

# Introduction

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## GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

by Sandra HAYES and Geneviève POURCIN

### INTRODUCTION

A fundamental tool in the theory of complex manifolds  $X$  is Riemann's Theorem on Removable Singularities of holomorphic functions which ensures that all functions holomorphic outside of a rare analytic subset of  $X$  and locally bounded on  $X$  can be extended to functions holomorphic on all of  $X$ . In other words, all weakly holomorphic functions on  $X$  are actually holomorphic. Although this theorem does not hold for arbitrary complex spaces, Oka [12] showed in 1951 that every complex space  $X$  can be modified to a complex space  $\tilde{X}$  for which Riemann's continuation theorem is valid, the so-called normalization of  $X$ .

Stein spaces  $X$  are complex spaces which can be completely described by the algebra  $\mathcal{C}(X)$  of global holomorphic functions. Since a complex space is Stein if and only if its normalization is Stein [11], it is natural to ask if the normalization  $\tilde{X}$  of a Stein space  $X$  can be constructed just from the holomorphic functions on  $X$ . Phrased differently, the question is whether the algebra  $\mathcal{C}(\tilde{X})$  of all holomorphic functions on  $\tilde{X}$  or equivalently, the algebra  $\tilde{\mathcal{C}}(X)$  of all weakly holomorphic functions on  $X$ , can be derived from the algebra  $\mathcal{C}(X)$  of holomorphic functions on  $X$ .

The purpose of this paper is to demonstrate that this is possible when  $X$  is irreducible:  $\tilde{\mathcal{C}}(X)$  is the topological closure of the integral closure  $\widetilde{\mathcal{C}(X)}$  of  $\mathcal{C}(X)$ . An example given in § 1 shows that  $\widetilde{\mathcal{C}(X)}$  is not in general topologically closed even if  $X$  is locally irreducible.  $\tilde{\mathcal{C}}(X)$  can also be obtained by taking the intersection of the localizations  $S_x^{-1} \widetilde{\mathcal{C}(X)}$  of the integral closure  $\widetilde{\mathcal{C}(X)}$  of  $\mathcal{C}(X)$  with respect to  $S_x := \{g \in \mathcal{C}(X) : g(x) \neq 0\}$  for every  $x \in X$  (see § 3).

The proof relies on an analytic and an algebraic theorem, namely Rossi's theorem [13] generalizing the Remmert quotient and the integral closure theorem of Mori-Nagata [7].

An analytic consequence of the construction presented here is that the normalization  $\tilde{X}$  of an irreducible Stein space  $X$  is  $\widetilde{\mathcal{O}(X)}$ -convex,  $\widetilde{\mathcal{O}(X)}$ -separable and has local coordinates by functions in  $\widetilde{\mathcal{O}(X)}$ . Some algebraic results are that  $\mathcal{O}(\tilde{X})$  is completely normal and that the two algebras  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  are always locally equal, i.e. their localizations at all maximal ideals in  $\mathcal{O}(X)$  are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

### 1. EXAMPLE OF A STEIN SPACE $X$ WITH $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let  $(X, \mathcal{O})$  be a complex space with normalization  $\pi: \tilde{X} \rightarrow X$ . Since  $\pi$  is surjective, the map  $\pi^*: \mathcal{O}(X) \rightarrow \mathcal{O}(\tilde{X})$ ,  $f \mapsto f \circ \pi$ , is injective and the holomorphic functions  $\mathcal{O}(X)$  on  $X$  can be considered to be a subring of the holomorphic functions  $\mathcal{O}(\tilde{X})$  on the normalization  $\tilde{X}$  of  $X$ ; this will be indicated by  $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$ . If  $X$  is irreducible and Stein, then  $\mathcal{O}(\tilde{X})$  contains the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  but does not always coincide with it, as will be shown in this section.

For an irreducible complex space  $(X, \mathcal{O})$ , the integral domain  $\mathcal{O}(X)$  is said to be *normal*, if it is integrally closed in its field of fractions  $Q(\mathcal{O}(X))$ , i.e.  $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$ . Recall that  $Q(\mathcal{O}(X))$  is the field of meromorphic functions  $M(X)$  on  $X$  when  $X$  is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras  $M(X)$  and  $M(\tilde{X})$  are isomorphic for every complex space  $X$  [8, p. 161].

The following characterization of normal irreducible Stein spaces  $X$  by their global function algebra  $\mathcal{O}(X)$  is essentially contained in [2, § 1, p. 35].

**THEOREM 1.** *An irreducible Stein space  $X$  is normal if and only if the integral domain  $\mathcal{O}(X)$  is normal.*

An analysis of the proof shows that even when  $X$  is just irreducible and normal,  $\mathcal{O}(X)$  is also normal. Theorem 1 implies

**COROLLARY 1.** *For an irreducible Stein space  $X$  with normalization  $\tilde{X}$ , the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  is contained in  $\mathcal{O}(\tilde{X})$ .*

The following example shows that there are functions  $f \in \mathcal{O}(\tilde{X})$  which are not integral over  $\mathcal{O}(X)$ . In this example,  $X := (\mathbb{C}, \mathcal{O})$  is an irreducible