# §3. Loop Groups

$x$ is a *cut point* (with respect to $p$) if there is a geodesic from $p$ to $x$ that minimizes arc length up to $x$ but no further. The *cut locus* is the set of cut points. Similarly a vector $X$ in the tangent space $T_p$ is a tangent *cut point* if $\exp_p X$ is a cut point along the geodesic $\exp_p(tX)$. The *tangent cut locus* is the set of all such points in $T_p$, and is homeomorphic to the unit sphere in $T_p$. When $M = G/K$ we take $p = 1$.

(2.26) THEOREM. *Let $G/K$ be a simply-connected symmetric space, with $G$ simple. Then the tangent cut locus is precisely the K-orbit in $\mathfrak{m}$ of the outer wall of the Cartan simplex $\Delta_m$. It is therefore canonically identified with the topological building of the associated real form $G_{\mathbf{R}}$.*

As usual, the assumption $G$ simple is just for convenience. We sketch the proof: the first assertion is a fairly easy consequence of Theorem (1.8), and is left to the reader. Now consider the building $\mathscr{B}_{G_{\mathbf{R}}}$. It is a quotient space of $G_{\mathbf{R}}/B_{\mathbf{R}} \times \Delta_0 = K/C_K t_m \times \Delta_0$, where $\Delta_0$ is a simplex of dimension (rank $G/K$)-1; we take $\Delta_0$ to be the outer wall of $\Delta_{\mathfrak{m}}$. For each $I \leqslant S_{G/K}$, let $\Delta_I$ temporarily denote the corresponding face of $\Delta_0$; *i.e.* $\{X \in \Delta_0 : \alpha_i(x) = 0 \; \forall \, i \in I\}$. Then the $K$-orbit of $\Delta_0$ in $\mathfrak{m}$, $K\Delta_0$, is also a quotient of $K/C_K t_{\mathfrak{m}} \times \Delta_0$. The relations are $(k_1 X) \sim (k_2 X)$ if $X \in \mathring{\Delta}_I$ and $k_1 = k_2 \bmod K_I$. But $K_I = K \cap \mathcal{O}_I$, so these relations are identical to the ones that define the building. □

## § 3. LOOP GROUPS

Let $LG$, $LG_{\mathbf{C}}$ denote the free loop spaces. Let $G_{\mathbf{C}}$ denote the group of loops which are restrictions of regular maps $\mathbf{C}^* \to G_{\mathbf{C}}$, and let $L_{alg}G = L_{alg}G_{\mathbf{C}} \cap LG$. Thus if we fix an embedding $G_{\mathbf{C}} \subset GL(n, \mathbf{C})$, $L_{alg}G$ consists of the loops $f$ in $LG$ admitting a finite Laurent expansion $f(z) = \sum_{k=-m}^{m} A_k z^k$, whereas $L_{alg}G_{\mathbf{C}}$ consists of the loops $f$ in $LG_{\mathbf{C}}$ such that both $f$ and $f^{-1}$ admit finite Laurent expansions. We will also write $\tilde{G}_{\mathbf{C}}$ for $L_{alg}G_{\mathbf{C}}$. In fact $\tilde{G}_{\mathbf{C}}$ is the group of points over $\mathbf{C}[z, z^{-1}]$ of the algebraic group $G_{\mathbf{C}}$. Its Lie algebra is the loop algebra $\tilde{g}_{\mathbf{C}}$ of regular maps $\mathbf{C}^* \to g_{\mathbf{C}}$. The integer $m$ in the above Laurent expansion defines a filtration of $\tilde{G}_{\mathbf{C}}$ by finite dimensional subspaces; we give $\tilde{G}_{\mathbf{C}}$ the corresponding weak topology.

Let $P$ denote the subgroup of $\tilde{G}_{\mathbf{C}}$ consisting of regular maps $\mathbf{C} \to G_{\mathbf{C}}$ (i.e. maps with nonnegative Laurent expansion, or $G_{\mathbf{C}[z]}$), and let $\tilde{B}$ denote the Iwahori subgroup: $\{f \in P : f(0) \in B^-\}$. Finally, let $\tilde{N} = L_{alg}N_{\mathbf{C}}$, and recall that $\tilde{W}$ can be regarded as a "subgroup" of $\tilde{G}_{\mathbf{C}}$, since $R \leqslant \mathrm{Hom}(S^1, T) \leqslant L_{alg}T$. More precisely, we have $\tilde{N}/T_{\mathbf{C}} = \hat{W}$, and $\tilde{W} \subset \hat{W}$.

The *affine root system* $\tilde{\Phi}$ is the set $\mathbf{Z} \times \Phi$. It can be thought of as a set of affine linear functionals on $t$, but for our purposes it is just a device for encoding combinatorial information about the affine Weyl group and $\tilde{G}_{\mathbf{C}}$. In particular, to each $(n, \alpha) \in \tilde{\Phi}$ we associate a root subalgebra $X_{n,\alpha}$ of $\tilde{g}_{\mathbf{C}}$ consisting of the regular maps $\mathbf{C}^* \to X_\alpha$ homogeneous of degree $n$. These subalgebras are one—dimensional, and are precisely the nontrivial eigenspaces of the following $T^{l+1}$ action: The constant loops $T^l$ act in the obvious way, and the extra $S^1$ factor acts by rotating the loops. We also have root subgroups $U_{(n,\alpha)} = \exp X_{n,\alpha} \leqslant \tilde{G}_{\mathbf{C}}$. One can easily check that $\tilde{W}$ (acting by left conjugation) permutes the root subgroups. The resulting action of $\tilde{W}$ on $\tilde{\Phi}$ is given by $(w\lambda) \cdot (n, \alpha) = (n + \alpha(\lambda), w\alpha)$ for $\lambda \in \hom(S^1, T)$, $w \in W$. The various additional structures associated with ordinary root systems can be defined here as well. The positive roots $\tilde{\Phi}^+$ are the $(n, \alpha)$ with $n \geqslant 1$ or $n = 0$ and $\alpha < 0$ (note these correspond to the Iwahori subgroup $\tilde{B}$); the remaining roots are negative. As in the finite case, the length of an element $\sigma$ in $\tilde{W}$ is equal to the number of positive roots taken to negative roots by $\sigma$ (in particular this latter number is finite, as is clear anyway from the above formula for the $\tilde{W}$ action). The simple affine roots are defined as the set of elements of $\tilde{\Phi}^+$ which are indecomposable with respect to addition: $(m, \alpha) + (n, \beta) = (m+n, \alpha+\beta)$ (if $\alpha+\beta$ is a root). Hence the simple roots are $(0, -\alpha), \cdots (0, -\alpha_l)$ and $(1, \alpha_0)$.

To each root $(n, \alpha)$, we can also associate a "little $SL_2$" subgroup generated by $U_{n,\alpha}$ and $U_{-n,-\alpha}$. In particular $\tilde{G}_{\mathbf{C},i}$ is the subgroup corresponding to the $i$th simple affine root, $0 \leqslant i \leqslant l$. Thus $\tilde{G}_{\mathbf{C},i} = G_{\mathbf{C},i}$ if $i \neq 0$, and $\tilde{G}_{\mathbf{C},0}$ corresponds to $(1, \alpha_0)$. For example, if $G = SU(2)$, $\tilde{G}_{\mathbf{C},0}$ is the subgroup of matrices $\begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix}$ with $ad - bc = 1$. We let $\tilde{G}_i = \tilde{G}_{\mathbf{C},i} \cap LG$. Again $\tilde{G}_i = G_i$ if $i \neq 0$. Note that for all $i$, evaluation at $z = 1$ gives an isomorphism $\tilde{G}_i \xrightarrow{\cong} G_i \cong SU(2)$.

(3.1) THEOREM. *Assume $G$ is simply-connected. Then $(\tilde{G}_{\mathbf{C}}, \tilde{B}, \tilde{N}, \tilde{S})$ is a topological Tits system satisfying the four axioms of* § 2.

*Proof.* That $(\tilde{G}_{\mathbf{C}}, \tilde{B}, \tilde{N}, \tilde{S})$ is a Tits system in the ordinary sense is essentially due to Iwahori and Matsumoto [16]. (They work over a complete local field $K$; here we take $K$ to be the field of infinite Laurent series bounded below. It is not hard to get from the Chevalley group $G_K$ to $G_{\mathbf{C}[z,z^{-1}]} = \tilde{G}_{\mathbf{C}}$.) See also Kac and Peterson [17].

Clearly $\tilde{B}$ and $\tilde{N}$ are closed subgroups and $\tilde{W}$ is discrete. For Axiom (2.11) we need to show that if $\tilde{W}$ is an irreducible affine Weyl group, and $I$ is a proper subset of $\tilde{S}$, then $\tilde{W}_I$ is finite. This is obvious since the elements of $I$ have a common fixed point (i.e. the intersection of the corresponding reflection hyperplanes is nonempty). In Axiom (2.12) we take $A_s = \tilde{G}_s$. We have $\tilde{G}_s\tilde{B} = \tilde{G}_{\mathbf{C},s}\tilde{B} = \tilde{B} \quad U_s s\tilde{B} = P_s$. In particular $P_s/\tilde{B} = \tilde{G}_s/(\tilde{G}_s \cap \tilde{B}) \cong SU(2)/T = \mathbf{C}P^1$, which also proves Axioms (2.20) and (2.21). □

(3.2) COROLLARY. *$\Omega_{alg}G$ is a CW-complex with cells of even dimension, indexed by $\operatorname{Hom}(S^1, T)$. The Poincaré series for its integral homology is $\sum_{\lambda \in \operatorname{Hom}(S^1,T)} t^{2\bar{l}(\lambda)}$, where $\bar{l}(\lambda)$ is the minimal length accuring in $\lambda W$. Identifying $\operatorname{Hom}(S^1, T)$ with $\tilde{W}^S$, the closure relations on the cells are given by the Bruhat order on $\tilde{W}^S$.* □

*Remark.* An explicit formula for $\bar{l}(\lambda)$ is given in [16], Prop. 1.25: $\bar{l}(\lambda) = (\sum_{\alpha > 0} |\alpha(\lambda)|) - |\{\alpha > 0 : \alpha(\lambda) > 0\}|$.

We will also need the "Iwasawa decomposition" (see [17], [27], [29]):

(3.3) THEOREM. $\tilde{G}_{\mathbf{C}} = \Omega_{alg}G \times P$. □

*Remark.* Note that (3.3) shows that the associated building, which we will be denoted simply by $\mathscr{B}_G$, is a quotient of $L_{alg}G/T \times \Delta$. The equivalence relation is then $(f_1T, X) \sim (f_2T, X)$ if $X \in \mathring{\Delta}_I$ and $f_1 = f_2 \bmod LG \cap P_I$.

## § 4. QUILLEN'S THEOREM FOR LOOP GROUPS

In this section we will give Quillen's proof of the following theorem.

(4.1) THEOREM. *Let $G$ be a compact Lie group. Then the inclusion $\Omega_{alg}G \to \Omega G$ is a homotopy equivalence.*

If $G$ is simply connected, let $\mathscr{B}_G$ denote the topological building associated to the algebraic loop group $L_{alg}G_{\mathbf{C}}$ as in § 2.

(4.2) THEOREM (Quillen). *$\Omega_{alg}G$ acts freely on $\mathscr{B}_G$, with orbit space $G$.*

*Proof of (4.1).* It is easy to reduce to the case when $G$ is simply connected. Since $B_G$ is contractible by Theorem 2.16, we conclude at once from Theorem (4.2) that $\Omega_{alg}G \to \Omega G$ is a weak equivalence. Since both spaces have the homotopy type of a $CW$-complex, the map is in fact a homotopy equivalence. □