1. Some results on isoclinic n-PLANES in R^{2n}

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1. SOME RESULTS ON ISOCLINIC n-PLANES IN R^{2n}

By a Euclidean (vector) m-space R^m, where m is a positive integer, we mean an m-dimensional vector space provided with a positive definite inner product. An r-plane $(1 \leq r \leq m-1)$ in R^m is an r-dimensional vector subspace of R^m provided with the induced inner product. In R^m, length of a vector, angle between two vectors, orthogonality between a k-plane and an r-plane, (orthogonal) projection of a vector on an r-plane, orthonormal bases and rectangular coordinates are defined in the usual way.

In an R^{2n}, let $\mathbf{A}$, $\mathbf{B}$ be any two n-planes. Then we say that $\mathbf{A}$ is *isoclinic* with $\mathbf{B}$ at angle θ if the angle between every nonzero vector in $\mathbf{A}$ and its projection on $\mathbf{B}$ is always equal to θ. It turns out that if $\mathbf{A}$ is isoclinic with $\mathbf{B}$ at angle θ, then $\mathbf{B}$ is isoclinic with $\mathbf{A}$ at the same angle θ. Therefore, in this case, we shall say that $\mathbf{A}$ and $\mathbf{B}$ are isoclinic at angle θ, or simply, $\mathbf{A}$ and $\mathbf{B}$ are isoclinic.

A set Φ of n-planes in R^{2n} is said to be a *maximal set of mutually isoclinic n-planes* if every pair of n-planes in Φ are isoclinic and Φ is not contained in a larger set of mutually isoclinic n-planes. It is easy to see from definition that if $\mathbf{A}$ is isoclinic with $\mathbf{B}$ at angle θ, then its orthogonal complement $\mathbf{A}^{\perp}$ is isoclinic with $\mathbf{B}$ at angle $\frac{\pi}{2} - \theta$. Consequently, if Φ is any maximal set of mutually isoclinic n-planes in R^{2n} and $\mathbf{A} \in \Phi$, then $\mathbf{A}^{\perp} \in \Phi$.

In his memoir [8] Wong determined, for each n, the dimensions of the maximal sets of mutually isoclinic n-planes in R^{2n}, the number of non-congruent maximal sets of a given dimension, and explicit equations of the n-planes in any maximal set of mutually isoclinic n-planes containing a given n-plane.

In the following, we summarize some of his results related to the problem studied in this paper.

THEOREM 1.1. (Wong [8, pp. 25-26]). *In* R^{2n} *provided with a rectangular coordinate system* $(x, y) \equiv ([x_1 \dots x_n], [x_{n+1} \dots x_{2n}])$, *any maximal set* Φ *of mutually isoclinic n-planes containing the n-plane* $\mathbf{O}: y = 0$ *(and consequently, also the n-plane* $\mathbf{O}^{\perp}: x=0$*) is congruent to the set of n-planes with equations*

$$(1.1) \qquad x = 0\,, \quad \text{or} \quad y = x(\lambda_0+\lambda_1 B_1+\ldots+\lambda_{p-1}B_{p-1})\,,$$

where $(\lambda_0, \lambda_1, \ldots, \lambda_{p-1})$ *are* p *real parameters and* $(B_1, \ldots, B_{p-1})$ *is a maximal set of* $n \times n$ *matrices satisfying the Hurwitz matrix equations*

$$(1.2) \quad B_h + B_h^T = 0\,, \quad B_h^2 = -I\,, \quad B_hB_k + B_kB_h = 0\,(h, k=1, \ldots; h\neq k)\,.$$

Here, by $(B_1, \ldots, B_{p-1})$ being a maximal set of matrices satisfying equations (1.2), we mean that $(B_1, \ldots, B_{p-1})$ is not a subset of another set containing more matrices satisfying equations (1.2).

REMARK. It is of some historical interest that equations (1.2) first appeared in the literature in 1923 in connection with the famous problem of A. Hurwitz [4] on composition of quadratic forms, and then reappeared in 1961 in a very different type of problem. For more information about these equations, we refer the reader to Wong's memoir [8] and J. A. Tyrrell-J. G. Semple's book [6].

A maximal set of mutually isoclinic n-planes in R^{2n} is said to be p-dimensional (or, of dimension p), if it contains p parameters $\lambda_0, \lambda_1, \ldots, \lambda_{p-1}$ as in Theorem 1.1. It can be proved (cf. [8, p. 54]) that the dimension of a maximal set of mutually isoclinic n-planes in R^{2n} is always $\leqslant n$, and that there exist maximal sets of dimension n in R^{2n} if and only if $n = 2, 4$, or 8. Moreover, we have

THEOREM 1.2. (Wong [8, p. 57]). *Let* Φ *be a p-dimensional maximal set of mutually isoclinic n-planes in* R^{2n}. *Then, through any point in* $R^{2n}\backslash O$, *there passes at most one n-plane of* Φ. *In order that through any point in* $R^{2n}\backslash O$, *there passes exactly one n-plane of* Φ, *it is necessary and sufficient that* $n = p = 2, 4$, *or* 8.

THEOREM 1.3. (Wong [8, pp. 62-64]). *Any p-dimensional maximal set of mutually isoclinic n-planes in* R^{2n}, *if regarded as a submanifold of the Grassmann manifold of n-planes in* R^{2n}, *is diffeomorphic with the p-sphere* S^p.

Since the unit sphere S^{2n-1} in R^{2n} is intersected by an n-plane in a great $(n-1)$-sphere, a consequence of Theorems 1.2 and 1.3 is

THEOREM 1.4. (Wong [8, pp. 65-66]). *In* R^{2n}, $n = 2, 4$, *or* 8, *the intersection of the unit sphere* S^{2n-1} *by any n-dimensional maximal set of mutually isoclinic n-planes furnishes a fibering of* S^{2n-1} *by* S^{n-1} *over* S^n.

The above three theorems direct our attention to the three special cases $n = 2, 4, 8$, for which we now prove:

THEOREM 1.5.

(i) *For* $n = 2$, *every maximal real solution of the Hurwitz matrix equations (1.2) is orthogonally similar to the maximal solution* $\{B_1\}$ *where*

$$B_1 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}. \tag{1.3}$$

(ii) *For* $n = 4$, *every maximal real solution of the Hurwitz matrix equations (1.2) is orthogonally similar to the maximal solution* $\{B_1, B_2, B_3\}$ *where*

$$B_1 = \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}, \quad B_2 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix}, \quad B_3 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}. \tag{1.4}$$

(iii) *For* $n = 8$, *every maximal real solution of the Hurwitz matrix equations (1.2) contains either* 3 *or* 7 *matrices. In the latter case, it is orthogonally similar to the maximal solution* $\{B_1, \ldots, B_7\}$, *where*

$$B_1 = \begin{bmatrix} & 1 & & & & & & \\ -1 & & & & & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \end{bmatrix}, \quad B_2 = \begin{bmatrix} & & 1 & & & & & \\ & & & -1 & & & & \\ -1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & -1 & & & \\ & & & & & -1 & & \end{bmatrix}, \tag{1.5}$$

$$B_3 = \begin{bmatrix} & & & 1 & & & & \\ & & 1 & & & & & \\ & -1 & & & & & & \\ -1 & & & & & & & \\ & & & & & & & 1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & -1 & & & \end{bmatrix}, \quad B_4 = \begin{bmatrix} & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \\ -1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{bmatrix},$$

$$B_5 = \begin{bmatrix} & & & & & 1 & & \\ & & & & 1 & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \\ & -1 & & & & & & \\ -1 & & & & & & & \\ & & & -1 & & & & \\ & & 1 & & & & & \end{bmatrix}, \quad B_6 = \begin{bmatrix} & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & -1 & & & & & \\ & & & 1 & & & & \\ -1 & & & & & & & \\ & -1 & & & & & & \end{bmatrix},$$

$$B_7 = \begin{bmatrix} & & & & & & & 1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ & & & -1 & & & & \\ & & -1 & & & & & \\ & 1 & & & & & & \\ -1 & & & & & & & \end{bmatrix}.$$

Proof. This theorem is a reformulation of Theorem 8.1 in [8, pp. 107-109]. In fact, if we denote by C_i the matrices used in Theorem 7.2 in [8, pp. 54-56] and by U the diagonal matrix $(1, -1, \ldots, -1)$ of order n, then we can easily verify that $B_i = UC_iU^{-1}$.

An immediate consequence of Theorem 1.1 and 1.5 is the following

THEOREM 1.6.

(i) *In* R^4, *every maximal set of mutually isoclinic* 2*-planes is of dimension* 2 *and is congruent to the set* Φ_2 *consisting of the* 2*-plane* $x = 0$ *and the* 2*-planes* $y = xB(\lambda)$, *where*

$$(1.6) \qquad B(\lambda) \equiv \lambda_0 + \lambda_1 B_1 = \begin{bmatrix} \lambda_0 & \lambda_1 \\ -\lambda_1 & \lambda_0 \end{bmatrix}.$$

(ii) *In* R^8, *every maximal set of mutually isoclinic* 4*-planes is of dimension* 4 *and is congruent to the set* Φ_4 *consisting of the* 4*-plane* $x = 0$ *and the* 4*-planes* $y = xB(\lambda)$, *where*

$$(1.7) \qquad B(\lambda) \equiv \lambda_0 + \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 \\ -\lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 \end{bmatrix},$$

(iii) *In* R^{16}, *every maximal set of mutually isoclinic* 8-*planes is of dimension* 4 *or* 8. *Every maximal set of dimension* 8 *is congruent to the set* Φ_8 *consisting of the* 8-*plane* $x = 0$ *and the* 8-*planes* $y = xB(\lambda)$, *where*

$$B(\lambda) \equiv \lambda_0 + \lambda_1 B_1 + \ldots + \lambda_7 B_7 = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \\ -\lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 & \lambda_5 & -\lambda_4 & -\lambda_7 & \lambda_6 \\ -\lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 & \lambda_6 & \lambda_7 & -\lambda_4 & -\lambda_5 \\ -\lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 & \lambda_7 & -\lambda_6 & \lambda_5 & -\lambda_4 \\ -\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_5 & \lambda_4 & -\lambda_7 & \lambda_6 & -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_6 & \lambda_7 & \lambda_4 & -\lambda_5 & -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_7 & -\lambda_6 & \lambda_5 & \lambda_4 & -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix} \tag{1.8}$$

In (1.6), (1.7) and (1.8) above, the λ_0 in $\lambda_0 + \lambda_1 B_1 + \ldots$ stands for the scalar matrix $\lambda_0 I$.

REMARK. The maximal set Φ_n of mutually isoclinic n-planes in R^{2n} in Theorem 1.6 is congruent to that in Theorem 7.2 in [8, pp. 54-56] under the orthogonal transformation

$$f:(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{2n})$$
$$\to (x_1, -x_2, \ldots, -x_n, x_{n+1}, -x_{n+2}, \ldots, -x_{2n}),$$

which obviously leaves invariant the n-planes $\mathbf{O}: y = 0$ and $\mathbf{O}^{\perp}: x = 0$. To see this, let us denote by Ψ_n the maximal set of mutually isoclinic n-planes in Theorem 7.2 in [8, pp. 54-56] and write the equations of these n-planes as $x = 0$ and

$$y = x(\lambda_0 + \lambda_1 C_1 + \ldots + \lambda_{n-1} C_{n-1}).$$

Then f sends Ψ_n to the set $f(\Psi_n)$ of mutually isoclinic n-planes with equations $x = 0$ and

$$yU = xU(\lambda_0 + \lambda_1 C_1 + \ldots + \lambda_{n-1} C_{n-1}),$$

i.e.,

$$y = xU(\lambda_0 + \lambda_1 C_1 + \ldots + \lambda_{n-1} C_{n-1})U^{-1},$$

where U is the diagonal matrix $(1, -1, \ldots, -1)$ of order n. But, as we have seen in the proof of Theorem 1.5, these equations are the same as $x = 0$ and

$$y = x(\lambda_0 + \lambda_1 B_1 + \ldots + \lambda_{n-1} B_{n-1}).$$

Therefore, $f(\Psi_n)$ is the set Φ_n of mutually isoclinic n-planes in our Theorem 1.6.

2. Some further results

From now on we shall confine our attention to n-dimensional maximal sets of mutually isoclinic n-planes in R^{2n}, and therefore, n has always the values 2, 4, or 8 unless stated otherwise.

In this section, we prove a few more theorems for use in § 3. In these theorems, the indices a, b have the range of values $(0, 1, ..., n-1)$; $B_0 = I$ is the identity matrix of order n; $B_1, ..., B_{n-1}$ are the $n \times n$ matrices listed in Theorems 1.5 and 1.6; $\lambda = (\lambda_a)$ is an ordered set of n real parameters; and

$$B(\lambda) \equiv \sum_a \lambda_a B_a, \qquad N(\lambda) \equiv \sum_a \lambda_a^2 .$$

Moreover, for any matrix M, we denote its transpose by M^T.

Theorem 2.1.

(i) $B(\lambda)B(\lambda)^T = N(\lambda)I.$

(ii) *If* $\lambda \neq 0$, *then*

$$B(\lambda)^{-1} = B(\lambda)^T/N(\lambda) = \sum_a \lambda_a B_a^T/N(\lambda),$$

so that if $\lambda \neq 0$, *the equation* $y = xB(\lambda)$ *is equivalent to the equation* $x = yB(\mu)^T$, *where* $\mu = \lambda/N(\lambda) \neq 0$.

(iii) $$\det B(\lambda) = +\left(N(\lambda)\right)^{n/2}.$$

(iv) *If* $N(\lambda) = 1$, *then* $B(\lambda) \in SO(n)$, *where* $SO(n)$ *is the set of all orthogonal matrices of order* n *and determinant* $+1$.

Proof.
$$B(\lambda)B(\lambda)^T = \left(\sum_a \lambda_a B_a\right)\left(\sum_b \lambda_b B_b^T\right) = \sum_{a,b} \lambda_a \lambda_b B_a B_b^T$$
$$= \sum_a \lambda_a^2 B_a B_a^T + \sum_{a<b} \lambda_a \lambda_b (B_a B_b^T + B_b B_a^T),$$

which, on account of the Hurwitz matrix equations (1.2), is equal to $(\sum_a \lambda_a^2)I = N(\lambda)I$. This proves (i), and also (ii). To prove (iii), we first note that since $B(\lambda)$ is a square matrix of order n, $\det B(\lambda)$ is a homogeneous polynomial of degree n in the λ_a's, and it follows from (i) that

$$(\det B(\lambda))^2 = \det\left(B(\lambda)B(\lambda)^T\right) = \left(N(\lambda)\right)^n .$$