

# 1. Geometry of the unit tangent bundle

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The paper is organized into the following sections:

1. *Geometry of the unit tangent bundle.* We describe the metric in two ways, and when the base space is a round sphere, we see that geodesics in its unit tangent bundle project to spherical helices on the sphere.
2. *Geodesics in  $US^2$ .* Some of the phenomena show up in this case.
3. *Helices in  $S^3$ .* Frenet equations, curvature, torsion and writhe.
4. *Sasaki's equations.* A general calculus for geodesics in the unit tangent bundle  $UM$  of any Riemannian manifold  $M$ .
5. *Proof of the Fundamental Constraint.* A blend of the Sasaki and Frenet equations.

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## 1. GEOMETRY OF THE UNIT TANGENT BUNDLE

Let  $M$  be an  $n$ -dimensional Riemannian manifold, and  $(p(t), v(t))$  a path in its unit tangent bundle  $UM$ . It is customary to give  $UM$  the Riemannian metric in which arc length  $s(t)$  along this path is given by the formula

$$s'(t)^2 = |p'(t)|^2 + |v'(t)|^2,$$

where

$p'(t)$  = tangent vector to the curve  $p(t)$  in  $M$ ,

$v'(t)$  = covariant derivative of  $v(t)$  along  $p(t)$  in  $M$ ,

and the norms of these vectors are measured in the given Riemannian metric on  $M$ .

When  $M$  is flat, and hence parallel translation is independent of path, the above metric on  $UM$  is simply the product metric of  $M \times S^{n-1}$ . So the constant speed geodesics in  $UM$ , for example, are just the paths  $(p(t), v(t))$  for which  $p(t)$  and  $v(t)$  are themselves constant speed geodesics in their respective spaces. In particular, each geodesic in  $UM$  certainly projects to a geodesic in  $M$ .

But when  $M$  is curved, the story is quite different. A geodesic in the unit tangent bundle  $UM$  need not project to a geodesic in  $M$ . We can already see this when  $M$  is a round two-sphere.

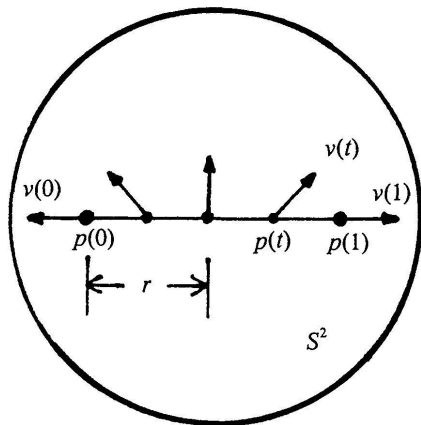


FIGURE 1

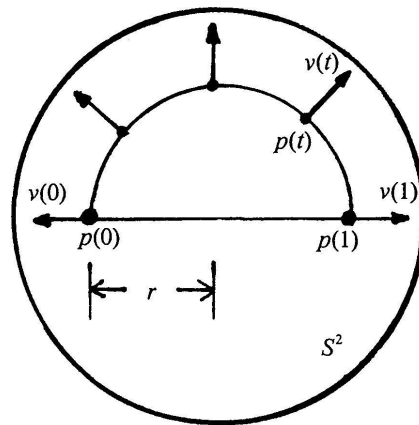


FIGURE 2

In each of Figures 1 and 2, we depict a path  $(p(t), v(t))$  in the unit tangent bundle  $US^2$  of a round two-sphere  $S^2$  of radius 1. Though the paths are different, their initial points are the same and their terminal points are the same.

In the first path, the point  $p(t)$  travels at constant speed along a geodesic of length  $2r$  on  $S^2$ . At the same time the tangent vector  $v(t)$  rotates at constant speed with respect to a parallel coordinate frame, turning through a total angle  $\pi$  from beginning to end. The length of this path  $(p(t), v(t))$  is

$$\sqrt{\pi^2 + 4r^2}.$$

If the base space were  $R^2$  instead of  $S^2$ , this path in the unit tangent bundle would be a geodesic, indeed a shortest connection between its endpoints.

In the second path, the point  $p(t)$  travels at constant speed along a semicircle of length  $\pi \sin r$ . At the same time the tangent vector  $v(t)$  rotates at constant speed with respect to a parallel coordinate frame, turning through a total angle somewhat less than  $\pi$  because of the curvature in the base space  $S^2$ . The savings is half of the area  $2\pi(1 - \cos r)$  inside the small circle. Hence the total angle that  $v(t)$  turns through is  $\pi \cos r$ . It follows that the length of this second path  $(p(t), v(t))$  is  $\pi$ .

So the second path is shorter than the first. Indeed, it is a minimizing geodesic in  $US^2$  between its endpoints, whose distance apart is therefore  $\pi$ .

Yet its projection on the base space  $S^2$  is a small circle, not a geodesic.

Immediately one asks: *which curves on  $S^n$  are projections of geodesics in  $US^n$ ?*

In answering this, we use another approach to the geometry of  $US^n$ , viewing it as the homogeneous space  $SO(n+1)/SO(n-1)$ . Here, the special orthogonal group  $SO(n+1)$  is given the usual bi-invariant Riemannian metric, and then the inner products in directions orthogonal to the cosets of  $SO(n-1)$  are transferred to the coset space  $SO(n+1)/SO(n-1)$ . This makes the projection map from  $SO(n+1)$  to  $US^n$  a Riemannian submersion. We leave it as an exercise to show that this Riemannian metric on  $US^n$  coincides with the one described earlier.

A geodesic in  $SO(n+1)$  which starts out orthogonal to one of the cosets of  $SO(n-1)$  remains orthogonal to all the cosets, and projects to a geodesic in  $SO(n+1)/SO(n-1) = US^n$ . Furthermore, all the geodesics in  $US^n$  are obtained this way.

Suppose, for example, that  $n = 3$ . If  $(p(t), v(t))$  is a geodesic in  $US^3$ , then by the above, there must be a geodesic  $h(t)$  through the identity in  $SO(4)$  such that

$$h(t)(p(0)) = p(t) \quad \text{and} \quad h(t)(v(0)) = v(t).$$

But every such geodesic  $h(t)$  in  $SO(4)$  consists of independent, constant speed rotations in a pair of orthogonal two-planes in four-space. Hence  $p(t)$  travels along a spiral on an invariant torus, that is, along a spherical helix.

Notice that the isometry  $h(t)$  which takes  $p(0)$  to  $p(t)$  and  $v(0)$  to  $v(t)$ , also takes the entire helix  $\{p(t)\}$  to itself. Hence it takes the Frenet frame of the helix at  $p(0)$  to the Frenet frame at  $p(t)$ . It follows that

$$v(t) = aT(t) + bN(t) + cB(t)$$

has constant coefficients with respect to this Frenet frame.

Beyond  $S^3$ , nothing new happens for geodesics: it is easy to see that every geodesic in  $US^n$  lies inside a totally geodesic submanifold  $US^3$ . Indeed, if  $(p, v)$  and  $(q, w)$  are nearby points on the geodesic, then the vectors  $p, v, q$  and  $w$  determine the corresponding  $S^3$ .

When it comes to proving the Fundamental Constraint, we will capitalize on this observation by restricting our attention to  $S^3$ .

We conclude: *the only curves on  $S^n$  which can be projections of geodesics on  $US^n$  are spherical helices (allowing great and small circles and points as special cases) which lie on great 3-spheres. All such spherical helices will appear in this way.*