

## 2. A Topological Lemma

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **22.09.2024**

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from  $\{u \in C^\infty(X), \int u dX_g = 0\}$  to  $\{v \in C^\infty(X), \int v dX_{g'} = 0\}$  ( $dX_{g'}$  denotes the volume form in the metric  $g'$ ) when  $\lambda = 0$ .

For completeness, let us indicate how, for instance theorem 0.2, can be reduced to equation (1) with  $\lambda = 0$ . It is quite straightforward. First of all we are given a cohomology class  $c \in H^2(X, \mathbf{R})$  such that there exists a Kähler form  $\omega$  in  $c$ ; let  $\rho$  be the Ricci form of  $\omega$ :  $\rho \in C_1(X)$ , the first Chern class of  $X$ .

Then we are given  $\gamma \in C_1(X)$  and hence  $f \in C^\infty(X)$  a real function (defined up to an additive constant), which measures the deviation for  $\omega$  from satisfying 0.2:

$$\gamma - \rho = \sqrt{-1} \partial \bar{\partial} f.$$

Now we look for another Kähler form  $\omega' \in c$ , i.e. we look for a smooth real function  $\varphi$  (also defined up to an additive constant), where

$$\omega' - \omega = \sqrt{-1} \partial \bar{\partial} \varphi$$

such that the Ricci form  $\rho'$  of  $\omega'$  coincides with  $\gamma$ .

In other words, we want  $\varphi$  to satisfy

$$\rho' - \rho \equiv \sqrt{-1} \partial \bar{\partial} \varphi,$$

or equivalently, if  $g$  and  $g'$  are the Kähler metrics respectively associated with  $\omega$  and  $\omega'$ ,

$$\partial \bar{\partial} \{-\text{Log det}(g'g^{-1})\} \equiv \partial \bar{\partial} \varphi$$

which immediately yields equation (1) with  $\lambda = 0$ :

$$-\text{Log det}(g'g^{-1}) = f,$$

since anyway  $f$  is only defined up to an additive constant.

As  $\omega$  and  $\omega'$  are cohomologous and closed, so are the corresponding volume forms, therefore  $X$  has same volume measured with the metrics  $g$  and  $g'$ ; this defines completely  $f$ , subject to the natural constraint mentioned above.

## 2. A TOPOLOGICAL LEMMA

In our setting, the continuity method becomes a "surjectivity method" since it is based on the following

LEMMA 2.1. Let  $A, B$  be metric spaces, with  $A \neq \emptyset$  and  $B$  connected. Let  $P: A \rightarrow B$  be a continuous map. Assume:

- (i)  $P$  is open,
- (ii)  $P$  is proper, that is, for any compact subset  $K$  in  $B$ ,  $P^{-1}(K)$  is compact. Then  $P$  is surjective.

*Proof.* We only need to prove that  $P(A)$  is closed. Let  $b_0$  be a point in  $\overline{P(A)}$ . Since  $B$  is a metric space, there exists a sequence  $(b_i)_{i>0}$  in  $P(A)$  converging to  $b_0$ . The subset  $K = \{b_0, b_1, b_2, \dots\}$  is compact, hence so is  $PP^{-1}(K)$ . The latter contains  $b_1, \dots, b_i, \dots$ , hence  $b_0$ , and it is obviously contained in  $P(A)$ . Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

### 3. LOCAL INVERSION

THEOREM 3.1. Let  $X$  be a smooth compact manifold,  $V$  and  $W$  smooth vector bundles on  $X$ ,  $U$  an open set in  $C^\infty(X, V)$ , and  $P: U \rightarrow C^\infty(X, W)$ , a smooth nonlinear elliptic partial differential operator. Let  $A$  and  $B$  be LCFC submanifolds of  $U$  and of  $C^\infty(X, W)$  respectively, such that the restriction  $P_A$  of  $P$  to  $A$ , sends  $A$  into  $B$ . Then the Jacobian criterion holds for  $P_A$ , namely, if the derivative of  $P_A: A \rightarrow B$  is invertible at  $\varphi_0 \in A$ , then  $P_A$  is a local diffeomorphism near  $\varphi_0$ .

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the classical (Banach) inverse function theorem combined with *elliptic regularity*.

*Remark 3.2.* The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

### 4. PROPERNESS

In view of (2), theorem 3.1 implies that  $P_\lambda$  is open. We want to apply lemma 2.1 in order to prove that  $P_\lambda$  is surjective from  $A_\lambda$  to  $B_\lambda$ . Since  $P_\lambda(A_\lambda) \neq \emptyset$  (it contains 0), and since  $B_\lambda$  is connected, this amounts to proving that  $P_\lambda$  is *proper*. Let us explain why *a priori* estimates imply properness.

Concerning subsets in  $A_\lambda$  we have