $\delta$  is minimal, i.e.:

if  $A = \sum_{P \in F, \gamma \in \mathbb{N}^n} d(\gamma, P) X^\gamma P$  then  $\delta \leq \max \{ \gamma + \deg(P) \mid d(\gamma, P) \neq 0 \}$ .

Suppose that  $\deg(L) < \delta$ . Then the lemma above yields

$$L = \sum_{P, Q \in F, \alpha \in \mathbb{N}^n} a(\alpha, P, Q) X^\alpha S(P, Q), \quad a(\alpha, P, Q) \in R$$

(note that for  $\beta, \gamma \in \mathbb{N}^n$  there is an  $\alpha \in \mathbb{N}^n$  such that  $S(X^\beta P, X^\gamma Q) = X^\alpha S(P, Q)$ ).

But according to (2) the  $S$ -polynomials are admissible combinations of  $F$  and clearly the same holds for the  $X^\alpha S(P, Q)$ 's. Since their degree is smaller than  $\delta$ , this is a contradiction to the minimality of  $\delta$ . Hence  $\deg(L) = \delta$ . But then  $\text{in}(A) = \text{in}(L) \in \langle \text{in}(F) \rangle$ .

3.6. THEOREM. Let  $J$  be the ideal generated by  $F$ . Then a Gröbner basis of  $J$  can be constructed (in a finite number of steps) by the following algorithm:

$$F_0 := F$$

$$F_{i+1} := F_i \cup (\overline{\{S(P, Q) \mid P, Q \in F_i\}} - \{0\})$$

( $\overline{S(P, Q)}$  is a rest of  $S(P, Q)$  after division by  $F_i$ ). If  $F_i = F_{i+1}$ , then  $F_i$  is a Gröbner basis of  $J$ .

*Proof.* By the preceding theorem we only have to show that there is a  $k \in \mathbb{N}$  such that  $F_k = F_{k+1}$ .

If  $F_i \subset F_{i+1}$  then  $\langle \text{in}(F_i) \rangle \subset \langle \text{in}(F_{i+1}) \rangle$ . Since the strictly ascending sequence  $\langle \text{in}(F_0) \rangle \subset \langle \text{in}(F_1) \rangle \subset \dots$  must be finite, there is a  $k \in \mathbb{N}$  with  $F_k = F_{k+1}$ .

3.7. *Example.* Consider the graded lexicographic ordering on  $\mathbb{N}^2$  and

$$F := \{P_1 := 2X_1X_2^2 - X_1, P_2 := 3X_1^2X_2 - X_2\} \subseteq \mathbb{Z}[X_1, X_2].$$

Then

$$\begin{aligned} F_0 = F \quad \text{and} \quad S(P_1, P_2) &= 3X_1P_1 - 2X_2P_2 = -3X_1^2 + 2X_2^2 \\ &= \overline{S(P_1, P_2)} = : P_3. \end{aligned}$$

So

$$F_1 = \{P_1, P_2, P_3\} \quad \text{and} \quad \overline{S(P_1, P_2)}^{F_1} = 0,$$

$$\overline{S(P_1, P_3)}^{F_1} = 4X_2^4 - 3X_1^2 = : P_4, \quad \overline{S(P_2, P_3)}^{F_1} = 2X_2^3 - X_2 = : P_5.$$

Therefore  $F_2 = \{P_1, P_2, P_3, P_4, P_5\}$  and all rests after division by  $F_2$  of  $S$ -polynomials are 0. Hence  $F_2$  is a Gröbner basis of the ideal generated by  $F$ .

See figure 6.

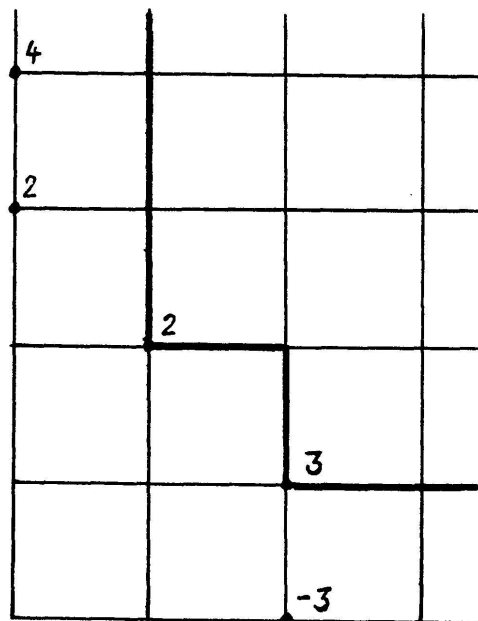


FIGURE 6.

3.8. *Remark.* Let  $G$  be a Gröbner basis of an ideal  $J$ . We shall say that  $G$  is "simplified" if all  $P \in G$  fulfill the following two conditions:

$$\text{lc}(P) \text{ generates the ideal } {}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle$$

and

$$\text{in}(P) \notin \langle \text{in}(G - \{P\}) \rangle.$$

It is easy to see that the elements of a simplified Gröbner basis have pairwise different degrees.

If  $R$  is a field then  $G$  is simplified iff the elements of  $G$  have pairwise different degrees and  $\deg(G)$  is the set of minimal elements (with respect to the natural partial ordering on  $\mathbb{N}^n$ ) in  $\deg(J)$ .

If  $G$  is not simplified, then in the following way we can construct (in a finite number of steps) a simplified Gröbner basis of  $J$ :

For every  $P \in G$  choose an admissible combination  $P'$  of  $G$  such that  $\deg(P) = \deg(P')$  and  $\text{lc}(P')$  generates the ideal

$${}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle.$$

Then  $G' := \{P' \mid P \in G\}$  is a Gröbner basis of  $J$ , since  $\langle \text{in}(J) \rangle = \langle \text{in}(G) \rangle \subseteq \langle \text{in}(G') \rangle \subseteq \langle \text{in}(J) \rangle$ .

If there is a  $P' \in G'$  with  $\text{in}(P') \in \langle \text{in}(G' - \{P'\}) \rangle$ , then  $G' - \{P'\}$  is a Gröbner basis, since then  $\langle \text{in}(G' - \{P'\}) \rangle = \langle \text{in}(G') \rangle = \langle \text{in}(J) \rangle$ .

Replace  $G'$  by  $G' - \{P'\}$ . After finitely many eliminations of this kind we obtain a simplified Gröbner basis.

In example 3.7. the Gröbner basis  $F_2$  is not simplified, since  $\text{in}(P_2) = -X_2 \text{in}(P_3)$  and  $\text{in}(P_4) = 2X_2 \text{in}(P_5)$ .  $\{P_1, P_3, P_5\}$  is a simplified Gröbner basis of the ideal generated by  $F_2$ .

#### 4. APPLICATION TO SYSTEMS OF ALGEBRAIC EQUATIONS

Let  $J$  be an ideal in  $R[X]$ , generated by a subset  $F \neq \{0\}$ .

4.1. We may consider  $F$  as a system of algebraic equations in  $n$  variables. We denote by  $K$  an algebraic closure of the quotient field of  $R$ .

Let  $Z(F)$  (resp.  $Z_K(F)$ ) be the set  $\{z \in R^n$  (resp.  $K^n$ )  $\mid P(z) = 0$  for all  $P \in F\}$  of common zeros in  $R^n$  (resp.  $K^n$ ) of the elements of  $F$ . Clearly  $Z(F) = Z(J)$  and  $Z_K(F) = Z_K(J)$ .